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Part II.

Decision support system for plant variety testing in the Czech Republic

Václav Adamec[1], David Hampel[2], Jitka Janová[3]

Abstract. Plant variety testing through field experiments and subsequent statistical evaluations provide necessary information for variety entry in national and international registers. A newly registered variety must possess unique properties making it distinct from other already registered varieties. It must be uniform, stable and provide original characteristics, which set it above the current level of production potential. Qualified decision-making about variety registration is conditioned upon availability of relevant information. The current paper describes the process of creating a support system for decision-making (DSS) to assist planning and evaluations of field experiments according to guidelines set by the National Plant Variety Office of Central Institute for Supervising and Testing in Agriculture. The decision support system shall represent a powerful instrument to provide assistance in the area of quantitative analysis, a part of the variety testing and evaluations. DSS generates optimum design of field experiments through suitable placement of varieties in field trial plans and secures the follow-up statistical analysis of the experimental results. The paper describes statistical and mathematical methods implemented in original or modified form and focuses on modules of the decision support system as a whole. Issues of validating the implemented algorithms are discussed. The DSS system was implemented in the MATLAB language and supplied with suitable user interface.

Keywords: decision support system, field experiments, NNA, plant variety testing, REML.

JEL classification: C13
AMS classification: 65C20

1 Introduction

During the next 50 years, new challenges are expected to appear in the sector of agriculture: increase food production, reduce soil and ecosystem degradation, efficiently use and protect water resources and cut emissions of greenhouse gases. Developing new plant varieties can help tackle these tasks. It can be accomplished by intensive breeding with the objective to bring new plant varieties with better resistance, higher yield and lower need of fertilizers, water and energy.

Listing in National Register is a legal requirement for new varieties of main plant species, which seeks to ensure that no variety can be marketed unless it is genuinely new, and it constitutes an improvement to the varieties already being marketed. For farmers, the National Listing guarantees value for cultivation, quality of the planting material, and also safety to human health, animals, plants and the environment. Legal framework for variety listing in National Register is provided by the International Treaty on Legal Protection of New Plant Varieties of 1991. In the EU it is determined by EC directive No. 2100/94 and in the Czech Republic by Act No. 408/2000 Coll., Act No. 147/2002 Coll. and Act No. 219/2003 Coll.

Authorities responsible for protecting plant variety rights are the Central Institute for Supervising and Testing in Agriculture (ÚKZÚZ) and the Ministry of Agriculture (MA). Testing and certification of

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newly developed or imported plant varieties in the Czech Republic is carried out by the National Plant Variety Office of the Central Institute for Supervising and Testing in Agriculture (NOÚ ÚKZÚZ), an organization of the MA. Objective of testing and certification is to verify that a new variety is distinct from others, uniform in its characteristics and stable in the long run. Impartial variety testing provides qualitative and quantitative information essential for entry in the National Register.

Administration of modern certification system, independent testing, advice on the varieties and identification of the most profitable planting material for farmers creates on the side of the ÚKZÚZ the need for efficient system capable of handling experimental design, processing primary data coming from the testing stations and performing statistical analyses and evaluations of test data. The objective of this project is development of computerized decision support system (DSS) generating design of variety trials, speedy statistical analyses of experimental data and presentation of the results in user-friendly format (Excel). DSS must be able to provide quality analyses of large datasets via graphical user interface and offer reliable transport of results to the end user. Open code must permit updates of the statistical procedures, if required. The DSS modules are programmed in modern MATLAB language and computing environment, thus allowing full utilization of computing power in current computer hardware.

The DSS system is expected to assist the ÚKZÚZ in the process of awarding registrations and breeder's rights of the varieties, provide valuable information about specific varieties to the farmers, the public and the companies processing and marketing agricultural commodities and recommend novel domestic and foreign crop varieties to be cultivated and marketed. Inevitably, DSS will promote economic viability of Czech farmers in the EU agricultural market and increase reputation of the Czech testing and certification system. DSS may also assist the MA in implementation of the country's and EU policies in agriculture.

DSS can be accessed via graphical user interface. The modules provide planning of field experiments, evaluation of a single trial or entire experimental series. As a result, the DSS modules can be used on data from several years or applied to isolated experiments. Main parts of the DSS are shown in Fig. 1.

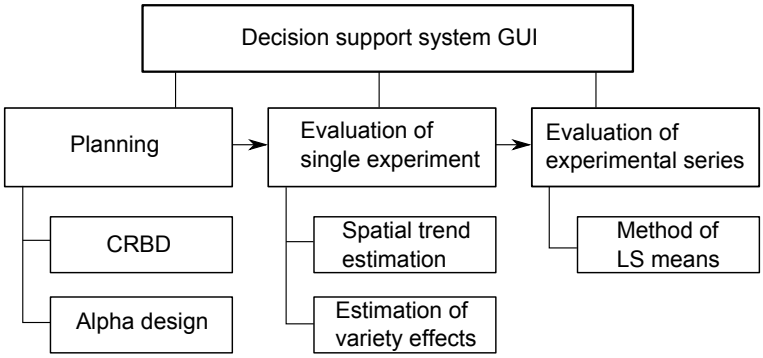

Figure 1 Modules of the DSS system

A typical application of DSS begins with construction of appropriate experimental design for several testing stations and a calendar year by a cultivar analyst. The experimental design is either the Completely Randomized Block Design (CRBD) or the Alpha Design [3], depending on the number of varieties in test. Following the field trial completion, the analyst enters the data and analyzes individual experiments. Approximate impact of soil heterogeneity on the crop yields is revealed by the Nearest Neighbor Analysis (NNA) of spatial trend [12]; final estimates of variety effects are obtained by AI-REML [2]. The analyst then combines estimates from the current year with historical results and evaluates the whole experimental series by the method of LS means. The output of the Fisher's Least Significant Difference (LSD) then allows to make inferences about means of specific varieties, testing stations, calendar years and testing systems. The following text focuses on estimation of spatial trend by the NNA and mixed model analysis of trial data by AI-REML with subsequent pair-wise comparisons.

2 Nearest Neighbor Adjustment for spatial trend

The NNA method of adjusting the crop variety yields for spatial heterogeneity of soil and water environment, i.e. spatial trend, was implemented in the second module. Computing algorithm of the Wilkinson's NNA method was described by Stroup *et al.* [11], who also provided own implementation in SAS code.

The Stroup's realization of NNA mimics functionality of the ANOFT software programmed in Visual Basic 3.0 by Erik Schwarzbach [8] in early 1980s.

In theory, identical performance of the named NNA variants is anticipated in crop yield adjustment for environmental heterogeneity and estimation of local environmental trend used for construction of the trend map. The trend map of the field experiment displays the estimated effects of the local soil and environmental trends on the observed crop yield. It is prepared in tabular and graphical form.

In the field experiment, the plots of rectangular shape are arranged linearly in block design. The varieties are assigned to the plots according to prepared design [3], a single variety per row. A variety must not be placed in adjacent plots nor in marginal plots only, since this arrangement would prevent separation of the variety effects on the observed yield from the environmental trend by statistical methods and void the whole experiment. The scheme in Fig. 2 illustrates the customary arrangement of the testing plots in the field experiment with three rows; y_{ijk} denotes observed yield of the k-th variety in i-th row and j-th column; e_{ijk} denotes deviation of the observed k-th variety yield in i-th row and j-th column from the mean yield of the k-th variety.

		$e_{i-1,j}$		
	$e_{i,j-1}$	y_{ijk}	$e_{i,j+1}$	
		$e_{i+1,j}$		

Figure 2 Arrangement of plots in the testing experiment

The NNA analysis by Wilkinson can be described in the following steps:

1. Firstly, arithmetic means of crop yield $\bar{y}_{..k}$ are calculated for every variety included in the testing experiment. Then, deviations of observed yield y_{ijk} from the mean yield of the corresponding k-th variety are received for all plots in the testing trial by the formula $e_{ijk} = y_{ijk} - \bar{y}_{..k}$.
2. Two auxiliary spatial trend variables are formed from the deviated observations. Within the i-th row, averaging two neighboring deviations produces variable κ for the observed yield y_{ijk}. For the marginal observations, means are obtained from two closest deviations within the same row. Variable κ is a reflection of the spatial trend within a row. It is expected to exert significant influence on the observed yield. The second variable ω is derived for observations of the middle rows by averaging two neighboring deviations from the rows on the right and left side of the observation y_{ijk}. This ω variable reflects the spatial trend of observations in the same column. Column trend is assumed to have marginal effect on the results of crop testing.
3. An auxiliary model of ANCOVA with no intercept $y_{ijk} = \alpha_k + \beta_1\kappa_{ij} + \beta_2\omega_{ij} + \varepsilon_{ijk}$ is constructed in the next step. It describes variability of observed yield y_{ijk} as a function of variety α_k, spatial trend κ_{ij} within a row, local trend ω_{ij} pertaining to column and random disturbance ε_{ijk}.
4. Estimated variety yields $\hat{\alpha}_k$ partially corrected for local trends and estimated errors $\hat{\varepsilon}_{ijk}$ are extracted there from. Error sum of squares (ESS) is calculated.
5. In the iterative variant of NNA, deviations of observed yield y_{ijk} from estimated expected yield of the varieties from the preceding round of iteration are calculated. Points 2 through 4 are then repeated until change in ESS from two subsequent iterations becomes negligible.

After convergence, estimated coefficients of variety yields from the Wilkinson method corrected for local trend can be compared to variety estimates from ANOFT software by paired t-test. In addition, F-test of concurrent restrictions for parameters of linear calibration model $Y_{NNA} = \beta_0 + \beta_1 Y_{ANOFT} + \varepsilon$ with $H_0 : \beta_0 = 0$ and simultaneously $\beta_1 = 1$, described in [1], can verify similarity of variety yield estimates received from the two methods.

Upon convergence, estimated deviations of spatial trend for rows and columns associated with the experimental plots are expressed on the relative scale. The diagram of the trend map then depicts the local trend deviations exceeding 5 % of the overall mean in absolute value. The authors' implementation of NNA in MATLAB was validated on datasets provided by the recipient institution. Testing data available from ÚKZÚZ were affected by severe, medium-size and near absent environmental trends. Estimates of trend-adjusted yield and trend map from the authors' NNA software and the ANOFT program by Erik Schwarzbach were available to evaluate performance of NNA applied in the current software.

3 Mixed model estimation by Average Information REML

The estimation procedure in linear models with fixed and random effects includes interconnected processes of estimating the fixed effects, predicting the random effects and estimation of variance components. Estimation of the fixed effects is often realized by OLS or GLS; prediction of random effects is based on theory developed early by C. R. Henderson and it is referred to as the Best Linear Unbiased Prediction (BLUP). Estimation of variance components can be realized by the method of Maximum Likelihood (ML), see [4], which in 1960s replaced Henderson's methods I, II and III. A few years later, the Residual Maximum Likelihood (REML) substituted ML, and currently, it is the superior method.

Theoretical foundations of REML can be traced back to Patterson and Thompson's paper [6]. REML maximizes the residual (restricted) likelihood derived from the error contrasts after filtering out the fixed effects by OLS or GLS. REML method corrects for the bias in ML estimates of variance components by accounting for the degrees of freedom attributed to the fixed effects. Residual Maximum Likelihood is a nonlinear function of the variance components; therefore, iterative algorithms must be applied to estimate the variances.

Computational algorithms of REML can be either derivative (based on first or second order derivatives of the natural logarithm of likelihood) or derivative-free (DF-REML), which use other principles. Average Information REML (AI-REML) belongs to the second-order derivative methods along with method of Newton-Raphson (NR) and Fisher-Scoring (FS). In estimation of variance components, however, AI-REML uses the average information matrix. Among the second-order derivative methods, AI-REML is the most computationally effective, especially when applied to large datasets. Obtaining estimated variance components in AI-REML requires less iteration in comparison with other derivative methods. For its favorable properties, AI-REML algorithm has been implemented in many software products, e.g. AS-REML or GenStat [2], [10].

Symbol n designates sample size; p indicates number of unique levels of varieties; k denotes number of random effects excluding error, and q_m denotes number of levels of the m-th random effect. We start with model equation with fixed effects of varieties and random effects of blocks and replications

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Z}\boldsymbol{u} + \boldsymbol{\varepsilon},$$

where $\boldsymbol{y}$ is column vector storing observations of the crop yield in experimental plots $n \times 1$, $\boldsymbol{X}$ is full column rank design matrix for the fixed effects of varieties without intercept $n \times p$, $\boldsymbol{\beta}$ is vector of fixed effects parameters for varieties $p \times 1$, $\boldsymbol{Z}$ is design matrix for random effects of blocks and replications without constant $n \times (q_1 + q_2)$, $\boldsymbol{u}$ is vector of random parameters of blocks and replications $(q_1 + q_2) \times 1$, $\boldsymbol{\varepsilon}$ is column vector of random disturbances $n \times 1$.

Expectation of the dependent variable is $E(\boldsymbol{y}) = \boldsymbol{X}\boldsymbol{\beta}$. Covariance matrix of the dependent variable is $D(\boldsymbol{y}) = \boldsymbol{H} = \boldsymbol{Z}\boldsymbol{G}\boldsymbol{Z}^T + \boldsymbol{R}$, where $\boldsymbol{G} = \bigoplus_{m=1}^{k} \sigma_m^2 \boldsymbol{I}_{q_m}$ is a block-diagonal covariance matrix[1] of other than error random effects and $\boldsymbol{R}$ is covariance matrix of the error terms. The design matrix $\boldsymbol{Z}$ for non-error random effects can be partitioned to matrices of the random effects, i.e. $\boldsymbol{Z} = [\boldsymbol{Z}_1 \boldsymbol{Z}_2]$. Multivariate distribution of the random effects is assumed

$$\begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{\varepsilon} \end{bmatrix} \sim N\left(\boldsymbol{0}, \begin{bmatrix} \boldsymbol{G} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R} \end{bmatrix}\right).$$

Absorption matrix $\boldsymbol{P}$ is defined in the form

$$\boldsymbol{P} = \boldsymbol{H}^{-1} - \boldsymbol{H}^{-1}\boldsymbol{X}\left(\boldsymbol{X}^T\boldsymbol{H}^{-1}\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T\boldsymbol{H}^{-1}.$$

Natural logarithm of the restricted likelihood for linear model with fixed and random effects (mixed model) can be written

$$\ell = -\frac{1}{2}\left(\ln|\boldsymbol{X}^T\boldsymbol{H}^{-1}\boldsymbol{X}| + \ln|\boldsymbol{H}| + (n-p)\ln\sigma_0^2 + \boldsymbol{y}^T\boldsymbol{P}\boldsymbol{y}/\sigma_0^2\right) + c.$$

Assuming vector of variances of random effects to be estimated is $\boldsymbol{\kappa} = (\sigma_0^2, \sigma_1^2, \sigma_2^2)$. Symbol σ_0^2 indicates error variance, σ_1^2 and σ_2^2 denote variances of blocks and replications, respectively and c is a constant.

[1] Note, that $\boldsymbol{A} \oplus \boldsymbol{B} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{B} \end{bmatrix}$ is the direct sum of $\boldsymbol{A}$ and $\boldsymbol{B}$.

For mixed model with three random effects of error, block and replication, the first partial derivatives of log-likelihood with respect to variance components can be written [2]

$$\frac{\partial \ell}{\partial \sigma_0^2} = -\frac{1}{2}\left((n-p)/\sigma_0^2 - \boldsymbol{y}^T\boldsymbol{P}\boldsymbol{y}/\sigma_0^4\right),$$
$$\frac{\partial \ell}{\partial \sigma_1^2} = -\frac{1}{2}\left(tr(\boldsymbol{P}\boldsymbol{Z}_1\boldsymbol{Z}_1^T) - \boldsymbol{y}^T\boldsymbol{P}\boldsymbol{Z}_1\boldsymbol{Z}_1^T\boldsymbol{P}\boldsymbol{y}/\sigma_0^2\right),$$
$$\frac{\partial \ell}{\partial \sigma_2^2} = -\frac{1}{2}\left(tr(\boldsymbol{P}\boldsymbol{Z}_2\boldsymbol{Z}_2^T) - \boldsymbol{y}^T\boldsymbol{P}\boldsymbol{Z}_2\boldsymbol{Z}_2^T\boldsymbol{P}\boldsymbol{y}/\sigma_0^2\right).$$

$\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ are the partial design matrices for random blocks and replications, respectively. In the next step, elements of the observed information matrix $-\frac{\partial^2 \ell}{\partial \kappa_i \partial \kappa_j}$ in Newton-Raphson or the expected information matrix $-E\left(\frac{\partial^2 \ell}{\partial \kappa_i \partial \kappa_j}\right)$ in Fisher-Scoring must be initiated. For estimating the mixed model by AI-REML, "average" information matrix $\boldsymbol{A}$ is assembled from elements

$$\boldsymbol{A}(\sigma_0^2,\sigma_0^2) = \frac{1}{2}\boldsymbol{y}^T\boldsymbol{P}\boldsymbol{y}/\sigma_0^6, \qquad \boldsymbol{A}(\sigma_0^2,\sigma_1^2) = \frac{1}{2}\boldsymbol{y}^T\boldsymbol{P}\boldsymbol{Z}_1\boldsymbol{Z}_1^T\boldsymbol{P}\boldsymbol{y}/\sigma_0^4,$$
$$\boldsymbol{A}(\sigma_0^2,\sigma_2^2) = \frac{1}{2}\boldsymbol{y}^T\boldsymbol{P}\boldsymbol{Z}_2\boldsymbol{Z}_2^T\boldsymbol{P}\boldsymbol{y}/\sigma_0^4, \qquad \boldsymbol{A}(\sigma_1^2,\sigma_1^2) = \frac{1}{2}\boldsymbol{y}^T\boldsymbol{P}\boldsymbol{Z}_1\boldsymbol{Z}_1^T\boldsymbol{P}\boldsymbol{Z}_1\boldsymbol{Z}_1^T\boldsymbol{P}\boldsymbol{y}/\sigma_0^2,$$
$$\boldsymbol{A}(\sigma_1^2,\sigma_2^2) = \frac{1}{2}\boldsymbol{y}^T\boldsymbol{P}\boldsymbol{Z}_1\boldsymbol{Z}_1^T\boldsymbol{P}\boldsymbol{Z}_2\boldsymbol{Z}_2^T\boldsymbol{P}\boldsymbol{y}/\sigma_0^2, \qquad \boldsymbol{A}(\sigma_2^2,\sigma_2^2) = \frac{1}{2}\boldsymbol{y}^T\boldsymbol{P}\boldsymbol{Z}_2\boldsymbol{Z}_2^T\boldsymbol{P}\boldsymbol{Z}_2\boldsymbol{Z}_2^T\boldsymbol{P}\boldsymbol{y}/\sigma_0^2.$$

Estimated components of variance are obtained by iteration using the scoring method [9]

$$\boldsymbol{\kappa}^{(j+1)} = \boldsymbol{\kappa}^{(j)} + \left(\boldsymbol{A}^{(j)}\right)^{-1}\frac{\partial \ell}{\partial \boldsymbol{\kappa}}(\boldsymbol{\kappa} = \boldsymbol{\kappa}^{(j)})$$

until convergence is reached. Symbol $(\boldsymbol{A}^{(j)})^{-1}$ indicates inverted average information matrix and $\frac{\partial \ell}{\partial \boldsymbol{\kappa}}(\boldsymbol{\kappa} = \boldsymbol{\kappa}^{(j)})$ a vector of first derivatives of the log-likelihood ℓ with respect to variance components $\boldsymbol{\kappa}$, evaluated at estimates received in j-th round. Suitable non-negative starting values of variance components $\boldsymbol{\kappa}^{(0)}$ are found by applying several iterations of first-order derivative EM-REML [5], known to be less sensitive to poor starting values, before switching to AI-REML. It is recommended, that AI-REML iteration to convergence be traced.

Upon convergence, estimates from two successive rounds satisfy the criterion $|\kappa_m^{(j)} - \kappa_m^{(j-1)}| < 10^{-6}$. Fixed effects for varieties are then estimated by OLS or GLS using

$$\hat{\boldsymbol{\beta}} = \left(\boldsymbol{X}^T\boldsymbol{H}^{-1}\boldsymbol{X}\right)^{-1}\boldsymbol{X}^T\boldsymbol{H}^{-1}\boldsymbol{y}.$$

In the calculation, covariance matrix $\boldsymbol{H}$ originates from the most recent round of iteration. Covariance matrix of the fixed effects of varieties $D(\hat{\boldsymbol{\beta}})$ is estimated by

$$D(\hat{\boldsymbol{\beta}}) = (\boldsymbol{X}^T\boldsymbol{H}^{-1}\boldsymbol{X})^{-1}\hat{\sigma}_0^2,$$

where $\hat{\sigma}_0^2$ denotes REML estimate of error variance from the last iteration. Levels of random effects associated with blocks and replications are predicted from

$$\hat{\boldsymbol{u}} = \boldsymbol{G}\boldsymbol{Z}^T\boldsymbol{H}^{-1}\left(\boldsymbol{y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}\right).$$

Fisher's Least Significant Difference $LSD_\alpha(r,s)$ for pairwise differences between mean yields of varieties r and s can be calculated

$$LSD_\alpha(r,s) = t_{1-\frac{\alpha}{2}}(\nu_\varepsilon)\cdot SE\left(\hat{\beta}_r - \hat{\beta}_s\right).$$

Standard errors of the differences between variety means are obtained from

$$SE\left(\hat{\beta}_r - \hat{\beta}_s\right) = \sqrt{D(\hat{\beta}_r) + D(\hat{\beta}_s) - 2Cov(\hat{\beta}_r,\hat{\beta}_s)}$$

using matching elements of $D(\hat{\boldsymbol{\beta}})$ matrix. Symbol ν_ε denotes residual degrees of freedom.

Value of log restricted likelihood, estimated effects of varieties, $LSD_\alpha(r,s)$ for pair-wise comparisons, variance components, predicted random effects and residuals from GenStat and R-software [7] were available to us to verify implementation of the AI-REML algorithm in MATLAB. Statistical methods and models, required output, testing data and results of the original procedures were provided by the recipient institution.

4 Conclusions

At present, plant variety yield data from testing experiments is statistically analyzed by existing software and purchased third-party software working mostly under MS-DOS operating system. It offers limited user comfort and prevents necessary updates to meet current demands for quality and fast data processing, since only precompiled code is now available. Decision support system (DSS) developed by the authors provides transfer of statistical procedures to modern programming platform represented by MATLAB, supports critical updates of the statistical methods, secures speedy analysis of large datasets and provides user-friendly environment and prompt transfer of results to the authorities and the farmers. After completion, DSS system will assist with generating quality decisions about variety registrations, protect breeder's rights, increase amount of information about variety production potential available to the government and the public and thus increase economic endurance of the Czech agriculture.

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International portfolio selection with Markov processes and liquidity constraints

Enrico Angelelli[1], Sergio Ortobelli[2], Tomáš Tichý[3], Daniele Toninelli[4]

Abstract. This paper proposes an ex-post comparison of portfolio selection strategies applied to 64 stock exchange markets during the period 2008-2011. Assets are considered according to three different classes of liquidity constraints which differ for their average traded daily value. Moreover, in order to simplify the computational complexity of the choice we preselected assets among the global market considering different return characteristics: the joint Markovian behavior of the returns; their association with market stochastic bounds and their ex-ante reward-risk performance. For each liquidity class of assets we compare the ex-post performance of three portfolio selection strategies: one of them is based on the maximization of the Sharpe ratio; the other two are based on the maximization of a reward-risk performance measure that considers a Markovian evolution of the portfolio returns. The ex-post analysis shows that the reward-risk functionals allow better performance in terms of final wealth and demonstrates, with respect to liquidity classes, different features of portfolio composition in terms of assets nationality.

Keywords: portfolio selection strategies, Markov chains, liquidity constraints.

JEL Classification: G11, G01, G17
AMS Classification: 60J22, 91B28, 91G10

1 Introduction

The last financial crises highlight how world financial markets are exposed to risks that are magnified from the close connections between the markets of different countries. Thus, it is clear that it becomes necessary to find instruments that can help investors in their portfolio choices, helping them to maximize their return and minimize the investments' risk. In this paper we carry on an analysis through a study based on data covering the last financial crises (from 2008 to 2011). In particular, the purpose of this paper is to compare, ex-post, three different portfolio selection strategies on three groups of stocks preselected within the global market, for a total of 64 stock exchange markets.

In order to simplify the computational complexity of the task, we preselected assets among the global stock market considering three different classes of liquidity constraints that differ for their average traded daily value. Moreover, we preselected several stocks for each portfolio selection problem, taking into account different return characteristics: the joint Markovian behavior of the portfolio returns (Angelelli and Ortobelli 2009a, 2009b); their association with market stochastic bounds (see also Ortobelli *et al.*, 2011); their ex-ante reward-risk performance. Among Markovian models we essentially distinguish two categories: (i) parametric models (see: Cont and Tankov, 2003; Cox et al., 1979), and (ii) non parametric models (e.g., Angelelli and Ortobelli 2009a; D'Amico, 2006). For the first category the Markovian hypothesis is used for diffusive models of the log returns; in the parametric models the time series are used to estimate the transition among the states. In our research we assume a non-parametric Markovian model for portfolio selection, describing the returns time evolution by means of a homogeneous Markov chain (for other nonparametric models, see also Ait-Sahalia 1996; Stutzer, 1996). With this methodology, the estimation of the future wealth distribution can be a heavy task; nevertheless, the computational complexity can be controlled by means of a Markov chain. Moreover, the states of Markov chain are chosen to induce a recombining effect on the future wealth (Angelelli and Ortobelli 2009a).

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For each liquidity class of assets we compare the ex-post performance of three portfolio selection strategies: one of them is based on the maximization of the Sharpe ratio; the other two are based on the maximization of a reward-risk performance that considers a Markovian evolution of the portfolio returns. The comparison of the ex-post final wealth obtained by the optimization of the reward-risk functionals remarks very different characterristics of international portfolio composition among the three liquidity classes of assets. Thus the results emphasize the importance of liquidity constraints in selecting international portfolios.

In the following section of the paper we introduce the reduction of the dimensionality of the portfolio problem within Markov processes and we provide more details about the Markovian behavior of wealth. The results of the empirical analysis are discussed in Section 3.

2 Criteria to reduce the dimensionality of the portfolio problem

Our analysis is applied to *n* risky assets, and in our simulation we assume that investors want to maximize the performance of their choices at a given future time *T* and that the portfolio returns can be described by a homogeneous Markov chain with *N* states defined on a filtered probability space $\left(\Omega,\Im,\left(\Im_t\right)_{0\le t\le\infty},P\right)$. This Markovian hypothesis is useful to forecast future behavior of the wealth. The vector of the positions taken by an investor in the *n* preselected risky assets is denoted by $x=[x_1,\ldots,x_n]'$ and the portfolio return during the period [*t*, *t*+1] is given by $z_{(x),t+1}=x'z_{t+1}=\sum_{i=1}^{n}x_i z_{i,t+1}$, where $z_{t+1}=[z_{1,t+1},\ldots,z_{n,t+1}]'$ is the vector of the gross returns. The objective is to maximize a given performance measure applied to the forecasted future wealth produced by an investment strategy.

In general the approach to future wealth can be considered static or dynamic. In the latter the gross returns are considered following a Markov chain. Therefore, the investor maximizes a functional of future wealth of the type $\max_{x\in S} f\left(W_T\left(z_{(x)}\right)\right)$, where $W_T\left(z_{(x)}\right)$ is the predicted wealth after *T* trading days obtained investing in the portfolio $z_{(x)}=x'z$. On the other side, with the static approach, the investor maximizes a functional of future wealth that is assumed independent from his temporal horizon [0, *T*], i.e. he solves problems of the type $\max_{x\in S} f\left(z_{(x),t+1}\right)$.

In our work we consider assets selected by both the approaches (static and dynamic). More into details, the selection is based on three steps: 1) consider a set of desirable ordering criteria; 2) assets are sorted by each criterion; 3) intersect top ranked assets so that a fixed number of assets are selected (about 100 assets satisfying the dynamic approach criteria and about 50 assets satisfying the static approach criteria). In particular, in our analysis, the selected assets satisfy some properties of wealth behavior and optimality criteria (Ortobelli et al. 2011), such as: 1) Timing (we want both to maximize the time taken for wealth to reach a given lower bound and to minimize the time to reach a given upper bound; 2) An asymptotic and a Markovian behavior of wealth; 3) Consistency with investors' preferences; 4) Association with market stochastic bounds.

A common problem with other portfolio selection strategies is the reduction of the dimensionality of the problem. If this is the target to be generally reached, other studies (for example, Papp *et al*., 2005, and Kondor *et al.*, 2007) have shown that the number of observations should increase proportionally with the number of assets, if we want to obtain a good approximation of the portfolio risk-reward measures. Thus in our research we have to find a compromise between a good statistical approximation of the historical series, and a reduced number of parameters. Usually, the dimensionality of a group of preselected assets can be reduced identifying a few common factors to approximate the asset returns. In particular, the principal component analysis (PCA) can be applied to the Pearson correlation matrix (that can be computed on the Markovian forecast of future returns). Therefore, the returns can be approximated applying a regression analysis on the first few components obtained by means of the PCA. Following the suggestions of Ortobelli and Tichy (2010, 2011), we can "reduce the dimensionality of the problem approximating the return series with a *k*-fund separation model (or other regression-type models) that depends on an adequate number (not too large) of parameters" (Ortobelli and Tichy, 2011). The first step of this analysis is the application of the Principal Component Analysis (PCA) to identify portfolios with the highest return variability (se also Biglova *et al.*, 2009). For this purpose, we work on the gross returns, replacing the 150 preselected time series $\left(z_i; i=1,...,n=150\right)$ with 150 uncorrelated ones $\left(R_i; i=1,...,n=150\right)$. Each z_i can be obtained as a linear combination of R_i. The reduction of dimensionality is obtained selecting, within the R_i group, only *s* time series with a significant dispersion measure; they are called f_i factors (*i* = 1, …, *s*). We know that these factors can be seen as a linear combination of z_i, as follows:

$f_i = \sum_{i=1}^{n} x_i z_i$, where $\sum_{k=1}^{n} x_k^2 = 1$. We can consider the remaining *n*-*s* time series as errors $\left(\varepsilon_j; j = s+1,...,n\right)$. Seen this all, the *n* selected time series can be viewed as a linear combination of factors f_i and a linear combination of factors with small dispersion measure (uncorrelated noise ε_i); the relation is sinthesized in the following formula:

$$z_i = \sum_{k=1}^{s} a_{ik} f_k + \sum_{j=s+1}^{n} a_{ij} R_j = \sum_{k=1}^{s} a_{ik} f_k + \varepsilon_i, \tag{1}$$

In our analysis we apply the PCA both to the Pearson correlation matrix and to the linear correlation matrix $Q = [\rho_{i,j}]$, where $\rho_{i,j} = O_{2,\Im_1}(z_i, z_j)$ for the same sigma algebra $\Im_1$ considered in Ortobelli and Tichy, 2011. Recall that the matrix $O_{p,\Im_1}(X,Y) = \text{cor}((X^{<\frac{p}{2}>} - \text{E}(X^{<\frac{p}{2}>} \mid \Im_1)), (Y^{<\frac{p}{2}>} - \text{E}(Y^{<\frac{p}{2}>} \mid \Im_1)))^{<\min(\frac{p}{2},2)>}$, where $\Im_1$ is a sub-sigma algebra of $\Im$ (i.e. $\Im_1 \subset \Im$) and X and Y are not $\Im_1$ measurable, refers to a logical extension of the Pearson correlation measure, since we obtain the Pearson correlation measure if $p = 2$ and $\Im_1 = \{\varnothing; \Omega\}$. Then we determine the 12 factors with the highest variability of the preselected gross returns for each linear correlation matrix (Pearson and Q). In particular, we assume that the two correlation matrixes are computed on the forecasted wealth $W_{20}(z_i)$ obtained investing in each asset z_i after 20 trading days and assuming a Markovian beahovior of the wealth. Finally we approximate the returns regressing the series on the 24 f_j factors (12 determined with the PCA on the Pearson correlation matrix and the other 12 determined with the PCA on the matrix $Q = [\rho_{i,j}]$), as follows:

$$z_i = b_{i,0} + \sum_{j=1}^{24} b_{i,j} f_j + \varepsilon_i. \tag{2}$$

2.1 The Markovian behavior of wealth

In our research we assume that each portfolio follows a Markov chain. If we define with $z_{(x),k}$ the *k*-th observed gross return of the portfolio $z_{(x)}$, and if we assume that the gross returns are ordered ($z_{(x)}^{(i)} > z_{(x)}^{(i+1)}$ for $i = 1,..., N-1$), the range of the portfolio gross return is: $(\min_k z_{(x),k}; \max_k z_{(x),k})$. This range can be divided into *N* intervals $(a_{(x),i}; a_{(x),i-1})$, such as: $a_{(x),i} = \left(\frac{\min_k z_{(x),k}}{\max_k z_{(x),k}}\right)^{i/N} \cdot \max_k z_{(x),k}$, $i = 0, 1, \ldots, N$. The return of each state, $z_{(x),k}$, can be computed as the geometric average of the extremes of the interval: $z_{(x)}^{(i)} = \sqrt{a_{(x),i} a_{(x),i-1}} = \max_k z_{(x),k} \left(\frac{\max z_{(x),k}}{\min z_{(x),k}}\right)^{\left(\frac{1-2i}{2N}\right)}$, where: $i = 1, 2, \ldots, N$. Consequently: $z_{(x)}^{(i)} = z_{(x)}^{(1)} u^{1-i}$, where: $u = \left(\frac{\max z_{(x),k}}{\min z_{(x),k}}\right)^{1/N} > 1$. Let now assume that the initial wealth W_0 at time 0 is equal to 1, and that, for each possible wealth W_t at time *t*, we have *N* possible different values $W_{t+1} = W_t z_{(x)}^{(i)}$ at time *t*+1 (*i* = 1, 2, …, *N*). Taking into consideration the recombining effect of the Markov chain, after *k* steps we would obtain $1 + k(N-1)$ values of wealth $W_k(z_{(x)})$. If we assume a homogeneous Markov chain, the transition matrix $P = [p_{i,j}]$ does not depend on time and the entries $p_{i,j}$ are estimated using the maximum likelihood estimates $\hat{p}_{i,j} = \frac{\pi_{ij}(K)}{\pi_i(K)}$, where $\pi_{ij}(K)$ is the number of observations (out of *K* observations) that transit from the *i*-th state to the *j*-th state and $\pi_i(K)$ is the number of observations (out of *K* observations) in the *i*-th state. Similarly to what made in Iaquinta and Ortobelli (2006), the distribution function of future gross returns can now be computed. In particular, the $(N-1)k+1$ dimensional vector $p^{(k)}$ represents the unconditional distribution of wealth $W_k(z_{(x)})$ at a given time *k* (*k* = 0, 1, 2, ..., *T*). This distribution can be computed by means of a sequence of matrixes $\left\{Q^{(k)}\right\}_{k=0,1,\ldots,T}$, where $Q^{(k)} = [q_{i,j}^{(k)}]_{\substack{1 \le i \le (N-1)k+1 \\ 1 \le j \le N}}$ and $q_{i,j}^{(k)}$ is the unconditional probability of obtaining a wealth $w_{(x)}^{(i,k)}$ and of being in the

$z_{(x)}^{(j)}$ state at time *k*. The $Q^{(k)}$ matrix is called *unconditional evolution matrix*. In the following analysis we assume to forecast wealth for 20 days assuming 9 states in the Markov chain.

3 An empirical ex-post comparison among international portfolio strategies

In this section we propose an ex-post comparison among portfolio models based on three different portfolio strategies during a period of 41 months (where one month contains 20 trading days) starting from December 2007 to February 2011. In particular, we compare the ex-post performance of three portfolio selection strategies applied to three different clusters of stocks preselected from various financial markets in the world[5] that contain more than 90000 stocks during the analysed period. We distinguish the traded assets for their level of liquidity. We consider the stocks whose average of daily value (expressed in USD) of contracts belongs to a given interval. The value of daily contracts is given by the formula:

$$\text{value of daily contracts} = (\text{closure price}) * (\text{daily volume}),$$

and the average of the value of daily contracts is computed on the last two years (500 trading days). The first group of stocks (with the biggest liquidity) consider the assets with average daily values bigger than 90 million USD. The second group, (defined "Setmid") is composed by stocks with average daily values included between 250,000 and 90 million USD. The third group includes stocks with average daily values between 10,000 and 250,000 USD.

Moreover, since we value the ex-post value of wealth of three models during a period with high volatility of the markets and presence of systemic risk, we introduce an "alarm" that tells us about the presence of systemic risk. The "alarm" is a simple rule that counts the assets whose average returns on the last 20 days is lower than the mean on the last 100 working days. If the number of these assets is higher than 65% of the assets we deduce that probably there is systemic risk on the market and 65% of the assets are losing value. Therefore, in presence of systemic risk we do not invest in the market for any strategy. When we do not observe the presence of systemic risk, we suggest to invest on the 150 preselected stock returns. In particular, when the proposed alarm does not find out the presence of systemic risk, we preselect the 150 stocks using the criteria suggested by Ortobelli *et al* 2011. Then we perform the approximation in formula (2) of the preselected returns. Finally, on the preselected approximated gross returns we maximize a reward-risk performance ratio assuming that no short sales are allowed and we cannot invest more than 10% in a single asset. Doing so, starting by an initial wealth $W_{t_0} = 1$, we evaluate the ex-post wealth sample paths for three different reward-risk portfolio strategies. The three reward risk performance measures we maximize are:

1) The Sharpe ratio (see, Sharpe 1994) $SR(x'z) = \dfrac{E\left(x'z - z_f\right)}{St.dev\left(x'z\right)}$, where the mean and the standard deviation of the portfolio *x'z* are approximated by the empirical mean and standard deviation of the last 125 trading days (we assume that the riskless is null, that is $z_f = 1$).

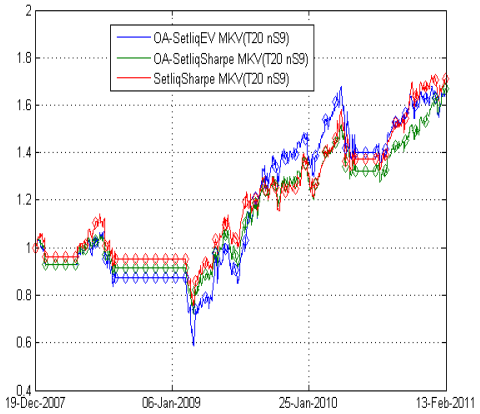
2) The OA-Sharpe ratio: $\dfrac{E\left(W_{20}\left(x'z\right)\right)-1}{St.dev.\left(W_{20}\left(x'z\right)\right)}$, where $W_T\left(x'z\right)$ is the predicted wealth obtained after $T = 20$ working days investing in the *i*-th asset (for any $i = 1, 2, \ldots, n$).

3) The OA-Expected value: $OA-EV\left(W_{20}\left(x'z\right)\right) = \dfrac{\sum_{t=1}^{20} E\left(W_t\left(z_{(x)}\right)/W_t\left(z_{(x)}\right) > 1.2\right)}{\sum_{t=1}^{20} E\left(W_t\left(z_{(x)}\right)/W_t\left(z_{(x)}\right) < 0.97\right)}$ that assumes as: reward measure the expected value of the future wealth conditional that this wealth is greater than a benchmark 1.2; as risk measure the expected value of the future wealth conditional that this wealth is smaller than a benchmark 0.97.

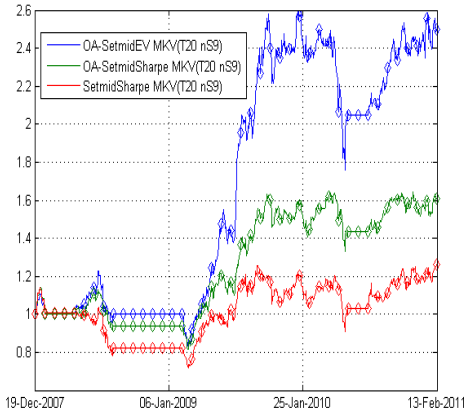
In Figure 1 we report the international diversification and the ex-post comparison of the sample paths of wealth obtained with the different portfolio strategies and for different liquidity constraints that we denote as: "Setliq", "Setmid" and "Setsmall" the strategies respectively with biggest, medium, smallest liquidity.

[5] Considered markets: Australia, New Zealand; Sao Paulo, Mexico, Buenos Aires, Bogotà, Caracas, Lima, Santiago; NYSE, AMEX, NASDAQ, Toronto; Ghana, Johannesburg, Cairo, Casablanca, Bombay, National India, Tel Aviv, Istanbul, Shanghai, Shenzen, Hong Kong, Corea, Taiwan, Bangkok, Indonesia, Kuala Lumpur, Philippine Stock Exchange, Singapore, Thailand, Kuwait City, Tokyo, London, SIX Swiss, Euronext (Amsterdam, Brussels, Lisbon, Paris), Berlin, Madrid, Milan, Helsinki, Athens, Cyprus, Dublin, Iceland, Vienna S. E., Copenhagen S. E., Stockholm, Oslo S. E., Bratislava, Kiev, Riga, Tallinn, Lithuania, Ljubljana, Prague, Belgrade, Zagreb, Bucharest, Budapest S. E., Sofia, Warsaw, Russian Trading System, Moskow.

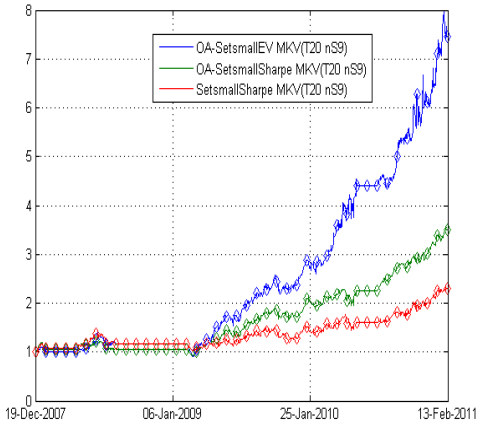
a) Ex post wealth with biggest liquidity



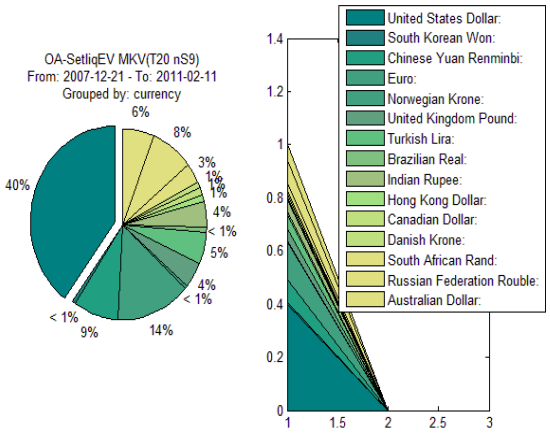
b) Ex post wealth with medium liquidity



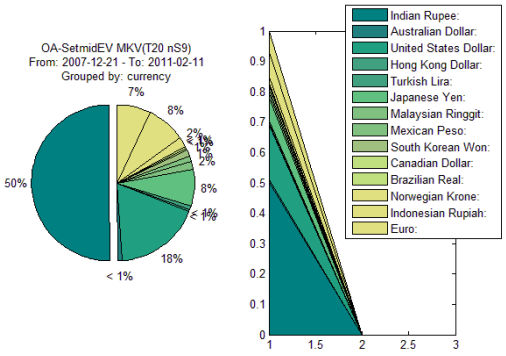
c) Ex post wealth with smallest liquidity



d) International portfolio diversification with OA-SetliqEV



e) International portfolio diversification with OA-SetmidEV



f) International portfolio diversification with OA-SetsmallEV

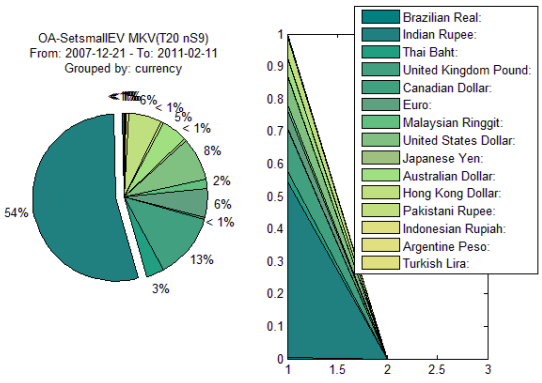


Figure 1 Ex-post wealth and portfolio diversification with different liquidity constraints

4 Conclusion

In the paper, we focused in more details on the problem of portfolio selection assuming internationally diversified portfolio and considering Markovian processes and liquidity constraints.

We have shown a better ex-post performance of the strategy based on the maximization of the OA-Expected value ratio. It clearly appears (in Figure 1 a) b) and c)) that both strategies based on the Markovian evolution of the returns present higher final wealth than the classic one based on the maximization of the Sharpe ratio.

Moreover, the proposed empirical analysis demonstrates that the alarm inserted to detect the presence of systemic risk works well, since it is able to stop, to identify and, to forecast the largest period of systemic risk of the recent crises (sub-prime crisis 2008-2009 and the country credit risk crisis 2010-2011).

Finally, we also discover that the turnover behavior does not change too much among the three strategies. In addition, the three strategies diversify among different assets but not among different countries for each class of liquidity. However, while the most liquid strategies invest most of the wealth in the most industrialized countries (US, Europe); the less liquid strategies invest most of the wealth in the BRIC countries (in particular India, China and Brazil).

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Inverse fuzzy eigenproblem in databases

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Abstract. Fuzzy algebra is defined as the real unit interval equipped by two binary operations: maximum and minimum, which are used analogously as addition and multiplication in the classical linear algebra. Fuzzy algebra is used in applications such as fuzzy control systems, or stability of discrete event systems. Eigenvectors of a fuzzy matrix correspond to steady states of a given discrete-events system.

The inverse eigenproblem is motivated by the task of finding all possible systems possessing a given steady state. In the contribution, the inverse problem is studied as a system of fuzzy equations, possibly with upper end lower bounds. Application of the inverse approach to efficient search in databases is presented.

Keywords: fuzzy algebra, max-min algebra, eigenproblem, database search

JEL classification: C44
AMS classification: 90C15

1 Introduction

By max-min algebra (sometimes called: fuzzy algebra) we understand a linear structure on a linearly ordered set with two binary operations maximum and minimum, used similarly as addition and multiplication in the classical linear algebra. Fuzzy algebra is used in many applications such as fuzzy control systems or stability of discrete event systems. Eigenvectors of a fuzzy matrix correspond to steady states of a given discrete-events system. Investigation of the max-min eigenvectors of a given matrix is therefore of great practical importance. The eigenproblem in max-min algebra has been studied by many authors. Interesting results were found in describing the structure of the eigenspace, and algorithms for computing the maximal eigenvector of a given matrix were suggested, see e.g. [3], [4], [5], [6], [7].

While the standard eigenproblem looks for the steady states of a system characterized by given transition matrix, the inverse eigenproblem is motivated by the task of finding all possible systems with given steady state vector. In this paper the inverse problem is studied for fuzzy linear systems. The problem is a special case of solving a system of max-min fuzzy equations, however, the direct way to inverse solution allows description of solutions with more details and with smaller computational complexity.

Application of the inverse fuzzy eigenproblem to databases helps to find suitable intervals for database search systems fulfilling some given conditions. This will narrow down the scanned field, and the problem may be solved for greater data sets. The model represents the database searching by the preferences of the user.

2 Notions and notation

Many applications can naturally be solved by using operations maximum and minimum on a linearly ordered set. The so-called max-min algebras were frequently studied by number of authors and various algorithms have been developed for solving problems analogous to those in classical linear algebra.

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Max-min algebra is defined as a triplet $(\mathcal{B}, \oplus, \otimes)$, where $\mathcal{B}$ is a linearly ordered set, and $\oplus = \max$, $\otimes = \min$ are binary operations on $\mathcal{B}$. The notation $\mathcal{B}(n,n)$ ($\mathcal{B}(n)$) denotes the set of all square matrices (all vectors) of given dimension n over $\mathcal{B}$. Operations $\oplus$, $\otimes$ are extended to matrices and vectors in a formal way.

The eigenproblem for a given matrix $A \in \mathcal{B}(n,n)$ in max-min algebra consists of finding a vector $x \in \mathcal{B}(n)$ (eigenvector) such that the equation

$$A \otimes x = x \tag{1}$$

holds true. By the eigenspace of a given matrix we mean the set of all its eigenvectors.

In the inverse eigenproblem the eigenvector x is considered as fixed, and the matrix A is unknown. It is clear that the equation (1) with fixed x and variable A is a special case of the equation (2) below, with fixed coefficients x, b. In the next two sections, the equation (2) with upper and lower bounds for the solution matrix A will be studied. The maximal (greatest) solution and the set of all minimal solutions will be described. The applications of the results are briefly presented in the last section.

3 The greatest solution

Definition 1. Let $x, b \in \mathcal{B}(n)$ and $\mathbf{A} = [\underline{A}, \overline{A}] \subseteq \mathcal{B}(n,n)$ be given. Then by *inverse interval problem* we understand linear equation

$$A \otimes x = b \tag{2}$$

w.r.t. the unknown matrix $A \in \mathbf{A}$.

Following matrix will play a crucial role in solving problem (2). For given $\mathbf{A} = [\underline{A}, \overline{A}] \subseteq \mathcal{B}(n,n)$, x and $b \in \mathcal{B}(n)$, the matrix $\hat{A}(x,b)$ is defined by its entries:

$$\hat{a}_{ij}(x,b) = \begin{cases} b_i & \text{when } \overline{a}_{ij} \otimes x_j > b_i, \\ \overline{a}_{ij} & \text{otherwise.} \end{cases}$$

Proposition 1. $\hat{A}(x,b) \leq \overline{A}$.

Proof. By contradiction: let exist $i, j \in N$ such that $\hat{a}_{ij}(x,b) > \overline{a}_{ij}$, then $\hat{a}_{ij}(x,b) = b_i < \overline{a}_{ij}$, contradiction. □

Proposition 2. $\hat{A}(x,b) \otimes x \leq b$.

Proof. By contradiction: let exist $i, j \in N$ such that $\hat{a}_{ij}(x,b) \otimes x_j > b_i$, but then $\hat{a}_{ij}(x,b) = b_i$, contradiction. □

Proposition 3. *If* $A \otimes x = b$, $A \leq \overline{A}$, *then* $A \leq \hat{A}(x,b)$.

Proof. By contradiction: let exist $i, j \in N$ such that $a_{ij} > \hat{a}_{ij}(x,b)$, then $\hat{a}_{ij}(x,b) = b_i$ and $x_j > b_i$, which yields $a_{ij} \otimes x_j > b_i$, contradiction. □

Proposition 4. *If* $b \leq \overline{A} \otimes x$, *then* $\hat{A}(x,b) \otimes x = b$.

Proof. By contradiction: let $i_* \in N$ be such that $\bigoplus_{j=1}^{n} \hat{a}_{i_* j}(x,b) \otimes x_j < b_{i_*}$, then it follows that $\hat{a}_{i_* j}(x,b) = \overline{a}_{i_* j}$ for all $j \in N$, thence $\bigoplus_{j=1}^{n} \overline{a}_{i_* j} \otimes x_j < b_{i_*}$, which is in contradiction with $b \leq \overline{A} \otimes x$. □

Proposition 5. *If* $\underline{A} \otimes x \leq b$, *then* $\underline{A} \leq \hat{A}(x,b)$.

Proof. By contradiction: let element $\hat{a}_{i_*j_*}(x,b)$ be such that $\hat{a}_{i_*j_*}(x,b) < \underline{a}_{i_*j_*}$, then $\hat{a}_{i_*j_*}(x,b) = b_{i_*}$ and $x_{j_*} > b_{i_*}$, from $\underline{a}_{i_*j_*} \otimes x_{j_*} > b_{i_*}$ it follows that $\bigoplus_{j=1}^{n} \underline{a}_{i_*j} \otimes x_j > b_{i_*}$, which is in contradiction with $\underline{A} \otimes x \leq b$. □

Remark 1. Preceding propositions 3 till 5 show that if there exists some solution of (2) in $\mathbf{A}$, then $\hat{A}(x,b)$ is also a solution and moreover it is the greatest one in $\mathbf{A}$. This finding justifies next definition.

Definition 2. For a given problem (2), a matrix $\hat{A}(x,b)$ introduced before will be called *the greatest solution.*

Remark 2. For $\mathbf{A} = [\underline{A}, \overline{A}] \subseteq \mathcal{B}(n,n)$ and $x \in \mathcal{B}(n)$ we will denote $\mathbf{A} \otimes x := [\underline{A} \otimes x, \overline{A} \otimes x]$.

Theorem 6. *There is at least one solution of (2) in* $\mathbf{A} = [\underline{A}, \overline{A}]$ *iff* $b \in \mathbf{A} \otimes x$.

Proof. Implication $\Rightarrow$ is clear, converse implication follows from preceding propositions. □

Theorem 7. *Let* $x, y \in \mathcal{B}(n)$ *and* $\mathbf{A} = [\underline{A}, \overline{A}] \subseteq \mathcal{B}(n,n)$ *be given. Then the following statements are equivalent:*

(i) There exists A *in* $\mathbf{A}$ *such that* $A \otimes x = A \otimes y$,

(ii) $\mathbf{A} \otimes x \cap \mathbf{A} \otimes y \neq \emptyset$,

(iii) $(\underline{A} \otimes x) \oplus (\underline{A} \otimes y) \leq (\overline{A} \otimes x) \oplus' (\overline{A} \otimes y)$.

Proof. $(i) \Rightarrow (ii)$: for such A clearly $A \otimes x \in \mathbf{A} \otimes x \cap \mathbf{A} \otimes y$, hence the intersection is nonempty.
$(ii) \Rightarrow (i)$: let us choose arbitrary but fixed $b \in \mathbf{A} \otimes x \cap \mathbf{A} \otimes y$ and construct matrix A with entries:

$$a_{ij} = \begin{cases} b_i & \text{when } \overline{a}_{ij} \otimes \max(x_j, y_j) > b_i, \\ \overline{a}_{ij} & \text{otherwise.} \end{cases}$$

We will show that A is a solution. Since $b \in \mathbf{A} \otimes x$, there exists for every $i \in N$ such index $j_* \in N$ that $\hat{a}_{ij_*}(x,b) \otimes x_{j_*} = b_i$ holds. Therefore either $x_{j*} = b_i \ \wedge \ \hat{a}_{ij_*}(x,b) \geq b_i$ or $x_{j_*} > b_i \ \wedge \ \hat{a}_{ij_*}(x,b) = b_i$. In both cases $a_{ij_*} \otimes x_{j_*} = b_i$. From the latter and the fact that $A \leq \hat{A}(x,b)$ it follows that

$$\bigoplus_{j \in N} a_{ij} \otimes x_j = b_i,$$

which holds for every $i \in N$, i.e. $A \otimes x = b$. By the same considerations we derive $A \otimes y = b$.
$(ii) \Leftrightarrow (iii)$: clearly $\mathbf{A} \otimes x \cap \mathbf{A} \otimes y = [(\underline{A} \otimes x) \oplus (\underline{A} \otimes y), (\overline{A} \otimes x) \oplus' (\overline{A} \otimes y)]$. □

4 Minimal solutions

Let us introduce some notation: Let $\mathbf{A} = [\underline{A}, \overline{A}]$, where $\underline{A}$ and $\overline{A} \in \mathcal{B}(n,n)$, and $x, b \in \mathcal{B}(n)$ for some $n \in \mathbb{N}$. Then for every $i \in N$ we define:

- $e(i) = \{j \in N |\ \underline{a}_{ij} \otimes x_j = b_i\}$,
- $k(i) = \begin{cases} \emptyset & \text{when } e(i) \neq \emptyset, \\ \{j \in N | x_j \geq b_i \wedge \underline{a}_{ij} < b_i \leq \overline{a}_{ij}\} & \text{otherwise,} \end{cases}$
- $l_i = \begin{cases} 1 & \text{when } e(i) \neq \emptyset, \\ |k(i)| & \text{otherwise.} \end{cases}$
- $l = \prod_{j \in N} l_j$, $L = \{1, 2, \ldots, l\}$.

In addition, let ϕ be n-tuple with elements satisfying

$$\phi(i) = \begin{cases} 0 & \text{for } k(i) = \emptyset, \\ j \in k(i) & \text{for } k(i) \neq \emptyset. \end{cases}$$

Then we define matrix $\check{A}(x, b, \phi)$ with entries $\check{a}_{ij}(x, b, \phi) = \begin{cases} b_i & \text{when } j = \phi(i), k(i) \neq \emptyset, \\ \underline{a}_{ij} & \text{otherwise.} \end{cases}$

Theorem 8. *Let* $\mathbf{A}$, x, b *a* l *be from the latter notation and moreover* $\underline{A} \otimes x \leq b$. *Then the following holds: equation (2) has in the interval* $\mathbf{A}$ *exactly* l *mutually incomparable minimal solutions. These solutions are in the form* $\check{A}(x, b, \phi)$ *for any* ϕ *from the latter notation.*

Proof. First, we will prove that $\check{A}(x, b, \phi) \otimes x = b$ for some ϕ. For ith entry of this product two cases may arise: Either $e(i) = \emptyset$ and $k(i) \neq \emptyset$, then for index $j_0 = \phi(i)$ must hold

$$\check{a}_{ij_0}(x, b, \phi) \otimes x_{j_0} = b_i \otimes x_{j_0} = b_i$$

thence

$$\bigoplus_{j \in N} \check{a}_{ij}(x, b, \phi) \otimes x_j \geq b_i.$$

If the last inequality was strictly greater, there would exist index j_1 such that $x_{j_1} > b_i$ and also $\check{a}_{ij_1}(x, b, \phi) = \underline{a}_{ij_1} > b_i$, which is in contradiction with the assumption $\underline{A} \otimes x \leq b$.

When $k(i) = \emptyset$, $e(i)$ is either nonempty, from which it follows that the product will give b_i from the similar considerations as in the last case, or it is empty too, i.e. $l = 0$. In this case we have

$$x_j > b_i \Rightarrow b_i < \underline{a}_{ij} \vee b_i > \overline{a}_{ij}, \quad x_j = b_i \Rightarrow b_i > \overline{a}_{ij} \geq \underline{a}_{ij}.$$

Since $\underline{A} \otimes x \leq b$ could not hold for $x_j > b_i \wedge b_i < \underline{a}_{ij}$, b_i has to be $b_i > \overline{a}_{ij}$ for $x_j > b_i$. In sum, from $x_j \geq b_i$ it follows that $b_i > \overline{a}_{ij}$. But then

$$(\overline{A} \otimes x)_i = \bigoplus_{j \in N} \overline{a}_{ij} \otimes x_j < b_i$$

and there exists no solution in the interval $\mathbf{A}$. This result corresponds with $l = 0$. That proves the equality $\check{A}(x, b, \phi) \otimes x = b$.

Now we prove $\check{A}(x, b, \phi) \in \mathbf{A}$:

$$\underline{A} \leq \check{A}(x, b, \phi) \Leftrightarrow \underline{a}_{ij} \leq \check{a}_{ij}(x, b, \phi) = \begin{cases} \underline{a}_{ij} & \text{inequality holds,} \\ b_i & \text{from } k(i) \text{ definition also holds,} \end{cases}$$

$$\overline{A} \geq \check{A}(x, b, \phi) \Leftrightarrow \overline{a}_{ij} \geq \check{a}_{ij}(x, b, \phi) = \begin{cases} \underline{a}_{ij} & \text{inequality holds,} \\ b_i & \text{from } k(i) \text{ definition also holds.} \end{cases}$$

Therefore we have l matrices $\check{A}(x, b, \phi)$ from the interval $\mathbf{A}$ that satisfy $\check{A}(x, b, \phi) \otimes x = b$. We will now show their incomparability: let us have $\underline{A}(x, b, \phi_1)$ and $\underline{A}(x, b, \phi_2)$, where $\phi_1 \neq \phi_2$. Then must exist such $i \in N$ that $\phi_1(i) \neq \phi_2(i)$. By mutual matrices's entries comparison

$$\begin{array}{rcccccl} \check{a}_{i,\phi_1(i)}(x, b, \phi_1) & = & b_i & > & \underline{a}_{i,\phi_1(i)} & = & \check{a}_{i,\phi_1(i)}(x, b, \phi_2), \\ \check{a}_{i,\phi_2(i)}(x, b, \phi_1) & = & \underline{a}_{i,\phi_2(i)} & < & b_i & = & \check{a}_{i,\phi_2(i)}(x, b, \phi_2), \end{array}$$

which yields their incomparability.

Finally, we will prove their minimality: Let $A \in \mathbf{A}$, $A \otimes x = b$ and A be at least at one element a_{ij} lesser than $\check{A}(x, b, \phi)$ for some ϕ. Then exists $(i_0, j_0) \in N \times N$ such that $a_{i_0,j_0} < \check{a}_{i_0,j_0}(x, b, \phi) = b_{i_0}$ (second case $\underline{a}_{i_0,j_0}$ can not arise, because $A \in \mathbf{A}$), thus $x_{j_0} \geq b_{i_0}$ together with $a_{i_0,j_0} \otimes x_{j_0} < b_{i_0}$ and $j_0 = \phi(i_0)$. From $\check{a}_{i_0,j_0}(x, b, \phi) \neq \underline{a}_{i_0,j_0}$ follows that $e(i_0) = \emptyset$ and $k(i_0) \neq \emptyset$. Then there exists $u \in k(i_0)$, i.e. $x_u \geq b_{i_0}$ and $b_{i_0} > \underline{a}_{i_0,u}$ such that $a_{i_0,u} = b_{i_0} > \underline{a}_{i_0,u} = \check{a}_{i_0,u}(x, b, \phi)$. Other i_0th row entries can not be less than $\underline{a}_{i_0,j}$, otherwise A would not be from $\mathbf{A}$, and even not greater, since then A could not be minimal.

Let us define $\phi'(i_0) = u$. This is possible, because from preceding considerations u is from $k(i_0)$. Since the same deduction can be applied to all rows of matrix A that contain elements less than corresponding elements of $\check{A}(x, b, \phi)$ and that otherwise we can define ϕ' in the same way as ϕ, for such ϕ' it holds that $A = \check{A}(x, b, \phi')$ and moreover $\phi' = \phi_\lambda$ for some $\lambda \in L$.

Any minimal matrix A is thus the same as some $\check{A}(x, b, \phi_\lambda)$ for appropriate $\lambda \in L$. These matrices are therefore minimal and the only minimal in the interval $\mathbf{A}$. □

Definition 3. For a given problem (2), a matrix $\check{A}(x, b, \phi)$ for some ϕ as introduced before will be called *a minimal solution.*

Remark 3. Number of minimal solutions is at most n^n.

Example 1. Let us define for all $i, j \in N$: $(\underline{A})_{ij} = 1$, $(\overline{A})_{ij} = 3$, $x_i = b_i = 2$. Then for each $i \in N$ is $e(i) = \emptyset, k(i) = N$ and $\phi : N \to N$. There exist n^n such mappings ϕ.

Remark 4. A solution set of (2) can be described as the union of intervals with a lower bound equal to $\check{A}(x, b, \phi)$ and an upper bound to $\hat{A}(x, b)$. These intervals are not disjoint, e.g. $\hat{A}(x, b)$ lies always in their intersection. Special cases of exactly one solution (i.e. $\check{A}(x, b, \phi) = \hat{A}(x, b)$) or of no solution are also possible.

By *inverse eigenproblem* we understand problem (2) with the right-hand side b equal to x. As it is easily seen, this constraint does not affect introduced algorithms for computing the greatest and minimal elements, therefore they can be used for inverse eigenproblems without modifications.

5 Applications in databases

Motivation of the inverse eigenproblem is to find the set of all possible systems corresponding to given conditions such as user preferences. The user preferences are written into vector, whereas the set of possible systems is represented by an interval matrix. In this case we have given the vector of preferences and we are looking for the matrix, which fulfil the conditions given by vector. The first step is to find the maximal matrix which accomplishes the conditions. The maximal matrix is uniquely determined. Next we decide the form of the lower bound - minimal matrix. Unlike the maximal matrix, the minimal matrix is not unique and the set of all minimal matrices (solutions of the inverse eigenproblem) may be very large. In spite of this fact, the solution set of the inverse eigenproblem consists of all matrices between the maximal and one of the minimal solutions.

Applications of the inverse eigenvector problem below is derived from the next two articles. In paper [1] a solution of recommendation system based on the user profile and heterogeneous components connected via web services was presented. The prototype developed under the code name Personal Recommendation (PRECO) is located on the following website http://preco.uhk.cz . The architecture of the system is guided by the need for centralized store of user preferences on one hand and on the other hand by the necessity to track the users' behaviour.

In response to this article the max-min inverse approach can be used as follows. Each database user is able to set the degree of importance of any discipline that is listed in the database. This importance depends on the focus orientation of the user. Inputs of interval matrix is given as the preference matrix of relevant disciplines. For any value of input a_{ij} the next notation is used: a_{ij} = rate of priorization of discipline i in comparison with discipline j. By using the inverse approach we are able to find any set of systems according to the requirements specified by the user. This will help us to narrow down the scanned field in database.

The next paper [2], talks about the idea to adapt principles and technologies from the BI concept to public health management. In this article the authors explore the decision support systems of support for Public Health in the Management of Biological Incidents using simulations, judgemental, and other methods.

In this application the input vector x can be represented by degree of intensity of manifested symptoms (conditions). A survey conducted by the authors, was arranged the necessary data, from which we can build two types of approaches. The first type is the largest accumulation, i.e. it assembles the largest possible array of possible measures of overlapping symptoms, conditions or circumstances. Second type is the smallest overlapping, i.e. it prepares a matrix of the lowest possible rates possible concurrence

of symptoms, conditions or circumstances. By using these types of the inverse approach a narrower interval overlapping symptoms can be found. The motivation is an acceleration of the search for potential biological incidents.

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A note on the methods of constructing weekly stock market returns

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Abstract. To avoid the non-synchronicity of daily stock market data, the weekly returns are commonly used by the empirical researchers. We focused three methods on construction of weekly returns: Wednesday-to-Wednesday, Friday-to-Friday, and returns calculated from weekly price averages. Using data from the CEE-3 (the Czech PX, the Hungarian BUX, and the Polish WIG) and three developed stock markets (the MSCI Germany, the Dow Jones Euro Stoxx 50, and the MSCI World) we show, that the statistical properties of weekly returns depend on the method used to construct the series. Simulations are also performed to observe the differences between variance and autocorrelations of the weekly returns.

Keywords: stock markets, weekly returns, statistical properties.

JEL Classification: C10, C80, G10
AMS Classification: 62-07, 62P20, 62M10, 91B84

1 Introduction

Because of non-synchronicity, researchers using and analyzing data from different stock markets often try to avoid the usage of daily closing prices. We distinguish between two types of non-synchronicity in daily stock market data (Baumöhl and Lyócsa [1]):

- The "non-synchronous trading effect I" – different number of observation between daily samples arises from the fact, that the stock markets are subject to different national, religious, and other holidays, and other unexpected events;
- The "non-synchronous trading effect II" – the fact that the national stock exchanges operate in different time zones induces cross-autocorrelations in closing prices.[4] Similarly, even within the stock markets with same trading hours, non-synchronicity occurs as the last trades of individual stocks may occur at different times.

According to Lo and MacKinlay [9], ignoring the non-synchronous trading effects can result in biased inference regarding the short-run behavior of prices. They proposed a data generating process (DGP) which models returns and allows for various forms of stochastic non-trading. Among others, using the proposed DGP they showed how non-trading can effect autocorrelation of the observed time-series. To cope with non-synchronous trading effects, Hamao et al. [7] disaggregated returns into open-to-close and close-to-open returns, while Forbes and Rigobon [6] used overlapping multiday returns.[5] Burns et al. [2] proposed to forecast the missing observations from values of known (presumably correlated) returns, therefore achieving synchronized returns.[6] The idea is that such a prediction, although not necessarily correct, will be an unbiased estimator of the true returns. They also show, that by ignoring non-synchronous trading effects, the correlations are understated, which has important implications in portfolio management (see also Martens and Poon [10] or Scherer [11]).

Probably the most popular approach to deal with non-synchronous trading effects involves using lower frequency data, i.e. weekly or monthly returns. The idea is, that time differences between closing prices are much smaller relative to 168 hours than to 24 hours (Martens and Poon [10]). Many authors who prefer using weekly returns claim that they mitigate or control the effects of non-synchronous trading, infrequent trading, bid-ask spreads, non-overlapping trading hours, spurious spill-over effects, or other microstructure frictions which can bias subsequent estimation.

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[4] The information set included in the closing prices of equities in Tokyo is obviously distinct from the information set at the end of the same trading day in New York.
[5] Thus introducing autocorrelation in the resulting series.
[6] The forecasts are based on a vector first order moving average model of returns with a GARCH covariance matrix.

There are several popular approaches used to construct weekly returns. While it is clear that the resulting series will differ with regard to their statistical properties, it is the goal of this paper to show, that these differences might be quite substantial. Unfortunately, in empirical studies, the method for data construction is rarely[7] specified.[8]

2 Data and methodology

Our dataset for empirical part of the paper covers six stock market indices, namely Polish WIG 20 (WIG), Czech PX, Hungarian BUX, MSCI Germany (GER), Dow Jones Euro Stoxx50 (STX) and MSCI World (MSCI) and a period from January 4, 1999 to September 28, 2012. Daily closing prices were obtained from the Thomson Reuters Datastream. Weekly returns were estimated using three methods, which can be grouped into conceptually two different approaches.

The *first approach* is a systematic sampling from the initial series of daily closing prices. Let P_n be the closing price of a stock market index in given day $n = 1, 2, ..., N$. Then assuming a week of five trading days, new series is created by sampling every fifth observation from the initial series of daily closing prices. If days from this subsample correspond to Fridays, the resulting series is denoted as S^F_t. Similarly, if these days correspond to Wednesday, the resulting series is denoted as S^W_t. The weekly (continuous) returns are then calculated as $ln(S_t/S_{t-1})$, where $t = 1, 2, \ldots, T$ corresponds to a given week and are denoted as *F-to-F* (Friday-to-Friday) and *W-to-W* (Wednesday-to-Wednesday). In case of *W-to-W* returns, if Wednesday prices were not available, they were substituted by prices in the following order: Thursday, Tuesday, Friday, and Monday. In case of *F-to-F* returns, if Friday prices were not available, they were substituted by prices in the following order: Thursday, Wednesday, Tuesday, and Monday. This approach corresponds to an intuitively appealing definition of a weekly return, where weekly stock market returns should correspond to a nominal continuous return acquired over five trading days.

The *second approach* takes averages over P_n of a given week t, i.e., $S^A_t = n(t)^{-1}\sum_{i=1}^{n(t)} P_{t,i}$, where $n(t)$ is the number of trading days (indexed as $i = 1, 2, \ldots n(t)$) in a given week t. The weekly returns are then calculated as before and denoted as *AV*. Using *AV* returns, one could argue, that average price is a representative value of prices in a given week. Therefore weekly returns from such series may also be regarded as representative returns. Note, that if one assumes, that stock prices can be regarded as a continuous-time stochastic process, then the resulting return is a return realized between prices from both weeks, but the time difference between them is stochastic, in contrast to *F-to-F* and *W-to-W* methods, where this difference is fixed and set to five trading days.

3 Results

3.1 Simulations

Knowing the DGP of the daily stock prices, one could derive the variance and autocorrelation of the resulting weekly returns. For example, assuming that daily closing prices follow $P_n = P_{n-1} + \delta_n$, with $\delta_n = \rho\delta_{n-1} + v_n$, where $v_n \sim N(0,\sigma^2)$ are *iid* and $|\rho| < 1$, we could derive the variance, autocovariance and autocorrelation of weekly re-

[7] For this version of the paper, we have performed a preliminary screening of 147 papers published via a selected publisher. In 79 cases (53.7%) the method used to construct weekly returns was not explicitly specified, although in 21 from these cases researchers used data from Datastream, where weekly returns correspond to the end of the week closing prices. Apart from that, end of the week prices were explicitly specified in 15 (10.2%) cases. Apart from few studies (e.g. Arab stock markets), end of the week prices refer to Friday closing prices. Corresponding returns are denoted as *F-to-F* (Friday-to-Friday). In 39 cases (26.5%) researchers used returns calculated from *W-to-W* (Wednesday-to-Wednesday) closing prices. Other days of the week were used in as much as 14 cases (9.5%). Note, that we are not concern by studies, where one is interested in in- or out-of sample forecasting of specific prices. Our sample includes studies, where weekly prices are used mostly due to non-synchronous trading effects. The use of *F-to-F* closing prices was rarely justified. The use of *W-to-W* prices was justified for various reasons (which depend on the markets): fewer holidays on Wednesdays compared to Fridays, higher trading volume on Wednesdays than in other trading days, elimination of the beginning-of-the-week and end-of-the-week effects. If Friday or Wednesday closing prices were unavailable, the last known closing prices are used, or a specific order of prices is specified (based on researchers choice and data availability, e.g. "*if Wednesday closing prices are unavailable we used Thursday, Tuesday, Friday or Monday closing prices*").

[8] If possible, the recommended approach is to use "synchronous closing prices", also called a "common window" (e.g. Schotman and Zalewska [12]; Égert and Kočenda [5]). This however requires high frequency data, which are not always available, particularly for analyzed periods longer than a decade. Even if daily data are available, in many instances, they are not required as for the problem under investigation, they can be too noisy.

turns. However, the assumptions put on v_n are rather simplistic. We have therefore decided to perform a simple simulation study, where the goal was to observe the differences between variance and autocorrelations of the simulated weekly returns. As defined in previous section, two approaches were compared.

The simulation was conducted as follows:

1. Using daily (continuously compounded) returns, we estimated ARMA(p, q)-GARCH(r, s) models. In this step, we have considered several specifications of the variance equation. For each of the following specifications; GARCH, CSGARCH, TGARCH, AVGARCH, NGARCH, FGARCH, APARCH, GJR-GARCH, NAGARCH we choose p, q (up to 5) and r, s (up to 2) such that no autocorrelation and conditional heteroskedasticity was indicated (using the Ljung-Box test) in the resulting standardized residuals. From these specifications, we chose the one, with the lowest BIC.[9]
2. Estimated coefficients were used to simulate returns.
3. Simulated returns were used to reconstruct the series of daily closing prices.
4. The simulated daily closing prices were used to draw weekly prices and calculate weekly returns using the two approaches as described in Section 2.
5. The variance and autocorrelation at lag 1 were computed.
6. Steps 2 – 5 were repeated 10000 times. The results may be found in Table 1 and Figure 1.

This simulation study revealed, that autocorrelation at first lag is higher for *AV* returns. Note, that when computing *AV* returns, the averaging is performed over non-overlapping trading days. One of the possible explanations of this autocorrelation is that averaging acts as a filtering process, thus reducing the noise of the series (note a lower variance in all series). This allows us to measure the underlying dependence among returns, which is otherwise masked by larger noise.

Panel A: CEE markets												
	WIG				PX				BUX			
	I. approach		*II. approach*		*I. approach*		*II. approach*		*I. approach*		*II. approach*	
	Mean	Std.	Mean	Std.	Mean	Std.	Mean	Std.	Mean	Std.	Mean	Std.
ACF(1)	0.0410	0.0500	0.2650	0.0440	0.0774	0.0563	0.3146	0.0501	-0.0125	0.0457	0.2224	0.0431
VAR	0.0011	0.0002	0.0007	0.0002	0.0011	0.0002	0.0008	0.0002	0.0012	0.0002	0.0008	0.0001
Panel B: Developed markets												
	GER				STX				MSCI			
	I. approach		*II. approach*		*I. approach*		*II. approach*		*I. approach*		*II. approach*	
	Mean	Std.	Mean	Std.	Mean	Std.	Mean	Std.	Mean	Std.	Mean	Std.
ACF(1)	-0.0024	0.0569	0.2323	0.0523	-0.0336	0.0672	0.2023	0.0615	0.0193	0.0548	0.2405	0.0500
VAR	0.0013	0.0005	0.0009	0.0004	0.0014	0.0018	0.0009	0.0011	0.0007	0.0004	0.0005	0.0003

Table 1 Mean and standard deviation of the autocorrelation coefficient and variance of simulated weekly returns

Notes: Std. – standard deviation, VAR – variance, ACF(1) – autocorrelation at lag 1.

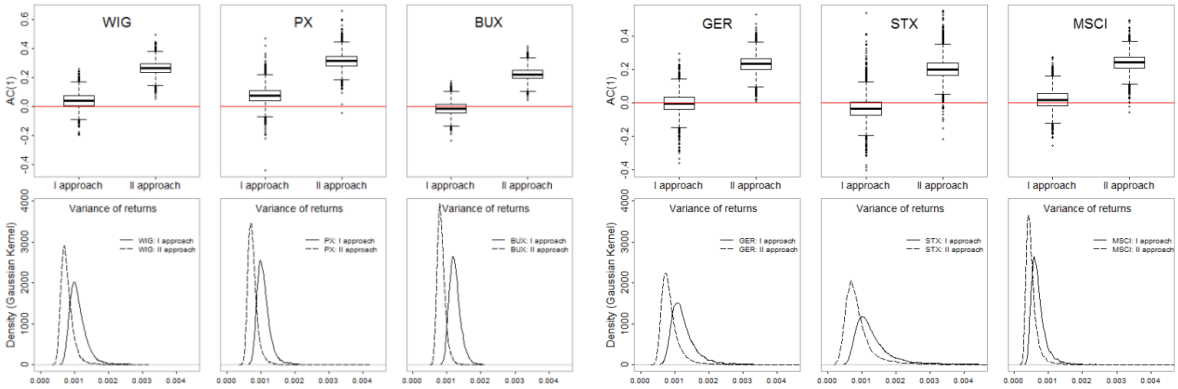

Figure 1 Simulated autocorrelation coefficients and variances of weekly returns

[9] For the sake of brevity, results from these models are not reported.

3.2 Empirical comparisons

Table 2 presents some descriptive statistics of the resulting weekly returns. These are in line with previous simulations. The standard deviation and extremes of returns for *AV* were lowest for each of the series. Similarly as in simulations, the systematically positive autocorrelation of *AV* returns was found. Small and positive first order autocorrelations suggest that *AV* returns have some short memory. This might have some implications for the random walk hypothesis (see Ding et al. [4]; Campbell et al. [3]). Therefore one may readily see that in this framework of weekly returns, the choice of the method of constructing weekly returns can already have some implications in empirical studies. Positive autocorrelation is not a rarity in financial time series. On the other hand, *W-to-W* or *F-to-F* returns have small autocorrelation, which might be more in line with the efficient market hypothesis and such series requires less filtering in a subsequent analysis.

Panel A: CEE markets									
	WIG			**PX**			**BUX**		
	AV	*W-to-W*	*F-to-F*	*AV*	*W-to-W*	*F-to-F*	*AV*	*W-to-W*	*F-to-F*
Mean	0.002	0.002	0.002	0.001	0.001	0.001	0.002	0.001	0.002
Std.	0.027	0.033	0.032	0.028	0.034	0.033	0.030	0.037	0.038
Min.	-0.128	-0.175	-0.171	-0.162	-0.170	-0.305	-0.176	-0.201	-0.269
Max.	0.093	0.125	0.120	0.135	0.133	0.156	0.095	0.151	0.152
ACF(1)	0.226	-0.037	0.042	0.240	-0.013	0.076	0.233	0.001	-0.003
JB	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Panel B: Developed markets									
	GER			**STX**			**MSCI**		
	AV	*W-to-W*	*F-to-F*	*AV*	*W-to-W*	*F-to-F*	*AV*	*W-to-W*	*F-to-F*
Mean	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Std.	0.028	0.035	0.035	0.027	0.034	0.033	0.022	0.025	0.026
Min.	-0.140	-0.165	-0.239	-0.121	-0.148	-0.251	-0.153	-0.165	-0.224
Max.	0.116	0.165	0.156	0.109	0.168	0.136	0.075	0.093	0.116
ACF(1)	0.143	-0.113	-0.039	0.101	-0.175	-0.083	0.154	-0.061	-0.022
JB	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 2 Descriptive statistics of weekly returns

Notes: Std. – standard deviation, Min. – minimum, Max. – maximum, ACF(1) – autocorrelation at lag 1, JB – the p-value of Jarque – Bera normality test.

Panel A: CEE markets									
	WIG			**PX**			**BUX**		
[100%]	$\frac{(Wr-Fr)}{Fr}$	$\frac{(Wr-AV)}{AV}$	$\frac{(Fr-AV)}{AV}$	$\frac{(Wr-Fr)}{Fr}$	$\frac{(Wr-AV)}{AV}$	$\frac{(Fr-AV)}{AV}$	$\frac{(Wr-Fr)}{Fr}$	$\frac{(Wr-AV)}{AV}$	$\frac{(Fr-AV)}{AV}$
1st qrt	-114.599	-28.918	-99.668	-113.676	-30.325	-87.651	-123.860	-36.498	-104.290
median	-33.706	11.816	-23.031	-33.925	12.537	-13.689	-44.580	10.004	-10.367
3rd qrt	43.893	60.315	49.302	45.237	62.934	63.028	27.720	53.410	64.231
IQR	158.491	89.233	148.970	158.913	93.259	150.678	151.580	89.907	168.521
Panel B: Developed markets									
	GER			**STX**			**MSCI**		
[100%]	$\frac{(Wr-Fr)}{Fr}$	$\frac{(Wr-AV)}{AV}$	$\frac{(Fr-AV)}{AV}$	$\frac{(Wr-Fr)}{Fr}$	$\frac{(Wr-AV)}{AV}$	$\frac{(Fr-AV)}{AV}$	$\frac{(Wr-Fr)}{Fr}$	$\frac{(Wr-AV)}{AV}$	$\frac{(Fr-AV)}{AV}$
1st qrt	-122.837	-34.460	-84.648	-130.875	-34.019	-116.234	-114.768	-28.049	-97.059
median	-39.449	11.388	-11.437	-41.251	13.303	-16.264	-39.442	14.831	-16.201
3rd qrt	37.013	53.036	72.706	37.548	64.393	58.321	36.199	51.214	63.890
IQR	159.850	87.496	157.354	168.423	98.412	174.555	150.967	79.263	160.949

Table 3 Descriptive statistics: Percentage of differences between weekly returns

Notes: Wr – W-to-W returns, Fr – F-to-F return, AV – weekly returns calculated from averages. 1st qrt – first quartile, 3rd qrt – third quartile, IQR – interquartile range.

Calculating percentage differences between weekly returns offers a more straightforward approach to comparison. As our dataset had many outliers, Table 3 presents quartiles.[10] *W-to-W* returns appear to be systematically smaller (from 33.706% to 44.580%) than *F-to-F* returns. *F-to-F* returns are also systematically larger than *AV*

[10] The presence of outliers is of no surprise, as in denominator, often very small (near zero) returns are present. This largely invalidates the use of moment based statistics like: mean, standard deviation, skewness, and kurtosis.

returns (from 10.367% to 23.031%). Finally, *W-to-W* returns appear to be systematically larger than *AV* returns (from 10.004% to 14.831%). Note, that this anomaly is different from day-of-the-week effects, although clearly it has implications on the series of weekly returns.

Perhaps even more important is the comparison of the variability of the percentage differences between weekly returns. Because of outliers, we have used interquartile range (IQR). Whenever *F-to-F* returns were used, the variability was large, at least compared to the variability of percentage differences between *W-to-W* and *AV* returns.

As previous analysis suggested, the differences between *W-to-W* and *AV* returns were smaller. We have performed a series of simple regressions to observe the dependence between different types of weekly returns. The regressions were in the following form: $Ar_t = \beta_0 + \beta_1 Fr_t + u_t$. Based on the estimates of the β_1 coefficients, we have tested whether the coefficient might be regarded as statistically different from 0, or in other instance different from 1. Inability to reject the second hypothesis would suggest that the differences between returns are not statistically significant. As errors from these regressions are probably autocorrelated and heteroskedastic, the standard errors of regression coefficients were calculated using HAC estimates of standard errors with kernel weights based on quadratic spectral and automatic bandwidth selection as proposed in Kiefer and Vogelsang [8]. Table 4 reports the results and the corresponding critical values, which are different from those derived from standard normal probability distribution.

As observed in Table 4, the dependence was highest among *W-to-W* and *AV* returns. Again, where *F-to-F* returns were involved, the dependence was smaller. All test rejected the null hypothesis, therefore all coefficients might be regarded as significantly different from 0, but at the same time, significantly different from 1, i.e., the differences between series are systematic.

Panel A: CEE markets									
	WIG			**PX**			**BUX**		
	Ar ~ Fr	*Ar ~ Wr*	*Wr ~ Fr*	*Ar ~ Fr*	*Ar ~ Wr*	*Wr ~ Fr*	*Ar ~ Fr*	*Ar ~ Wr*	*Wr ~ Fr*
Estimate of β_1	0.633	0.767	0.636	0.612	0.774	0.629	0.592	0.757	0.617
Std. error	0.049	0.019	0.062	0.033	0.020	0.053	0.025	0.013	0.026
H_0: $\beta_1 \leq 0$ [*t*-stat]	12.918	40.368	10.291	18.545	38.700	11.868	23.680	58.231	23.731
H_0: $\beta_1 \geq 1$ [|*t*-stat|]	7.490	12.263	5.890	11.758	11.300	7.000	16.320	18.692	14.731
b	0.279	0.191	0.434	0.257	0.160	0.375	0.216	0.180	0.372
5% critical values	2.732	2.341	3.670	2.627	2.173	3.297	2.442	2.261	3.297
Panel B: Developed markets									
	GER			**STX**			**MSCI**		
	Ar ~ Fr	*Ar ~ Wr*	*Wr ~ Fr*	*Ar ~ Fr*	*Ar ~ Wr*	*Wr ~ Fr*	*Ar ~ Fr*	*Ar ~ Wr*	*Wr ~ Fr*
Estimate of β_1	0.600	0.754	0.590	0.556	0.722	0.523	0.600	0.794	0.585
Std. error	0.016	0.012	0.026	0.014	0.017	0.016	0.008	0.014	0.012
H_0: $\beta_1 \leq 0$ [*t*-stat]	36.585	62.833	22.692	39.714	42.471	32.688	74.074	55.139	48.750
H_0: $\beta_1 \geq 1$ [|*t*-stat|]	24.390	20.500	15.769	31.714	16.353	29.813	49.383	14.306	34.583
b	0.322	0.160	0.453	0.329	0.159	0.470	0.326	0.194	0.441
5% critical values	2.966	2.173	3.816	2.966	2.173	3.947	2.966	2.341	3.670

Table 4 Regression coefficients: dependence between different types of weekly returns

Notes: Ar ~ Fr – this notation means that: "weekly returns calculated from averaged prices are explained by weekly returns calculated from Friday prices" (and analogously in other cases). Std. error – standard errors, b – corresponds to the proportion of the total sample size used as a bandwidth. 5% critical values are asymptotic critical values from Kiefer – Vogelsang [8].

4 Conclusion

Based on a simulation study and empirical data, we found evidence of qualitatively distinctive statistical properties of the weekly returns of selected stock market indices. Our main conclusion is that using *AV* returns introduces small and positive autocorrelation into the series, while *W-to-W* and *F-to-F* returns do not display this autocorrelation, which is usually found in daily stock returns. This is probably due to larger variance of the *W-to-W* and *F-to-F* weekly returns.

It is still premature to recommend one method over other. Clearly, if one is interested in in- or out-of-sample forecasting, *AV* returns might not be suitable at all. On the other hand, in most of the empirical studies, when one is interested in verifying certain relationships, *AV* returns have following advantages over *W-to-W* or *F-to-F* returns: (1) *AV* returns smear out the volatility of daily returns, which is lower compared to *W-to-W* or *F-to-F*

returns as well, (2) *AV* returns are less prone to day-of-the-week effects, (3) *AV* returns are less prone to non-synchronous trading effect II, which is still present for both *W-to-W* or *F-to-F* weekly returns.

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Generalization of geometric median

Přemysl Bejda[1]

Abstract. In the paper are suggested new robust estimators of location and variance. It is proved that these estimators have a breakdown point one half. The used method comes from a geometric median. In the first step it is shown that we can employ one half of observations and the estimate stays robust in the sense of the breakdown point. In the second step we show that we can add even more observations which are in some sense close to the geometric median and still get robust results. The robustness is proved in both steps for a multidimensional case. Since we can employ more observations and stay robust in the sense of the breakdown point, we enlarge the used information in comparison to other robust estimators like median and therefore get better results. We combine the advantage of the robust estimator and the classical mean. Our estimators are compared by simulation study with classical estimators like mean, median or alpha windsorised estimator. The comparison is done for different distributions like normal, Cauchy or exponential. We also consider the case, when the normal distribution is with some probability contaminated by another distribution. In the last section are shown some illustrations of the approach.

Keywords: breakdown point, contamination, geometric median, robustness

JEL classification: C14
AMS classification: 91G70

1 Generalized geometric median

We will consider any normed vector space.

Further we consider the concept of a breakdown point which was introduced in [1]. We employ the definition which can be found in [3].

Consider $S_{\boldsymbol{x}_i} = \sum_{j=1}^{n} \|\boldsymbol{x}_i - \boldsymbol{x}_j\|$.

Definition 1. Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ are any observations from some normed vector space. By a geometric median we mean the number

$$\hat{\boldsymbol{x}} = \underset{i=1,\ldots,n}{\operatorname{argmin}} \sum_{j=1}^{n} \|\boldsymbol{x}_i - \boldsymbol{x}_j\|. \tag{1}$$

If there are more observations which satisfies (1), we take an average of only two of them, which are different i.e. $\boldsymbol{x}_i \neq \boldsymbol{x}_j$.

Remark 1. It is not necessary to take the average of only two observations to gain geometric median. It is done only for purpose of the following proposition.

The standard definition of geometric median is slightly different from the ours.

Remark 2. By $\|\cdot\|_q$, $q \in \mathbb{N}$ we understand a norm in $\mathbb{R}^p$ such that any $\boldsymbol{x} \in R^p$ satisfies

$$\|\boldsymbol{x}\|_q = \sqrt[q]{|x_1|^q + \cdots + |x_p|^q}.$$

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And we denote $\|\boldsymbol{x}\|_\infty = \max_{i=1,\dots,p} |x_i|$.

In special one dimensional case are all norms $\|\cdot\|_q$ equal. The distance between x and y is for us $\|x-y\|$.

Proposition 1. *For any norm $\|\cdot\|_q$ and one dimensional case is the geometric median $\hat{x}$ equal to a median x^{m}.*

Proof. We know that the median can be find by $\operatorname{argmin}_{a\in\mathbb{R}} \sum_{i=1}^{n} |x_i - a|$. We order the observations. In case of n odd we get the observation in the middle. So this observation has to be as well the geometric median.

If n is even then the median is the average of two observations in the middle of our ordered observations (for simplicity we neglect some degenerated cases). Denote these two observations $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ and let $\boldsymbol{x}^1 \leq \boldsymbol{x}^2$. Now we will compute $S_{\boldsymbol{x}^1}$ and $S_{\boldsymbol{x}^2}$. Denote D^1 the sum of distances between $\boldsymbol{x}^1$ and all observations less than $\boldsymbol{x}^1$. Similarly D^2 the sum of distances between $\boldsymbol{x}^2$ and all observations greater than $\boldsymbol{x}^2$. $D^3 = \|\boldsymbol{x}^2 - \boldsymbol{x}^1\| = |\boldsymbol{x}^2 - \boldsymbol{x}^1|$. Then $S_{\boldsymbol{x}^1} = D_1 + D_3 + \frac{n-2}{2} D_3 + D_2 = S_{\boldsymbol{x}^2}$. So the distances are same.

We know that if we remove $\boldsymbol{x}_{(n)}$ then $S_{\boldsymbol{x}^1}$ is the smallest. If we return $x_{(n)}$ back then for all $\boldsymbol{x}_i \leq \boldsymbol{x}^1$ is $S_{\boldsymbol{x}^1} \leq S_{\boldsymbol{x}_i}$, because the distance between $\boldsymbol{x}^1$ and $\boldsymbol{x}_{(n)}$ is smaller than between other these observations and $\boldsymbol{x}_{(n)}$. The same is true for $\boldsymbol{x}^2$ and observations greater than this one. So we get that the value $S_{\boldsymbol{x}^1} = S_{\boldsymbol{x}^2}$ is the smallest.

We should note that in this proof is necessary to use the last sentence from the definition of our case of the geometric median. □

Now we will deal with the breakdown point of $\hat{\boldsymbol{x}}$. According to the definition of the breakdown point we try to find when $\|\hat{\boldsymbol{x}}\| = \infty$ if enough observations satisfy $\|\boldsymbol{x}\| = \infty$.

Proposition 2. *The break down point of geometric median $\hat{\boldsymbol{x}}$ is one half.*

Proof. Let $k < \lceil n/2 \rceil$ denote the number of observations, which can be contaminated (i.e. $\|\boldsymbol{x}\| = \infty$). We propose that k points can be arbitrary far from the rest of points and our estimator stay among the original set. Denote by K_1 the set of points which were changed and by K_0 the set of points which stayed unviolated. We will proceed by a contradiction. Let us assume that $\hat{\boldsymbol{x}} \in K_1$. We denote $\boldsymbol{x}^l = \operatorname{argmin}_{i\in K_0} \|\hat{\boldsymbol{x}} - \boldsymbol{x}_i\|$, $A = \min_{i\in K_0} \|\hat{\boldsymbol{x}} - \boldsymbol{x}_i\|$, $B = \sum_{x_i \in K_1} \|\hat{\boldsymbol{x}} - \boldsymbol{x}_i\|$ and $a = \max_{i,j\in K_0} \|\boldsymbol{x}_i - \boldsymbol{x}_j\|$. Observations in K_1 are arbitrary far from the original observations. Let then $A \geq \min_{\boldsymbol{x}_i\in K_0, \boldsymbol{x}_j \in K_1} \|\boldsymbol{x}_i - \boldsymbol{x}_j\| > (n-1)a$. Now compute

$$S_{\hat{\boldsymbol{x}}} = \sum_{\boldsymbol{x}_i \in K_0} \|\hat{\boldsymbol{x}} - \boldsymbol{x}_i\| + \sum_{\boldsymbol{x}_i \in K_1} \|\hat{\boldsymbol{x}} - \boldsymbol{x}_i\| \geq (n-k)A + B.$$

For any $\boldsymbol{x} \in K_0$

$$\begin{aligned} S_{\boldsymbol{x}} = \sum_{\boldsymbol{x}_i \in K_0} \|\boldsymbol{x} - \boldsymbol{x}_i\| + \sum_{\boldsymbol{x}_i \in K_1} \|\boldsymbol{x} - \boldsymbol{x}_i\| \leq \\ (n-k-1)a + \sum_{\boldsymbol{x}_i \in K_1} \|\boldsymbol{x} - \hat{\boldsymbol{x}} + \hat{\boldsymbol{x}} - \boldsymbol{x}_i\| \leq \\ (n-k-1)a + k\|\boldsymbol{x} - \hat{\boldsymbol{x}}\| + B \leq \\ (n-k-1)a + k\|\boldsymbol{x} - \boldsymbol{x}^l\| + k\|\boldsymbol{x}^l - \hat{\boldsymbol{x}}\| + B \leq \\ (n-k-1)a + ka + kA + B = (n-1)a + kA + B. \end{aligned}$$

We get the contradiction if $S_{\boldsymbol{x}} < S_{\hat{\boldsymbol{x}}}$. This holds when $(n-1)a < (n-2k)A$. But $n-2k$ is according to the definition of k at least 1. So it is enough when $(n-1)a < A$. But this is the contradiction. □

Let us now derive another kind of estimators from the geometric median.

Definition 2. Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ are any observations from some normed vector space. Let in a set L is one half of observations which distances from geometric median are less than a distance of any other observation which does not lie in the L. By a G2 estimator we mean the number which is computed as an average of $\lfloor n/2 \rfloor$ observations lying in L. Denote this by $\tilde{\boldsymbol{x}}$.

Proposition 3. *The break down point of $\tilde{\boldsymbol{x}}$ is one half.*

Proof. Once more it can be contaminated at most $k < \lceil \frac{n}{2} \rceil$. L is the same set as in the definition. For such k we know that $\hat{\boldsymbol{x}} \in L$ and it is not an outlier. For a contradiction we suppose that exist $\boldsymbol{x}_u \in L$, which is an outlier. Let K_0 is a set of not outlying observations and K_1 is a set of outlying observations and $A = \max_{\boldsymbol{x}_i \in K_0} \|\boldsymbol{x}_i - \hat{\boldsymbol{x}}\|$. Let $\boldsymbol{x}$ be the estimate which was gained from the sample, where were not any outlying observations. Put $B = \min_{\boldsymbol{x}_i \in K_1} \|\boldsymbol{x}_i - \boldsymbol{x}\|$, $b = \|\boldsymbol{x} - \hat{\boldsymbol{x}}\|$. Now $\|\boldsymbol{x}_u - \boldsymbol{x}\| \leq \|\boldsymbol{x}_u - \hat{\boldsymbol{x}}\| + \|\hat{\boldsymbol{x}} - \boldsymbol{x}\|$ from that follows $\|\boldsymbol{x}_u - \hat{\boldsymbol{x}}\| \geq B - b$. Since the outlier can be arbitrarily far from $\boldsymbol{x}$, we can choose B large enough i. e. $B - b > A$ to get the contradiction. □

The set L can be employed for constructing different kinds of estimators. We can take for instance once more the median from the set or $r = \max_{\boldsymbol{x}_i, \boldsymbol{x}_j \in L} \|\boldsymbol{x}_i - \boldsymbol{x}_j\|$ can serve as an estimator of inter quartile range (from the similar estimator we can, under assumption of normality, easily derive an estimator of variance) etc.

Definition 3. Let $r = \max_{\boldsymbol{x}_i, \boldsymbol{x}_j \in L} \|\boldsymbol{x}_i - \boldsymbol{x}_j\|$. $b > 0$. Let the set E contains all observations $\boldsymbol{x}$ such that $\min_{\boldsymbol{x}_i \in L} \|\boldsymbol{x}_i - \boldsymbol{x}\| \leq br = a$.

Proposition 4. *If we make any estimator $\tilde{\boldsymbol{x}}$ from observations in E then the estimator has the break down point one half. It means that the estimator $\tilde{\boldsymbol{x}}$ is a function of the observations from E. The function has to be bounded on any finite interval.*

Proof. Because of the property of the function, from which is the estimator constructed, we have to investigate only the situation when observation from E tends to infinity. The rest of the proof is based on the same idea as the previous proof, so we skip it. □

Remark 3. The last proposition instruct us, how to include more observations and so increase efficiency. But a should not be too large. We can for instance employ theoretical quantiles of a normal distribution to find an appropriate constant a. Let us denote the constant a as a constant of widening.

Remark 4. Thanks to the properties of a norm we can easily show that our estimators (we employ an average on the set L and E respectively) are scale and location equivariant.

2 One dimensional case and simulation study

Let us now consider only one dimensional case. We want to compute a mean from observations in E, but we have to gain the set.

Let X is a random variable with a mean μ, a standard deviation σ and let $q_{\mu,\sigma}(\alpha)$ denotes its quantile function. Then for any continuous distribution G (in case that the second moment is finite) the following ratio is constant for every μ and σ

$$K_{G,\alpha} = \frac{q_{\mu,\sigma}(1 - \alpha/2) - q_{\mu,\sigma}(\alpha/2)}{q_{\mu,\sigma}(0.75) - q_{\mu,\sigma}(0.25)}$$

This comes from the fact $\sigma q_{0,1}(\alpha) + \mu = q_{\mu,\sigma}(\alpha)$.

Let us further suppose that $X \sim \mathrm{N}(\mu, \sigma)$. We construct estimate of $q_{\mu,\sigma}(0.75) - q_{\mu,\sigma}(0.25)$ such that we put $\mathrm{RIQR} = \mathrm{argmax}_L(x) - \mathrm{argmin}_L(x)$. In this manner we construct a robust estimator

of inter quartile range RIQR and it has according to the proposition 3 the breakdown point one half. We put the constant of widening as $a = K_{\mathrm{N}(0,1),\alpha}\mathrm{RIQR}$.

Otherwise than in the definition 3 we take all observations which distance from their median is less than a. Let us denote the set GM3S. The only difference is that we do not take into account all observations from the set L, but only one from these observations. So the proposition 4 stays valid. This approach gives also better results in practise. We can employ other symmetric distributions than normal.

All employed methods and a generation of random samples were implemented in R. For each situation we have 1000 samples and in each sample 100 observations. The estimators used in simulation study are: mean, median, $\alpha-$winsorised mean, truncated mean, GM2 and GM3, where GM2 and GM3 are means from the sets L and GM3S respectively.

During the simulation study we truncate 20 percent of observations from each side for the $\alpha-$winsorised mean. We do the same for the the $\alpha-$truncated mean. And we employ $\alpha = 0.01$ for the GM3.

In the following tables is employed a function $\sum_{i=1}^{N} |\widehat{\boldsymbol{x}}_i - \mu|$, where N is a number of samples (1000), $\widehat{\boldsymbol{x}}_i$ is some estimate (mean, median, etc.) from sample i, and μ is a parameter of location, which is known (it is usually a median of the not contaminated distribution).

In the first table is a normal distribution $\mathrm{N}(0,1)$ contaminated by some other distribution with probability p.

An expression $\mathrm{U}(a,b)$ denotes a uniform distribution on the interval (a,b).

Distribution	Mean	Med	Trim	Winsor	GM2	GM3
$p = 5\%$						
$\mathrm{N}(0,100)$	187	106	90	89	123	85
Cauchy	247	99	86	86	115	86
$\mathrm{U}(-10,10)$	131	108	92	90	123	89
$p = 10\%$						
$\mathrm{N}(0,100)$	270	104	93	94	122	87
Cauchy	481	101	89	88	116	88
$\mathrm{U}(-10,10)$	170	115	102	103	130	95
$p = 40\%$						
$\mathrm{U}(-20,10)$	1960	277	546	1062	184	155

Table 1: A normal distribution contaminated by a distribution from the first column with a probability p.

Distribution	Mean	Med	Trim	Winsor	GM2	GM3
$\mathrm{N}(0,1)$	78	100	85	83	117	84
$\mathrm{N}(0,100)$	774	974	830	811	1132	824
Cauchy	5101	121	138	168	121	135
$\mathrm{U}(-10,10)$	476	803	638	561	1034	476
t_5	104	105	94	95	119	98
t_{10}	91	104	91	91	119	92
Laplace	118	87	95	105	94	99

Table 2: Different symmetric distributions.

In the table 2 are employed some symmetric distributions always with parameter of location equal to 0 (for Laplace distribution is scale equal to 1).

2.1 Conclusion

From the simulation study is visible that GM3 is robust and employ enough observations. Under assumption of normality gives GM3, $\alpha-$trimmed and $\alpha-$winsorised means almost the same results, but for more contaminated cases is our estimator much better.

A disadvantage of GM3 can appear when we set wrongly the widening constant (for different distribution), but the table 2 shows that this does not violate the results too much (especially when our choice was rather conservative for normal distribution). In case of asymmetric distribution we could spread the set L asymmetrically.

GM2 is good only in cases of really bad contamination or for some heavy tailed distributions. But it serves as the first step for GM3.

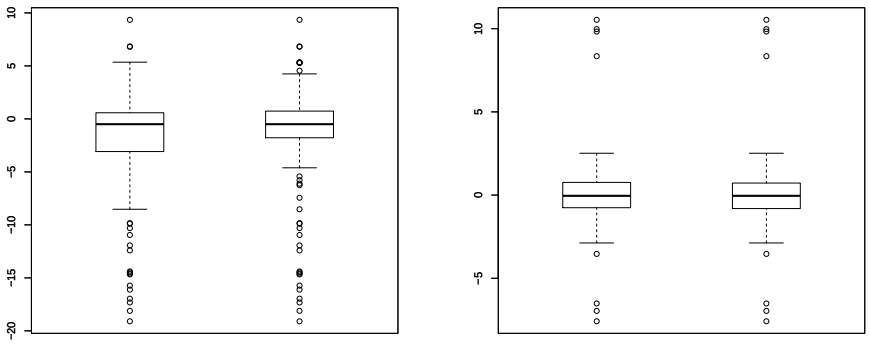
Computational complexity for both GM2 and GM3 is $n \ln n$ as for other robust estimators. But in practise it is more time consuming than computation of median.

The theoretical equivalent of GM2 and GM3 is not easy to grasp.

3 Illustrations

3.1 Boxplot

In boxplots we can employ instead of IQR the RIQR estimator. As whiskers can be taken borders, which are given by constant of widening and are used for construction of GM3 (the last and the first observation from GM3S). To get proper results it is necessary to have symmetric density of a not contaminated distribution. In a case of really bad contamination we are able to classify outliers better.



(a) $p = 0.4$, n=100, U$(-20, 10)$ (b) $p = 0.1$, n=100, N$(0, 100)$

Figure 1: On the left side is the classical boxplot and on the right side is our modification. In both cases we contaminate N$(0, 1)$ with a probability p by the observations with a distribution which is stated under the figures.

In the figure 1 is visible that the standard boxplot and our method give very similar results in

case of mild contamination. When the contamination is higher than 25 % then our boxplot is still able to find outliers in comparison to the standard boxplot.

3.2 Outlying shortfall

We can construct a similar statistic to an expected shortfall. Very brief overview about a topic of VaR and expected shortfall can be found e.g. in [2].

Let r_t is a logarithmic return in time t. We suppose that r_t are independent and has a normal distribution contaminated by some other distribution $r_t \sim p \cdot G + (1-p) \cdot \mathrm{N}(\mu, \sigma^2)$. We are interested in how much we can loss because of the contamination. We take a negative value of a mean from all observations which are less than argmin(GM3S).

As a small illustration of this approach we employ EUR/USD hourly rates from 1. 8. 2012 to 3. 11. 2012.

Results:

- A 95 % VaR is 0.00140406. As the VaR we take a negative value of a 5 % quantile from observed returns.
- A 95 % expected shortfall is 0.00219346. As the expected shortfall we take a negative value of an average of all returns lower than the 5 % quantile.
- An outlying shortfall (with a constant of widening with $\alpha = 0.01$) is 0.002376221.
- A 99 % VaR is 0.002517572.
- A 99 % expected shortfall is 0.003608807.

A disadvantage of this approach is its worse interpretation. We can also determine how often we suffer a loss because of outlying observations. In our case it is 3.91 %.

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The Use of Belief Functions for the Detection of Internet Auction Fraud

Ladislav Beránek[1], Václav Nýdl[2]

Abstract. Currently, Internet auction portals are a standard part of business activities on the Internet. Anyone can easily participate in online auctions, either as a seller or a buyer (bidder), and total turnover on Internet auction portals attains billions of dollars. However, the amount of fraud in these Internet auctions is also related to their popularity. To prevent discovery, fraudsters perform normal trading behaviors and disguise themselves as honest members. It is therefore not easy to detect fraud in online auctions. Users must rely on information that is on the web auction systems available, for example user's ratings (reputation), descriptions of the offered items, time frames of different activities and records of transactions. This paper presents a set of characteristics that characterize fraudulent behavior on the internet auction – the sale of counterfeit or stolen goods. To evaluation, whether or not this fraudulent behavior occurs, an approach based on belief functions is applied. Experimental results show that this approach gives good results in detection of this type of fraud on online auctions.

Keywords: e-commerce, fraud, belief function, online auction

JEL classification: L81
AMS classification: 68T37

1 Introduction

Currently, many users participate in online auctions organized by many different Internet online auction systems operating on Internet infrastructure. Aukro [1] (Czech auction company with a turnover of USD 250 million and 2.5 million users in January 2012) is an example of such Internet auction system. Internet auctions allow their users to buy or sell a great amount of products and services. A large number of users use Internet auctions even as their main means of trading. For example Aukro states [1] that from 2.5 million users 9,300 users are professional dealers.

On the other hand, the amount of frauds on online auction is also increasing. The most common frauds are incorrect (purposely) description of goods, undelivered goods, irredeemable payments, sale of stolen goods, and more. Fraudsters are attracted by low admission costs and high profit potential.

This paper examines the characteristics of Internet auction fraud on Czech Aukro online auction systm [1] with a focus on specific fraudulent behaviour - the sale of stolen goods. The rest of this paper is organized as follows: section 2 presents the basic principles of the theory of belief functions. Section 3 summarizes some related work concerning the use of this theory in the field of Internet auctions. Section 4 presents our approach and describes key definitions of belief functions to represent fraudulent behavior. Section 5 presents our experimental results and model verification. Section 6 describes some interesting conclusions and directions for further research.

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2 Background and Related Work

The online auction sellers and bidders are not in physical contact and bidders can not even physically see the auctioned items. This situation provides opportunities for cheating [6, 7].

From the perspective of the seller, online auctions bring along following risks [9, 11], particularly:

- Bidder will not to pay for the goods supplied.
- Bidder wrongly claims that the goods were not delivered.

From the perspective of the bidder:

- The seller refuses to send the goods.
- The description of the auctioned object is false.
- Seller sends a different goods or goods that does not match by its quality [8].
- Goods intended for auction is a fake or stolen.

Fraudulent behavior is on online auctions relatively widespread. The main reasons is that it is relatively easy to be carried out:

- Online auction participants are largely anonymous – they act under a pseudonyms. Internet auction systems use different methods for verifying the identity of users. These methods, however, may not be sufficiently reliable.
- Operators of online auctions insufficiently monitor course of auctions.
- The laws are often unclear with regard to specific situations or to the diversities of legal systems in different countries.

3 The Suggestion of Our Model

We used the approach similar to [4, 5, 12] and we performed statistical analysis of 28 real cases of online auctions on Aukro.cz auction system [1] in order to find out the characteristics of online auction fraud related to the sale of counterfeit or stolen goods. These cases were chosen on the basis of the complaints referred to in various internet forums, for example, [2, 3]. Users on these forums complained that somebody had stolen some their thing (radio from their car) and this thing appeared shortly after the theft on some Internet auction. We evaluated 8 auctions as auctions in which the stolen goods are sold. Following characteristics of online auction offering stolen goods observed:

1. Stolen goods are sold at inadequate low prices (at least about 20% less than the price of legitimate goods).
2. Fraudsters prefer to sold for fixed price.
3. A variety of goods are sold via fraudulent account (such as car accessories, footwear, sporting goods etc.).
4. Activities of respective fraudulent account is very short (often less than ten days).
5. In most of the cases, the goods are sold within several days of creating the account.
6. Fraudsters have accounts on multiple auction systems, the value of their reputational score is not high.

We will discuss these characteristics in the following paragraphs. We will define the corresponding belief functions. We will perform combination of respective belief functions and we will formulize conclusion whether the respective seller sells stolen goods on respective Internet auction account or not.

We have chosen the following attributes of sale of stolen goods on the basis of our analysis. (the other ones are too difficult to verify or to express them mathematically):

1. Inadequate low price;
2. Goods are sold mostly at fixed price;
3. A variety of goods being sold.

In the following text, we will denote $\Theta_i = \{stolen_i, \neg stolen_i\}$ as a frame of discernment [10] concerning the our belief that the seler i sells stolen goods.

Inadequate low price

This attribute shows that the seller i sells stolen good for lower price than it is the average price of legitimate good. The belief functions have the following form:

$$\begin{aligned} m_L(\{stolen_i\}) &= \begin{cases} v_{LA}\frac{\overline{P}-P_i}{\overline{P}} & \text{for } P_i \leq \overline{P} \\ 0 & \text{for } P_i > \overline{P} \end{cases} \\ m_L(\{\neg stolen_i\}) &= 0 \\ m_L(\Theta_i) &= \begin{cases} 1 - v_{LA}\frac{\overline{P}-P_i}{\overline{P}} & \text{for } P_i \leq \overline{P} \\ 1 & \text{for } P_i > \overline{P} \end{cases} \end{aligned} \tag{1}$$

where v_L is the weight of this evidence. We can intuitively read this weight as a reliability of this evidence, P_i – is the average price at which the seller sells certain goods. $\overline{P}$ is the average price of this goods offered through online auction system.

With this equation, we have expressed the inadequate low price of goods offered by the seller i. Usually, the lower the price of goods offered by a seller i, compared to the average price of respective goods j, the higher the suspicion that the seller offers stolen goods. Therefore, we assume that the equation reflecting the offering of legitimate goods, does not show that the seller does not offer "*stolen*", i.e. $m_L(\{stolen_i\}) = 0$.

Goods are sold mostly at fixed price

The sellers (fraudsters) want to sell his goods as quickly as possible. They want to get rid of it as quickly as possible. Therefore they prefer to sell goods at a fixed price (on Internet auction systems it is the option "buy now"). When a seller sells goods at fixed price, the auction ends and the seller does not have to wait to the end of the auction. Belief functions will have the following forms:

$$\begin{aligned} m_F(\{stolen_i\}) &= v_F\frac{N_{Fi}}{N_i} \\ m_F(\{\neg stolen_i\}) &= 0 \\ m_F(\Theta_i) &= 1 - v_F\frac{N_{Fi}}{N_i} \end{aligned} \tag{2}$$

where N_{Fi} is the number of goods of the seller i for the fixed price, N_i is the total number of goods sold by this seller.

It is valid that the higher the number of goods sold at fixed price, compared to the total number of goods sold by this seller, the higher the suspicion that this seller sells "*stolen*" goods. Therefore, we also assume that the given equation does not indicate that the seller does not sells "*stolen*" goods, i.e. $m_F(\{stolen_i\}) = 0$. The parameter v_F is in these equations the weight of evidence. We can intuitively interpret this weight as the reliability of the given evidence.

A variety of goods being sold

Let's suppose that the seller sells "*stolen*" goods. This fact has an effect that he sells the goods that he "gets". The variety of goods being sold is then higher than at the average proper seller. The belief functions of this attribute have the following form:

$$\begin{aligned} m_V(\{stolen_i\}) &= \begin{cases} v_V \frac{V_i - \overline{V}}{V_i} & \text{for } \mathrm{V}_i \geq \overline{V} \\ 0 & \text{for } \mathrm{V}_i < \overline{V} \end{cases} \\ m_V(\{\neg stolen_i\}) &= 0 \\ m_V(\{\Theta_i\}) &= \begin{cases} 1 - v_V \frac{V_i - \overline{V}}{V_i} & \text{for } \mathrm{V}_i \geq \overline{V} \\ 1 & \text{for } \mathrm{V}_i < \overline{V} \end{cases} \end{aligned} \tag{3}$$

where V_i is the amount of different types of goods sold by seller i, $\overline{V}$is the amount of different types of goods sold by proper sellers in a respective category. The v_V parameter is the weight of evidence. We can intuitively interpret this weight as the reliability of the given evidence.

It is valid that the more amount of different goods the seller sells, compared to the average types of goods sold by proper seller, the higher the suspicion that the seller sells a "fake" goods. Therefore, we also assume that the given equation does not indicate that the seller does sell "*stolen*" goods, i.e. $m_V(\{stolen_i\}) = 0$.

Combination of characteristic signs (proofs) of shill behavior

One characteristic alone is not enough to identify fraudulent behavior. This, once we have obtained the belief functions expressing our belief regarding fraudulent behavior, we combine them in a consistent manner to get a more complete assessment of what the whole group of signs indicates. The combination of belief functions is done with the help of the Dempster's combination rule [10]. We express the assumption that a given seller i sells "*stolen*" goods with the help of belief function $m(\{stolen_i\})$. We calculate the value $m(\{stolen_i\})$ using the combination of single belief functions expressing appropriate evidence:

$$m(\{stolen_i\}) = (m_L \oplus m_F \oplus m_V)(\{stolen_i\}) \tag{4}$$

The operator $\oplus$ is the Dempster's rule of belief function combination [10].

We perform the combination of multiple proofs according to the Dempster's rule – first we combine two belief functions, then we combine the result with the third belief function, fourth belief function and so forth.

Categorization of users according to the resulting belief functions representing the behavior - selling of stolen goods.

When constructing belief functions, we assume that they do not reflect that a given seller does not sell stolen goods. It is valid that $m(\{stolen_i\}) = 0$. After calculating the belief value that the seller i shows the character of illegal behavior (selling stolen goods), the value $m(\{stolen_i\})$ is assigned to the seller i as a measure which indicates the strength of the conviction that the user i sells stolen goods [4]. Now, we will categorize users into categories according to the view that a certain user i sells stolen goods. We will classify them into two categories: "*Seller sells stolen goods*" and "*Proper user*".

We have to define threshold η. We will categorize according to these threshold as follows: if $m(\{stolen_i\}) \geq \eta$ then the user i sells stolen goods. If $m(\{stolen_i\}) < \eta$ then the user i is a proper user. The threshold η will be qualified on the basis of statistical evaluations of analyzed auctions.

4 Case Study and Analysis of Results

To demonstrate the feasibility of the suggested reputation mechanism, we tested our methodology using real auction data from Aukro.cz [1]. We explored the bidding history of 28 auctions and various Internet discussions dedicated to selling stolen goods on online auctions. We investigated past auctions hosted by a particular seller. We evaluated 8 auctions as auctions in which the stolen goods are sold.

We explored mainly prices, number of auction carried out for fixed prices, number of various goods sold by sellers. We had to investigate all information manually because Aukro does not have (in contrast

to eBay any API interface enabling automatic gathering of information. We calculated the degree of belief that the seller sells "stolen" goods. We expressed belief functions defined on the section 4 on the basis of observed values. Then, we calculated the total degree that the seller sells "stolen" goods.

Some typical results of our exploration of auctions on Aukro are presented in Tables I and II. The basic masses assigned for evidence specified in our model and the resulting values are shown in Table II.

Table I. Chosen auction data collected from Czech online auction Aukro [1]

Seller i	Average price at which the seller i sells certain goods [in CZK]	Average price of this goods through on-line auction system [in CZK]	Number of goods which the seller i sells at fixed price	Total number of goods sold by seller i	Amount of different type of goods offered by seller i	Average amount of different type sold goods sold by sellers in respective category
D***r	1500	2525	2	2	2	2
O***2	700	1850	1	1	1	2
m***k	1250	1600	1	2	2	2
d***l	650	750	2	6	6	2
2***j	1420	1540	1	7	5	2
b***s	1200	1450	0	8	5	2

The values of belief of selling stolen goods are calculated using equations (1), (2), (3) and (4). Calculations are presented in Table II. The weights of evidence v_L, v_F and v_V were set in agreement with our experiments at the level 0.9, 0.7 and 0.8. We consider the characters "Inadequate low price", as the most predicative. The character "Goods sold at fixed prices" is the less reliable in determining a selling of fake goods. These values correspond to our survey of the relevance of single described characteristics from examined auctions. The value of threshold η was set on the basis of our experiments at the levels 0.80. This value correspond to results of our exploration of the relevance of single characteristics from examined auctions.

Table II. The basic masses assigned to single "selling stolen goods" characteristics ((1), (2), (3) and (4).

Seller i	$m_L(\{stolen\})$	$m_F(\{stolen\})$	$m_V(\{stolen\})$	$m(\{stolen\})$	$m(\Theta)$	Result
D***r	0.405941	0.7	0	0.821782	0.178218	Seller sells stolen goods
O***2	0.621622	0.7	0	0.886486	0.113514	Seller sells stolen goods
m***k	0.21875	0.35	0	0.492188	0.507813	Proper seller
d***l	0.133333	0.233333	0.533333	0.689926	0.310074	Proper seller
2***j	0.077922	0.1	0.48	0.568468	0.431532	Proper seller
b***s	0.172414	0	0.48	0.569655	0.430345	Proper seller

Values of $m(\{stolen\})$ express our belief that the seller sells stolen goods. On the other side, values $m(\Theta)$ represent our uncertainty or rather ignorance concerning the classification if the seller sells legal goods or if the seller sells stolen goods. The values of $m(\{stolen\})$ are high at the sellers D***r and O***2. They are greater than the threshold η.

5 Conclusion and Future Work

In our work, we presented a computational model of trust to seller on online auctions. It is based on the seller's rating obtained after performed transactions and his possible fraudulent behavior. We verified our model on Czech online auction Aukro. We performed a number of experiments on this auction. We make certain that we can increase the prediction of trust to seller by using reputation (rating information) as a basis which is completed with evaluation of possible fraudulent behavior (shilling) as additional information. Nevertheless we are also aware that the mathematical formalization of parameters used in our model (especially the parameters v_L, v_F, v_V) is necessary to increase the practical usefulness of our model.

In our future work, we want to define these parameters by the help of mathematical formulas. We will perform further statistical analyses of online auctions to verify these formulas and values of the parameters of our model.

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Average rate of return of pension or investment funds based on original, stochastic and continuous price index

Jacek Białek[1]

Abstract. In this paper we consider the problem of the proper construction of the average rate of return of pension (or investment) funds. The economic postulates for this kind of measure were formulated by Gajek and Kałuszka [6]. It is easy to show (see Białek [5]) that the definition of the average return given by these authors (and an original definition) can be expressed by using a chain price index. In our paper we propose and discuss an analogical continuous-time formula. We consider the situation when the prices and the number of the participating units are stochastic processes. We also show in a simulation study that all presented formulas and the continuous formula approximate each other.

Keywords: average rate of return of funds, price index theory

JEL Classification: C43, G12, G23
AMS Classification: 62P20

1 Introduction

There is a number of measures of the efficiency of open pension (or investment) funds (see Białek [3] or Białek [4]). The measures should be properly defined – it means that all changes of fund's assets, connected with any investment, should have impact on the given measure. The information about the average return of the group of funds is very important both for fund clients and fund managers. Firstly, it allows to compare the financial outcome of the given fund to the rest of funds. It may be helpful to customers in making a decision about money allocation. Secondly, having the knowledge about the average returns of investment funds from different sectors (manufacturing, agricultural, service etc.) provides important information about the financial situation within these sectors. And finally, in case of pension funds we can find legal regulations defining the *minimal rate of return* of funds based on the average rate of return. For example, in the Polish law regulations (The Law on Organization and Operation of Pension Funds, Art. 173, Dziennik Ustaw Nr 139 poz. 934, Art. 173; for the English translation see *Polish Pension…*, 1997, see [21]) half of the average return of a group of funds or the averege return minus four percentage points (depending on which of these values is higher) determines a minimal rate for any pension fund. In case of a deficit the *weak* fund has to cover it. It is always a very dangerous situation for the fund[2]. Under the Polish law the average return of a group of n pension funds is defined as:

$$\bar{r}_0(T_1,T_2)=\sum_{i=1}^{n}\frac{1}{2}r_i(T_1,T_2)\cdot\left(\frac{A_i(T_1)}{\sum_{i=1}^{n}A_i(T_1)}+\frac{A_i(T_2)}{\sum_{i=1}^{n}A_i(T_2)}\right) \tag{1}$$

where $r_i(T_1,T_2)$ denotes the rate of return of the $i-$th fund during a given time period $[T_1,T_2]$ and $A_i(t)$ denotes the value of $i-$th fund's assets at time t. Since 2004 the results of funds for the last 36 months have been verified twice a year. Unfortunately, the measure defined in (1) does not satisfy some economic postulates given by Gajek and Kałuszka [6]. Moreover, considering an even number of funds, where half of them have the return rates equal to 50% and the rest of funds have the return rates equal to (-50%), we should get the real average return rate on the level 0%. But using formula (1) we get 12.5 %. In our opinion, this is an argument for searching new definitions of the average rate of return of a group of funds. Our propositions for a discrete time can be found in Białek's paper [4]. In this work we propose and discuss an analogical formula for the continuous time. We consider the situation when the prices and the number of the participating units are some stochastic processes. We also show in a simulation study that presented discrete formulae and the continuous formula approximate each other.

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[2] In Poland, in 2001 and 2002 Bankowy Fund did not reach the minimal rate of return.

2 Economic postulates and discrete formulas of average return

At first sight the problem of constructing the average rate of return of funds seems to be straightforward. But if we look at postulates of Gajek and Kałuszka [6], which are quite natural and economically legitimate, we have to verify this opinion. The above-mentioned authors propose seven economic postulates (see also Białek [2]) and they prove that the Polish measure described by (1) violates four of them. Moreover, Gajek and Kałuszka [7] propose their own definition of the average rate of return of funds:

$$\bar{r}_{GK}(T_1,T_2)=\prod_{t=T_1}^{T_2-1}(1+\sum_{i=1}^{n}A_i^*(t)r_i(t,t+1))-1, \tag{2}$$

where

$$A_i^*(t)=\frac{A_i(t)}{\sum_{i=1}^{n}A_i(t)}=\frac{p_i(t)q_i(t)}{\sum_{i=1}^{n}p_i(t)q_i(t)}, \tag{3}$$

and $p_i(t)$ denotes the value of the participation unit of the $i-$th fund at time t, $q_i(t)$ denotes the number of units of the $i-$th fund at time t. In this part of the paper we treat the group of funds as an aggregate that contains n commodities (funds) with prices $p_i(t)$ and quantities $q_i(t)$, where $t\in[T_1,T_2]$. Let us denote by $P^L(t,t+1)$ the Laspeyres price index and by $P^{LL}(t,t+1)$ the logarithmic Laspeyres price index as follows (see von der Lippe [24])

$$P^L(t,t+1)=\frac{\sum_{i=1}^{n}q_i(t)p_i(t+1)}{\sum_{i=1}^{n}q_i(t)p_i(t)}, \tag{4}$$

$$P^{LL}(t,t+1)=\prod_{i=1}^{n}\left(\frac{p_i(t+1)}{p_i(t)}\right)^{A_i^*(t)}. \tag{5}$$

The definition (2) can be written with the use of the Laspeyres chain index. In fact we have (see Białek [5])

$$\prod_{t=T_1}^{T_2-1}P^L(t,t+1)-1=\prod_{t=T_1}^{T_2-1}\left(1+\sum_{i=1}^{n}\frac{q_i(t)p_i(t)}{\sum_{i=1}^{n}q_i(t)p_i(t)}\cdot\frac{p_i(t+1)-p_i(t)}{p_i(t)}\right)-1=\bar{r}_{GK}(T_1,T_2)\cdot \tag{6}$$

Białek [3] proposes another definition of the average return. His formula $\bar{r}_B$ can be written with the use of the logarithmic Laspeyres chain index (see Białek [5])

$$\bar{r}_B(T_1,T_2)=\prod_{t=T_1}^{T_2-1}P^{LL}(t,t+1)-1=\prod_{t=T_1}^{T_2-1}\prod_{i=1}^{n}\left(\frac{p_i(t+1)}{p_i(t)}\right)^{A_i^*(t)}-1=\prod_{t=T_1}^{T_2-1}\exp\left(\sum_{i=1}^{n}A_i^*(t)\ln\frac{p_i(t+1)}{p_i(t)}\right)-1\cdot \tag{7}$$

In the papers by Gajek and Kałuszka [7] or Białek [2], [3] we can find proofs, that measures $\bar{r}_{GK}$ and $\bar{r}_B$ satisfy all the above-mentioned postulates from Gajek and Kałuszka. It can be shown that $\bar{r}_B(T_1,T_2)\le\bar{r}_{GK}(T_1,T_2)$. Moreover, if $p_i(t+1)\approx p_i(t)$ for each i and $t\in[T_1,T_2]$, then $\bar{r}_B(T_1,T_2)\approx\bar{r}_{GK}(T_1,T_2)$. Gajek and Kałuszka claim that the Polish measure defined in (1) overestimates the real value of the average rate of return of funds. These authors consider not only the discrete stochastic model but they also propose continuous (deterministic and stochastic) measures (see Gajek and Kałuszka [8]). In the next part of the paper we present an original, stochastic and continuous measure of the average return. It seems to be a natural next step in using the chain index theory for constructing the average rate of return of funds.

3 Continuous time stochastic model

Let $\{p_i(t): t \geq 0\}$ denote the stochastic process of the price of unit of $i-$th fund ($i = 1,2,...,n$) defined on a probability space $(\Omega, \Im, P)$ and let $\{q_i(t): t \geq 0\}$ denote the stochastic process of the number of units of $i-$th fund defined on the same probability space. Let $F = \{\Im_t : t = 0,1,2,...\}$ be a filtration, i.e. each $\Im_t$ is an $\sigma-$algebra of Ω with $\Im_0 \subseteq \Im_s \subseteq \Im_t \subseteq \Im$ for any $s < t$. Without loss of generality, we assume $\Im_0 = \{\emptyset, \Omega\}$. The filtration F describes how the information about the market is revealed to the observer. We assume additionally that processes $p_i(t)$ and $q_i(t)$ are progressively measurable with respect to the family $\{\Im_t : t \geq 0\}$. In practice, the price and quantity processes have positive values. Thus in finance, the processes of share prices are often described by the geometric Brownian (Wiener) motion[3] (also known as exponential Brownian motion) as follows (see Koo [15])

$$dp_i(t) = \alpha_i p_i(t)dt + \beta_i p_i(t)dW_i(t), i \in \{1,2,...,n\}, \tag{8}$$

where the percentage drift α_i and the percentage volatility β_i are constants, $W_i(t)$ denotes the standard Wiener process. For an arbitrary initial real value $p_i(0)$ the stochastic differential equation (8) has the analytic solution (under Ito's interpretation, see Ito [13])

$$p_i(t) = p_i(0)\exp((\alpha_i - \frac{\beta_i^2}{2})t + \beta_i W_i(t)), i \in \{1,2,...,n\}. \tag{9}$$

Thus the price processes described in (9) have always positive values and additionally $p_i(t)$ is log-normally distributed (see Oksendal [18]). Let us assume that not only prices of units are described by the geometric Wiener process but also processes of number of units of funds are described as follows

$$dq_i(t) = \gamma_i q_i(t)dt + \theta_i q_i(t)dW_i(t), i \in \{1,2,...,n\}, \tag{10}$$

and thus

$$q_i(t) = q_i(0)\exp((\gamma_i - \frac{\theta_i^2}{2})t + \theta_i W_i(t)), i \in \{1,2,...,n\}, \tag{11}$$

Under above assumptions and significations we propose the following definition of the average rate of return of a group of funds on a time interval $[T_1, T_2]$

$$R_P(T_1,T_2) = \exp[\int_{T_1}^{T_2}(\sum_{i=1}^{n} A_i^*(t)\alpha_i + \frac{1}{2}\sum_{i=1}^{n} A_i^*(t)\beta_i\theta_i - \frac{1}{2}\sum_{i=1}^{n}(A_i^*(t))^2\beta_i^2 - \frac{1}{2}\sum_{i=1}^{n}(A_i^*(t))^2\beta_i\theta_i)dt + \sum_{i=1}^{n}\int_{T_1}^{T_2} A_i^*(t)\beta_i dW_i(t)] - 1 \tag{12}$$

where the integral on the right side of a formula (12) is the Ito integral (see Karatzas and Shreve [14]). Let us notice that if we reduce in (12) the random factor connected with the Wiener process taking $\beta_i(t) = 0$, and thus $\alpha_i(t) = dp_i(t)/dt$, then we obtain

$$R_P(T_1,T_2) = \exp(\int_{T_1}^{T_2}\sum_{i=1}^{n} A_i^*(t)\alpha_i dt) - 1 = \exp(\int_{T_1}^{T_2}\sum_{i=1}^{n} A_i^*(t)dp_i(t)) - 1 = P^{Div}(T_1,T_2) - 1, \tag{13}$$

where P^{Div} denotes the continuous Divisia price index (see Banerjee [1], Hulten [12]).

[3] Geometric Brownian Motion is used to model stock prices in the Black–Scholes model and is the most widely used model of stock price behavior (see Hull [11]).

If $\beta_i(t) = 0$ we also have from (9)

$$\frac{p_i(T_2)}{p_i(T_1)} = \exp[\alpha_i(T_2 - T_1)]. \tag{14}$$

It can be shown that from (13) and (14) we obtain

$$R_p(T_1, T_2) = \prod_{i=1}^{n} \left(\frac{p_i(T_2)}{p_i(T_1)}\right)^{w_i} - 1 = P^{CD} - 1, \tag{15}$$

where

$$w_i = \frac{\int_{T_1}^{T_2} A_i^*(t)dt}{T_2 - T_1} \quad , \quad \sum_{i=1}^{n} w_i = \frac{1}{T_2 - T_1} \sum_{i=1}^{n} \int_{T_1}^{T_2} A_i^*(t)dt = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} \sum_{i=1}^{n} A_i^*(t)dt = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} dt = 1 , \tag{16}$$

and P^{CD} is the well known Cobb-Douglas price index (see von der Lippe [24]). From (6), (7), (13) and (15) we conclude that the stochastic proposition of the average return rate (12) seems to be well-constructed. It can be also shown (the proof is omitted) that in general, stochastic case one of the most important postulates (from Gajek and Kałuszka) holds, namely

$$R_A(T_1, T_2) + 1 = (R_P(T_1, T_2) + 1)(R_Q(T_1, T_2) + 1), \tag{17}$$

where $R_A(T_1, T_2)$ and $R_Q(T_1, T_2)$ denote respectively the relative change in net assets and the number of clients of funds.

4 Simulation study

Let us take into consideration a group of $n = 4$ funds, the time horizon of observations $T = 1$ and the following parameters of prices of units and numbers of units processes

(a) the case of prices of units (see (8)):

$$\alpha_1 = 0{,}3,\ \beta_1 = 0{,}25,\ \alpha_2 = -0{,}12,\ \beta_2 = 0{,}05,\ \alpha_3 = 0{,}55,\ \beta_3 = 0{,}3,\ \alpha_4 = -0{,}45,\ \beta_4 = 0{,}07,$$

(b) the case of numbers of units (see (10)):

$$\gamma_1 = 0{,}25,\ \theta_1 = 0{,}05,\ \gamma_2 = -0{,}45,\ \theta_2 = 0{,}1,\ \gamma_3 = 0{,}7,\ \theta_3 = 0{,}25,\ \gamma_4 = 0{,}3,\ \theta_4 = 0{,}03.$$

Without loss of generality we assume that $p_i(0) = q_i(0) = 1$ for each $i \in \{1,2,...,4\}$. Some realization of the average return rate $R_P(0,t)$ for $t \in [0,1]$ is presented in Fig.1. The generated values of index $R_P(0,1)$ for each of $i-th$ realization of price and quantity processes ($R_{Pi}(0,1) : i = 1,2,...,100$) are presented in Fig. 2.

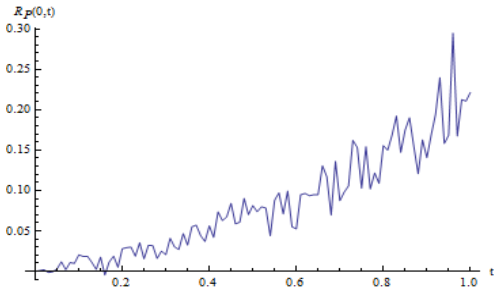

Figure 1 Realization of $R_P(0,t)$ process

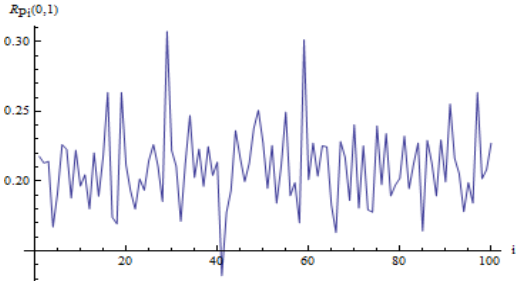

Figure 2 Generated values of index $R_P(0,1)$

Next we compare the stochastic version of the average rate of return of funds $R_P(0,1)$ with average rates of return $\bar{r}_{GK}(0,1)$ and $\bar{r}_B(0,1)$ for which we divide the time interval $[0,1]$ into ten subintervals of the same length. The results of our comparisons for $n = 10000$ generated realizations[4] of prices of units and numbers of units processes are presented in Tab.1 (to read more about estimation of mean value and variance and the bias of this estimation see Żądło [23], Małecka [17] or Papież, Śmiech [19]).

Parameter	$R_P(0,1)$	$\bar{r}_B(0,1)$	$\bar{r}_{GK}(0,1)$
Mean	0,233	0,432	0,479
Standard deviation	0,020	0,216	0,238
Median	**0,232**	**0,241**	**0,273**
Median deviation	0,014	0,141	0,130

Table 1 Basic parameters of average return rates

5 Conclusions

The form of the R_p measure seems to be proper – in the deterministic case, where $\beta_i(t) = 0$, it can be expressed by using some known chain indices (see (13) or (15)). The well-constructed $\bar{r}_{GK}$ and $\bar{r}_B$ have the same property in a discrete version (see (6) and (7)). Moreover, the stochastic definition R_p and the mentioned measures seems to approximate each other (see our simulation study). But let us notice that only $R_P(0,1)$ rate has a small volatility in our research (see Tab. 1). We suppose that some extreme realizations of price or (and) quantity processes lead to extreme values of the considered chain indices. Using medians for our comparison and thus ruling out these extreme realizations we obtain quite good approximations (for example: 0,232 in case of $R_P(0,1)$ and 0,241 in case of $\bar{r}_B(0,1)$).

Apart from measuring fund return rates, the proposed formula might find use in measuring mean efficiency of any class of investable assets, and in particular mutual funds. If the class is too broad to be completely enumerated, a representative sample might be chosen according to various criteria of representativeness discussed by Kruskal [16] and Gamrot [10]. The measure could be also used to calculate a premium rate in motor insurance (see Szymańska [22]).

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Measuring European railway efficiency using DEA approach.

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Abstract. There has been a great deal of interest in analysing deregulating and restructuring policies in the European railway sector. A large amount of studies is based on data envelopment analysis (DEA) for evaluating such policies. One of the main reasons why DEA gain so much popularity is, that it allows to measure efficiency, when production process presents the structure of multiple inputs and outputs, without knowing information about prices. On the other hand the complete freedom in choosing weights for inputs and outputs can cause overestimation of efficiency levels for some DMUs. The objective of this paper is to evaluate relevance of this fact in railway sector. For this purpose it's performed common CCR, BCC and SBM model and after a deeper insight into the results assurance region method is drawn up. Finally efficiency levels of selected models and resulting rankings of European countries are compared. Analysis is performed on the national rail system data for the year 2008 for 23 European countries using passenger and freight services as outputs and length of lines,staff, freight and passenger transport stock as inputs.

Keywords: DEA, railway, efficiency, CCR, BCC, SBM, assurance region method

JEL classification: C670,L920
AMS classification: 90B06,90B10,90B90

1 Introduction

During the 90's the European rail industry began to be reformed and deregulated in the hope of boast efficiency by promoting competition. Generally we can say that reforms go through two levels: by vertical dimension, when infrastructure separated from operation and horizontal dimension when service on rails has been enable to more operators. But different counties gradually introduced reforms differently with respect to the extent and time frame. These structural changes in the big industry as the railway sector is, created room for empirical investigation of the impact of reforms. Each such analysis starts by measuring efficiency of countries. Then in the analysis of the second stage obtained efficiency levels are regressed on dummy variables identifying reforms and other control variables. There are two common methodology for measuring efficiency levels. The first one is deterministic non parametric approach known under the name DEA (Data Envelopment Analysis) and the second one is based on the estimation of stochastic parametric distance function, called SFA (Stochastic Frontier Analysis). In this paper I will pay attention to the former.

The main reasons for which DEA has gained so much popularity is that it allows to handle multiple outputs and inputs without knowing information about prices and about functional form for the production function. On the other hand, these undemanding model assumptions bear its drawbacks. One of the most known disadvantage is connected with the fact, that DEA method is deterministic and so very sensitive to outliers. Another problem, which however empirical papers doesn't address, is related to the selection of the weights for evaluation of the efficiency level. DEA allows each item to select weights that maximise its own efficiency score. But sometimes we can find large differences in weights from item to item and even DEA can admit to set up values of certain inputs or outputs to zeros. Then such a selection doesn't have to appropriately reflect reality and by freedom in choosing weights efficiency levels of some countries may be overestimated. The aim of the paper is to assess the relevance of consequences of this fact.

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For this purpose, at first common DEA CCR, BCC and not so known SBM model are performed. Then assurance-region model is drawn up and obtained efficiency scores are compared.

2 Data

The production process in railway sector is very complex making measurement of performance complicated. Since railways transport both passengers and freight, two standard outputs are used: passenger-km transported (pkm) for passenger transport and tonnes-km transported (tkm) for freight transport [1]. In terms of input variables, the following are considered: mean annual staff strength (l) recalculated to full time equivalents, length of lines (d) measured as number of km of railway infrastructure and finally a proxy for capital stock (kf) indicating the number of wagons in freight transport and (kp) embodying the number of railcars and coaches in passenger transport. The analysis is performed for 23 European countries on the sample for the year 2008. Data information was taken from the reports published by International Union of Railways. Since in some countries there are more than one operator in freight or passenger transport, data of all companies were summed up for each country.

	Mean	**St, dev.**	**Min**	**Max**
pkm	14788	23754	78	86664
tkm	17335	21575	743	91178
d	8605	9039	699	33862
l	50717	59502	2938	240008
kp	3987	4978	123	18671
kf	23294	27303	1498	119916

Table 1 Summary statistics

3 Methodology and results

Data envelopment analysis was at first introduced in [2]. It's linear programming method which try to identify piecewise linear production possibility frontier. On its basis efficiency score of decision making units, here represented by countries, is determined. Efficiency score denoted θ is then defined as ratio of weighted outputs to weighted inputs, where the weights are selected in a manner to maximize resulting score. θ can also be interpreted as a ratio to which a given country could proportionally reduce its inputs without losing any outputs[2]. Though this method enable to work with the production process of multiple inputs and outputs and doesn't require any information about prices neither functional form of production, it's implicitly assumed, that a constructed production function is achievable for all countries.

3.1 CCR,BCC and SBM model

CCR is the most standard model, which production possibility set is based on the constant return to scale assumption. The linear programming model for each country is defined as follows:

$$\begin{aligned} \max uy_0 \quad & \text{s.t.} & & & \min \theta \quad & \text{s.t.} & (1)\\ vx_0 &= 1 & \iff & & Y\lambda &\geq y_0 & (2)\\ uY - vX &\leq 0 & & & \theta x_0 - X\lambda &\geq 0 & (3)\\ u,\, v &\geq 0 & & & \lambda &\geq 0, & (4) \end{aligned}$$

[1] A possible alternative in place of passenger-km and tonnes-km is to use train-km. But the former is preferred by the vast majority of studies, because it takes into consideration also allocation efficiency and not only the technical one.

[2] This type of model is called input-orientation and is more convenient for railway industry. Also there exist a output-oriented form of the model, when the efficiency score is interpreted as how much we can raise outputs without altering inputs.

where y_0 is the column vector of outputs for country 0, $y_0 = (pkm_0, tkm_0)$, x_0 is the column vector of inputs for country 0, $x_0 = (l_0, d_0, kf_0, kp_0)$, Y is the output matrix of dimension 2×23 one column for each country, X is the input matrix of dimension 4×23, one column for each country and u, v are the row vectors of output/input weights, which are unknown. Finally θ represents an efficiency score and λ's are the Lagrange multipliers of linear problem (and also searched coefficients of dual problem)

On the left hand side the model is in the multiplier form, which could be be informally interpreted as the effort to maximize weighted output (efficiency score) under the limitation, that at given weights any country cannot achieve greater efficiency score than 1, eq.(2). Its dual problem is on the right hand side, which can be seen as maximal possibility to radially lower inputs guaranteeing the same output (eq. 2) assuming that the combination $(\theta x_0, y_0)$ remain insight production possibility set (eq. 3). In other words it can be represented as a non-negative combination of inputs and outputs of DMUs (here captured by vector λ). Banker, Charnes and Cooper in [1] enhanced the model to implementation variable return to scale from decreasing to increasing. Formally it's simple done by adding one restriction to dual problem in the following form:

$$\sum_{i=1}^{23} \lambda_i = 1. \tag{5}$$

This enables to take only convex combinations of DMUs for construction of production possibility frontier. Since these two models in dual form differ only in that one constraint, production possibility set of the BCC model is the subset of production possibility set of the CCR model and so resulting efficiency score of the BCC model cannot be smaller than that of the CCR model. This fact can also be supported by the results contained in the first two columns of the table 2. Empirical studies in railway sector are based on the one of these two models [3]. Eventually they compare both approaches as e.g.[3]. From their results *Cantos at al* claim that the differences between two methods are especially notable for small countries. Authors explain the fact in the way, that there are economies of scale in small railway systems which fail in exploiting them. But as is stated in [5], a DMU that has a minimum input value for any input item, or a maximum output value for any output item, is BBC-efficient. Thus countries whit extreme values of any input factor are considered as efficient without knowing anything about what reality about efficiency score in fact is. In that study, it concerns Germany and Macedonia. Especially in the case of Macedonia, on the basis of CCR model it's hard to believe, that this country would lie on production possibility frontier. In the deeper look into result, specifically into reference sets[4], it could be possible to notice, that a vast majority of small countries (such as Bosnia and Herzegovina, Greece, Moldova and Slovenia) have Macedonia in their reference set in BCC model but not in the CCR model. This could be a more important reason than variable returns to scale, why small countries have bigger efficiency score in BCC model.

Now let's turn look on the SBM (slack based measure) model, which was introduced by Tone in [8]. Beside the CCR model [5] it take into computation of efficiency scores moreover values of slacks. Slacks are output shortfalls or input excesses which after radial projection onto production possibility set remain. To ensure CCR or SBM efficiency these slacks have to be zero. SBM efficiency score is defined by following product formula:

$$\rho = \left(\frac{1}{4} \sum_{i=1}^{4} \frac{x_{i0} - s_i^-}{x_{i0}} \right) \cdot \left(\frac{1}{2} \sum_{j=1}^{2} \frac{y_{j0} + s_j^+}{y_{j0}} \right)^{-1}, \tag{6}$$

where the first term evaluates the mean proportional reduction rate of inputs and the second evaluates the mean proportional expansion rate of outputs. After some transformations, SBM model can be expressed in linear form as follows:

[3] Either they prefer the CCR model e.g.[4] or to BCC model e.g.[6].

[4] Reference set is a set of countries with which a particular country is evaluated, in other words a set of those countries which are connected with non-zero λ^*.

[5] It can be also implemented in BCC model, but due to greater credibility of CCR results, the SBM model was implemented in this form.

$$\min t - \frac{1}{4}\sum_{i=1}^{4}\frac{S_i^-}{x_{i0}} \quad \text{s.t.} \quad 1 = t + \frac{1}{2}\sum_{j=1}^{2}\frac{S_i^+}{y_{j0}}$$
$$tx_0 = X\Lambda + S^-$$
$$ty_0 = Y\Lambda - S^+$$
$$\Lambda \geq 0,\ S^- \geq 0,\ S^+ \geq 0,\ t > 0,$$

where S^-, S^+ and Λ are defined as $S^- = ts^-$, $S^+ = ts^+$ and $\Lambda = t\lambda$ and where s^- denotes input slack, s^+ output slack and t convenient positive constant. All other letters have the same meaning as that in problem describe by (1-4).

The results of the efficiency score can be found in fourth column in the table 2. Incorporating slacks results in lower efficiency scores of the SBM model in comparison with the CCR model. But for some countries as Croatia and Turkey the drop is more significant and on the other hand in the Czech Republic, Slovakia and Slovenia the drop was gentle and accordingly they slightly jumped upwards in the ranking, see 7th column of the table 2.

DMU	θ_{CRR}	θ_{BCC}	θ_{SBM}	r_C	r_B	r_S	$\frac{u_{pkm}}{u_{tkm}}$	$\frac{v_{kp}}{v_{kf}}$	$\frac{v_l}{v_{kp}+v_{kf}}$	$\frac{v_d}{v_{kp}+v_{kf}}$	θ_{AR_C}	θ_{AR_B}	r_{AR_C}	r_{AR_B}	$\frac{v_d}{v_{kp}+v_{kf}}$	$\frac{v_l}{v_{kp}+v_{kf}}$	$\frac{v_{kp}}{v_{kf}}$	$\frac{v_{pkm}}{v_{tkm}}$	AR_{C2}	r_2
AUS	0.79	0.79	0.67	11	15	10	4.76	1.2E10	0.00	0.35	0.62	0.67	10	12	0.30	0.20	20.00	2.08	0.61	10
BAH	0.22	0.73	0.17	23	17	23	9.01	3.5E09	0.00	0.00	0.17	0.48	23	16	0.30	0.20	0.20	0.43	0.16	23
BEL	1.00	1.00	1.000	1	1	1	0.6	1.10	0.01	1.18	0.86	1.00	7	1	0.30	0.20	0.20	0.43	0.84	6
BLG	0.32	0.37	0.25	22	22	22	9.01	2.3E10	0.00	0.00	0.22	0.27	22	23	0.30	0.20	0.20	0.43	0.21	22
CRO	0.71	0.84	0.45	12	14	13	9.01	2.1E11	0.00	0.00	0.38	0.49	15	15	0.3	0.20	20.00	2.08	0.34	15
CZE	0.36	0.36	0.33	19	23	17	4.76	1.1E09	0.00	0.35	0.33	0.33	17	21	0.3	0.20	20.00	2.08	0.32	17
ESP	1.00	1.00	1.00	1	1	1	37.75	258.13	0.12	0.05	0.94	0.97	5	8	0.21	0.29	1.82	5.00	0.88	5
FIN	1.00	1.00	1.00	1	1	1	4.07	79.94	0.09	0.02	0.91	0.92	6	9	0.20	0.30	20	1.92	0.84	7
FRA	1.00	1.00	1.00	1	1	1	9.9	0.64	0.02	0.00	1.00	1.00	1	1	0.30	0.20	0.55	3.06	1.00	1
GER	0.88	1.00	0.72	9	1	9	4.76	1.5E09	0.00	0.35	0.69	1	9	1	0.30	0.20	20	2.08	0.65	9
GRE	0.42	0.59	0.30	17	18	18	5.86	1.1E11	0.08	0.00	0.34	0.52	16	14	0.24	0.26	20.00	5.00	0.32	16
HUN	0.49	0.55	0.40	15	19	15	0.35	0.46	0.06	0.00	0.43	0.46	13	17	0.30	0.20	0.2	0.43	0.39	14
ITA	0.93	0.94	0.80	8	11	8	3.6E07	6.8E07	0.20	0.25	0.83	0.89	8	10	0.30	0.20	20.00	5.00	0.78	8
LIT	1.00	1.00	1.00	1	1	1	0.66	236.84	0.11	0.27	1.00	1.00	1	1	0.27	0.23	12.53	0.64	1.00	1
MKD	0.33	1.00	0.27	21	1	21	9.01	1.0E11	0.00	0.00	0.28	1	19	1	0.30	0.20	0.20	0.43	0.26	19
MOL	0.34	0.77	0.28	20	16	20	4.76	4.9E09	0.00	0.35	0.23	0.39	21	19	0.30	0.20	0.20	0.43	0.24	21
POL	0.55	1.00	0.44	14	1	14	9.01	3.0E06	0.00	0.00	0.42	0.6	14	13	0.30	0.20	20.00	2.08	0.39	13
POR	1.00	1.00	1.00	1	1	1	2.61	64.47	5.12	0.38	1.00	1.00	1	1	0.20	0.30	10.73	4.00	1.00	1
RO	0.45	0.45	0.30	16	20	18	9.01	2.3E10	0.00	0.00	0.28	0.28	20	22	0.30	0.20	20.00	2.08	0.26	20
SLO	0.65	0.86	0.57	13	13	11	0.71	2.57	0.00	0.00	0.58	0.79	11	11	0.30	0.20	0.20	0.43	0.55	11
SUI	1.00	1.00	1.00	1	1	1	818.63	0.28	0.39	3.36	1.00	1.00	1	1	0.30	0.20	1.32	1.37	1.00	1
SVK	0.39	0.43	0.34	18	21	16	4.76	1.5E13	0.00	0.35	0.32	0.34	18	20	0.30	0.20	0.20	0.43	0.32	18
TUR	0.86	0.88	0.52	10	12	12	9.01	3.5E10	0.00	0.00	0.45	0.45	12	18	0.21	0.29	20.00	1.97	0.40	12

Table 2 Efficiency levels, ranking and weights

3.2 Assurance-region model

Above models allow for computation efficiency scores select both for inputs and outputs arbitrary non-negative weights. So this great flexibility is associate with the weakness, when can be huge differences in weights assigned to individual countries which may distort relative comparisons and also bias efficiency levels upward. Even for inefficient countries DEA enable for some inputs or outputs assign weight equal to zero. One way how to neutralise this problem is to add constraints for weight control. This type of model is called assurance-region method and first it was introduced in 1986 in [7].

But the task to select admissible bounds for the ratios of weights is not so simple. As is recommended in [5], it's possible to use of the knowledge of experts or of the evaluated weights of CCR model for preferable efficient countries. In the look on selected ratios of weights contained in columns 8–11 in table 2, it can be seen that both great relative magnitude in weights and many zeros are present in performed CCR model. Among efficient countries is really not easy to find some representative ones. With the help of the mean values of selected variables (table 1), given weights from the CCR model and some theoretical background, following bounds were set up. Bounds were selected in a very relaxed way, rather from the point of view to prevent unrealistic weighting. It was imposed so that ratio of weights of passenger and tonne kilometre would be between 0.2 and 5 ($0.2 \leq \frac{u_{pkm}}{u_{tkm}} \leq 5$); ratio of weights of passenger and freight capital stock would be between 0.2 and 20 ($0.2 \leq \frac{v_{kp}}{v_{kf}} \leq 20$); ratio of weights of staff and distance of traffic

lines to total capital stock would be between 0.2 and 1 ($0.2 \leq \frac{v_l}{v_{kp}+v_{kf}} \leq 1$, $0.2 \leq \frac{v_d}{v_{kp}+v_{kf}} \leq 1$) and finally the sum of staff and lines to total capital to be at least 0.5 ($0.5 \leq \frac{v_l+v_d}{v_{kp}+v_{kf}}$). The linear programming problem for each country is formulated as follows:

$$\begin{aligned} \min \theta \quad \text{s.t.} \quad & Y\lambda + Q\tau \geq y_0 \\ & \theta x_0 - X\lambda + P\pi \geq 0 \\ & \lambda \geq 0,\ \pi \geq 0,\ \tau \geq 0, \end{aligned}$$

where added restrictions in linear representation are contained in matrices P and Q, so that

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 & -1 \\ -1 & 1 & 0.2 & -1 & 0.2 & -1 & 0.5 \\ 0.2 & -20 & 0.2 & -1 & 0.2 & -1 & 0.5 \end{pmatrix}, \quad Q = \begin{pmatrix} -1 & 1 \\ 0.2 & -5 \end{pmatrix}.$$

In comparison of the efficiency scores of the AR-CCR with CCR model, following facts can be noticed. First efficiency scores of AR-CCR model are always lower. But this consequences is imposed by adding restriction, which can limit achieving highest possible score if the corresponding bounds don't lie within bounds. In average efficiency level drop about 18 %, with a standard deviation of 14 which is not negligible and also not uniform change. On the other hand the influence on the ranking of countries is not so great. It can be found only small changes (for example Czech Republic and Slovenia jump by two places up otherwise Croatia and Romania droped by three and four places down). Also Belarus, Spain and Finland lost the position of efficient countries, so that production possibility function has changed. Finally in 4 columns to the end of the table 2, ratios of weights of AR-CCR model are referred, where it's confirmed, that in a majority of cases added constraints for ratios of weights are binding.

In the 13th column of the table 2 results of AR-BCC model are shown for completeness and the corresponding ranking is listed two columns further. It is worth noting here, that in the case of Poland, dramatic drop of efficiency level between AR-BCC and BCC model is present. Though Poland has efficiency score equal to 1, it isn't BCC efficient, since slacks aren't zero and in fact very high. Thus some linear combination of countries of reference set of Poland (BEL, LIT, FRA) caused by chance, that Poland lies on the edge of the production possibility frontier though with a high output shortfall and input excesses. But bounds for ratios of weights don't enable such as fitting and consequently more realistic scores are returned.

Further another AR-CCR model was performed with more restrictive bounds to evaluate sensitivity of efficiency scores on given bounds. Following restriction has changed to the these ones: $2 \leq \frac{v_{kp}}{v_{kf}} \leq 10$, $0.2 \leq \frac{v_d}{v_{kp}+v_{kf}} \leq 1$, $1 \leq \frac{v_l+v_d}{v_{kp}+v_{kf}}$. Results, efficiency scores and ranking, can be found in last two columns of table 2. By a minor modification of restriction, the results didn't change significantly and also the rankings remain the same to the swapping of Belorus to Finland. So efficiency scores are rather sensitive to the inclusion of bounds but their minor modification doesn't have practically any impact.

	CCR	**BCC**	**SBM**	**AR-CCR**	**AR-BCC**	**AR-CCR(2)**
CCR	1.000	0.748	0.987	0.967	0.722	0.968
BCC	0.748	1.000	0.732	0.740	0.906	0.745
SBM	0.987	0.732	1.000	0.975	0.730	0.980
AR-CCR	0.967	0.740	0.975	1.000	0.776	0.999
AR-BCC	0.723	0.906	0.730	0.776	1.000	0.775
AR-CCR(2)	0.968	0.745	0.980	0.999	0.775	1.000

Table 3 Spearman rank correlation coefficients

For final summarization of results Spearman rank correlation coefficient were calculated and are contained in the table above (table 3). All the coefficients lie above 0.7, thus the choice of DEA method is not so crucial, nevertheless has non negligible impact on efficiency scores and for precise analysis should

definitely taken into account. Also this empirical railway study shows, that it does matter in the choice between CCR and BCC model.

4 Conclusion

Nowadays due to many reforms and structural changes in European railway industry, measuring efficiency levels becomes actual and starting point for further analysis. This paper discussed and compared standard techniques based on the DEA method. The most common in this industry is CCR and BCC model. Some authors claim that BCC model is better because it enable to handle variable returns to scale. But as this study shown this advantage is questionable and one have to be careful if some inefficient small or big country don't create production possibility frontier and consequently don't raise artificially resulting efficiency scores for some countries. Also SBM model, which incorporates input excesses and output shortfalls was performed to give more precise results. Resulting efficiency scores didn't change significantly but some corrections can be found.

The main objective of this paper was to check the impact of freedom in choosing weight for inputs and outputs by adding lower and upper bounds for selected ratios of weights. At first empirical results show, that for a majority of countries selected weights, which maximize efficiency scores, are far from reality. By imposition of some though relaxed bounds, the efficiency score decrease in average nearly by 20%. Even though efficiency scores changed significantly, the ranking of European countries don't change dramatically. However to a gentle modification of ratios of weights the results are robust. So it depends mainly on introducing of bounds than on their concrete values. If we want to measure railway efficiency, it would be reasonable to pay attention to good model specification, since results are relatively sensitive to it. But if the efficiency scores serves only as one step of complete analysis in railway industry, choice of model specification could probably not have a significant impact on final conclusions. This fact, however, should be further scrutinized.

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Responses to Monetary Policy Shock in the Czech Republic

Sára Bisová[1]

Abstract. The concept of VAR models is widely used approach for the econometric analyses of the monetary policy transmission mechanism. Dynamic models are crucial in the monetary policy analyses due to significant lags between an action on one hand and the appropriate effects in the economy on the other hand. This paper estimates the responses of the key macroeconomic indicators to an exogenous monetary policy shock in the Czech economy using the VAR models. We estimate an identified VAR using recursive and non-recursive identification schemes (assuming the small open economy). To avoid the possibility of a price puzzle effect, due to mixing policy regimes in the sample period, we try to use data from single monetary policy regime starting 1998, when the Czech central bank switched to the inflation targeting regime. The model is based on the set of variables concerning output, consumer price index, short term interest rate and exchange rate, all measured at quarterly frequency. The importance is stressed on inclusion of the interest rate and the exchange rate channels as they are fundamental in the inflation targeting transmission mechanism. In the empirical analysis, the obtained results indicate a relatively high sensitivity to the identification scheme in SVAR.

Keywords: Choleski decomposition, identification, impulse response functions, monetary policy, SVAR.

JEL Classification: C32, C51, E52, E58
AMS Classification: 62M10, 91B84

1 Introduction

The concept of VAR models is frequently used for the econometric modeling of the macroeconomic stabilization policies. We estimated three identified VAR models for the analysis of the monetary transmission mechanism using the recursive and non-recursive identification techniques. The advantage of the non-recursive assumption lies in the fact that we can employ presumed structure of the economy and its monetary transmission mechanism. Most studies stress the importance of the interest rate channel and the exchange rate channel, in some papers also the money channel, during the transmission mechanism of monetary policy. We include two crucial transmission channels in the model – the interest rate channel and the exchange rate channel.

Using impulse response functions (IRF) for the real Czech data we quantified the effects of an unexpected monetary policy shock on the analyzed variables, particularly the price level (target variable). The sample period covers the single monetary policy regime starting 1998[2], when the Czech central bank switched to the inflation targeting regime, although earlier data are available. The reason is excluding policy regime switching during the sample period to avoid possible puzzle effects in estimated outputs, due to mixing policy regimes in the sample period. Regime switching VAR models were employed for example in [3][3]. We analyze the sensitivity to the identification scheme in SVAR using three different structures of identification matrix in the same model.

The paper is organized as follows: section 2 states theoretical background of VARs, data are presented in section 3 and empirical analysis in section 4. Conclusions are summarized in section 5.

2 VAR models

The unrestricted VAR approach in the field of macroeconomic analyses was introduced in [11]. Opposed to structural VARs, this concept avoid any a priori assumptions based on short-run or long-run restriction derived from the economic theory.

The structural VAR (or identified VAR) approach allows us to identify the IRF by imposing a priori restrictions on the covariance matrix of the structural errors (a priori short-run restrictions on contemporaneous

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[2] In 1998 the Czech central bank accepted inflation targeting regime.
[3] The authors specified a regime switching VAR model for the U.S. fiscal policy analysis to allow differences among historical episodes with different economic environments.

effects of shocks and long-run restrictions to employ long-run impact of shocks to identify the IRF). The structural form (SVAR) without intercepts can be defined as follows [6]

$$\mathbf{A}\mathbf{y}_t = \mathbf{\Pi}(L)\mathbf{y}_{t-1} + \mathbf{B}\mathbf{u}_t, \tag{1}$$

where $\mathbf{u}_t$ is a structural disturbance vector generated by vector white noise process, with identity covariance matrix $E(\mathbf{u}_t\mathbf{u}_t^{\mathrm{T}}) = \mathbf{\Lambda}$, where diagonal elements are variances of structural shocks, $\mathbf{\Pi}(L)$ is a polynomial matrix in the lag operator and $\mathbf{y}_t$ is a vector of m endogenous variables. The elements outside the diagonal in matrix $\mathbf{B}$ may be non-zero, therefore, some of the shocks can influence more endogenous variables of the system [6].

We can estimate the **reduced form**

$$\mathbf{y}_t = \mathbf{A}^{-1}\mathbf{\Pi}(L)\mathbf{y}_{t-1} + \mathbf{v}_t, \tag{2}$$

where $\mathbf{v}_t$ is a disturbance vector of reduced form with a covariance matrix $E(\mathbf{v}_t\mathbf{v}_t^{\mathrm{T}}) = \mathbf{\Sigma}$. For the disturbance vectors $\mathbf{u}_t$ and $\mathbf{v}_t$ holds the following formulae [6]

$$\mathbf{v}_t = \mathbf{A}^{-1}\mathbf{B}\mathbf{u}_t \quad \text{or} \quad \mathbf{A}\mathbf{v}_t = \mathbf{B}\mathbf{u}_t \tag{3}$$

and

$$E(\mathbf{v}_t\mathbf{v}_t^{\mathrm{T}}) = \mathbf{A}^{-1}\mathbf{B}^{\mathrm{T}}E(\mathbf{u}_t\mathbf{u}_t^{\mathrm{T}})\mathbf{B}^{\mathrm{T}}\mathbf{A}^{-1} \quad \text{or} \quad \mathbf{\Sigma} = \mathbf{A}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{\Lambda}\mathbf{B}^{\mathrm{T}}\mathbf{A}^{-1}. \tag{4}$$

The matrix $\mathbf{A}$ identifies the relation between the structural disturbances $\mathbf{u}_t$ and the reduced form shocks $\mathbf{v}_t$. There are many identification techniques based on imposing restriction on the parameters of the system mentioned above.

For structural VAR we employ AB-model, see (3), where $\mathrm{var}(\mathbf{u}_t) = \mathbf{\Lambda}$ is unit matrix. In order to get a just identified system we need $(m^2 - m)/2$ restrictions. Usually the **recursive identification** scheme, so-called **Choleski decomposition**, is used to obtain a just identified VAR, where the matrix $\mathbf{A}$ is lower triangular and the matrix $\mathbf{B}$ is diagonal, see [11]

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_{21} & 1 & 0 & \cdots & 0 \\ a_{31} & a_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 \\ a_{m1} & a_{m2} & a_{m3} & \cdots & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_{11} & 0 & 0 & \cdots & 0 \\ 0 & b_{22} & 0 & \cdots & 0 \\ 0 & 0 & b_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & b_{mm} \end{bmatrix}. \tag{5}$$

We can also use a **non-recursive scheme**, where the matrix $\mathbf{A}$ takes different form from (5), see empirical part of this paper in section 3 or e.g. [1], [2], [5], [7], [8], [10]. This approach allows us to employ more assumptions originated in the economic theory about the relations between analyzed variables.

The estimation of VAR models is usually followed by construction of impulse response functions (IRF). IRF are substantial in the macroeconomic policy analyses, especially in case of analyzing and anticipating the fiscal and monetary policy effects. The order of endogenous variables in vector $\mathbf{y}_t$ influences the IRF when using (5). For deriving and interpretation of IRF see [9].

3 Data

For the analysis, four macroeconomic indicators were chosen. The set of endogenous variables includes output (gross domestic product in constant prices of 2005), consumer price index (2005=100), short term interest rate (3M PRIBOR) and exchange rate (Euro/ECU exchange rate in national currency), all measured at quarterly frequency. Similar to [2], [5], [10] we include one exogenous variable – world commodity price index. The reason of considering the world or foreign variables lies in the need to avoid a misinterpretation of domestic monetary shock in case of reaction to a foreign shock[4]. To reduce the possibility of puzzle effects, in response to mixing policy regimes in the sample period[5], we use the data from single monetary policy regime starting 1998, as men-

[4] For example oil shocks, foreign policy shocks etc. [5]
[5] Observed for example in [2].

tioned above. The sample period considered for the analysis consists of Q1 1998 – Q4 2012 (60 observations). Data series were obtained from Eurostat[6] (interest rate, exchange rate), CSO[7] (CPI and GDP) and The World Bank[8] (commodity price index). The abbreviations for variables used in following tables and graphs are *GDP* for gross domestic product, *CPI* for consumer price index, *IR* for interest rate, *ER* for exchange rate and *COM* for commodity price index.

GDP and CPI were seasonally adjusted by X12 ARIMA method in order to reduce the number of parameters to be estimated in the model. For the estimation EViews 7 and GRETL 1.9.5 software were used.

Table 1 shows the ADF tests of all the above mentioned variables in levels. All the variables are non-stationary and therefore should be transformed to reach the stationarity for the application in VAR. In order to eliminate the non-stationarity, differences between logarithms[9] were computed (for their useful interpretation as growth rates)

$$\Delta \ln(y_t) = \ln y_t - \ln y_{t-1} = \ln \frac{y_t}{y_{t-1}}, \tag{6}$$

where y_t represents each of the analysed variables in period *t*. All the time series are stationary in first differences – see table 1.

LEVEL	t-statistic	Prob.*	D-LOG	t-statistic	Prob.*
GDP	-1.083167	0.7170	***GDP***	-5.192390	0.0001
CPI	-0.485191	0.8864	***CPI***	-4.590175	0.0004
IR	-2.154578	0.2248	***IR***	-4.068376	0.0022
ER	-1.381421	0.5855	***ER***	-6.496229	0.0000
COM	-0.754072	0.8241	***COM***	-6.255058	0.0000

*MacKinnon (1996) one-sided p-values
Notes: Test critical values -3.54 (1% level), -2.91 (5% level), -2.59 (10% level), IR not in logs

Table 1 ADF tests for variables in levels

Figure 1 shows graphs of the time series in levels (*GDP* and *CPI* are seasonally adjusted as mentioned above). We can clearly see the non-stationarity of all variables in their level form.

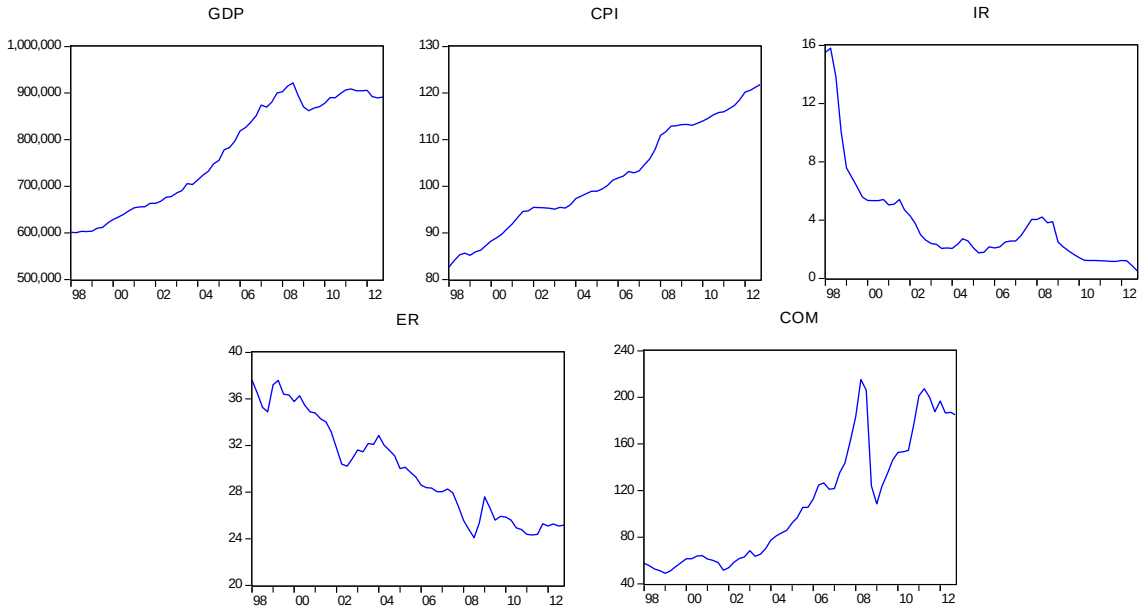


Figure 1 Time series plots

[6] http://epp.eurostat.ec.europa.eu/portal/page/portal/statistics/search_database.
[7] Czech statistical office - www.czso.cz.
[8] http://econ.worldbank.org.
[9] Except for interest rate which is in percentage form and therefore transformed directly into first differences.

4 Application

Many studies using VAR methodology for analyzing monetary policies do not include money stock in the models, see [5], [7] and [10][10]. Oppositely, a number of studies use money supply in the vector of endogenous variables, see [1], [2], [8]. In this study we construct the four-variable VAR excluding the money stock. The extended five-variable model is the subject of upcoming studies.

The four-variable SVAR was estimated applying the recursive and non-recursive identification schemes. The non-recursive scheme takes the following form

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & 0 \\ 0 & a_{32} & 1 & a_{34} \\ a_{41} & a_{42} & a_{43} & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_{11} & 0 & 0 & 0 \\ 0 & b_{22} & 0 & 0 \\ 0 & 0 & b_{33} & 0 \\ 0 & 0 & 0 & b_{44} \end{bmatrix}, \tag{7}$$

where parameters a_{ij} are unconstrained and the vector of endogenous variables is transformed, as mentioned in section 3

$$\mathbf{y}_t = (gdp_t, cpi_t, ir_t, er_t).^{11} \tag{8}$$

We alternatively use the form (7) with the modification used in [5], where $a_{32} = 0$ instead of $a_{31} = 0$.

The non-recursive identification scheme (7) assumes simultaneity between the interest rate and the exchange rate[12] and that the monetary authority considers contemporaneous prices rather than output, when making monetary policy decisions concerning stimulation of interest rates in the inflation targeting regime. In [5] authors hold the assumption of the forward-looking monetary authority, which considers product rather than prices ($a_{32} = 0$ instead of $a_{31} = 0$). In this study, we compare the three models described above. We use world commodity price index as an exogenous variable and intercepts in all models. The recursive identification scheme uses the same vector of endogenous variables (8) and the structure (5).

Firstly we analyze the lag length in the VAR model. The calculated lag-length criteria for the four-variable VAR are listed in table 2.

LAG	LogL	LR	FPE	AIC	SC	HQ
0	493.6188	NA	2.52e-13	-17.65887	-17.36689	-17.54596
1	526.1082	57.89029	1.39e-13	-18.25848	-17.38255*	-17.91975*
2	545.6839	32.03295*	1.23e-13	-18.38851	-16.92863	-17.82396
3	562.8328	25.56738	1.22e-13*	-18.43028*	-16.38645	-17.63992
4	573.4610	14.29979	1.56e-13	-18.23495	-15.60716	-17.21876

*indicates lag order selected by the criterion
Notes: LR: sequential modified LR test statistic (each test at 5% level), FPE: Final prediction error, AIC: Akaike information criterion, SC: Schwarz information criterion, HQ: Hannan-Quinn information criterion.

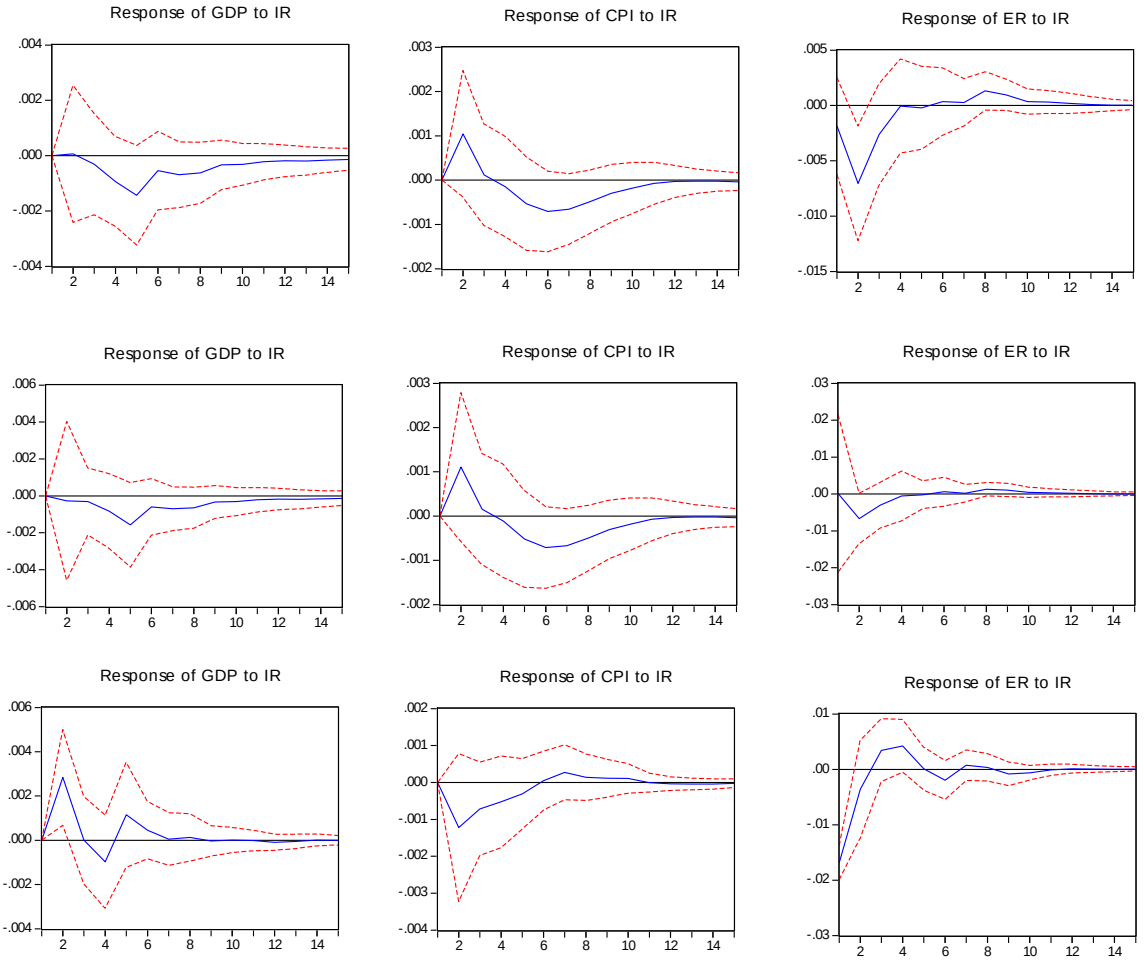
Table 2 VAR lag order selection criteria

The VAR(3) model was estimated as the AIC recommends (the inclusion of dynamics is desirable - there is a significant delay between the initial stimulus of the central bank and the change in the target variables in the monetary transmission mechanism – between 12 and 18 months. For the purpose of estimation, the reduced form and the OLS estimator were applied.[13]

Figure 2 shows selected responses to positive exogenous shock in the interest rate (monetary restriction) using the recursive identification scheme (the first row), the non-recursive identification scheme described in (7) (the second row) and the modification of (7) replacing $a_{32} = 0$ for $a_{31} = 0$ (the third row). In the columns are given responses of product, prices and exchange rate, respectively, to one standard deviation monetary policy

[10] The reason is usually a weakening effect of the money supply in the monetary transmission.
[11] Output, price level, interest rate and exchange rate, respectively.
[12] See for example [2], [5].
[13] The diagnostic control of the estimated model suggests the presence of heteroskedasticity of the error terms (joint test). Therefore, the robust standard errors were computed.

shock (domestic). Our assumptions about responses to an exogenous monetary contraction are following: the price level declines, the output does not increase and the exchange rate appreciates under the flexible exchange rate regime[14].



Notes: the recursive identification scheme in the first row, the non-recursive schemes in the rows 2 and 3.

Figure 2 IRF, four-variable VAR(3)

Using the first and second benchmark models, the results suggest similar responses to an exogenous monetary policy shock. On the other hand, the third version of the identification provides different behavior.

In the first two models, the contractionary monetary policy shock is followed by an immediate exchange rate appreciation with a slight depreciation after around two years in the first model – one possible explanation can lead to delayed overshooting puzzle[15], on the other hand, the confidence intervals are relative wide and therefore bring some uncertainty to our interpretation. The prices start to fall after about 3 quarters of persistence in response to the monetary policy shock with a bottom after approximately 6 quarters that is consistent with the assumed targeting horizon between 12 and 18 months. The output falls in response to monetary policy tightening with the bottom after about one year - it is in line with our assumptions.

In contrary to the first and second model, the third benchmark model shows different results. The interpretations of output and exchange rate are somewhat difficult while changing sings of responses over the horizon. Nevertheless, in the case of exchange rate there is an immediate appreciation after the exogenous positive shock in interest rate. The counterintuitive results in responses of output can be observed over the longer term horizon. The prices are falling, except for a slight increase in period between 8 to 10 quarters. The minimum is reached in about 2 quarters.

[14] While almost no theory predicts an increase in output and prices in response to an exogenous contractionary monetary policy shock, there is a number of theories concerning the exchange rate behavior, see [8].
[15] See for example [8].

The first two models exhibit similar results corresponding with our assumptions and with most studies providing similar analyses (e.g. [2], [5]). We can conclude that the responses are close to expected movements of analyzed variables. The third model provides different results, less consistent with our assumptions and difficult to interpret in case of some responses. We do not get similar results to [5] - one possible reason consists in using different definition of product – the authors use output gap instead of GDP growth (ex-post data), see [5]. We can conclude that the IRF are very sensitive to the structure of matrix **A** which identifies the relation between the structural disturbances and the reduced form residuals.

5 Conclusion

Using the structural VAR model, we analyzed the transmission mechanism of monetary policy in the Czech economy. We estimated three alternative four-variable VAR(3) models, imposing zero restrictions on some contemporaneous structural parameters. In the first model we employed just-identified VAR using the usual recursive identification scheme. In the second model we assumed the authority which considers contemporaneous prices when making monetary policy decisions about interest rates and in the third model we considered output instead of prices. The second and third models also assume simultaneous relation between the interest rate and the exchange rate. Compared to our previous analyses (e.g. [4]), and provided results, we used data period from Q1 1998 to Q2 2013 to eliminate puzzle effects due to mixing policy regimes, although earlier data are available. We also included one exogenous variable in the model not to misinterpret domestic monetary shock with the reaction to foreign shocks.

Considering the analyzed models, the second benchmark model with the assumed structure of the economy is preferred. In this model, an unexpected rise in interest rate (a contractionary monetary policy shock) implies a temporary fall in output that peaks around 5 quarters after the shock, a decrease in prices, although prices tend to be more persistent, and exchange rate appreciation. The third model does not give us persuasive results - the wave-shaped responses are difficult to interpret. In all the models, confidence intervals are relatively wide. There is important to note, that our model concentrates only on a basic part of the real system. The extended models - e.g. a five-variable SVAR and a regime switching VAR - are subjects of upcoming analyses.

Acknowledgements

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Input-output interactions in a DSGE framework

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Abstract. Despite their (heavily advertised) microeconomic foundations, DSGE models are usually implemented for economic analyses using country-level aggregate data. However, national statistical offices of the EU members create national accounts that contain detailed information about individual industries, such as the Input-Output tables. In this paper, we present a preliminary study of a multi-sector extension of a canonical RBC model that allows to use data on the input-output structure with multiple sectors. We formulate a simple baseline model that allows for an arbitrary number of sectors with an arbitrary input-output structure. The practical obstacle to using the model in practice lies in the need to find approximate steady-state values of the variables. In the general case, finding a solution to the steady-state problem is diffucult; however, we do provide an analytic solution to a special symmetric case, which can sometimes provide a close enough approximation for non-symmetric problems as well.

Keywords: DSGE models, Input-Output tables, National Accounts

JEL classification: E16, E17
AMS classification: 91B51

1 Introduction

In today's macroeconomic analyses, Vector Autoregressions (VAR), Vector Error Correction models (VEC) or DSGE models are typically used, with DSGE models being the newest of the three approaches, and one that is becoming increasingly popular with researchers and regulators alike. For instance, the Czech National Bank has developed a DSGE model "g3" [1], which is used for both the interpretation of current economic phenomena and for economic forecasting. A key feature that makes DSGE models so favourable is the fact that they are built on microeconomic foundations, and thus should be resistant to the Lucas critique. In our opinion, however, the idea of DSGE models' microfoundations is at odds with the aggregate nature of data that are typically used for model calibration; country-level data are almost invariably used, even though industry level-data are available in all EU member states' national accounts and presented e.g. in the Input-Output tables [4].

It has to be noted that some of the existing models do in fact divide the *firms* sector into a small number of "industries", typically distinguishing either between *manufacturers* and *retailers*, or, as in case of the Hubert model that has been developed and used by the Ministry of Finance of the Czech Republic (see [5], [6]), between *manufacturers*, *retailers*, and *importers*. Nevertheless, in such settings, (i) the model comprises a very small number of industries, (ii) the distinction between the individual industries does not follow the structure of national accounts, and (iii) the input-output structure is limited to a one-sided relationship between retailers and manufacturers. The aim of this paper is to formulate a simple baseline model that allows for an arbitrary number of sectors with an arbitrary input-output structure.

To the best of our knowledge, the only existing studies that develop a similar type of a model are the works of Bouakez et al [3], [2]. Even though we were undoubtedly inspired by these works, our model differs from that of Boukaez et al in several aspects. Firstly, their model is built to utilize the structure of the U.S. national accounts – which differs from that in both the Czech Republic and the EU as a whole; most importantly, the U.S. national accounts collect data on the capital flows among

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the individual sectors (the Capital Flow Tables), which allow for a simple incorporation of the capital formation proces into the DSGE model. As we aim to apply our model to Czech data in future, we had to resort to a different, less straightforward method of combining the input-output processes and capital stock creation, which proved to complicate the equilibrium derivation. Secondly, while the models of Bouakez et al take on the New Keynesian perspective, ours is mostly in line with the real business cycle (RBC) literature.

2 The model

The model economy consists of a single infinitely-lived household and J firms, representing one sector each. The household is the sole labour supplier to all firms, and the firm's production can be either consumed by the household, used as a material input in other sectors, or used in the form of a capital investment. In our treatment, we use the social-planner formulation of the model, which should give identical equilibrium conditions as the competitive equilibrium formulation, but with a lower notational burden, see [7]. The social planner's aim is to maximize the expected sum of households (discounted) utilities, i.e. to maximize

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t U(C_t, H_t) = \mathbb{E}\sum_{t=0}^{\infty}\beta^t\big[\log C_t - \phi\log H_t\big],$$

where U is the (instantaneous) utility function, C_t and H_t are the consumption and labour-time bundle in period t and ϕ is a non-negative parameter (labour disutility). The consumption bundle is defined as a weighted Dixit-Stiglitz aggregate of sector-specific consumptions:

$$C_t \equiv \Big[\textstyle\sum_j \gamma_j C_{jt}^{\theta}\Big]^{1/\theta},$$

where C_{jt} is the quantity consumed of goods from sector j in period t, γ_j is the relative weight of the goods from sector j in the consumption bundle (γ_j are nonnegative and sum to one), and θ is a parameter greater than 1, such that $\theta/(\theta-1)$ is the elasticity of substition among the sectors' production.

Similarly, the labour time bundle is defined as

$$H_t \equiv \Big[\textstyle\sum_j H_{jt}^{\upsilon}\Big]^{1/\upsilon},$$

where H_{jt} are hours worked in sector j and period t and υ is a parameter less than 1 representing the substition among hours spent working in individual sectors. Note that in this setting the household prefers diversity in labour time.

The production function in sector j has the form of a Cobb-Douglas function with constant returns to scale:

$$Y_{jt} = (e^{Z_t}H_{jt})^{\alpha_{jH}}K_{j,t-1}^{\alpha_{jK}}M_{jt}^{\alpha_{jM}},$$

where Z_t is an exogenous aggregate labour productivity shock, $K_{j,t-1}$ is the capital stock in sector j at the beginning of period t (i.e. capital stock determined by the end of period $t-1$), M_{jt} is a bundle of all material flows to sector j from other sectors, and the α's are sector-specific output elasticities that sum to one. The material bundle is defined as

$$M_{jt} \equiv \Big[\textstyle\sum_i \mu_{ij}M_{ijt}^{\theta}\Big]^{1/\theta},$$

where M_{ijt} is the quantity of sector i's output being supplied (as an input) to sector j, and μ_{ij} is the weight of sector i's output in sector j's material (input) bundle. The labor productivity shock is assumed to follow the standard AR(1) process,

$$Z_t = \rho Z_{t-1} + \varepsilon_t, \qquad \varepsilon_t \sim \mathcal{N}(0,\sigma^2),\ \text{i.i.d.} \tag{1}$$

The budget constraint is given as

$$Y_{jt} = C_{jt} + I_{jt} + \textstyle\sum_k M_{jkt},$$

where I_{jt} denotes capital investment in sector j and period t defined as

$$I_{jt} \equiv K_{jt} - (1-\delta)K_{j,t-1},$$

where δ is the depreciation rate.

Comments on alternative model specifications and calibration. The original sector-specific data (and, most notably, Input-Output tables) presented by the Czech Statistical Office are divided into 81 industries. Technically, such a number of sectors would probably prove too high for the resulting model to be computationally tractable. Therefore, it seems necessary to aggregate these industries in order to reduce the number of sectors to an acceptable level. Following the CZ-NACE classification, the 81 industries can be aggregated into as few as 19 sectors. As for calibration, the μ_{ij} parameters are supposed to be taken directly from Input-Output tables, and the γ_j parameters will be calibrated from the consumer price index. Moreover, we need the parameters of the Cobb-Douglas production function; we intend to use estimates based on the index number approach. The θ parameter can be estimated as the ratio of gross value added in current prices and compensation of employees. These data are published in the national accounts.

3 Solving the model

3.1 Equilibrium conditions.

First, we form the Lagrangian as[2]

$$\begin{aligned} L &= \mathbb{E}\sum_{t=0}^{\infty}\beta^t\Big[\log C_t - \phi\log H_t - \textstyle\sum_j \lambda_{jt}\Big(-Y_{jt} + C_{jt} + I_{jt} + \sum_k M_{jkt}\Big)\Big] \\ &= \mathbb{E}\sum_{t=0}^{\infty}\beta^t\Big[\log C_t - \phi\log H_t + \textstyle\sum_j \lambda_{jt}\Big(-Y_{jt} + C_{jt} + K_{jt} - (1-\delta)K_{j,t-1}\Big) - \sum_i\sum_j \lambda_{it}M_{ijt}\Big]. \end{aligned}$$

The first-order conditions are obtained by setting the partial derivatives of the Lagrangian (w.r.t. the decision variables and Lagrange multipliers) to zero. After the elimination of Lagrange multipliers and some obvious algebraic manipulation, we can list the equilibrium conditions as follows:

$$\begin{aligned} \alpha_{jH}\,\gamma_j\,Y_{jt}\,C_{jt}^{\theta-2}\,C_t^{-\theta} &= \phi\,(H_{jt}/H_t)^{\upsilon} && \text{for all } j, \\ \gamma_i\,C_{it}^{\theta-2}\,M_{ijt}^{1-\theta} &= \gamma_j\,\alpha_{jM}\,\mu_{ij}\,Y_{jt}\,C_{jt}^{\theta-2}\,M_{jt}^{-\theta} && \text{for all } i,j, \\ C_{jt}^{\theta-2}C_t^{-\theta} &= \beta\,\mathbb{E}_t\Big[C_{j,t+1}^{\theta-2}\,C_{t+1}^{-\theta}\Big(\alpha_{jK}(Y_{j,t+1}/K_{jt}) + 1 - \delta\Big)\Big] && \text{for all } j, \\ Y_{jt} &= (e^{Z_t}\,H_{jt})^{\alpha_{jH}}\,K_{j,t-1}^{\alpha_{jK}}\,M_{jt}^{\alpha_{jM}} && \text{for all } j, \\ Y_{jt} &= C_{jt} + I_{jt} + \textstyle\sum_k M_{jkt} && \text{for all } j, \\ Z_t &= \rho Z_{t-1} + \varepsilon_t, \\ \varepsilon_t &\sim \mathcal{N}(0,\sigma^2),\ \text{i.i.d.} \end{aligned}$$

Altogether, this gives $(J^2 + 4J + 1)$ equations. Note that if one wants to use a software package like Dynare, symbolic expressions containing loops and sums are not supported in the code. Even for J as low as 3, writing down these equations in hand is a very tedious task, and typing errors are likely to creep into the code. For these reasons, we put up a Matlab script that creates the appropriate Dynare code for an arbitrary number of sectors; we will gladly provide the Matlab script to anyone interested at an email request.

3.2 Steady state conditions

In the analysis of equilibrium dynamics, one has to find the steady state first. The steady state conditions are easily obtained from the equilibrium conditions by dropping the time subscript (t) and the error term

[2] We deliberately omitted the constraint (1) and its Lagrange multiplier in order to save space; note that Z_t is an exogenous random process, so it does not enter any first-order conditions w.r.t. the decision variables. The equilibrium conditions given below, however, do include the constraint on Z_t.

(ε_t). After a slight algebraic manipulation[3] we obtain a system of non-linear equations

$$\alpha_{jH}\,\gamma_j\,Y_j\,C_j^{\theta-2}\,C^{-\theta} = \phi\,(H_j/H_t)^{\upsilon} \qquad \text{for all } j, \tag{2}$$

$$\gamma_i\,C_i^{\theta-2}\,M_{ij}^{1-\theta} = \gamma_j\,\alpha_{jM}\,\mu_{ij}\,Y_j\,C_j^{\theta-2}\,M_j^{-\theta} \qquad \text{for all } i,j, \tag{3}$$

$$1 = \beta\Big(\alpha_{jK}(Y_j/K_j) + 1 - \delta\Big) \qquad \text{for all } j, \tag{4}$$

$$Y_j = H_j^{\alpha_{jH}}\,K_j^{\alpha_{jK}}\,M_j^{\alpha_{jM}} \qquad \text{for all } j, \tag{5}$$

$$Y_j = C_j + I_j + \textstyle\sum_k M_{jk} \qquad \text{for all } j. \tag{6}$$

3.3 Steady state – symmetric case

Consider now a special symmetric case where for all j, $\alpha_jK = \alpha_K$, $\alpha_jH = \alpha_H$, $\alpha_jM = \alpha_M$, $\gamma_j = \gamma$, and $\mu_{ij} = \mu$ for all $i \neq j$, $\mu_{ij} = 0$ otherwise. In this case, the analytic steady-state solution has the form given in the proposition below.

Proposition 1. *Let*

$$R = \frac{\alpha_K}{1/\beta - 1 + \delta}, \quad S = \mu^{1/\theta}\alpha_M(J-1)^{(1-\theta)/\theta}, \quad T = R^{-\frac{\alpha_K}{\alpha_H}}S^{-\frac{\alpha_M}{\alpha_H}}.$$

Then, in the steady state of the symmetric problem, for all j,

$$Y_j = \frac{\phi}{\alpha_H}C_j^2\,, \tag{7}$$

$$M_{ij} = \frac{\alpha_M}{J-1}Y_j\,,$$

$$K_j = R\,Y_j\,, \tag{8}$$

$$C_j = \frac{\alpha_H}{\phi(1-\delta R-\alpha_M)}\,, \tag{9}$$

$$H_j = S\,Y_j. \tag{10}$$

Proof. It is useful to derive the following identities first:

$$H^{\upsilon} = \textstyle\sum_j H_j^{\upsilon} = JH_j^{\upsilon}\,, \quad \text{and so} \qquad (H_j/H)^{\upsilon} = 1/J\,, \tag{11}$$

$$C^{\theta} = \textstyle\sum_j \gamma C_j^{\theta} = \gamma J C_j^{\theta}\,, \quad \text{and so} \qquad (C_j/C)^{\theta} = 1/(\gamma J)\,, \tag{12}$$

$$M_j^{\theta} = \textstyle\sum_{i\neq j}\mu M_{ij}^{\theta} = \mu(J-1)M_{ij}^{\theta}\,, \quad \text{and so} \qquad (M_{ij}/M_j)^{\theta} = 1/[\mu(J-1)]\,, \tag{13}$$

the last equation holds for all $i \neq j$. Now, plugging (11) and (12) into (2) immediately yields $\alpha_H Y_j = \phi C_j^2$, a simple rearrangement of (7). In (3), the terms γ_i, γ_j and C_i, C_j can be cancelled due to symmetry, and plugging in (13) gives $M_{ij} = \frac{\alpha_M}{J-1}Y_j$. Note that summing this formula across the second subscript yields

$$\textstyle\sum_k M_{jk} = \sum_{k\neq j}\frac{\alpha_M}{J-1}Y_k = \alpha_M Y_j\,. \tag{14}$$

The equation $K_j = RY_j$ follows immediately from (4). Next, note that

$$M_j = \Big[\sum_i \mu M_{ij}^{\theta}\Big]^{1/\theta} = \Big[\mu(J-1)\left(\frac{\alpha_M}{J-1}Y_j\right)^{\theta}\Big]^{1/\theta} = S\,Y_j.$$

The equation (9) can be obtained by plugging (7), (8) and (14) into (6). From (5) and the previous results, we have

$$Y_j = H_j^{\alpha_H}(R\,Y_j)^{\alpha_K}(S\,Y_j)^{\alpha_M},$$

which gives (10), and completes the proof. □

It is easily seen that the equations in Proposition 1 follow a recursive scheme in the decision variables – C_j is expressed in terms of the model parameters only, Y_j is in terms of C_j, and the rest is given in terms of Y_j.

[3] To be specific, we cancelled equal terms on both sides of the third equation, and noted that in the equilibrium, $Z = \rho Z$, and hence $Z = 0$.

4 Conclusions

As we pointed out in the introduction, this paper was primarily meant to show that DSGE models can be formulated to work with a disaggregated structure of the economy, and start the academic discussion that will hopefully lead to creating DSGE models that explain some of the real-life phenomena that were not captured by the extant models. We carried out first steps in this direction by devising a simple RBC-based DSGE model that can accomodate an arbitrary number of sectors, the interactions of which are described by the input-output tables produced by the national accounts along the accepted EU accounting standards. In chapter 3, we derived the equilibrium conditions that can be e.g. entered into the widely-used Dynare software package.

However, even in this simple setting, there are some practical obstacles that need to be tackled before the model can be used with real-life data. Concretely, efficient numeric solutions of the steady state are required. As of yet, we have only obtained some preliminary results, namely the characterization of the steady state in the special case of a symmetric problem, described in Proposition 1. In future, we aim to test to what extent can these results be used to find initial guesses for the numeric algorithms that solve the problem in the general case.

Other goals for future reseach include calibrating the model on Czech data and extending the baseline models to a more elaborate setting; these extensions might be as follows:

- more sector-specific parameters (such as the depreciation rate, technology shocks etc.),
- the introduction of the government and foreign sectors,
- explicit treatment of the labour market and its imperfections,
- incorporating possible price, labour, and capital adjustment costs.

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Fuzzy multiple criteria evaluation method

Adam Borovička[1]

Abstract. The article deals with new a proposed fuzzy multiple criteria evaluation method. This approach enables to include some uncertainties in a decision making process. Criteria values and weights of criteria may be expressed in a vague, uncertain form. A decision maker sets an importance of criteria linguistically. Some criteria values can be stochastic, random variables with some probability distribution, respectively. The criteria values and weights are expressed as triangular fuzzy numbers. The method requires the ranking of fuzzy numbers which is made via modified McCohone's approach. The algorithm is computationally namely based on preference relations among alternatives and also differences in (non)fuzzy evaluations of alternatives. The proposed multiple attribute decision making method is able to choose primarily one effective alternative. Finally, the approach is clearly applied in the investment decision making process in the capital market with open shares funds.

Keywords: fuzzy number, membership function, uncertainty.

JEL Classification: C44, G11
AMS Classification: 90-08, 91B28

1 Introduction

Many uncertainties can occur in a decision making process, even stochastic character of criteria values, vague preferences of a decision maker etc. To make the decision making procedure more real these factors should be included in used methodical approaches.

When we want to choose some alternative from the set some multiple criteria evaluation method can be applied. There are many methods which offer full ranking of alternatives and mostly are not able to involve some uncertain, vague input information. But we are trying to choose only one effective alternative and mainly take into account uncertainties in the decision making process.

The proposed fuzzy method uses the basic concepts of fuzzy set theory in order to express the uncertainties quantitatively. It evaluates the alternatives on the basis of preference relations between alternatives constructed via some operations with fuzzy weights of criteria and also makes provision for the differences in criterial values, the distance from the ideal alternative is computed.

The reason of the proposal of the fuzzy multi-criteria evaluation method is the particular decision making situation. It is a choice of one open shares funds from each set in the capital market. For this purpose, we projected the method which could include the stochastic elements and vague information from the decision maker. Further, we focused on its computation complexity to be able to apply it in practice without bigger obstructions.

Of course, some illustrative investment practical application of this method is included at the end of the article.

2 Fuzzy multiple attribute decision making method

The proposed method uses fuzzy numbers in order to express the uncertainties. The approach evaluates the alternatives by means of preference relations and also takes into account differences in the criteria values. Modified McCahone's approach is employed for ranking of fuzzy valuations and the modified concept of Hamming distance is applied to compute distance between fuzzy numbers. The introduced method does not offer a full ranking of all alternatives, but chooses only one effective.

Fuzzy multiple criteria evaluation methods will be described in terms of several following steps.

Step 1: Given the matrix $Y = (y_{ij})$, where y_{ij} $(i = 1,2,...,p;\ j = 1,2,...,k)$ represents valuation of *i-th* alternative by *j-th* criterion. Some of them can be stochastic, the criteria values are random variables with some probability distribution, respectively. It will be approximated by normal probability distribution $N(\mu, \sigma^2)$.

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Then the vague valuation is expressed as triangular fuzzy number[2], or fuzzy number with triangular membership function as follows

$$T_X = (\mu - 3\sigma, \mu, \mu + 3\sigma),$$

where μ is a mean and σ denotes a standard deviation of values of random variables *X*. This function covers 99,7 % of the set. To make the problem easier the nonlinear probability density function is replaced by linear form in parts as a triangular probability density function. This simplification is clearly displayed in the following figure (Figure 1).

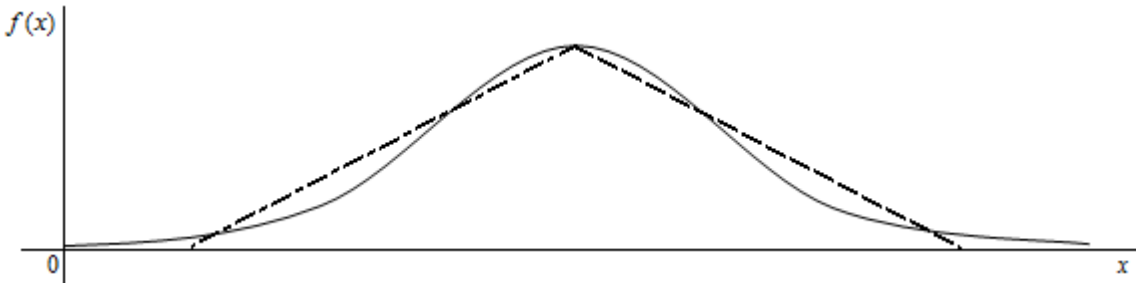

Figure 1 Comparison of two probability density functions

If the criteria values are not vague, uncertain, they can also be expressed as triangular fuzzy numbers with the same value of all three elements as follows

$$T_A = (a, a, a),$$

where *a* is the certain criteria value. Or as mentioned below, this transformation to fuzzy number is not necessary.

An importance of particular criteria is stated via weights formulated as triangular fuzzy numbers. The fuzzy numbers can be set on the basis of linguistic references about criteria importance from a decision maker. In other words, concrete value from interval $\langle 0,1 \rangle$ with some tolerance is assigned to each criterion (see more [2]).

Step 2: In the second step, we must find out definite ranking for all alternatives according to each criterion. This is a little problem in the case of fuzzy numbers. For this procedure, the McCahone's approach is applied with some smaller modifications and improvements.

Modified McCahone's approach

This approach compares the fuzzy numbers via the fuzzy maximum and fuzzy minimum [8]. Given *n* fuzzy numbers $F_1, F_2, ..., F_n$. The fuzzy max, or the membership function of fuzzy max can be formulated as

$$\mu_{\max}(x) = \sup_{x = x_1 \vee x_2 \vee ... \vee x_n} [\mu_{F_1}(x_1) \wedge \mu_{F_2}(x_2) \wedge ... \wedge \mu_{F_n}(x_n)] \quad \forall x, x_1, x_2, ..., x_n,$$

where $\mu_{F_1}(x_1), \mu_{F_2}(x_2), ..., \mu_{F_n}(x_n)$ are the membership functions of fuzzy numbers $F_1, F_2, ..., F_n$. We can analogously specify fuzzy min as

$$\mu_{\min}(x) = \sup_{x = x_1 \wedge x_2 \wedge ... \wedge x_n} [\mu_{F_1}(x_1) \wedge \mu_{F_2}(x_2) \wedge ... \wedge \mu_{F_n}(x_n)] \quad \forall x, x_1, x_2, ..., x_n.$$

The fuzzy max and fuzzy min are depicted in the following two graphs (Figure 2).

[2] *Triangular fuzzy number* is a convex fuzzy set with triangular shape of membership function (see more [4] or [9]). *Fuzzy set* is such a set whose elements are included with certain grade of membership (see more [3] or [10]). *Membership function* measures the grade of set membership. It takes the value from interval $\langle 0,1 \rangle$. The higher value denotes the higher degree of set membership [5].

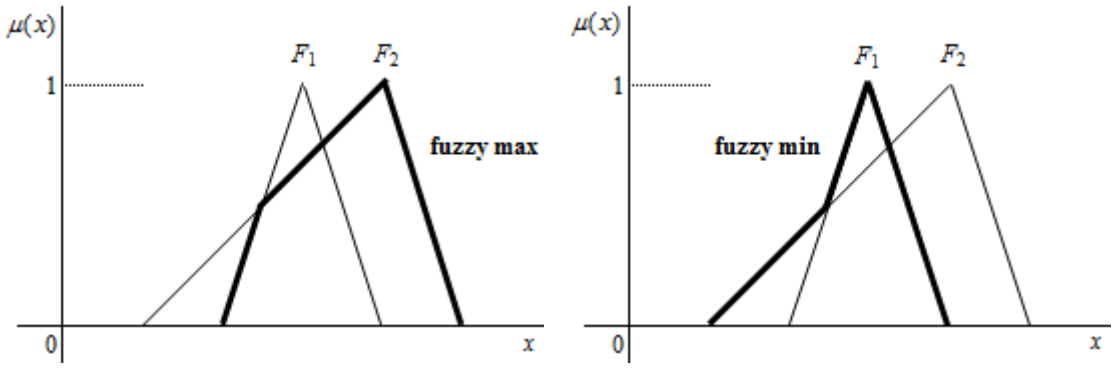

Figure 2 The fuzzy max a fuzzy min

Now we compute the contribution of fuzzy number F_i $(i = 1, 2, ..., n)$ toward the fuzzy max by the following formula

$$P(F_i) = \frac{\int_{S(F_i)} [\mu_{\max}(x) \wedge \mu_{F_i}(x)]\, dx}{\int_{S(F_i)} \mu_{F_i}(x)\, dx},$$

where $S(F_i)$ is the definition scope of *i-th* fuzzy number F_i.

Similarly, the contribution of fuzzy number F_i $(i = 1, 2, ..., n)$ toward the fuzzy min is defined by

$$N(F_i) = \frac{\int_{S(F_i)} [\mu_{\min}(x) \wedge \mu_{F_i}(x)]\, dx}{\int_{S(F_i)} \mu_{F_i}(x)\, dx}.$$

The contribution of fuzzy number F_i toward to fuzzy max, or fuzzy min is displayed by the following two graphs (Figure 3).

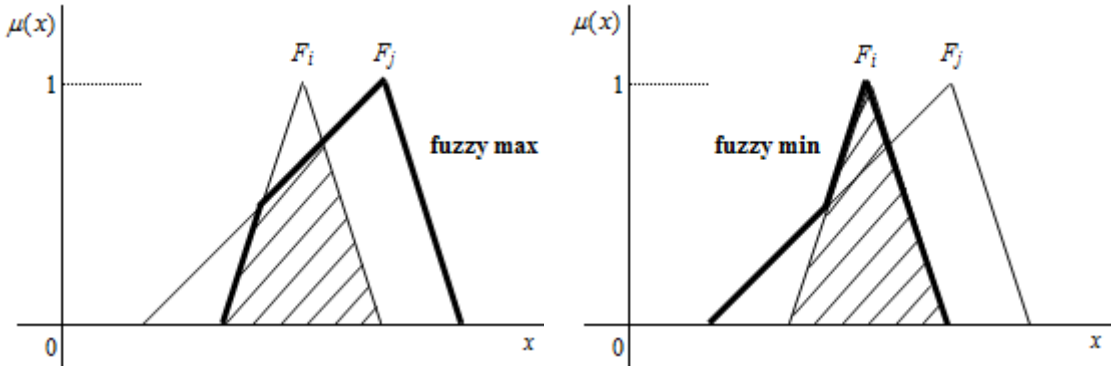

Figure 3 The contribution of F_i toward the fuzzy max and fuzzy min

In the next step, we rank the fuzzy numbers according to $P(F_i)$ descending and $N(F_i)$ upwardly. Two rankings are compared. If both raking orders are identical, the algorithm stops. If not, we will pick the fuzzy numbers sharing the same positions and perform pairwise comparison via the rules described in the following section.

We calculate the composite index

$$CP(F_i) = \frac{P(F_i)}{P(F_i) + N(F_i)}.$$

And now all fuzzy numbers sharing the same position are ranked descending according to this indicator. If the index is identical for more fuzzy numbers, it is not possible to distinguish them. Then we use the second rule, thus we compare the absolute sum of $P(F_i)$ and $N(F_i)$. One of the following relations must hold

$$\text{if } P(F_k)+N(F_k) > P(F_l)+N(F_l), \text{ then } F_k > F_l,$$
$$\text{if } P(F_k)+N(F_k) < P(F_l)+N(F_l), \text{ then } F_k < F_l,$$
$$\text{if } P(F_k)+N(F_k) = P(F_l)+N(F_l), \text{ then } F_k = F_l.$$

The problem of described method is that the locations of fuzzy numbers are not always considered. The indiscrimination case can be as follows. Generally, if we have a set of fuzzy numbers $\{F_1, F_2, ..., F_n\}$, where F_1 is the fuzzy min and F_n is the fuzzy max. Both of them do not overlap with other fuzzy numbers. It is obvious, that $F_2, F_3, ..., F_{n-1}$ cannot be distinguished by McCahone's approach. The indiscrimination case can be illustrated as follows (Figure 4).

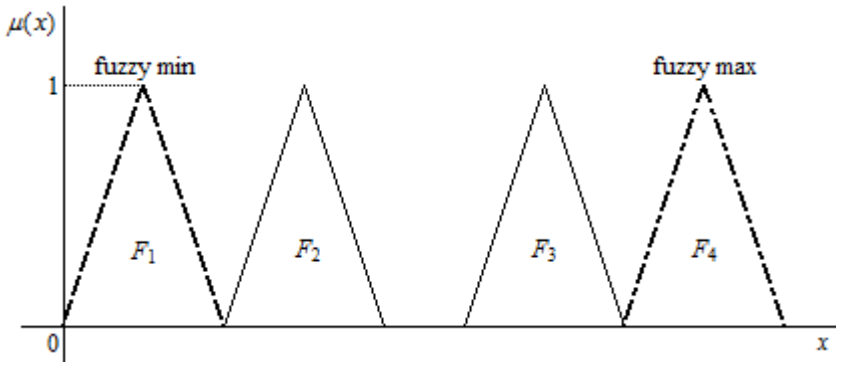

Figure 4 The indiscrimination case of McCahone's method

In this ominous situation, it holds $P(F_2) = ... = P(F_{n-1}) = N(F_2) = ... = N(F_{n-1}) = 0$. It is no doubt, that these fuzzy numbers will share the same position according to this algorithm and it is not possible to differentiate them even if they can unambiguously different. We suggest to apply the described McCahone's approach once again only for these fuzzy numbers to rank them in terms of common positions and then make a full ranking of all fuzzy numbers.

Step 3: And now we specify the set $I_{iRj} = \{r \mid y_{ir} \geq y_{jr}; i, j = 1, 2, ..., p; i \neq j\}$ containing the indices of criteria *r* according to which the alternative *i* is evaluated better or equally than the alternative *j*. Then we can formulate the matrix $\mathbf{S} = (s_{ij})$, where

$$s_{ij} = \sum_{q \in I_{iRj}} v_q \qquad i, j = 1, 2, ..., p, i \neq j,$$
$$s_{ij} = - \qquad i, j = 1, 2, ..., p, i \neq j.$$

The element of the matrix s_{ij} may be interpreted as the volume of the preference of the *i-th* alternative against the *j-th* alternative expressed as fuzzy number[3]. For each *i-th* alternative the average preference relation is calculated as follows

$$s_i^{average} = \frac{\sum_{\substack{j=1 \\ i \neq j}}^{p} s_{ij}}{p-1}.$$

Finally, the threshold value is computed as the following fuzzy number

$$s = \frac{\sum_{\substack{j=1 \\ i \neq j}}^{p} s_{ij}}{p(p-1)} = \frac{\sum_{i=1}^{p} s_i^{average}}{p}.$$

[3] The basic computational operations with fuzzy numbers are described in [6]. If some fuzzy numbers contain negative values, then positive constant is added to transform all values to nonnegative values in order to make the basic operations with fuzzy numbers easier.

Step 4: Firstly, we choose the alternatives which satisfy the following formula

$$s_i^{average} \geq s \,.$$

Because both sides of the constraint are represented by fuzzy number, the fuzzy ranking method (McCahone's approach) is applied once again. Secondly, the distance from ideal alternative is computed for choice alternatives. In the case of nonfuzzy valuation y_{ij} $(i = 1, 2, ..., p; j = 1, 2, ..., k)$, the distance from the best value by *j-th* criterion is as follows

$$d_{ij}^{nonfuzzy} = H_j^{nonfuzzy} - y_{ij}^{nonfuzzy} \,,$$

where $H_j^{nonfuzzy} = \max_i(y_{ij}^{nonfuzzy})$ is the best valuation according to *j-th* criterion.

If the criteria values are fuzzy numbers, then we use the modified concept of Hamming distance in order to find out the distance from the best valuation. The Hamming distance between two fuzzy number F_i and F_j is defined as [7]

$$d(F_i, F_j) = \int_{-\infty}^{+\infty} | \mu_{F_i}(x) - \mu_{F_j}(x) | \, dx \;\; .$$

To calculate the representative distance from the best value we will distinguish the parts, where $\mu_{F_i}(x) > \mu_{F_j}(x)$ and $\mu_{F_i}(x) < \mu_{F_j}(x)$. Let's denote fuzzy number F_j as the best valuation and F_i as a particular valuation whose distance from the best one is calculated. Specify

$$X^1 = \left\{ x; \mu_{F_i}(x) \geq \mu_{F_j}(x) \right\}, \quad X^2 = \left\{ x; \mu_{F_i}(x) < \mu_{F_j}(x) \right\}.$$

Then the distance of valuation from the best one will be defined as follows

$$d(F_i, F_j) = \int_{X^2} | \mu_{F_i}(x) - \mu_{F_j}(x) | \, dx - \int_{X^1} | \mu_{F_i}(x) - \mu_{F_j}(x) | \, dx \,.$$

In the case of fuzzy valuation, we should mark d_{ij}^{fuzzy} as the distance of *i-th* alternative from the best valuation by *j-th* criterion that is computed via the previous formula.

Finally, the general distance from ideal alternative for *i-th* chosen alternative is computed as

$$d_i = \sum_{j=1}^{k} \left[\frac{d_{ij}^{nonfuzzy}}{\max_i(d_{ij}^{nonfuzzy})} + \frac{d_{ij}^{fuzzy}}{\max_i(d_{ij}^{fuzzy})} \right].$$

The partial distance by each criterion *j* is standardized with the view of their comparability. The alternative with least distance from the ideal one is chosen as effective. This alternative is always nondominated. It is also possible to choose more alternatives which then could be dominated by effective one.

3 Practical application in the capital market with open shares funds

A potential investor decided to invest some money in open shares funds offered and managed by Investment company Česká spořitelna. There are three groups of shares funds – mixed, bond and stock funds. For illustrative instance, the group of bond funds is analysed. Thus we have five bond open shares funds – Sporoinvest, Sporobond, Trendbond, Korporátní dluhopisový and High Yield dluhopisový. The investor evaluates the investment alternative according to three main criteria – return, risk and costs.

We have 6-year time series of monthly returns of each open shares fund. According to statistic test, the distribution of returns can be described by Student's probability distribution because of its heavy tails and greater kurtosis. The distribution of returns is also usually negatively skewed. For simplification, we expect that the probability distribution is normal. As mentioned above, the probability density function is approximated by triangular shape. Then the return is expressed as a triangular fuzzy number. The risk is set as a standard deviation of monthly returns in last six years. The costs are represented by entry fee and TER (total expense ratio) containing (e. g.) management or license fee.

The preferences about criteria importance is described linguistically by the investor. The risk is very important, the return is important and the costs are little important. Each linguistic expression is transformed to a quantitative formula by means of fuzzy numbers. All necessary data is in the following table (Table 1).

Shares fund	Return	Risk	Costs
Sporoinvest	(-1.17, 0.06, 1.29)	0.41	1.07
Sporobond	(-3.7, 0.3, 4.30)	1.33	2.10
Trendbond	(-6.02, 0.18, 6.38)	2.07	2.74
Korporátní dluhopisový	(-9.63, 0.38, 10.38)	3.34	2.74
High Yield dluhopisový	(-13.15, 0.33, 13.81)	4.49	2.56
Weights	*(0.25, 0.3, 0.35)*	*(0.45, 0.55, 0.65)*	*(0.12, 0.15, 0.18)*

Table 1 The data about bond open shares funds

Shares fund Sporoinvest is the best in criteria risk and costs. According to modified McCahone's approach, Sporobond is the best in return. To choose one effective alternative the proposed fuzzy method is employed. Before the measuring distance from ideal alternative, two funds are chosen – Sporoinvest, Sporobond. Sporoinvest has the shortest distance from the ideal alternative, so it is the effective alternative according to applied introduced fuzzy method. Anyway it was quite expectable, because it dominates in the most important criterion.

This methodical approach can be applied in each group of shares funds and then the final investment portfolio can be created by appropriate methods (see more [1] or [2]).

4 Conclusion

The main contribution of this paper is a new fuzzy multiple criteria evaluation method which is able to contain some uncertain, vague information in the decision making process. The proposed approach makes use of the fuzzy set theory to quantify the uncertain elements, concretely triangular fuzzy numbers. Then the stochastic criteria values or vague information about importance of criteria may be included. This approach takes into account differences in criterial values and also computes preference relations between alternatives. In terms of the methodical procedure, the ranking of fuzzy number is done by McCahone's approach in the modified form and distance between fuzzy numbers is computed by means of the modified concept of Hamming distance.

In the case of fuzzy methods, the problem can be found in the fuzzy ranking procedure. It is obvious, that making ranking of fuzzy numbers may not be easy. So the result is also based on used fuzzy ranking approach. The disadvantage of the proposed method (in some practical decision making situation) can be the fact that it does not offer the full ranking of alternatives. But we are using this method in the concrete situation in the capital market when we want to choose only one alternative from several groups of open shares funds to make a portfolio.

Acknowledgements

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Consumption tax as a tool for greening transport

Botlíková Milena[1], Botlík Josef[2], Václavínková Klára[3]

Abstract. Development of transport is one of the region's competitiveness assumptions and its development is bound with European conventions. Strategic documents of the development of transport networks are based on the objectives of the White Paper, which is promoting environmentally friendly transport system, where is an appeal to create preconditions for an attractiveness of rail transport and the promotion of public transport. One of the steps to transfer traffic from road to rail and public transport was an increase of consumption taxes on fuels and lubricants, which led to an increase of fuel prices. Whether there is any relationship between the change of excise duty and change of preferred transport habits based on the performance of transport, registration of vehicles and questioning will be determined on the basis of statistical analysis such as correlation and regression[4], etc.

Keywords: taxes, consumer, automotive, consumer.

JEL Classification: C44, C01, C51
AMS Classification: 62P20, 62J12, 91B70

Introduction

In the Czech Republic are present various taxes which are state revenue and some tax acts as a regulator. Among the taxes with regulatory character belongs excise duty on fuels and lubricants. The excise tax on fuels and lubricants is an indirect tax, which is reflected in the price of fuel (1). This tax makes up 1.6% of the state budget and 9% of the total revenue of this tax is part of the State Fund for Transport Infrastructure (SFTI).

$$Fuel\ price = price + dealer\ excise\ duty + VAT \tag{1}$$

apply: Price …… dealer cost of buying = selling price of fuel, the cost of transportation. etc.
Excise……duty A fixed rate by the Ministry of Finance.

VAT …… value added tax-fixed by the Ministry of Finance

Efforts to increase the revenues of the state budget and the lack of financial resources SFDI, the development and renewal of road space, led government to increase the excise tax on fuels and lubricants in 2010. The structure of increased excise taxes is:

- Increase of excise tax on gasoline from11.84 CZK to 12.84 CZK,
- Increase of excise duty on diesel from 9.95 CZK to 10.95 CZK.

Increase of excise taxes on fuels and lubricants, together with the increase of VAT in the same year 2010 from 19% to 20% led to a significant increase of fuel prices.

The impact of increased excise taxes on fuels and lubricants

Increase of excise tax should be a tool for the creation of financial resources for rehabilitation and modernization of the transport area in the Czech Republic. After development of transport infrastructure in 2007, considerable problems occurred in financing of transport infrastructure. European funds were replaced by national; many buildings were slowed or stopped and conserved after 2009. After an increase of excise taxes SFDI items does not increase (see Figure 1). In the case of excise taxes on fuels and lubricants clearly acted Lafer curve and revenues from excise tax decreases. According to a study of Economics, which dealt with an issue of excise duty, it was revealed a need to reduce the excise duty. The reason for this reduction is clearly supportive and will not only help the state budget, but also domestic carriers, retailers and households.

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[4] Paper is based on research by SGS 5/2013, Silesian University in Opava, School of Business Administration in Karvina

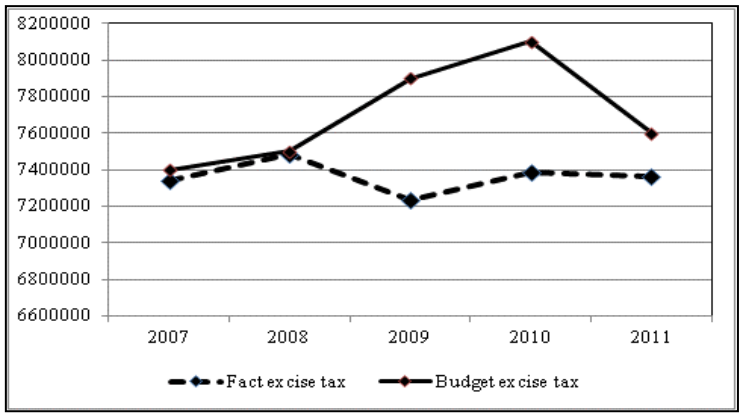

Figure 1 Excise tax SFTI 2007 - 2011

2.1 Excise tax as a means of greening transport

Another argument of the government was that the increase in excise duty PHM would not be the only tool to increase financial resources, but also a tool for increasing the use of environmentally-friendly transport and routing of the ecological behavior in transport, efforts to increase the use of public and rail transport, increase of availability of vehicles etc .. Indirect form should be to reduce congestion on the roads, reducing the environmental impacts resulting from transportation - emissions of CO2 and other greenhouse effects, etc.

Due to the increase of the real price of fuel it is expected that there will be a decrease in consumption as a result of improving the technical parameters of vehicles, restricting the use of vehicles with high capacity to change to another transport. If you want to increase the use of environmentally friendlier transport, then it should be a subsequent increase, for example [date 7]:

- Improving the performance of public transport, the number of passengers.
- Improving the Performance of railway passenger traffic, number of passengers.
- Improving the registration number of vehicles with alternative power.
- Reducing the registration of passenger vehicles.
- Reducing the number of trips by sharing transport.
- Reducing emissions REZZO 4[5]

2.2 Analysis of the impact of excise taxes on transportation habits

To assess the effect of increased excise tax and the increase in fuel prices on traffic patterns relationship between fuel price and individual characteristics was investigated. In an analysis, which is based on research SGS 5/2013, was determined using the relationship between each modality and the change in fuel prices due to excise duty. The analysis was made on the basis of the evaluation closeness of relationships between the characteristics and the price of fuel for various government interventions. Dependence was investigated in the case where was no increase of excise taxes in 2010, only VAT (Petrol_1; Diesel_1) was increased and then the relationship between characteristics and fuel price without state intervention was examined(Petrol_2; Diesel_2)

Correlation was used to determine the direct relationship between the amount of excise tax, which is reflected in the price of fuel, and individual characteristics. When data do not show a normal distribution and is not expected linear dependence, it is appropriate to use Spearman (ordinal) correlation coefficient. For qualified dependency evaluation and assessment which depends on the data with normal distribution Pearson's coefficient (1) can be used, which is more concise. Given that the data on the basis of testing SPSS a normal distribution can be the coefficient of leakage evaluation depending used.

$$r = \frac{n * \left(\sum_{i=1}^{n} x_i y_i \right) - \left(\sum_{i=1}^{n} x_i \right) \left(\sum_{i=1}^{n} y_i \right)}{\sqrt{\left(n \sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i \right)^2 \right) \left(n \sum_{i=1}^{n} y_i^2 - \left(\sum_{i=1}^{n} y_i \right)^2 \right)}} \quad (1)$$

[5] REZZO4 - Register of Emissions and Air Pollution Sources - Number 4 - mobile device or other internal combustion engines that pollute the air, especially road and motor vehicles, rail vehicles, vessels and aircraft

Based on the analysis, it was tested whether individual variables are correlated variables, if the correlation coefficient is equal to zero, ie, if true than: H_0: $r = 0$ the independence of two variables against the alternative hypothesis H_1: $r \neq 0$, the dependent variables and whether the individual coefficients correlation considered statistically significant, and thus applies the value of the test criterion T (2) is greater than the critical value of the coefficient $t_{(n-2)(\alpha)}$ (3).

$$T = \left| \frac{r}{\sqrt{(1-r^2)}} \right| * \sqrt{(n-2)} \tag{2}$$

$$|T| > \text{critical value } t_{(\alpha, n-2)} = 2{,}06 \tag{3}$$

apply: *n –Number of observations*

The values of the correlation coefficient in the range -1 < y < 1 can be interpreted on the basis of the claim as follows: [4,5] (Figure 2). The negative value of r indicates the relationship contradictory, if the there is growth of one character (X, respectively. Y) there will be a decrease of the second (X, respectively. Y). A positive correlation coefficient represents an increase of values for both variables, ie., With the growth of one variable (X respectively. Y) the second will fluctuate (X, respectively. Y).

Correlation coefficient	Interpretation
$\|r\| \geq 1$	absolute dependence (functional)
$1 > \|r\| \geq 0,9$	very high
$0,9 > \|r\| \geq 0,7$	high
$0,7 > \|r\| \geq 0,4$	medium
$0,4 > \|r\| \geq 0,2$	low dependence
$0,2 > \|r\| \geq 0,00$	low useless
$\|r\| = 0$	complete independence

Figure 2 The interpretation of correlation coefficient

Based on the analysis and interpretation of SPSS (see Figure 3) by Chráska was found functional dependence between the 95 octane petrol and bus, city bus, tram and underground transport. It was found that, at the current a tax burden exhibit different magnitudes greater dependence than if the tax burden was maintained at the original level. If the price of petrol 95 octane and diesel increases, the number passengers in bus and tram transport will decrease. If the fuel prices increase, there will be increase of passenger stations. The analysis confirmed the correlation between the price of fuel performance shipments (million passenger-kilometers) of railway and metro.

		BUS	RAILWAY	BUS MHD	TROLEJBUS	TRAMVAJ	METRO
PETROL 95	Pearson Correlation	-0.414	-0.179	**-0.754**	-0.148	**-0.648**	**0.673**
	Sig (2-tailed)	0.029	0.381	0.000	0.458	0.000	0.000
	N	28	28	28	28	28	28
DIESEL	Pearson Correlation	-0.307	-0.081	**-0.661**	-0.029	**-0.599**	**0.632**
	Sig (2-tailed)	0.112	0.681	0.000	0.882	0.001	0.000
	N	28	28	28	28	28	28
PETROL_1	Pearson Correlation	-0.442	-0.117	**-0.681**	-0.134	**-0.568**	**0.580**
	Sig (2-tailed)	0.018	0.555	0.000	0.496	0.002	0.001
	N	28	28	28	28	28	28
DIESEL_1	Pearson Correlation	-0.314	-0.008	**-0.569**	-0.001	**-0.509**	**0.530**
	Sig (2-tailed)	0.104	0.967	0.002	0.996	0.006	0.004
	N	28	28	28	28	28	28
PETROL_2	Pearson Correlation	**-0.448**	-0.100	**-0.660**	-0.129	**-0.546**	**0.555**
	Sig (2-tailed)	0.017	0.612	0.000	0.512	0.003	0.002
	N	28	28	28	28	28	28
DIESEL_2	Pearson Correlation	-0.314	0.010	**-0.545**	0.007	**-0.485**	**0.503**
	Sig (2-tailed)	0.104	0.961	0.003	0.972	0.009	0.006
	N	28	28	28	28	28	28

Figure 3 The relationship between the price of fuel and the number of persons on selected modalities

* Bus - bus lines and irregular (national and international), ** Bus_1 - bus service within IDOS (urban transport)

With the increase of price of fuel there will be an increase in traffic of railway and metro. Higher correlation dependencies were waylaid between the performance of rail transport in case of maintaining excise duty on the border of CZK 11.84 for petrol and 9.95 for diesel and maintenance of VAT at 19%. In the case of bulk power stations were established interdependence between the price of fuel (petrol and diesel) with excise duties and VAT, whose development was accompanied by changes.

On the contrary, the case of rising fuel prices will lead to a reduction in performance of trolleybus transport. Higher dependence between gasoline price and performance of trolleybus transport was demonstrated in the case of oil prices from the treated excise taxes in the case of gasoline excise original and modified, but the original VAT (see Figure 4).

		PER BUS	PER RAILWAY	PER BUS MHD	PER TROLEJBUS	PER TRAMVAJ	PER METRO
PETROL 95	Pearson Correlation	-0.062	0.379	-0.185	**-0.550**	-0.125	**0.442**
	Sig (2-tailed)	0.755	0.047	0.376	0.002	0.528	0.019
	N	28	28	28	28	28	28
DIESEL	Pearson Correlation	-0.100	0.392	-0.107	**-0.455**	-0.082	**0.485**
	Sig (2-tailed)	0.613	0.039	0.589	0.015	0.677	0.009
	N	28	28	28	28	28	28
PETROL_1	Pearson Correlation	-0.088	**0.416**	-0.213	**-0.568**	-0.139	0.392
	Sig (2-tailed)	0.657	0.028	0.277	0.002	0.480	0.039
	N	28	28	28	28	28	28
DIESEL_1	Pearson Correlation	-0.128	**0.423**	-0.120	**-0.447**	0.089	**0.437**
	Sig (2-tailed)	0.516	0.025	0.542	0.017	0.653	0.020
	N	28	28	28	28	28	28
PETROL_2	Pearson Correlation	-0.093	**0.424**	-0.219	**-0.564**	-0.142	**0.377**
	Sig (2-tailed)	0.636	0.025	0.262	0.002	0.470	0.048
	N	28	28	28	28	28	28
DIESEL_2	Pearson Correlation	-0.133	**0.429**	-0.124	**-0.442**	-0.090	**0.423**
	Sig (2-tailed)	0.499	0.023	0.530	0.018	0.649	0.025
	N	28	28	28	28	28	28

Figure 4 The relationship between fuel price and performance of selected modalities

In the event of an increase of fuel prices was seen positive correlation between the price of fuel and registrations of new cars with diesel engines and drive vehicles LPG, CNG (see Figure 5).

		PETRO CAR	DIESELCAR	ECOCAR	OTHERCAR	OLD CAR
PETROL 95	Pearson Correlation	0.094	**0.658**	**0.779**	-0.272	-0,312
	Sig (2-tailed)	0.633	0.000	0.000	0.161	0,105
	N	28	28	28	28	28
DIESEL	Pearson Correlation	0.094	**0.539**	**0.756**	-0.139	-0,211
	Sig (2-tailed)	0.634	0.003	0.000	0.480	0,282
	N	28	28	28	28	28
PETROL_1	Pearson Correlation	0.092	**0.583**	**0.760**	-0.244	-0,215
	Sig (2-tailed)	0.641	0.001	0.000	0.211	0,272
	N	28	28	28	28	28
DIESEL_1	Pearson Correlation	0.091	**0.445**	**0.724**	-0.092	-0,102
	Sig (2-tailed)	0.646	0.018	0.000	0.642	0,605
	N	28	28	28	28	28
PETROL_2	Pearson Correlation	0.091	**0.652**	**0.752**	-0.235	-0,188
	Sig (2-tailed)	0.644	0.002	0.000	0.228	0,337
	N	28	28	28	28	28
DIESEL_2	Pearson Correlation	0.090	**0.420**	**0.712**	-0.078	-0,074
	Sig (2-tailed)	0.650	0.026	0.000	0.692	0,709
	N	28	28	28	28	28

Figure 5 The relationship between fuel price and new car registrations by type of fuel

This clarified the annual growth of each new car registrations (see Figure 6). Although, there was a presumption that the increased price of fuel will be restricted in the use of vehicles with higher consumption, where you can include the vehicle's age, dependency between registration of old vehicles and fuel price has not been demonstrated.

in %	Petrol	Diesel	CNG	LPG	E85	Biodiesel	Elektric	Hybrid	Other
2006	69,25	28,13	0	0	0	0	0	0	2,62
2007	69,52	27,73	0	0	0	0	0	0	2,73
2008	70,97	23,91	0,05	0	0,01	0	0	0	5,07
2009	64,88	31,77	0,06	0,01	0,02	0	0	0	3,25
2010	57,28	40,00	0,09	0,04	0,02	0	0	0	2,57
2011	57,80	39,54	0,11	0,07	0,20	0	0,03	0	2,25
2012	55,79	41,38	0,27	0,30	0,34	0	0,05	0,21	1,67

Figure 6 Development of new car registrations by fuel used

Quality of correlation dependence was analyzed by comparing the test criteria T and critical values *t* of for statistically significant we can consider the interdependence, $| T | \geq t\ (n-2)\ (\alpha)$ ie. $| T | \geq 2.06$, correlation or rejected. Based on the calculations, we can confirm the correlation to be statistically significant at the significance level $\alpha = 0.05$ for all selected variables (see Figure 7).

	Petrol 95_2 x Bus	Petrol 95x Bus_1	Petrol 95 x Tramvaj	Petrol 95 x Metro	Diesel x Bus_1	Diesel x Tramvaj	Diesel x Metro	Petrol 95_2 x perf railway	Petrol x perf trolejbus	Petrol x perf metro	Diesel_2 x perf railway	Diesel x perf trolejbus	Diesel x perf metro	Petrol 95 x Diesel Car	Petrol 95 x Eco Car	Diesel x Diesel Car	Diesel x Eco Car
Test crieteria *T*	2.56	5.58	4.34	4.63	4.49	3.81	4.16	2.39	3.48	2.51	2.42	2.61	2.83	4.46	6.33	3.26	5.84
Critical values *t*	> 2.06																

Figure 7 Test the statistical significance of the correlation between the different variables

Confirmation of the existence of relationships between the price of fuel and transport behavior of individual consumers was based on a test of independence of two characters performed using a pilot survey.

Fuel price/How high fuel costs, by forcing you to changing traffic patterns	SHARING	ALTERNATIVE TRANSPORTATION	FUEL ALTERNATIVE	OTHER
Non specified	1	2	2	3
35 - 39	1	1	1	1
40 - 43	1	1	1	1
44 - 47	3	4	1	5
> 47	2	4	1	12

Figure 8 Pilot survey

Reaching respondents were not forced to deal with changes in transportation habits at the current price level, however, if there was an increase in fuel prices over 47 CZK they would consider their current modes of transportation (see Figure 8).

Confirmation or rejection of the relationship between fuel price and changes in travel behavior was based on the determination of hypotheses $H_0 = p_i = p_j$ independence of characters, comparing the test criterion and a critical value. If the value of *G* belonged to the interval (4), then the alternative H_1 reject the values of independence and pays between variables on doing. Since G = 6.93, falls within the field admission means. that lies in the interval (5), the hypothesis was the independence of two characters taken. It was therefore confirmed the relationship between the price of fuel and possible changes in transport practices. [2]

$$G = \sum_{i=1}^{J} \frac{(n_i - \psi_i)^2}{\psi_i} \tag{4}$$

Critical value

$$\chi^2_{(r-1)(s-1)}(\alpha) = 21.02$$

when: *r = 4; s = 5; α = 0.05*

Industry adoption $H_{o:}$ $\left\langle 0; \chi^2_{(r-1)(s-1)}(\alpha)\right\rangle = \left\langle 0; 21.02\right\rangle$ (5)

Field rejection of $H_{o:}$ $\left\langle \chi^2_{(r\ \ 1)(s\ \ 1)}(\alpha); +\infty \right\rangle = \left\langle 21.02; +\infty\right)$

Conclusion

This analysis did not find a causal relationship. There were found only association relationships. It has been shown that if the X (or Y) changed also Y (or X) would change, but it was not confirmed whether this change was caused by X or Y. There is no confirmation that it affects fuel prices preference for modality or modalities demand affecting prices.

In general it can be said that the majority of selected variables has demonstrated correlation dependence. Positive correlation dependence was observed in the number of shipments and transport operations underground, even if state intervenes in fuel prices. A positive correlation was also demonstrated in rail transport performance, higher dependence was demonstrated in the case with no changes in excise duty and VAT in 2010. For other quantities, if there was proven correlation dependence, there was dependent negative character, with rising fuel prices will be reduced the usage of bus, city bus, trolleybus and tram system, which can cause increasing congestion, reducing security and other problems associated with transport within cities. A positive aspect is that the increasing fuel prices lead to greening the fleet.

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The Empirical Analysis of the Taylor Rule

Milan Bouda[1]

Abstract. This paper deals with the empirical analysis of the Taylor rule. This rule is used in all New Keynesian DSGE models. The Taylor rule represents the behavior of the central bank. Currently, there are many versions of Taylor rules. This paper deals with the Taylor rule which is used by the Ministry of Finance of the Czech Republic in the model which is called HUBERT. It is a DSGE model that quantifies impacts of fiscal policy. The Taylor rule consists of gap and steady state variables. The empirical time series must be transformed into the steady state variables using the Hodrick-Prescott filter. The parameter estimates are performed using two different methods. The first approach uses the instrumental variables technique. It is necessary to estimate the compound expression which is present in the Taylor rule and in the second is the whole Taylor rule estimated. For both steps is used the Least-Squares method. The second approach estimates the whole Taylor rule at once using the Marquardt algorithm. It is the Least-Squares method which allows estimating the non-linear parameters. These two approaches are compared and the Marquardt algorithm is chosen as the best one. The paper verifies the quality of the Taylor rule and the results of the parameter estimates might be used for the calibration of the New Keynesian DSGE model of the Czech Republic.

Keywords: Taylor rule, H-P filter, HUBERT, DSGE.

JEL Classification: C01
AMS Classification: 91B84

1 Introduction

Monetary policy is the process by which the monetary authority of the Czech Republic controls the supply of money, often targeting a rate of interest for the purpose of promoting economic growth and stability. In economics, a Taylor rule is a monetary policy rule that stipulates how much the central bank should change the nominal interest rate in response to changes in inflation, output, or other economic conditions. In particular, the rule stipulates that for each one-percent increase of inflation, the central bank should raise the nominal interest rate by more than one percentage point. This aspect of the rule is often called the Taylor principle. For further details see Taylor [15].

The paper deals with the empirical analysis of the Taylor rule used in the model which is used by the Ministry of Finance (MF) of the Czech Republic. The MF uses a simple DSGE model of the Czech economy called HUBERT, see Štork, Závacká and Vávra [14]. The model describes the behavior of four basic agents in the economy: households, firms, government and world. Although Hubert is rather a simple version of standard DSGE models it incorporates standard features of New Keynesian economics such as imperfect competition, habit formation of households, nominal and real rigidities.

The main goal of this paper is to analyze the time series which are present in the Taylor rule. This rule is used in each New Keynesian model to predict the behavior of the central bank. It is necessary to analyze the Taylor rule and estimate its parameters. These estimated values can be used in future as a calibrated values of the New Keynesian model of the Czech Republic.

2 Taylor rule

The Taylor rule was developed by Taylor [15] and discussed by Svensson [13]. The rule takes the following form

$$i_t = (1-\phi_i)\left[\bar{i} + \lambda_\pi \hat{\pi}_t + \lambda_y \hat{y}_t\right] + \phi_i i_{t-1}, \tag{1}$$

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where i_t is short-term nominal interest rate, $\bar{i}$ steady state value of short-term interest rate, $\hat{\pi}_t$ deviation of inflation rate from its steady state (target) value, $\hat{y}_t$ output gap, λ_y output gap weight, λ_π inflation weight, ϕ_i interest rate smoothing parameter.

According to Srour [12], there are many reasons for interest rate smoothing. First, the behaving of the central bank is important for investors and smoothing of interest rate ϕ_i can reduce volatility of a term premium and therefore volatility of long-term interest rates and other financial market instruments. Second, the central bank has usually limited information about the shocks hitting the economy. Third, many shocks are serially correlated. According to Levin et al. [8], simple monetary policy rules with a high degree of interest rate smoothing $(\phi_i \to 1)$ are also surprisingly robust against model uncertainty and misspecification. Unfortunately, this is probably a characteristic feature for large closed economies only. Coté et al. [4] find that the most robust rule is the original $(\phi_i \to 1)$ for small open economy. Much worse, Natvik [10] shows that extending a DSGE model for a fiscal block can lead to a serious determinacy problem. From this point of view, a cautionary note should be made for straightforward application of Taylor rules.

3 Hodrick-Prescott filter

The Hodrick–Prescott (HP) filter is a mathematical tool used in macroeconomics, especially in a real business cycle theory to separate the cyclical component of a time series from raw data. It is used to obtain a smoothed-curve representation of a time series, one that is more sensitive to long-term than to short-term fluctuations. The adjustment of sensitivity of the trend to short-term fluctuations is achieved by modifying a multiplier λ. The filter was popularized in the field of economics in the 1990s by economists Robert J. Hodrick and Nobel Memorial Prize winner Edward C. Prescott, see Hodrick and Prescott [6].

The H-P filter allows to separate trend and business cycle from data using the following optimization problem

$$\min_{\left\{y_t, y_t^{trend}\right\}} \sum_{t=1}^{T} \left(y_t\right)^2 + \lambda \sum_{t=1}^{T} \left[\left(y_{t+1}^{trend} - y_t^{trend}\right) - \left(y_t^{trend} - y_{t-1}^{trend}\right)\right]^2 \tag{2}$$

subject to

$$y_t + y_t^{trend} = \log\left(Y_t\right), \tag{3}$$

where T is the last period. We require smooth trend which corresponds to data. This compromise is given by the parameter λ which weight the following expression

$$\left[\left(y_{t+1}^{trend} - y_t^{trend}\right) - \left(y_t^{trend} - y_{t-1}^{trend}\right)\right]^2. \tag{4}$$

The most important is a choice of λ parameter. Hodrick and Prescott [6] recommend for quarterly data $\lambda = 1600$. The smoothed GDP by H-P filter is very often called potential product. The difference between the original and smoothed data is interpreted as output gap. The H-P filter enables to quantify the steady state variables and the output gap. This approach was also performed by Bjørnland [3], Proietti [11] and Billmeier [2].

4 Data

Data are obtained from ARAD (database of the Czech National Bank) and Czech Statistical Office. In Table 1 is a list of all variables which are present in the Taylor rule. In Table 1 is also a short description of these variables and their source. There is a time period from Q1 2000 to Q3 2012 (51 observations). Following three variables are used as a source: PRIBOR 3M (in %), Consumer Price Index ($t-12=100$) and Gross Domestic Product (in millions) is in previous year average prices and it is seasonally adjusted. The target value of inflation is taken from the Czech National Bank. These time series are used in the following data transformation which transforms this time series to steady state or gap time series.

variable	description	source/transformation
i_t	short-term nominal interest rate	PRIBOR 3M
$\bar{i}$	steady state value of short-term interest rate	H-P filtered PRIBOR 3M
$\hat{\pi}_t$	deviation of inflation rate from its steady state (target) value	(Inflation) – (Inflation target set by the CNB)
$\hat{y}_t$	output gap in %	GDP minus H-P filtered GDP divided by H-P filtered GDP

Table 1 Description and source of all variables

5 Empirical analysis

This chapter is focused on empirical analysis of the Taylor rule. First, it is necessary to quantify the steady state and gap variables. This quantification is performed by the Hodrick-Prescott filter. The transformed variables are displayed in Figure 1, Figure 2 and Figure 3. Next, these variables are used in the empirical analysis.

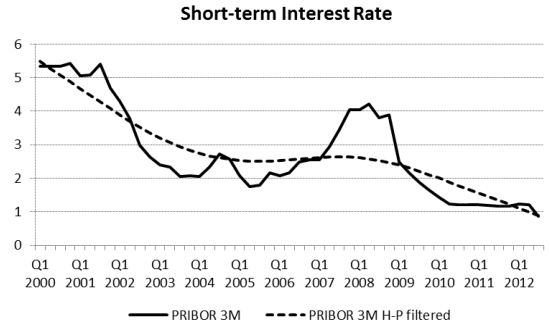

Figure 1 H-P filtered PRIBOR 3M

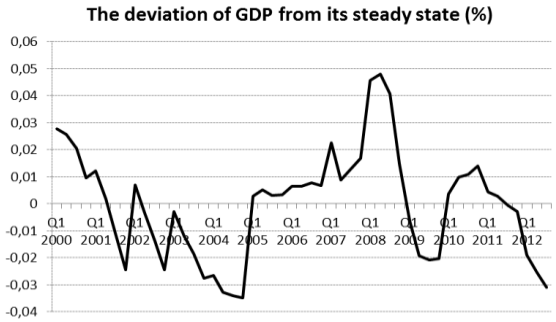

Figure 2 Output gap

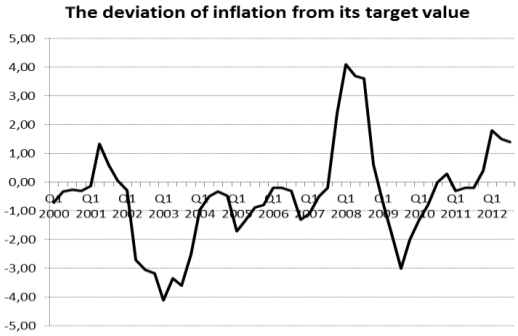

Figure 3 Deviation of inflation rate from its steady state

The empirical analysis verifies the quality of the Taylor rule. The verification is performed using two methods. First, model is estimated by the two steps instrumental method and the second approach is the estimates Taylor rule by the Marquardt method which allows estimating the nonlinear parameters.

Before the model is formulated, the stationarity of all variables is tested. The stationarity may be verified just by looking at the data or with the exact statistical tests, see Greene [5] or Arlt and Arltová [1]. The method used in this paper is KPSS test, for details see Kwiatkowski et al. [7]. The null hypothesis says that the time series is stationary. The Table 2 contains results of the KPSS test. In column KPSS are results of used time series and in column KPSS there are results, there are first differences of used time series. Level of significance is set to 5%. This test proves that variables i_t and $\bar{i}$ are not stationary. Yet it was decided to use original data and do not use first differences. Both presented estimation methods do not assume stationarity of time series and if the time series were modified the economic interpretation would be lost.

variable	KPSS	KPSS for differenced data
i_t	non-stationary	stationary
$\bar{i}$	non-stationary	stationary
$\hat{\pi}_t$	stationary	stationary
$\hat{y}_t$	stationary	stationary

Table 2 KPSS test for all variables

The (1) is estimated in two steps. The goal of the first step is to estimate the composite expression of (1). The following expression is estimated using the Ordinary Least Squares (OLS)

$$i_i = const + \alpha \bar{i} + \lambda_\pi \hat{\pi}_t + \lambda_y \hat{y}_t + \varepsilon_t, \tag{5}$$

$$i_i = -0.52 + 1.25\bar{i} + 0.30\hat{\pi}_t + 1.83\hat{y}_t. \tag{6}$$

The second step is the estimation of the Taylor rule

$$i_t = const + (1-\phi_i)\tilde{i}_t + \phi_i i_{t-1} + \varepsilon_t, \tag{7}$$

$$i_t = -0.06 + 0.50\tilde{i}_t + 0.50 i_{t-1}, \tag{8}$$

where the $\tilde{i}_t$ is represented by the fitted values from the (6). The estimated Taylor rule can be written in this form

$$i_t = (1-0.5)\left[\bar{i} + 0.30\hat{\pi}_t + 1.83\hat{y}_t\right] + 0.5 i_{t-1}. \tag{9}$$

Both equations (5) and (7) are estimated using the OLS in SAS.

The second method of the estimation of the Taylor rule is one step method. The estimation of all parameters in the (1) is performed using the SAS software and the Marquardt algorithm. This is an algorithm for least-squares estimation of nonlinear parameters. For details see Marquardt [9]. The results of the estimation procedure are in the Table 3.

parameter	description	estimate	std. error	lower	upper	pr > |t|
ϕ_i	interest rate smoothing parameter	0.6213	0.0675	0.4855	0.7571	<0.0001
λ_π	inflation weight	0.3462	0.0702	0.2049	0.4874	<0.0001
λ_y	output gap weight	6.1907	5.7605	-5.3979	17.7793	0.2880

Table 3 Parameter estimates

Figure 4 and Figure 5 show the short-term interest rate and fitted values. It is obvious that the Taylor rule works well in the Czech economy. The bad fact is the poor estimate of λ_y. Root Mean Square Error is in case of two step (instrumental) method equal to 0.071 and in case of Marquardt estimation method is equal to 0.065.

The Ministry of Finance of the Czech Republic calibrates parameters as follows (Table 4)

parameter	description	estimate	HUBERT
ϕ_i	interest rate smoothing parameter	0.6213	0.70
λ_π	inflation weight	0.3462	1.50
λ_y	output gap weight	6.1907	0.25

Table 4 Parameter estimates vs. Calibrated values in DSGE model HUBERT

As one can see, except of interest rate smoothing parameter ϕ_i the values are completely different.

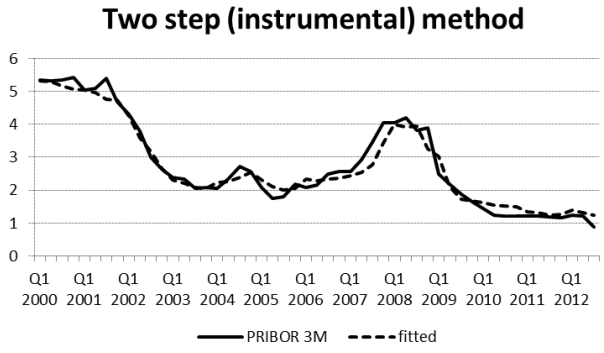

Figure 4 Instrumental method and its fit to data

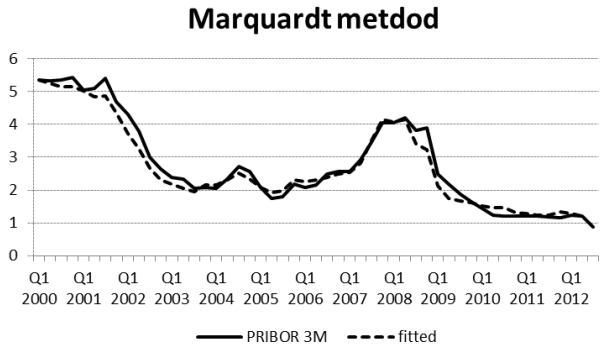

Figure 5 Marquardt method and its fit to data

6 Conclusion

The paper deals with the empirical analysis of the Taylor rule. This rule is very common in the New Keynesian models. The Taylor rule exists now in a lot of forms, but in the paper is analyzed the Taylor rule discussed by Svensson [13]. The rule is implemented in the DSGE model HUBERT which is used by the Ministry of Finance of the Czech Republic and its purpose is to analyze the impacts of fiscal policy. In the estimation of the DSGE model one has to calibrate values of the parameters which are in the Taylor rule. These parameters of the model have to be calibrated and this paper provides the deeper analysis of the rule and its fit to data.

First chapter describes the Taylor rule and its origin and purpose. Next chapter discusses the data sources and necessary data transformation. This chapter contains the description of all variables and parameters. Third chapter briefly describes Hodrick-Prescott filter and this filter is then applied on time series and one get as a result the gap or steady state variables. The last chapter deals with the empirical study. The estimation of parameters is performed using two different methods. The first is the two step instrumental variable method. The purpose of the method is to gain the estimate of the parameters inside the composite member of the Taylor rule. Next, in the second step the complete equation is estimated. The latter method estimates the equation at once. This method uses the Marquardt algorithm which allows estimation of non-linear equations. The quality of the final fit to the short-term interest rate is almost the same for both methods. According to Root Mean Square Error is the Marquardt algorithm better than two step instrumental variable method. Finally, values used by the Ministry of Finance in calibration of their DSGE model differ from parameter estimates which are obtained by Marquardt algorithm in this paper. According to this finding it is recommended to consider values used by the Ministry of Finance.

Acknowledgements

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Diversification-consistent DEA-risk tests – solution techniques and an empirical comparison

Martin Branda[1]

Abstract.

In this paper, we will propose numerically tractable formulations of the diversification-consistent DEA tests, which generalize traditional DEA tests as well as mean-risk models. We employ general deviation measures to measure risk of the investment opportunities. We will compare strength of the tests and give characterizations of efficient and inefficient investment opportunities. US industry representative portfolios will be ranked using the proposed DEA tests.

Keywords: Data envelopment analysis, diversification, deviation measures, efficiency, US industry representative portfolios

JEL classification: C44
AMS classification: 90C15

1 Introduction

Data Envelopment Analysis (DEA) was introduced by Charnes, Cooper and Rhodes [11] as a tool for testing efficiency of decision making units with the same structure of consumed inputs and produced outputs. Many generalizations have been proposed since than taking into account various aspects of production theory and practical applications, see, e.g., Cooper et al. [13]. A special attention has been given to applications of DEA models in finance. Murthy et al. [19] accessed in their seminal work the efficiency of mutual funds based on their mean-risk profile and transaction costs. More precisely, the indicators, which are preferably minimized by investors (e.g. risk, transaction costs), served as the inputs, and those, which are maximized (e.g. expected return), were used as the outputs. DEA model with variable return to scale introduced by Banker et al. [1] was employed. Similar efficiency tests based on DEA were proposed also by Basso and Funari [2], Branda and Kopa [8], Chen and Lin [12], where also other inputs and outputs were considered.

Several authors realized that the standard DEA tests employed in finance do not take into account dependencies between the assets. Namely, risk measures used as the inputs were combined linearly in the dual formulations, which does not correspond to proper diversification. It was even shown by Branda [6] that ignoring diversification leads to weaker tests which identify significantly higher number of efficient investment opportunities. DEA tests with diversification were introduced by Briec et al. [10], Joro and Na [14], Lozano and Gutiérrez [16], Branda [3]. However, these models were limited to particular inputs and outputs. Recently, Lamb and Tee [15] employed positive parts of coherent risk measures as the inputs and return measures as the outputs and introduced a general class of diversification-consistent tests. Several extension were proposed by Branda [6] who introduced input and input-output oriented tests focusing on the strength of the tests. Moreover, he avoided cutting the negative part of risk measures and suggested general deviation measures (Rockafellar et al. [20]) to quantify riskiness as the inputs. The resulting models can be seen as generalizations of Markowitz mean-risk models [17, 18], where variance and semivariance were used to quantify risk and which both belong to the class of general deviation measures. Branda and Kopa [9] even showed equivalence between particular new DEA tests and stochastic dominance efficiency tests.

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This paper is organized as follows. In Section 2, we will propose basic notation and axiomatic definitions of deviation and return measures. Standard DEA tests and diversification-consistent extensions are proposed in Section 3. In Section 4, we employ the DEA tests to access efficiency of 46 US industry representative portfolios.

2 Preliminaries and notation

In this section, we introduce basic notion and review the axioms of deviation and return measures, which will serve as the inputs and outputs in the following DEA tests. Let $\mathcal{X}$ be a set of random returns of available investment opportunities corresponding to one single asset or to a portfolio consisting of many assets. We consider n assets and denote R_i the return of i-th asset which is a real random variable defined on the probability space $(\Omega, \mathcal{A}, P)$. We will use the set with no short sales enabling full diversification: $\mathcal{X} = \{\sum_{i=1}^{n} R_i x_i : \sum_{i=1}^{n} x_i = 1, x_i \geq 0\}$. Other choices of the set are also possible, e.g. with limited number of assets, allowing short sales, borrowing or including proportional and fixed transaction costs, cf. Branda [4, 6].

Functionals $\mathcal{D} : \mathcal{L}_2(\Omega) \to [0, \infty]$ are called general deviation measures if they satisfy:
(D1) translation invariance: $\mathcal{D}(X + C) = \mathcal{D}(X)$ for all X and constants C,
(D2) positive homogenity: $\mathcal{D}(0) = 0$, and $\mathcal{D}(\lambda X) = \lambda \mathcal{D}(X)$ for all X and all $\lambda > 0$,
(D3) subadditivity: $\mathcal{D}(X + Y) \leq \mathcal{D}(X) + \mathcal{D}(Y)$ for all X and Y,
(D4) nonnegativity: $\mathcal{D}(X) \geq 0$ for all X, with $\mathcal{D}(X) > 0$ for nonconstant X.
Note that the axioms (D2) and (D3) imply convexity. The main examples are standard deviation and semideviations.

Functionals $\mathcal{E} : \mathcal{L}_p(\Omega) \to (-\infty, \infty]$ are called return measures if they satisfy:
(E1) translation equivariance: $\mathcal{E}(X + C) = \mathcal{E}(X) + C$ for all X and constants C,
(E2) positive homogenity: $\mathcal{E}(0) = 0$, and $\mathcal{E}(\lambda X) = \lambda \mathcal{E}(X)$ for all X and all $\lambda > 0$,
(E3) superadditivity: $\mathcal{E}(X + Y) \geq \mathcal{E}(X) + \mathcal{E}(Y)$ for all X and Y,
(E4) monotonicity: $\mathcal{E}(X) \geq \mathcal{E}(Y)$ when $X \geq Y$.
The space $\mathcal{L}_p(\Omega)$ is selected so as the measures are finite, usually $p = 1$. The axioms (E2) and (E3) imply concavity of the functional. It is obvious that expectation fulfills the axioms. Moreover, coherent risk measures multiplied by a negative constant can be used as return functionals, see Lamb and Tee [15].

3 DEA-risk tests

In this section, we will formulate several DEA tests, which will be employed in the numerical study to access efficiency of US market portfolios. We start with the standard model with variable return to scale. We can simply demonstrate its drawbacks and motivate an introduction of the DEA tests with diversification. The efficiency is expressed by the optimal value of the corresponding DEA tests – we say that an investment opportunity is efficient if the optimal value is equal to 1, otherwise we say that it is inefficient.

Traditional mean-risk (mean-deviation) efficiency can be defined as follows: an investment opportunity is efficient if there is no other investment opportunity with higher or equal expected return and lower of equal risk (deviation) with at least one inequality strict. This approach to efficiency is extended by the following DEA tests where K deviation measures and J return measures are employed at the same time.

3.1 Traditional input oriented tests

We are going to access efficiency of an investment opportunity R_0. We assume that the benchmark R_0 is not constant, which implies that the employed deviation measures are positive. The standard DEA test with variable return to scale introduced by Banker et al. [1] can be formulated in the dual form as

a linear program as follows:

$$\begin{aligned}
\theta_{VRS}(R_0) &= \min_{\theta, x_i} \theta \\
\text{s.t. } \sum_{i=1}^{n} x_i \mathcal{E}_j(R_i) &\geq \mathcal{E}_j(R_0),\ j = 1, \ldots, J, \\
\sum_{i=1}^{n} x_i \mathcal{D}_k(R_i) &\leq \theta \cdot \mathcal{D}_k(R_0),\ k = 1, \ldots, K, \\
\sum_{i=1}^{n} x_i &= 1,\ x_i \geq 0,\ 1 \geq \theta \geq 0.
\end{aligned}$$

This dual formulation shows the main problem of the standard DEA tests in finance. It compares benchmark deviation with a linear combination of assets deviations, which has no financial or economical meaning. This drawback is removed by the diversification consistent (DC) tests proposed in the following parts.

3.2 Input oriented diversification-consistent tests

The input oriented test can be extended to take into account diversification. This can be simply done by interchanging the deviation and sum operators resulting into

$$\begin{aligned}
\theta_I(R_0) &= \min_{\theta, x_i} \theta \\
\text{s.t. } \mathcal{E}_j\left(\sum_{i=1}^{n} R_i x_i\right) &\geq \mathcal{E}_j(R_0),\ j = 1, \ldots, J, \\
\mathcal{D}_k\left(\sum_{i=1}^{n} R_i x_i\right) &\leq \theta \cdot \mathcal{D}_k(R_0),\ k = 1, \ldots, K, \\
\sum_{i=1}^{n} x_i &= 1,\ x_i \geq 0,\ 1 \geq \theta \geq 0.
\end{aligned}$$

It can be shown that for convex deviation measures and concave return measures this test is stronger than the standard one, i.e. for any benchmark R_0 it holds $\theta_{VRS}(R_0) \geq \theta_I(R_0)$ for arbitrary $R_0 \in \mathcal{X}$, see Branda [6]. We can obtain a mean-deviation efficiency test by restricting the number of considered inputs and outputs to one per class.

3.3 Input-output oriented diversification-consistent tests

We assume that $\mathcal{E}_j(R_0)$ is positive for at least one j. An input-output oriented test where inefficiency is measured with respect to the inputs and outputs separately can be formulated as follows

$$\begin{aligned}
\theta_{I-O}(R_0) &= \min_{\theta, \varphi, x_i} \frac{\theta}{\varphi} \\
\text{s.t. } \mathcal{E}_j\left(\sum_{i=1}^{n} R_i x_i\right) &\geq \varphi \cdot \mathcal{E}_j(R_0),\ j = 1, \ldots, J, \\
\mathcal{D}_k\left(\sum_{i=1}^{n} R_i x_i\right) &\leq \theta \cdot \mathcal{D}_k(R_0),\ k = 1, \ldots, K, \\
\sum_{i=1}^{n} x_i &= 1,\ x_i \geq 0,\ \varphi \geq 1,\ 1 \geq \theta \geq 0.
\end{aligned}$$

The benchmark investment opportunity R_0 is efficient if and only if there is no other investment opportunity with lower or equal deviations and higher or equal returns with strict inequalities for at least one group. The optimal value can be seen as a ratio of minimal necessary improvement (=decrease) of inputs and minimal necessary improvement (=increase) of outputs to reach the efficiency.

The following reformulation was derived by Branda [7]. If we set $1/t = \varphi$ and substitute $\tilde{x}_i = tx_i$, $\tilde{\theta} = t\theta$, and $\tilde{\varphi} = t\varphi$, the decision variables $\tilde{\varphi}$ and t can be omitted resulting into an input oriented DEA test with nonincreasing return to scale (NIRS):

$$\begin{aligned}
\theta_{I-O}(R_0) = \min_{\tilde{\theta},\tilde{x}_i} & \tilde{\theta} \\
\text{s.t. } \mathcal{E}_j\left(\sum_{i=1}^n R_i\tilde{x}_i\right) &\geq \mathcal{E}_j(R_0),\ j=1,\dots,J, \\
\mathcal{D}_k\left(\sum_{i=1}^n R_i\tilde{x}_i\right) &\leq \tilde{\theta}\cdot\mathcal{D}_k(R_0),\ k=1,\dots,K, \\
\sum_{i=1}^n \tilde{x}_i &\leq 1,\ \tilde{x}_i\geq 0,\ 1\geq\tilde{\theta}\geq 0.
\end{aligned}$$

Note that it is important for the reformulation that all inputs $\mathcal{D}_k$ and all outputs $\mathcal{E}_j$ are positively homogeneous. Since the general deviation measures are convex and the return measures are assumed to be concave, we obtained a convex programming problem. Similar class of input oriented tests was proposed by Lamb and Tee [15], where the assumption of NIRS was suggested for financial applications. Note that the input-output oriented test is stronger than the input oriented test, i.e. we obtain the relation between the optimal values $\theta_I(R_0) \geq \theta_{I-O}(R_0)$ for arbitrary $R_0 \in \mathcal{X}$.

4 Numerical results

In this section, we employ the DEA tests proposed above to access efficiency of 46 US industry representative portfolios observed monthly from January 2002 to December 2011, see Branda and Kopa [9] for details. We consider discretely distributed returns r_{is}, $s = 1,\dots,S$ with equal probabilities $1/S$, in our case $S = 120$. For general continuous distributions we can obtain similar problems using sample approximation technique, cf. Branda [5]. CVaR deviation on level α can be then formulated as

$$\mathcal{D}_\alpha^S\left(\sum_{i=1}^n R_i x_i\right) = \min_{\xi\in\mathbb{R}} \frac{1}{S}\sum_{s=1}^S \max\left\{(\sum_{i=1}^n x_i r_{is} - \xi), \frac{\alpha}{1-\alpha}(\xi - \sum_{i=1}^n x_i r_{is})\right\}$$

We consider $\mathcal{D}_\alpha^S$ for various levels $\alpha_k \in (0,1), k = 1,\dots,K$, in particular $\alpha_k \in \{0.75, 0.9, 0.95, 0.99\}$ and $K = 4$, as the inputs and the expectation as an output, i.e. $J = 1$ and $\mathcal{E}_1(X) = \mathbb{E}X$. This leads to the following linear programming problem for the input-output oriented test:

$$\begin{aligned}
\theta_{I-O}(R_0) = \min_{\theta,x_i,u_{sk},\xi_k} & \theta \\
\text{s.t. } \sum_{i=1}^n \mathbb{E}[R_i]x_i &\geq \mathbb{E}[R_0], \\
\frac{1}{S}\sum_{s=1}^S u_{sk} &\leq \theta\cdot\mathcal{D}_{\alpha_k}^S(R_0),\ k=1,\dots,K, \\
u_{sk} &\geq \left(\sum_{i=1}^n x_i r_{is} - \xi_k\right),\ s=1,\dots,S, k=1,\dots,K, \\
u_{sk} &\geq \frac{\alpha_k}{1-\alpha_k}\left(\xi_k - \sum_{i=1}^n x_i r_{is}\right),\ s=1,\dots,S, k=1,\dots,K, \\
\sum_{i=1}^n x_i &\leq 1,\ x_i\geq 0,\ i=1,\dots,n.
\end{aligned}$$

Similar reformulations were obtained by Branda [6, 7] for the input oriented and input-output oriented DEA tests with general probabilities. However, no numerical comparison of these approaches was reported in the previous papers.

We solved the DEA problems by the CPLEX 12.1 solver using the modeling system GAMS 23.2. Selected efficiency scores can be found in Table 1. We selected only the seven portfolios which are

efficient according to the VRS DEA test which is the weakest. The stronger DC tests identified only the portfolio Coal as efficient. Most of the VRS efficient portfolios are highly ranked also by the DC tests. The only exception is Drugs, which is ranked by the input-output oriented DC test very badly. We also computed correlations between the optimal values. Perhaps surprisingly we observed higher correlation between the efficiency scores of the VRS and input oriented test (0.97) than between the DC tests (0.70). The simplest explanation is the common orientation on inputs. Ranking of all representative portfolios can be found in Table 2.

	Food	Smoke	Hshld[1]	Drugs	Mines	Coal	Meals
VRS	1.00	1.00	1.00	1.00	1.00	1.00	1.00
DC Inp	0.93	0.87	0.87	0.91	0.83	1.00	0.86
DC I-O	0.65	0.87	0.55	0.27	0.83	1.00	0.84

Table 1 Efficient industry representative portfolios and scores

5 Conclusions

In this paper, we have compared several approaches to efficiency of investment opportunities based on DEA. The traditional DEA test with variable return to scale has been shown much weaker that the extensions with diversification. These diversification consistent tests identified only one portfolio as efficient. However, the resulting rankings are significantly different. The diversification-consistent tests can be also seen as superefficiency models for the standard VRS DEA tests saying how much the benchmark inputs and outputs need to be improved to reach the efficiency frontier. Future research will be devoted to multiperiod–dynamic extensions.

	Agric	Food	Soda	Beer	Smoke	Toys	Fun	Hshld	Clths	Hlth
VRS	18	1	17	8	1	30	42	1	13	26
DC Inp	19	2	21	8	4	32	42	4	13	25
DC I-O	13	8	11	14	2	37	27	15	7	42
	MedEq	Drugs	Chems	Rubbr	Txtls	BldMt	Cnstr	Steel	FabPr	Mach
VRS	21	1	22	36	46	39	38	45	44	26
DC Inp	15	3	27	34	45	38	39	46	44	31
DC I-O	29	35	18	25	38	26	30	35	34	19
	ElcEq	Autos	Aero	Ships	Guns	Gold	Mines	Coal	Oil	Util
VRS	31	40	20	10	28	14	1	1	11	9
DC Inp	33	41	23	18	30	11	7	1	12	9
DC I-O	21	41	15	9	22	5	4	1	6	12
	Telcm	PerSv	BusSv	Comps	Chips	LabEq	Paper	Boxes	Trans	Whlsl
VRS	24	29	25	35	41	33	23	15	16	19
DC Inp	19	29	24	36	40	35	22	16	14	17
DC I-O	40	32	39	23	45	33	28	10	17	20
	Rtail	Meals	Insur	RlEst	Fin	Other				
VRS	12	1	34	43	37	32				
DC Inp	10	6	28	43	37	25				
DC I-O	24	3	43	31	44	46				

Table 2 Ranking of the industry representative portfolios

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Multiple criteria evaluation of the activities criticalness in the Project management

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Abstract. The primary aim of project management is to ensure completion of the project under many constraints as scope, time, quality and budget and optimal allocation of necessary recourses. The crucial question is to find out what can cause delaying of the project due date or failure of project objectives. Some authors try to analyse activity parameters called criticalness.
Criticalness usually expressed by Criticality Index reflects the risk of influencing project due date by this activities. Another view via project simulation is how often a particular activity should be element of the critical path. All of these methods are primary derived from activity duration, type of links between activities and type of constraints. In the reality many different criteria (soft or hard) should influence criticalness of the activity. Criticalness in our approach is defined in a slightly different way.
We derive this parameter from the character of activity duration (known or random), resource assignments (in right time and right place), activity costs and number of parallel activities. In this work we suppose different multiple criteria approaches to criticalness calculations and we compare multiplicative and additive models and DEA.

Keywords: Project management, activity, criticalness, multiple criteria, evaluation.

JEL Classification: O22, C44
AMS Classification: 90B50, 90B99

1 Introduction

The principal purpose of project management is a successful implementation of a project: finishing the project within planned time, with planned costs and with a fulfilled objective. The success of the project is conditioned by the level of recognition of weak spots in the project. The criticalness of project activities, such as the level of amount and strength of weak spots in the project, is not only given by the surroundings and environment of the project but also by internal arrangement and structure of the project.

The project risk deals with risk management, however, without a quantitative view of the sequence and arrangement of project activities and its other quantitative qualities. In the area of risk management a number of different techniques and approaches have been derived in order to reduce the risk of a project and its partial activities [10]. The criticality of project activities is often defined from a time perspective only, using stochastic approaches ([1]; [4]), fuzzy sets methods ([3]; [12]) or using the findings of a network analysis ([7]; [2]). Authors [7] mention that ignoring the impact of noncritical activities which may easily become critical, is the most frequent criticism of project time analysis methods. Another point of view of activities criticalness is given by the structure of relations in the project. In [1] or also [11] deal with a stochastic analysis of a project network where the criticality of activities in the project is derived from the relation between activity duration and the whole project, and on the basis of a number of resources used for an activity and the whole project.

A deeper insight into the issue is presented by [1], who defines the criticality of the activity as uncertainty given by activity duration and a number of resources. A different approach to a project network has been adopted by [9], who analyse the growth or decline in the criticality of activities in the project during network structure changes. A number of preceding and following activities always has a direct impact on the criticality of the activity. Authors [9] claim that if a project network is composed of several parallel on-going chains, a lower variability and criticality of activities in the project can be expected. The criticalness of activities in the project and search for its weak spots needs to be accepted from various perspectives. It is impossible to expect a generally valid principle to determine the criticality of activities. Each project and each situation in the project is unique.

The main aim of the paper is to introduce heuristic based approach finding weak points in project completion process, identifying key activities for meeting project objectives. In this work we deal activities criticalness iden-

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tification, we suppose different multiple criteria approaches to criticalness calculations and we compare multiplicative and additive models and DEA.

2 Indicators of activities criticalness and multiple criteria decision making methods

Estimation of overall criticalness of the project activities are based on the multiple criteria decision making methods using five indicators of the criticalness.

- **Time criticalness**

As a measure of the time criticalness of the activities can be used or time duration or indicators related to the project duration.

$$t_I \quad \text{or} \quad ct_I = \frac{t_I}{T} \tag{3}$$

where
t_I, t_K are the duration of the activities I, K,
T is the project duration,
ct_I is the time criticalness of the activity I,

- **Probability of Critical path**

This indicator evaluates the probability the activity will lie on critical pass related to the project structure.

$$p_{1j} = \frac{1}{h_1} \text{ and } p_{ij} = \frac{\sum_k p_{ki}}{h_i}, k \neq 1 \tag{1}$$

where
p_{1j}, p_{ki}, p_{ij} are the probability that the activities $1j$, ki, ij will be on the critical path,
h_i is the number of activities starting in the node i,

- **Slack criticalness**

The measure of the slack criticalness can be slacks or indicator related to maximal slack.

$$s_I \quad \text{or} \quad cs_I = \frac{\max_K s_K - s_I}{\max_K s_K} \tag{4}$$

where
s_I, s_K are the slack of the activities I, K,
cs_I is the slack criticalness of the activity I,

- **Cost criticalness**

The cost criticalness of the activities is based on the cost or indicators related to the project cost.

$$c_I \quad \text{or} \quad cc_I = \frac{c_I}{C} \tag{5}$$

where
C is the total project cost,
c_I, c_K are the cost of the activities I, K,
cc_I is the cost criticalness of the activity I,

- **Work criticalness**

As the indicator of the work criticalness of the activities can be used the work amount or indicators related to the project work.

$$v_I \quad \text{or} \quad cv_I = \frac{v_I}{V} \tag{6}$$

where
v_I, v_K are the total amount of work of the activities I, K,
cv_I is the work criticalness of the activity I,
V is the total amount of work of the project.

These indicators are used as input for the multiple criteria decision making approaches. We create and use multiplicative and additive multiple criteria models [5] and DEA model without outputs for evaluation of the project activities criticalness.

- **Multiplicative model**

In this approach all criteria evaluations are multiplied using formula $w_I = \prod_p r_{Ip}$, where w_I is global evaluation of the activities criticalness, r_{Ip} is evaluation of each components of activities criticalness and p is number of used components of criticalness.

- **Additive model**

In this approach all criteria evaluations are multiplied using formula $w_I = \sum_p r_{Ip}$, where w_I is global evaluation of the activities criticalness, r_{Ip} is evaluation of each components of activities criticalness and p is number of used components of criticalness.

- **Additive model with weights**

In this approach all criteria evaluations are multiplied using formula $w_I = \sum_p u_p r_{Ip}$, where w_I is global evaluation of the activities criticalness, r_{Ip} is evaluation of each components of activities criticalness, p is number of used components of criticalness and u_p is the weight of the p component of criticalness.

- **Data Envelopment Analysis Method**

Data Envelopment Analysis (DEA) serves for evaluation of units against the best unit. The DEA is a non-linear programming model for the estimation of an efficiency of units, based on the relationship between multiple outputs and multiple inputs. The DEA measure of the efficiency of any DMU is obtained as the maximum of a ratio of weighted outputs to weighted inputs, subject to the condition that the similar ratio for every units is less than or equal to 1. The simplest DEA model assumes constant returns to scale, this model is called the CCR model, according to its authors, Charnes, Cooper, and Rhodes [6]. Primal model is:

$$\Phi_H = \sum_{j=1}^{n} u_{jH} y_{jH} \rightarrow MAX$$

subject to

$$\begin{aligned} &\sum_{i=1}^{m} v_{iH} x_{iH} = 1 \\ &-\sum_{i=1}^{m} v_{iH} x_{ik} + \sum_{j=1}^{n} u_{jH} y_{jk} \leq 0,\ k = 1, 2, \ldots, p, \\ &u_{jH} \geq 0,\ j = 1, 2, \ldots, n, \\ &v_{iH} \geq 0,\ i = 1, 2, \ldots, m. \end{aligned} \tag{7}$$

Supposed DEA model has five inputs t_I, p_I, s_I, c_I, v_I and one output y_I meaning an expected criticalness of the activity I. At the beginning of the calculation all expected activities criticalness are set as $y_I = 1$. (The output of the DEA model without outputs is always unitary [8]) It means, that we suppose the equal criticalness of all activities. Using the CCR model, the real criticalness of a particular activity I is calculated using the primal output oriented model:

The relative efficiencies of the output orientated DEA model generally show how much an output must be increased. The more an output should increase, the higher criticalness this activity has, i. e. higher inefficiency means higher criticalness of particular activity. Therefore, these activities represent critical parts of a project, which require more attentions of a project manager. Those activities that lie on the efficient frontier do not represent critical parts of a project.

Authors used Efficiency Measurement System (EMS) SW for the calculation of the DEA model.

3 Evaluation of the project activities criticalness

To evaluate a project as a whole, from the point of view of its activity criticalness, is not easy. An unambiguous and fully sufficient approach is still non-existent. Activities on a critical path are considered the most threatening activities. The finding of a critical path and a follow-up analysis of reserves does not always suffice. In practice it can often be seen that even activities off a critical path have an extreme impact on the course and success of a project. To demonstrate the contribution of derived indicators of project activity criticalness, the following small-scale project is used as an illustrative example.

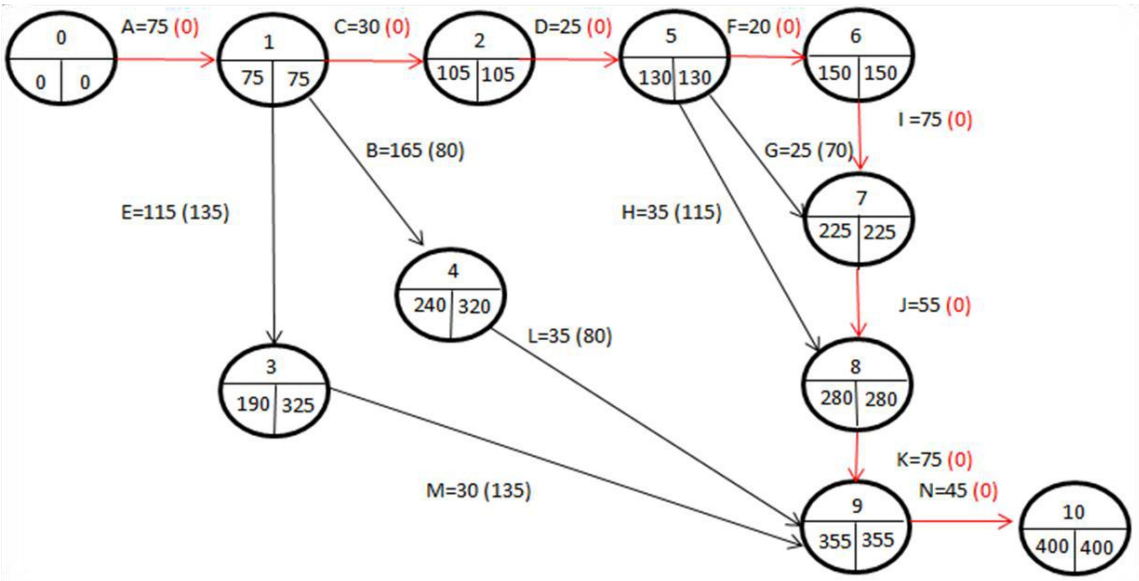

Figure 1 Small-scale project with the indication of a critical path.

A critical path of a project (**Figure 1**) is composed of activities A, C, D, F, I, J, K and N. If we derive proposed criticalness indicators for all activities (**Table 1**), some activities, which do not lie on a critical path, can be regarded, as concerns their values, as a greater threat to the project than activities which lie on a critical path. Several activities have much higher criticalness towards the project than a critical path method reveals.

Task	Days	Time criticalness	Probability of critical path	Slack	Slack criticalness	Cost	Cost criticalness	Work	Work criticalness
A	75	0.188	1	140	1	1350	0.121	450	0.049
B	165	0.413	0.33	60	0.407	990	0.088	495	0.053
C	30	0.075	0.33	140	1	630	0.056	630	0.068
D	25	0.063	0.33	140	1	175	0.016	175	0.019
E	115	0.288	0.33	5	0	690	0.062	345	0.037
F	20	0.050	0.11	140	1	60	0.005	60	0.006
G	25	0.063	0.11	70	0.481	150	0.013	150	0.016
H	35	0.088	0.11	25	0.148	210	0.019	210	0.023
I	75	0.188	0.11	140	1	1575	0.141	1575	0.170
J	55	0.138	0.22	140	1	1155	0.103	1155	0.125
K	75	0.188	0.33	140	1	2025	0.181	2025	0.218
L	35	0.088	0.33	60	0.407	875	0.078	875	0.094
M	30	0.075	0.33	5	0	360	0.032	180	0.019
N	45	0.113	1	140	1	945	0.084	945	0.102
MADM 1		**MAX**	**MAX**		**MAX**		**MAX**		**MAX**
DEA	**INPUT**		**INPUT**	**INPUT**		**INPUT**		**INPUT**	

Table 1 Activity criticalness indicators, input values for MADM.

Activities B and L which are not on a critical path and have a time reserve have, in regard to MADM (**Table 2**), higher criticalness values than activities that lie on a critical path. The order of activities is determined according to a total rank. In particular, activity B surpassed in its values other three activities on a critical path. The significance and impact of activity B towards the whole project is much higher than it can be assumed from the results of a critical path method.

Task	Multiplicative model	Rank	Aditive model	Rank	Additive model with weights	Rank	CCR-O super	Rank	Suma of ranks	Total rank
A	0.00110	2	2.357	1	0.395	1	282%	1	5	1
K	0.00245	1	1.917	3	0.325	3	266%	2	9	2
N	0.00097	3	2.299	2	0.384	2	183%	5	12	3
J	0.00039	5	1.585	5	0.252	5	200%	4	19	4
I	0.00049	4	1.608	4	0.260	4	100%	9	21	5
B	0.00026	6	1.292	8	0.220	7	243%	3	24	6
C	0.00009	7	1.529	6	0.236	6	133%	6	25	7
D	0.00001	9	1.427	7	0.211	8	117%	8	32	8
L	0.00009	8	0.997	10	0.174	9	130%	7	34	9
F	0.0000002	12	1.172	9	0.161	10	40%	13	44	10
E	0	13	0.716	11	0.136	11	100%	9	44	10
G	0.0000007	10	0.684	12	0.101	12	90%	11	45	12
H	0.0000006	11	0.387	14	0.065	14	56%	12	51	13
M	0	13	0.457	13	0.088	13	37%	14	53	14

Table 2 Results of MADM models and the rank of activities based on criticalness.

Deeper consideration should be given to the model DEA (CCR-O super) which introduces certain sturdiness and better results interpretation to activity criticalness evaluation. The appearing value *100%* for some activities can be interpreted as the activity which is at its inputs (Time criticalness, Probability of Critical path, Slack criticalness, Cost criticalness, Work criticalness) adequately critical toward a project and which is at its limit of criticalness towards the project and can be regarded as critical. Therefore, a significant finding could again be value *243%* for activity B which does not lie on a critical path, has a time reserve and is the third most critical activity towards the project based on the model CCR-O super.

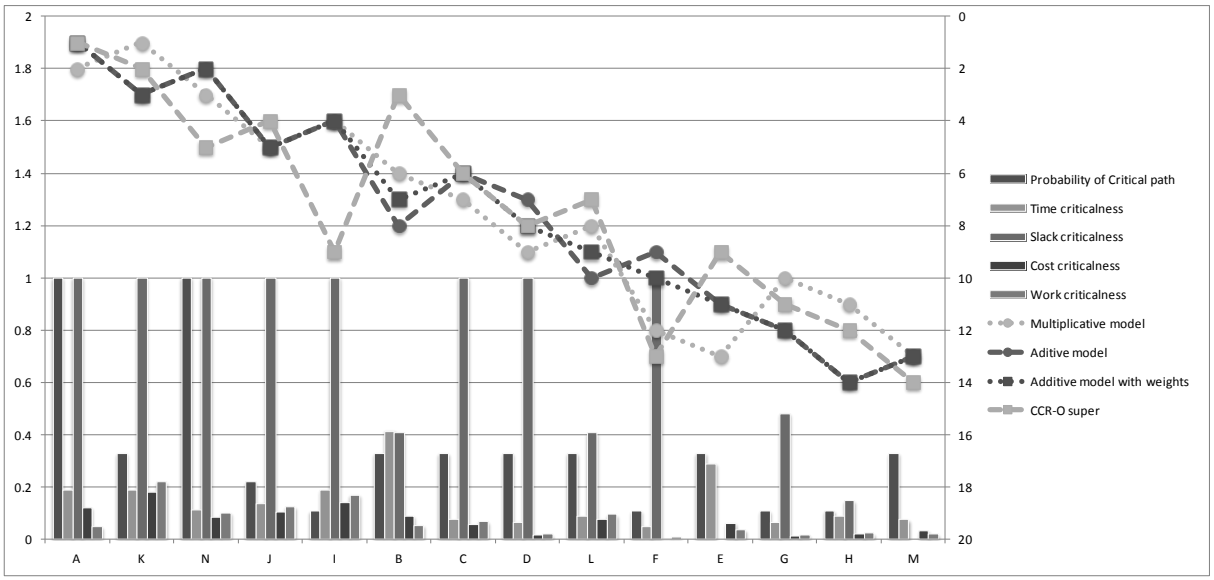

Figure 2 Comparison of activity criticalness indicators and MADM model results.

The final values of indicators and the rank obtained using MADM can be further illustrated in a graph (**Figure 2**), which shows that activity criticalness in the project is versatile and across activities descending, and it does not really correspond to the results of a critical path method.

4 Conclusion

The paper proposes indicators of project activity criticalness for the evaluation of activities from the perspective of their significance towards a project as a whole. Moreover, the use of the indicators as inputs for multiplicative, additive and DEA model was carried out. The calculated values of the indicators and the results of models for a small-scale project, used as an illustrative example support the fact that some activities which do not lie on a critical path, have a higher criticalness towards a project than activities that lie on a critical path. The significance and impact of activities in a project cannot be derived solely from the incidence of activities on a critical path.

Acknowledgements

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On Graphical Optimization of Linear Programming Models in the Column Space

Helena Brožová[1], Milan Houška[2]

Abstract. The necessary conditions for representing and solving a linear programming model graphically are well known. The model should contain up to two decision variables (a number of constraints could be unlimited, but finite) or up to two constraints (a number of decision variables could be unlimited, but finite). In the first case, we solve the model graphically in a so called "Row Space", where the axes of the graph represent decision variables. In the other case, we solve the model in a so called "Column Space", where the axes represent individual constraints.
In this paper, we focus on the optimization of linear programming models in the Column Space. There is a standard procedure to solve it, but it can be used if and only if all cost coefficients in the objective function are positive or zero. We make the procedure more general and show how to carry out the graphical optimization in the Column Space correctly, even if at least one cost coefficient is negative. We also demonstrate the proposed algorithm on a numerical example.

Keywords: linear programming, graphical optimization, column space.

JEL Classification: C61
AMS Classification: 90C05

1 Introduction

Linear programming (LP) is one of the most important areas in Operational Research. Many authors use the LP models to solve practical problems in various application domains, e.g. in economics [1], industry [2, 3], agriculture [4] or logistics [5, 6]. Linear programming models are also used as auxiliary models in other branches of Operational Research, e.g. in Project Management [7], Data Envelopment Analysis [8, 9] or Game Theory [10, 11].

Linear programming models of a small size (up to two decision variables or two constraints) could be solved using a graphical optimization technique. The graphical optimization procedure for the LP models containing two variables in the Row Space is commonly used and always described in textbooks dealing with Operational Research. Surprisingly, the procedure for the LP models containing two constraints and more than two decision variables (Column Space) is usually omitted there. The procedure is described in Simmonard [12] partially, but we have not found any other sources describing the procedure systematically. Withal, the graphical optimization techniques play important role at least when one starts dealing with the LP models [13].

We noticed that the procedure for the graphical optimization of the LP models in the Column Space is incomplete. It does not cover all cases, which may occur when a practical problem is solved. According to our best knowledge, the problem of negative cost coefficients in the objective function has not been mentioned and solved in literature. In this paper we enhance the standard procedure of the Column Space to be correct also for such coefficients, and provide necessary proofs. Finally, we demonstrate our approach on an illustrative numerical example.

2 Optimization in the Column Space

Let's have a general linear programming model containing two constraints

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$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ &z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \rightarrow MAX \\ &x_j \geq 0,\ j = 1,2,\ldots,n \end{aligned} \tag{1}$$

If the constraints are formulated as inequations, they have to be transformed to equations through slack variables added into the constraints. Then, the model is solvable in the Column Space using the following procedure [12].

Step 1: Elimination of different cost coefficients in the objective function

We transform the column vectors from the **A** matrix to ones, which reflect different cost coefficients of their variables. For *n* variables, we receive *n* transformed vectors denoted as $\boldsymbol{\alpha}_j$ as

$$\boldsymbol{\alpha}_j = \left(\frac{a_{1j}}{c_j}; \frac{a_{2j}}{c_j} \right)^T,\ j = 1,2,\ldots,n \tag{2}$$

After this operation we change the sense of the **A** matrix coefficients from "how many units of constraints are satisfied with one unit of the variable" to "how many units of constraints are satisfied with an amount of units of the variable obtained for one unit of the objective function".

Step 2: Identification of feasible solutions

Lemma 1. A feasible solution exists, if the cone of the column vectors exists that contains the right hand side vector.

Proof. If the column vectors $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ create feasible solution, the real numbers $x_1, x_2 \geq 0$ exist that $\mathbf{b} = \boldsymbol{\alpha}_1 x_1 + \boldsymbol{\alpha}_2 x_2$. Graphically, the cone of the vectors $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ has to contain the vector **b**. Any feasible solution does not exist, when no cone of the column vectors contains the right hand side vector.

Step 3: Optimization

Suppose that all coefficients of the objective function are positive and at least one feasible solution exists. If all column vectors lay in one half-plane and the feasible solution exists, the optimal solution exists. The column vectors $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ create feasible solution $\mathbf{b} = \boldsymbol{\alpha}_1 x_1 + \boldsymbol{\alpha}_2 x_2$ for the real numbers $x_1, x_2 \geq 0$. The equations $\mathbf{b} = \boldsymbol{\alpha}_1 x_1 + \boldsymbol{\alpha}_2 x_2$ for all feasible pairs of vectors have finite solutions and the problem is to select such a pair of vectors, for which the objective function $x_1 + x_2$ has the optimal value.

Lemma 2. Let the vectors $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ create a feasible solution, the coefficients of the objective function **c** = **1** and $\mathbf{b} = \boldsymbol{\alpha}_1 x_1 + \boldsymbol{\alpha}_2 x_2$ for $x_1, x_2 \geq 0$, then the objective function value is $z(\mathbf{x}) = \frac{\|\mathbf{b}\|}{\|\mathbf{p}\|}$, where the vector **p** is vector with the ending point P which is the intersection point of vector **b** and the abscissa between the ending points of the vectors $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$.

Proof. Suppose now that the vector **b** lies on the axis *X*, so **b** = (b_1, 0).

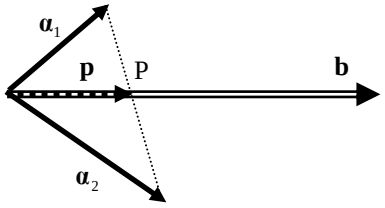

Figure 1 Combining the vectors

For the point *P* it holds $x_1 + x_2 = \frac{b_1}{p_1}$.

General case is transformed using rotation. As the rotation is isometry, for the general vectors **b** and **p** the following formula holds

$$z(\mathbf{x}) = x_1 + x_2 = \frac{\|\mathbf{b}\|}{\|\mathbf{p}\|} \tag{3}$$

This formula is also fulfilled regardless to the direction of vectors $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ and **b**.

The Lemma 2 shows that the optimal solution has to be selected according to the point P. If the point P is moving far from origin of coordinates system in the direction of the vector **b**, the value $z(\mathbf{x})$ is decreasing. The value $z(\mathbf{x})$ is increasing if the point P is moving to the origin.

Note: For all negative coefficients in the objective function this lemma is formulated similarly. The value of the objective function is $z(\mathbf{x}) = -\frac{\|\mathbf{b}\|}{\|\mathbf{p}\|}$ *If the point P is moving far from origin of coordinates system in the direction of the vector -**b**, the value z(**x**) is decreasing. The value z(**x**) is increasing if the point P is moving to the origin.*

3 Mixed negative and positive cost coefficients

A problem appears, when some of the cost coefficients in the objective function are positive and some of them are negative. In this case we cannot use the above-given procedure correctly. In Step 1, we would change the direction of the vector $\boldsymbol{\alpha}_j$, when the cost coefficient $\mathbf{c}_j$ is negative. It has negative impact on the Step 2, because we could omit some feasible combinations of variables, which apparently violate the Lemma 1, but in fact they are feasible. That is why we propose the following modification of the standard algorithm.

Step 1: Elimination of different cost coefficients in the objective function

Regarding to the Step 2, we should avoid redirecting the $\boldsymbol{\alpha}_j$ vectors. Thus we change the Eq. 2 to

$$\boldsymbol{\alpha}_j = \left(\frac{a_{1j}}{|c_j|}; \frac{a_{2j}}{|c_j|} \right)^{\mathrm{T}}, \; j = 1, 2, \ldots, n \tag{4}$$

and change the graphical notation of the vectors for the variables of the negative cost coefficients. Suppose that the cost coefficient of the variable x_1 is negative and the cost coefficient of the variable x_2 is positive. The vector $\boldsymbol{\alpha}_1'$ describes that the cost coefficient of variable x_1 is negative; the vector $\boldsymbol{\alpha}_1$ is shifted, so that the ending point of this vector is in the origin of the coordinates.

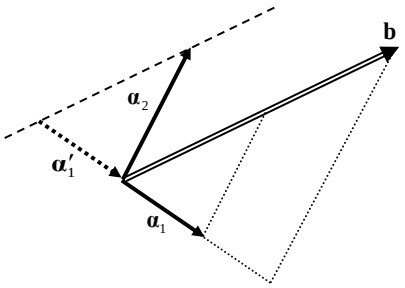

Figure 2 Graphical representation of the vectors of negative cost coefficients

After this modification, we can apply the Step 2 of the original algorithm to find all feasible solutions. But the Step 3 requires revisions as follows.

Step 3: Optimization

First we have to redirect the $\boldsymbol{\alpha}_j$ vectors for the variables of the negative cost coefficients. As the feasible solutions are determined with vectors of the equal orientation like in the **A** matrix, now we have to distinguish vectors increasing the objective function value (of the positive cost coefficient) or decreasing it (of the negative cost coefficient). Thus we recalculate back

$$\boldsymbol{\alpha}_j = \left(\frac{a_{1j}}{c_j}; \frac{a_{2j}}{c_j} \right)^{\mathrm{T}}, j = 1, 2, \ldots, n \tag{5}$$

Lemma 3. Let the vectors $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ create a feasible solution, the coefficients of the objective function $c_1 = 1$ and $c_2 = -1$ and $\mathbf{b} = \boldsymbol{\alpha}_1 x_1 + \boldsymbol{\alpha}_2 x_2$ for $x_1, x_2 \geq 0$, then the objective function value is $z(\mathbf{x}) = -\frac{\|\mathbf{b}\|}{\|\mathbf{p}\|}$, where the vector **p** is vector with the ending point P which is the intersection point of the vector **b** and the straight line defined by the ending points of the vectors $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$.

Proof. The vector **b** lies on the axis *X*, so **b**=(b_1, 0).

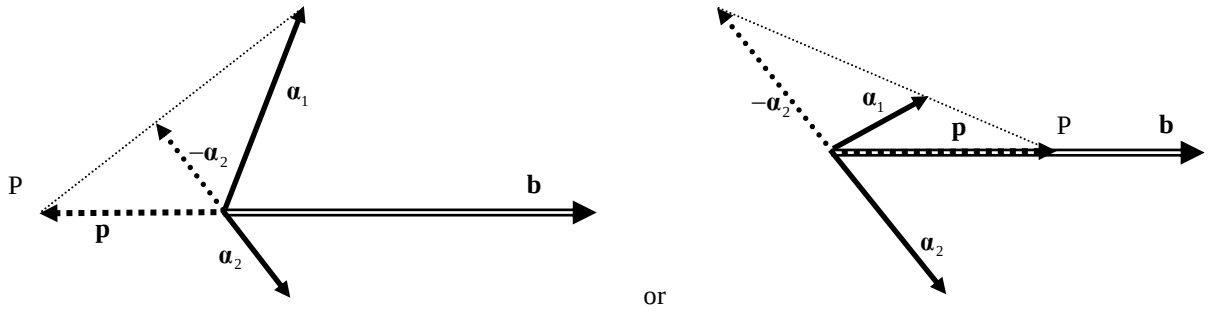

Figure 3 Combining the vectors for optimization

For the point *P* holds $x_1 - x_2 = \frac{b_1}{p_1}$. This value is positive or negative according to the value p_1.

If the point P does not exist, $x_1 - x_2 = 0$, the value p_1 is increasing without limits.

General case is received after isometry transformation - rotation. For the general vectors **b** and **p** the following formula depends on the direction of these vectors. If **b** and **p** have the same direction, the objective function value is

$$z(\mathbf{x}) = x_1 - x_2 = \frac{\|\mathbf{b}\|}{\|\mathbf{p}\|} \tag{6}$$

If **b** and **p** have the opposite direction, the objective function value is

$$z(\mathbf{x}) = x_1 - x_2 = -\frac{\|\mathbf{b}\|}{\|\mathbf{p}\|} \tag{7}$$

If the point P does not exist, $z(\mathbf{x}) = x_1 - x_2 = 0$.

Based on the Lemma 3 we can formulate a simple rule to order individual feasible solutions according to the values of the objective function. In descending order

- in the Quadrant I: from the nearest point P to the origin of coordinates system to the most distant one;
- then in the Quadrant III: from the most distant point P to the origin of coordinates system to the nearest one.

4 Numerical example

Using the above-described algorithm, we solve the following model

$$\begin{aligned} &4x_1 + 2x_2 - 0.5x_3 - x_4 \leq 16 \\ &2x_1 - x_2 + 2x_3 - x_4 \geq 6 \\ &z = 8x_1 + 2x_2 - 10x_3 - x_4 \rightarrow MAX \\ &x_{1,2,3,4} \geq 0 \end{aligned}$$

with added slack variables

$$\begin{aligned} &4x_1 + 2x_2 - 0.5x_3 - x_4 + s_1 = 16 \\ &2x_1 - x_2 + 2x_3 - x_4 - s_2 = 6 \\ &z = 8x_1 + 2x_2 - 10x_3 - x_4 + 0s_1 + 0s_2 \rightarrow MAX \\ &x_{1,2,3,4} \geq 0, s_{1,2} \geq 0 \end{aligned}$$

Step 1: Elimination of different cost coefficients in the objective function

Using the Eq. 4 we obtain

$$\boldsymbol{\alpha}_1 = \begin{pmatrix} 0.5 \\ 0.25 \end{pmatrix}; \boldsymbol{\alpha}_2 = \begin{pmatrix} 1 \\ -0.5 \end{pmatrix}; \boldsymbol{\alpha}_3 = \begin{pmatrix} -0.05 \\ 0.2 \end{pmatrix}; \boldsymbol{\alpha}_4 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}; \mathbf{s}_1' = \begin{pmatrix} \infty \\ 0 \end{pmatrix}; \mathbf{s}_2' = \begin{pmatrix} 0 \\ -\infty \end{pmatrix}$$

Step 2: Feasible combinations of variables in basis

The feasible combinations of the variables are highlighted in Figure 4.

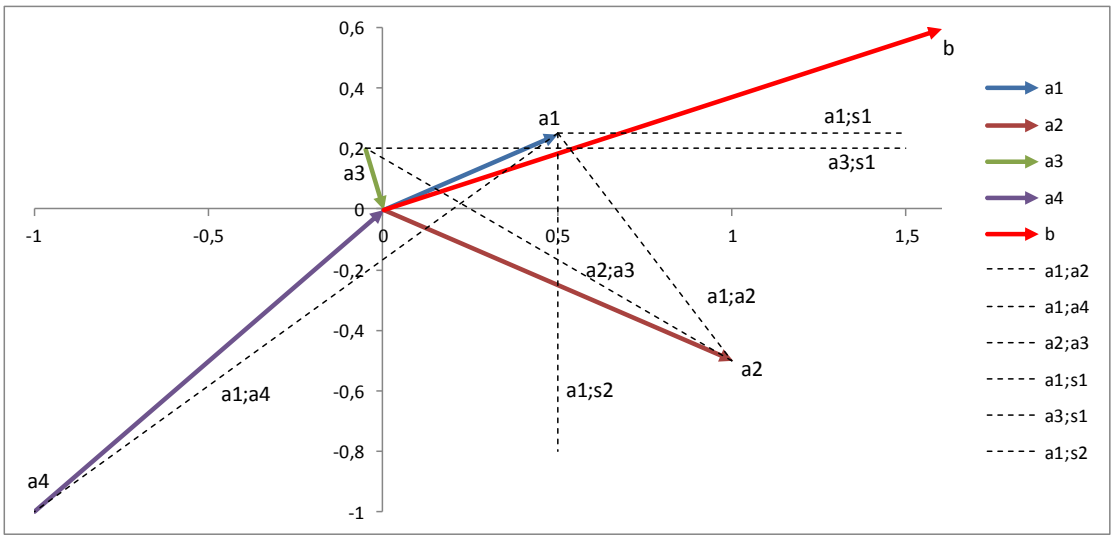

Figure 4 Feasible solutions

Step 3: Optimization

As the cost coefficients c_3 and c_4 are negative, the vectors $\mathbf{a_3}$ and $\mathbf{a_4}$ have to be redirected before optimization. Multiplying these vectors by (-1) we obtain

$$a_1 = \begin{pmatrix} 0.5 \\ 0.25 \end{pmatrix}; a_2 = \begin{pmatrix} 1 \\ -0.5 \end{pmatrix}; a_3 = \begin{pmatrix} 0.05 \\ -0.2 \end{pmatrix}; a_4 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; s_1' = \begin{pmatrix} \infty \\ 0 \end{pmatrix}; s_2' = \begin{pmatrix} 0 \\ -\infty \end{pmatrix}$$

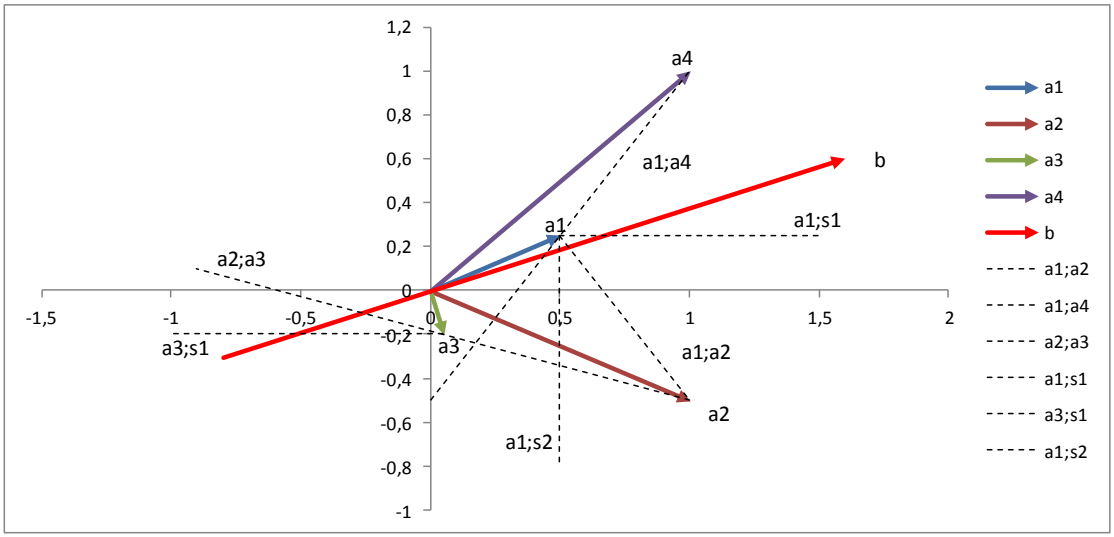

Figure 5 Optimization

Based on Figure 5, we can simply order the feasible basic solutions subject to descending values of the objective function

- in Quadrant I: $\mathbf{a_1}$; $\mathbf{a_4} - \mathbf{a_1}$; $\mathbf{s_2} - \mathbf{a_1}$; $\mathbf{a_2} - \mathbf{a_1}$; $\mathbf{s_1}$ (the closer to [0;0] the higher value of the objective function);
- in Quadrant III: $\mathbf{a_3}$; $\mathbf{s_1} - \mathbf{a_2}$; $\mathbf{a_3}$ (the closer to [0;0] the lower value of the objective function).

The vector of basic solution maximizing the objective function value is

$$\mathbf{x_B} = (x_1; 0; 0; x_4; 0; 0)^T$$

where the values of the variables x_1 and x_4 are calculated from the linear equations system

$$4x_1 - x_4 = 16$$
$$2x_1 - x_4 = 6$$

so $x_1 = 5$; $x_4 = 4$ and objective function value $z^{opt.} = 36$.

5 Conclusion

In this paper we provide the extension of the algorithm for graphical optimization of linear programming models in the Column Space. We showed how to solve the models, when both positive and negative values of the cost coefficients in the objective function occur. Of course, another way to overcome the problem is to construct the dual model, solve it in the Row Space, and finally re-interpret it back in terms of the primal model. We hardly recommend this way. We understand the graphical optimization in the Column Space as a regular algorithm, so we prefer to improve its imperfections rather than stop using it.

The authors know that there is another problem to solve: how to identify and recognize cases, when the objective function of the model is unbounded. The solution would be similar, but there are some difficulties, which could not be described and overcome in this paper. This is the topic for future work.

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Empirical analysis of labour markets: An International comparison

Jan Brůha[1], Jiří Polanský[2]

Abstract. The goal of this paper is to document and summarize main stylized facts about labour market data in transition and developed countries. First, we report various correlations among labour market variables at different leads and lags and at various frequencies (growth rates and cycles), and compare them across countries. Second, we investigate the time-varying nature of these features paying attention to times of financial distress. We conclude that there are robust relations among the variables across countries and times at business cycle frequency, that the cyclical comovement between real output and selected labor market indicators is strong. On the other hand, correlations in growth rates are not similar across countries and times. The paper concludes with our assessment of what these findings imply for dynamic general-equilibrium macroeconomic models with labour market.

Keywords: Labour market modeling, stylised facts, time-varying correlations

JEL classification: E24, J21, J30
AMS classification: 62P20

1 Introduction

The goal of this paper is to document and summarize main stylized facts about labour market data in several countries. This may be useful for various reasons. First, one may be interested to know which data features are robust across countries and which depends on countries' characteristics, such as labour market regulation. Second, the results can be used to constructing a set of empirical checks when constructing structural models with explicit labour market blocks. Prominent examples of such structural models are Dynamic Stochastic General Equilibrium (DSGE) models with labour market frictions. In this paper, we are interested mainly in the second question and seek to find facts that are robust across countries and times and which a successful structural model should replicate.

An alternative check of a structural model is a comparison of its impulse responses with those based on theoretical models, mostly often structural vector autoregression (SVAR) models. We do not opt for this alternative. Firstly, this has been done countlessly before, and secondly, realistic and credible identification of SVARs is notoriously difficult if possible at all.[1] Hence we instead focus on correlation analysis to characterise stylized patterns at various frequencies (mainly at cycles), and their stability. Thus, we report various correlations among labour market variables at different leads and lags and compare them across countries. Then, we investigate the time-varying nature of these features paying a special attention to times of financial distress.

We find that there are robust relations among some variables across countries and times at business cycle frequency, that the cyclical comovement between real output and selected labor market indicators (mostly unemployment and hours) is strong and that these cyclical features hold not only in normal times but also during the times of financial crises. On the other hand, we find little evidence for such a robust relation between wages and output or wages and unemployment or hours. This implies that the structural models should pay more attention to fit the first type of the feature, as its more quantitative important and more stable in time and in space. This paper is a part of our ongoing research focus and in

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[1]See [4] or [1] who provide examples of impossibility of identification of SVAR models in economies with realistic structures.

future we will extend presented results by a more detailed analysis of data and more extensive discussion of results.

2 Data

Up to now, we were able to obtain data from 33 countries for the analysis. When selecting countries, we want to cover a majority of European economies (transition as well as developed) and some other developed (OECD) countries. We have also data on Turkish economy, as it is now a market economy with its rapid economic development. Such a variety allows us to provide as robust analysis as possible due to different structural characteristics of labour market in different countries and also to deal with short time series in some countries. Some countries have relatively short times series (less than 10 years). However, if we find that some feature is present in most countries regardless their degree of development and regardless the time span of available data, such a feature should be judged as important, even if its identification in a single given country would be difficult.

All data come from statistical offices and national banks' databases. For each country, we collect real GDP, real consumption, real investment, various employment time series, total hours worked, unemployment rate and its level, nominal wages, compensation of employees, labour force and participation rates. However, we were unable to collect all data for all countries. Most data have quarterly frequency, some data were available at monthly frequency only. In that case, we have transformed these monthly data to quarterly data. Our all analyses are thus done on the quarterly data.

All in all, we were able to collect data on these countries: Austria, Australia, Belgium, Bulgaria, Canada, Cyprus, the Czech Republic, Denmark, Estonia, Finland, France, Germany, Hungary, Iceland, Ireland, Italy, Japan, Latvia, Lithuania, Malta, Netherlands, Norway, New Zealand, Poland, Portugal, Romania, Slovakia, Slovenia, Spain, Sweden, the United Kingdom, the United States, Turkey.[2]

3 Correlation analysis

As a first step in our analysis, we report various sample correlations among labour market variables at different leads and lags and at various frequencies (trends and cycles), and compare them across countries. Gap time series have been acquired by the Hodrick-Prescott filter [5] filter with the usual value of the smoothing parameter for quarterly frequency $\lambda = 1600$.[3] We carry out the filtration for all time series, including unemployment, in order to filter out some structural changes in the labour market (which can be recently relevant for example for Germany).

Figure 3 presents correlation analysis among selected variables at the -6 to 6 leads-lags interval. The figure contains the statistics for all countries in the sample. In each panel, the bold line depicts the median of the sample correlations at the selected lag, the dark shadow area is the interquartile range, and the light shadow area range based on all available countries.[4] If the dark (or even light shadow) area is thin, it means that the underlying correlation is very similar for all countries in the sample. The first row in this figure shows correlations between GDP at and hours worked, employment level (persons) and unemployment rate. In the second row, the first subplot presents correlation between wage bill and the GDP. Next two panels show correlations between nominal wages and employment level and unemployment rate, respectively. The third row in the figure presents correlations between real consumption and GDP, unemployment rate and the wage bill. The next three rows are analogous for growth rates (trends).

There are two main interesting findings derived from Figure 3. First, there are robust relations among the variables across countries at business cycle frequency and that the cyclical comovement between real output and some of labor market indicators is strong. This is particularly true for hours worked, employment and unemployment. The output-wage comovement is weak (if it is present at all) and the degree and the sign of this comovement vary across countries. On the other hand, at lower frequencies, all the relations are relatively weak. This is corroborated by the fact that the comovements is apparently much weaker for growth rates (which implicitly contain the movements across all frequencies), while they

[2] The data come from Eurostat, Australian Bureau of Statistics, Statistics Canada, OECD National Accounts Statistics, OECD Main Economic Indicators, Statistics New Zealand, Statistics Korea, Turkish Statistical Institute, and FRED.

[3] We check the results also using the band pass Christiano-Fitzgerald filter [3] and the results are essentially the same.

[4] Since not all data are available for all countries, the sample of countries can be different for each panel. E.g. since the hours worked are not available for some countries, the first panel is based on less countries than the second etc.

are strong for gap variables. The implication for developing of a DSGE model with an elaborate labour market block, it is much more important to capture these cyclical patterns than correlations in growth rates.

An interesting question is the typical lag where the correlation between any two gaps peaks (i.e., it is the largest in the absolute value). Figure 3 presents a histogram with these correlation peaks for all countries. E.g. the correlation between the real GDP gap and the gap in hours worked is maximal for 10 countries with no lag, while for another 10 countries it is maximal when the gap in hours lags the output gap by one quarter. From the first row of Figure 3, it is apparent that for most countries, employment and unemployment lagged the output gap by one or two quarters. For a smaller subset of countries, the correlation is largest for contemporaneous variables[5]. Again, for the correlation of wages with the rest of variable, we do not find any interesting pattern (see the second row in Figure 3).

Figure 3 plots the boxplots of the (base 10) logarithm of the standard deviations of gaps in variables relative to the standard deviations of the output $\varsigma_x^{rel} = \log_{10}\left(\frac{\sigma_x^{gap}}{\sigma_{GDP}^{gap}}\right)$, where σ_x^{gap} is the sample standard deviations of the cyclical part of the time series x. The boxplots are organized as follows: on each box, the central red mark is the median, the edges of the box are the 25th and 75th percentiles, the whiskers extend to the most extreme datapoints considered to be not outliers[6], and the outliers are plotted individually by red crosses. The logarithmic transformation was chosen for better readibility of the picture. Obviously $\varsigma_x^{rel} < 0$ means that the gap in the respective variable is less volatile than the output gap, while $\varsigma_x^{rel} > 0$ means the opposite. If $\varsigma_x^{rel} \cong 1$, the gap in the variable x is about 10times volatile than the output gap.

Again, some robust facts emerge. The investment and unemployment gaps are much more volatile than the output gaps in all countries. The consumption gap has on average the same volatility as the output gap, while the gaps in employment, hours worked and labour force is typically less volatile than the output gap, even if there are few outliers here. Nevertheless, for a typical economy in the sample, the volatility of these three variables is lower.

The results for the gaps in wages and wage bills are diverse and as in the case of correlation we cannot characterize a typical country in the sample: there is no systematic pattern of the relative volatilities for these countries. We also looked whether there is a relation between relative volatilities of wage gaps and employment or hour gaps, but we have not found any systematical pattern[7].

4 The stability of correlation in time

Recently with the advances in computational power, researchers started using the sophisticated models with time-varying parameters. Many times, the researchers conclude that the parameters are not stable, especially with the financial and macroeconomic crises after 2009. In lights of these findings, the natural question therefore is whether the facts reported in the previous part of the paper are stable in times.

To answer this question, we employ the approach suggested by [2]. This approach is based on the computation of the recursive correlations:

$$\rho_{it}^{s_1 s_2} = \frac{\sum_{\tau=t-s_1}^{t+s_2}(x_\tau - \bar{x}_t^{s_1 s_2})(y_\tau - \bar{y}_t^{s_1 s_2})}{\sqrt{\sum_{\tau=t-s_1}^{t+s_2}(x_\tau - \bar{x}_t^{s_1 s_2})^2}\sqrt{\sum_{\tau=t-s_1}^{t+s_2}(y_\tau - \bar{y}_t^{s_1 s_2})^2}},$$

where $\rho_{it}^{s_1 s_2}$ is the correlation between variables x and y in country i, centered at time t, $\bar{x}_t^{s_1 s_2}$ is the recursive mean of the variable x ($\bar{x}_t^{s_1 s_2} = \frac{1}{s_1+s_2+1}\sum_{\tau=t-2_1}^{t+s_2} x_\tau$) and analogously for $\bar{y}_t^{s_1 s_2}$.

Given a binary indicator z_t, one can test whether the recursive correlations $\rho_{it}^{s_1 s_2}$ are systematically different in times when $z_t = 1$ comparing to times when $z_t = 0$. The details of the test are described in [2]. Alternatively, one could ran a regression of $\rho_{it}^{s_1 s_2}$ on z_t. We apply this procedure for the case of recent crisis and hence it is a binary variable equal to one to the post Lehman times and zero otherwise.

[5] Note that for some countries, the correlations seem to be strongest for 5 or 6 quarter lags. This is a spurious result caused by previous business cycle.

[6] Outliers are defined as observations larger than $P_{75} + 1.5(P_{75} - P_{25})$ or smaller than $P_{25} - 1.5(P_{75} - P_{25})$, where P_{25} and P_{75} are the 25th and 75th percentiles, respectively.

[7] The idea was that countries with high wage volatility could have lower unemployment or employment gap volatility as the shocks to the economy would be absorbed by prices (i.e. wages) rather than quantities (employment). This conjecture does not prove.

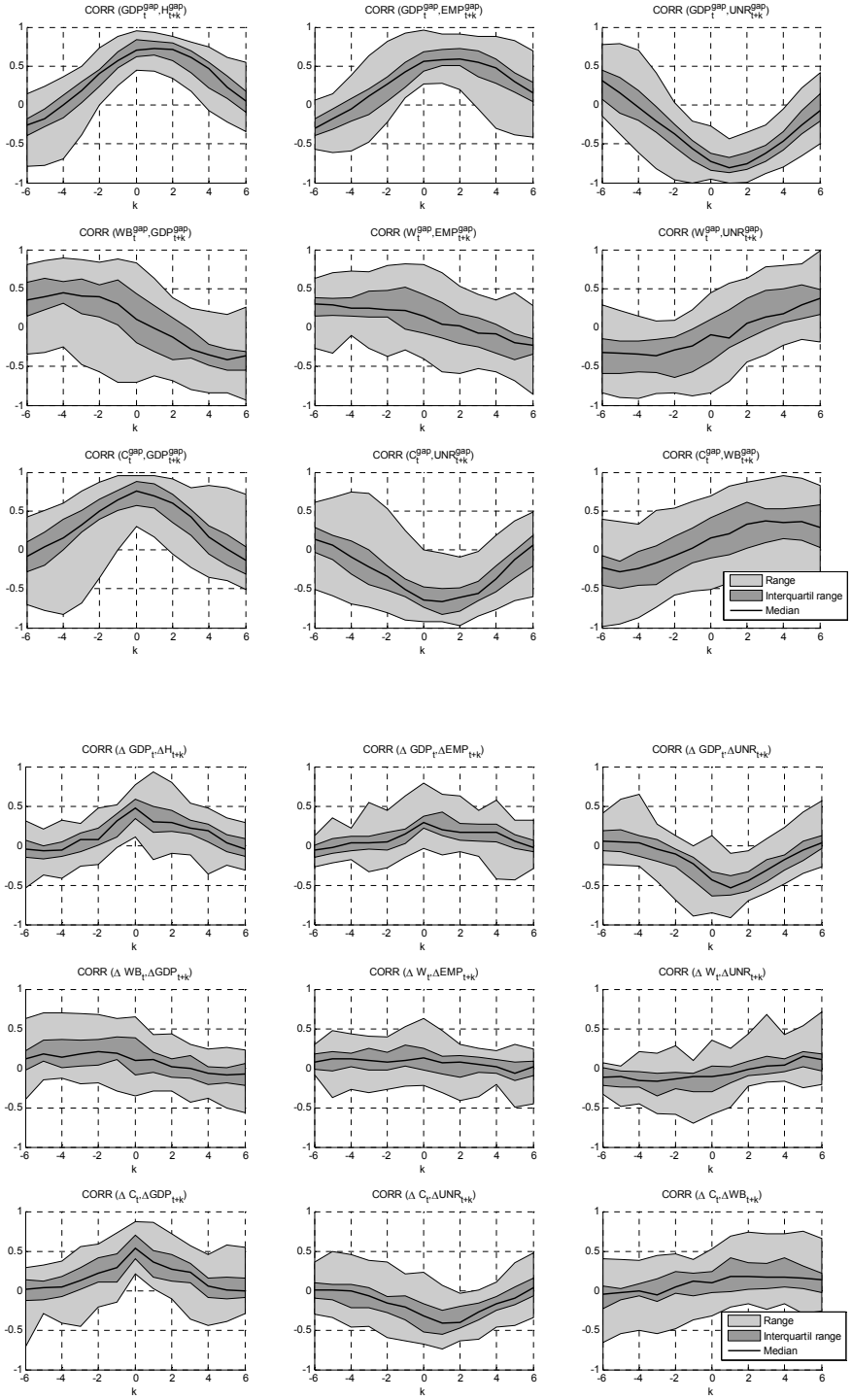

Figure 1 Correlation among labour market data

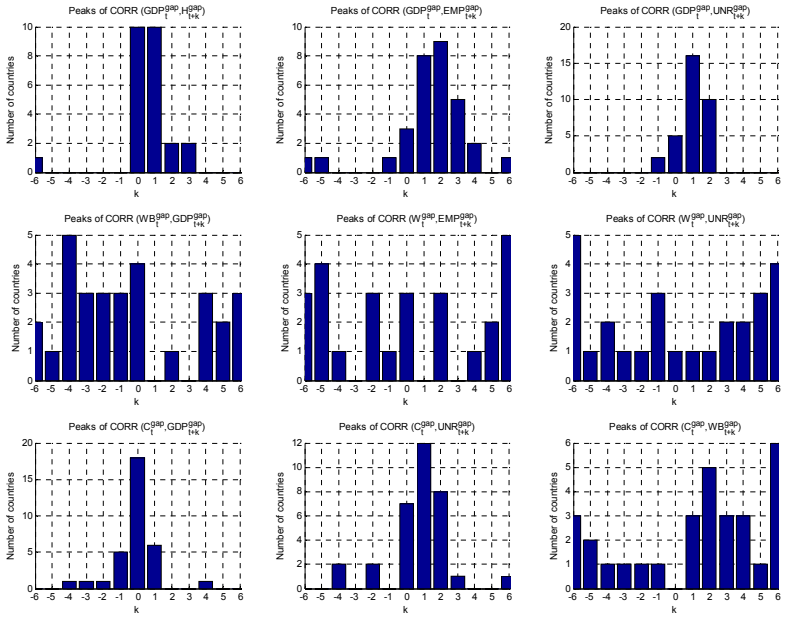

Figure 2 Peak of correlation among labour market data

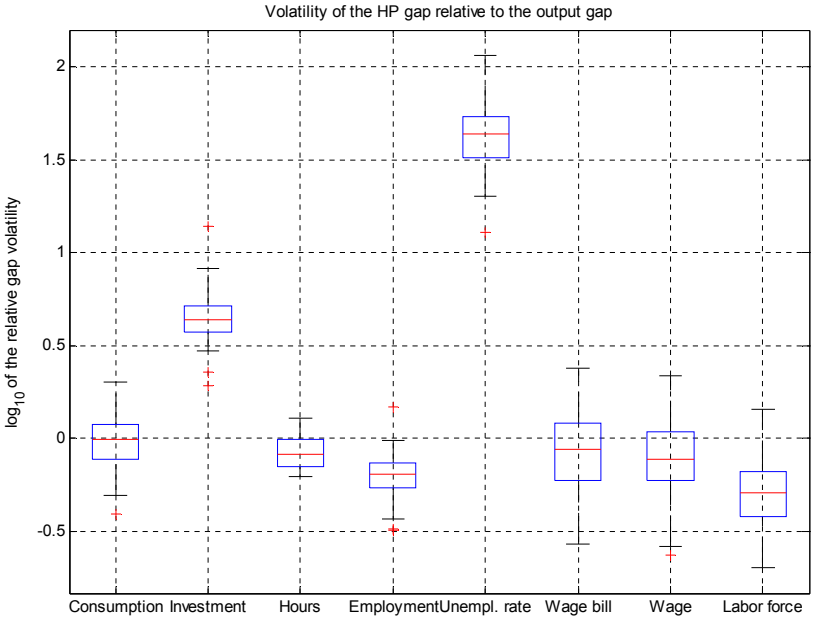

Figure 3 Relative volatilities of the cyclical components

If there were really a time variation after the beginning of the crisis, one would expect a dependency of $\rho_{it}^{s_1 s_2}$ on that indicator z_t.

We did this test for all countries in our sample for the correlation between output gap and unemployment gap. We do not find any evidence of the dependency of this recursive correlation on such an indicator. There is some, but weak, dependency between the correlation in growth rates. Therefore, we conclude that the stylized facts based on cyclical data are stable even during the recent crisis, but that the crisis can slightly affect the correlations in growth rates. This latter finding is probably caused by the change in low frequency component in data and is subject to our future research.

5 Conclusion

In this paper, we seek for robust stylized facts about labour markets in developed countries. Our findings are following. First, there are subsets of variables which correlates with the real output (and consumption) gap at business cycle frequencies. These are the hours worked, employment and unemployment. The correlation in growth rates are weaker, but still present. The gap in these labour market variables typically lag the output gap by one or two quarters. Moreover, contrary to correlations of growth rates, the correlation in gaps between these variables seem to be stable in time, even during the recent recession. The same relationships were not discovered for wages: the correlations of the wage gap and the output gaps, or between the wage gap and the gaps in the rest of labour market variables are small, and different between countries.

We can conclude that we find a set of robust stylized facts that seem to hold in different countries and that are stable in time. Therefore, the modelers dealing with structural models with elaborate labour market blocks should try to replicate these facts as these facts probably represent the true economic mechanisms in advanced economies.

Acknowledgements

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Influence of Demographic Data aggregation on Accuracy of Facility Location

Cebecauer Matej[1]

Abstract. When we are solving location problems in spatially large geographical areas we are dealing with huge numbers of possible candidate locations and serviced customers. The problem is that real infrastructure networks at the state level are too big and it is not possible to compute corresponding decision problems in a reasonable time. The commonly used approach for solving such problem sizes is to use data aggregation where all customers are aggregated into predefined points. The aim of this paper is to investigate influence of aggregation to results of optimization problems. First, we generate primary network based on the method called spatial decomposition. This method does not remove any data from network, but instead it breaks down the network into smaller parts while keeping the connectivity among all adjacent segments. To control the degree of aggregation we vary the size of cells covering the geographical area. The maximal size of cells corresponds to the macroscopic data model. Furthermore, we compare the results of p-median problem for different aggregation levels and evaluate the influence of data aggregation on the accuracy of location decisions.

Keywords: aggregation, network design, networks and graphs, p-median.

JEL Classification: C44
AMS Classification: 90C15

1 Introduction

The aim of this paper is to investigate and show the influence of spatial aggregation on the accuracy of facilities location. The spatially large realistic geographical areas contain huge numbers of possible candidate locations and customers. The reason of aggregation is that original problem is too big to be solved in a reasonable time. The idea behind aggregation is reduction of size of the problem. The result of commonly used aggregation of geographical areas is the fact that towns and villages are spatially non-dimensional points and that a certain point represents all of inhabitants of a town, but in the reality 20 000 people do not live in one point, but they exists in a certain space. It is obvious that this fact can have influence on solution. Hillmans [5] described aggregation errors in measuring distance between customers and facilities. And we describe aggregation errors with answering the question of influence of spatial aggregation on the accuracy of *p*-median facilities location.

The paper is structured as follows. In section 2 we describe commonly used aggregation, which we call macroscopic level and we propose an algorithm to the layout population and how to obtain different aggregated levels. Section 3 deals with the proposition of experiments and we introduce *price of aggregation* and *distortion of aggregation*, which express the aggregation errors. Finally, in section 4, we shall summarize our conclusions.

2 Data model

For our investigation we need microscopic geographical data. The free source of this kind of data is server OpenStreetMap which we describe in this section. Then we propose in [2] an algorithm for the layout of population to the two-dimensional space and how to obtain different levels of aggregated population used in this paper. The result of creation of the data model in this section is a network or graph usable for optimization problems.

2.1 OpenStreetMap

The project OpenStreetMap(OSM), born at University College London in July 2004, was founded by Steve Coast. OSM is an open source map server. All data and added information are available for free and cover the whole world. OSM contains realistic actual data from the whole world an allows to get enormous number of large-scale heterogeneous networks such as roads, railways, shops, restaurants, hospitals, fire/police stations, etc. You can see [1],[4] for more information.

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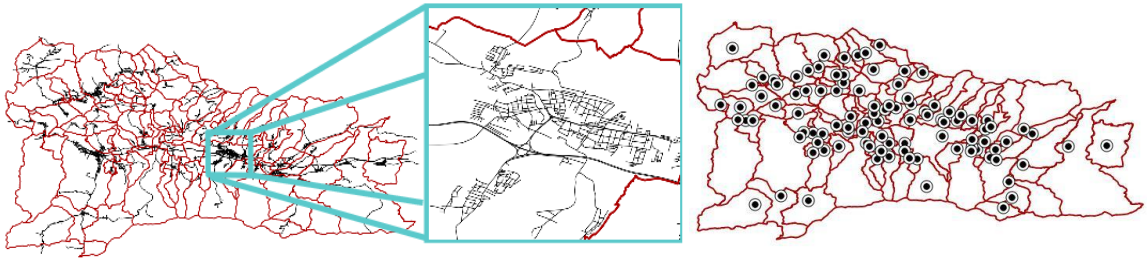
2.2 Microscopic and Macroscopic level

The commonly used data model in location problems is of macroscopic level with customers represented by towns and villages. In this part we describe process how to generate macroscopic level from microscopic level obtained from OSM.

First, we obtain microscopic data about road infrastructure from OSM, because this infrastructure has the highest density and the best connection to real customers in their houses. This obtained data form *microscopic level*. Further we aggregate this microscopic data to towns and villages. To make this aggregation automatic we have to know areas that belong to each towns or villages. This information about land use of towns can be obtained from OSM as well, just like the main point describing the coordinates and population in the towns. With this data we carry out decomposition method which we explained in more details in [2]. The result of decomposition method are two levels of namely macroscopic level and *microscopic level* divided into segments which represent microscopic data of all towns and villages on the macroscopic level. The only part of decomposition method in this work we need is the macroscopic level, because we work with problems of such a size that we are able to compute them as one entity on a microscopic level, so that we do not need microscopic segments.



(a) (b) (c)

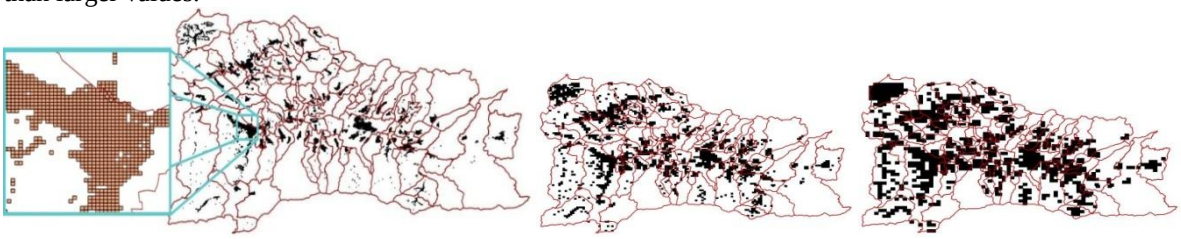
Figure 1 (a) The microscopic level with administrative border of towns and villages. (b) The zoom in microscopic level with roads infrastructure. (c) The macroscopic level with towns and villages points.

2.3 Layout of population

Once we have microscopic level segments, we also need to layout the population. We know the town population as well as the roads in segment. Furthermore, from the OSM we can get buildings and town residential areas that are situated in each segment. There are several different ways of allocating population that were described in [2]. It is not possible to consider every building or person as a node, because we would get a very large graph, and we could not easily find optimal solution in a reasonable time. So we introduce the technique with name *Grid Estimation Of Population* (GEOP) [2]

1. If there are no buildings or residential areas in a segment, then we put all population into main segment node (town point in macroscopic level) and stop. Otherwise we continue to the next step.
2. We create a spatial grid with items of constant size (size is arbitrary = size of square cells).
3. We divide all objects of the segment (buildings, residential areas) into grid cells based on coordinates.
4. We remove all grid cells that are empty (the result is land which we can consider settled).
5. Afterwards we allocate people into grid cells with regard to the number of buildings and fractions of residential areas in these cells.
6. Finally we decide about the location of the main nodes of grid cells and we connect them to the road network.

GEOP allows creating data models with different size of *Side Of Square Cells* (SOSC), so we can simply calibrate the size of grid cells. Each separate cell represents aggregated population, we call that aggregated customers. In the Fig. 3. you can see results for different values of SOSC. Smaller values of SOSC are closer to reality than larger values.



(a) (b) (c)

Figure 2 (a) The aggregated level with aggregated customers of SOSC(100). (b) The aggregated level with aggregated customers of SOSC(500). (c) The aggregated level with aggregated customers of SOSC(1000).

3 Experiments

In this section we propose experiments to illustrate the influence of demographic data aggregation on location of facilities. For these experiments we generate networks which are obtained from data models with different SOSC we described above.

3.1 Networks and input data for experiments

We decided to generate three different microscopic geographical areas based on administrative regions. Each area consists of one macroscopic level and ten microscopic levels with different sizes of aggregated customers (SOSC). It is important to note that towns in macroscopic level are also aggregated customers. The first generated area consists of three administrative regions of Slovak republic (SR), more specifically districts Ružomberok, Dolný Kubín and Liptovský Mikuláš (LM_DK_RK). The second area consists of only one district Žilina (ZA) and the last area consists of two districts Martin and Turčianske Teplice (MN_TT). More details about these areas are in Table 1.

Geographical area	population	number of towns and villages
LM_DK_RK	166 269	105
ZA	156 623	53
MN_TT	118 472	68

Table 1 Information about geographical areas

Further we apply GEOP with different SOSC on the microscopic areas. The results are microscopic levels with a different number of aggregated customers, for more details see Table 2. We decided to use 100 meters as the minimal value of SOSC. We consider this microscopic level, where SOSC equals 100 meters, as the most detailed layout of reality from all SOSC, which we are considering in these experiments and we are able to compute the optimal solution in reasonable time. We denote this SOSC as *SOSC(100)*.

	SOSC										
Geographical area	**2000**	**1000**	**900**	**800**	**700**	**600**	**500**	**400**	**300**	**200**	**100**
LM_RK_DK	505	901	980	1 108	1 271	1 435	1 757	2 178	2 983	4 830	12 119
ZA	239	453	520	574	674	781	989	1 256	1 785	3 307	8 320
MN_TT	284	506	552	607	677	769	965	1 190	1 620	2 647	6 787

Table 2 Information about number of aggregated customers in microscopic levels with different values of SOSC

3.2 Proposition of experiments

To illustrate the influence of the aggregation on the population we propose experiments for locating the facilities in these networks with different aggregated customers. In these networks we are locating different amount of facilities, from ten *(10f)* upwards. For illustration ten facilities represent very realistic number of ambulances for regions of these sizes. Candidates for facilities location are all the aggregated customers for each network. In each location problem for all networks we compute the optimal solution with the exact method that was proposed by Garciá and col. [3]. They called this method, *ZEBRA*. This method is very fast and can compute larger *p*-median problems in a reasonable time, faster than any other exact method before. We compute weighted *p*-median of location problem so that we can investigate the influence on aggregation of population. If we use an aggregation there are two basic errors which we investigate in this paper. These errors have influence to the sum of all distances among customers and their closest facilities that represent the value of the optimal solution.

To describe this errors we define ***F*** as a set of locations of facilities from specific aggregated problem ***X*** and the function ***TO(F,Y)*** that gives back the sum of all distances among aggregated customers from the aggregated problem ***Y*** to their closest facilities from the set ***F*.** This function *TO* makes the mapping from the location of facilities from one aggregated problem to the aggregated customers from another aggregated problem. Next we define function ***SUM(X)*** which returns the sum of all distances among aggregated customers from the aggregated problem *(X)* to their closest facilities located in the aggregated problem *(X)*.

The first is the error in the locating facilities, because in the aggregated problem we have less number of aggregated customers and candidates to locating facilities which are covering the spatial larger areas. Consequence of this fact is that it has *influence on accuracy of facility location.* We introduce ***the price of aggregation (POA)*** which expresses the percentual overpayment with which we overpay *SUM(Y)* of the aggregated problem if we are using the aggregation *X*. The value of POA represents the contrast between *TO(F,Y)* and *SUM(Y)* proportional to *TO(F,Y)*. It is important to note that *Y* aggregation has to be on the lower level of aggregation than *X* aggregation. The POA is defined as following :

$$POA = \frac{TO(F,Y) - SUM(Y)}{TO(F,Y)} \tag{1}$$

The POA is a number between 0 and 1. If the value of POA is closer to 0 then the efficiency measure is closer to *SUM(Y)* and overpayment is minimized. Values closer to 0 are preferred because they offer lower overpayment. We consider *Y* as the lowest aggregation that we denote as SOSC(100). It is important to note that POA also represents how much in percentage we can improve our solution *F* if we use lower aggregation *Y*.

The second error represents the *influence on measuring of distances*. This error is incurred by the aggregation and the fact that the result of aggregation generates spatial larger aggregated customers who are represented as a one point. For example 300 customers in one aggregated customer have the same distance to their closest facility and of course that have the same facility. These circumstances influence the error in the measuring distances. We named this error ***the distortion of aggregation (DOA)*** and it expresses the percentual distortion of the *SUM(X)* with regard to *TO(F,Y)*. The formula of DOA is :

$$DOA = \frac{TO(F,Y) - SUM(X)}{TO(F,Y)} \tag{2}$$

The DOA is a number between 0 and 1. If DOA has a value near 0 then the measure of the distortion is closer to aggregated problem *Y*. Values closer to 0 are preferred because they describe aggregated problem *Y* better. The value 1 is the maximal distortion; it means that the solution of the aggregated problem for all customers is without the costs. This is the lower-bound of the problem and it is not possible to achieve a better solution of this aggregated problem *X*. In this situation the POA can tell us how big the overpayment for this setting is.

The next attribute describing the impact of the aggregation is *the contrast between average distances X* and *Y*. We define function *AD(SUM(X))*, which returns a weighted average distance of the *SUM(X)* We named this attribute ***influence of aggregation (IOA)***. The IOA is the contrast between *AD(X)* and *AD(Y)*.

$$IOA = AD(SUM(X)) - AD(TO(F,Y)) \tag{3}$$

The value of IOA represents how far in the weighted average distance is *SUM(X)* from the *TO(F,Y)*. The aggregated problem *Y* is in our experiment denote as *SOSC(100)*.

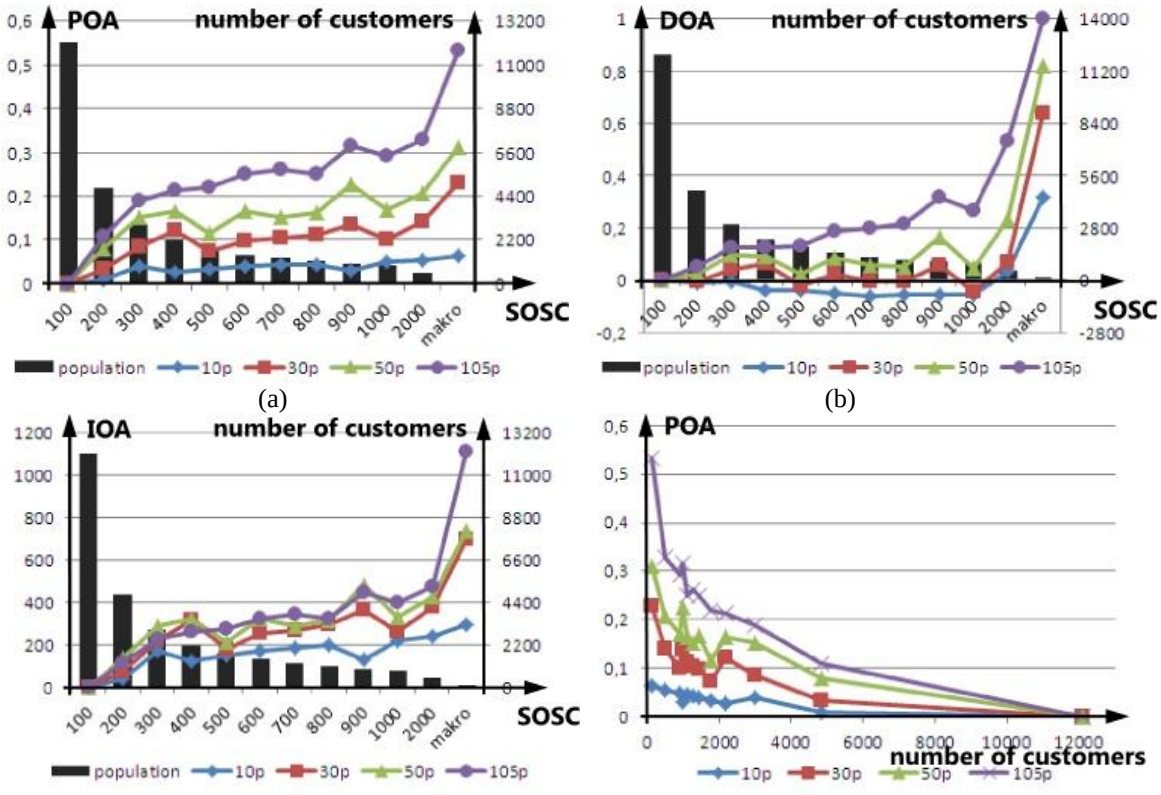


Figure 3 The results of the experiments on network LM_RK_DK. The histograms in graphs represent the number of aggregated customers in each SOSC. (a) The POA, (b) DOA and (c) IOA according to SOSC and the (d) POA according to aggregated customers.

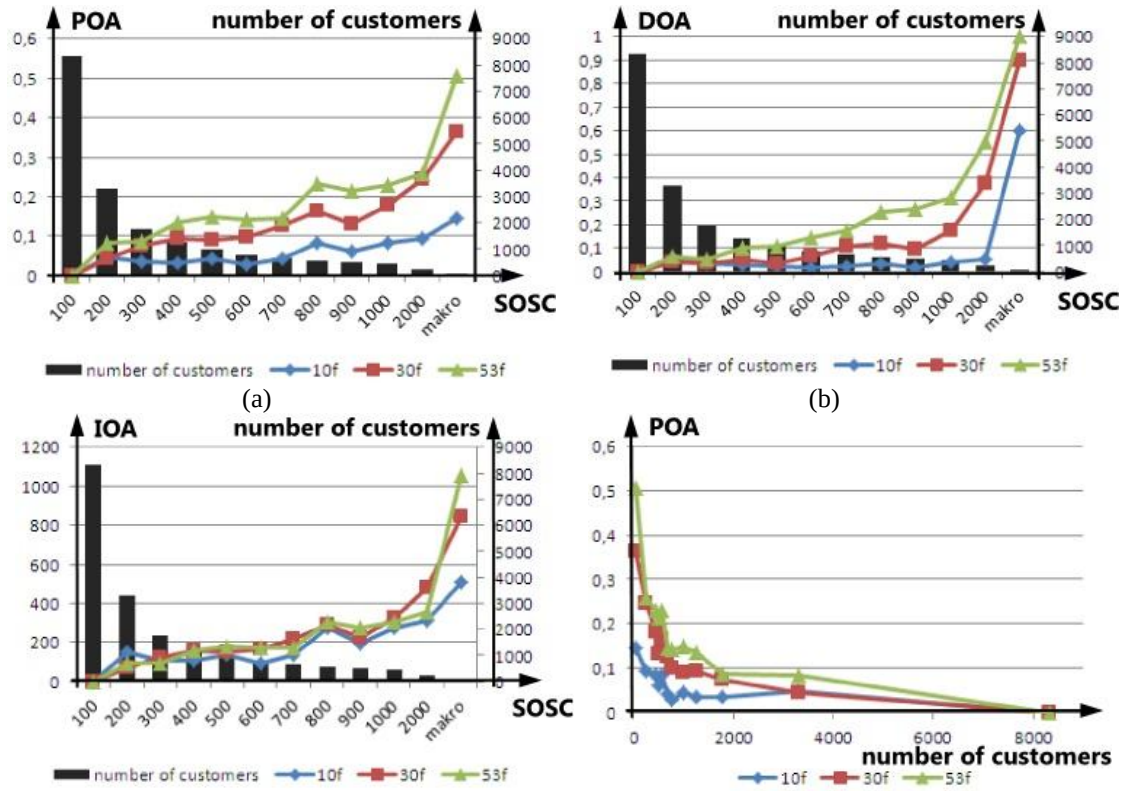


Figure 4 The results of the experiments on network ZA. The histograms in graphs represent the number of aggregated customers in each SOSC. (a) The POA, (b) DOA and (c) IOA according to SOSC and the (d) POA according to aggregated customers.

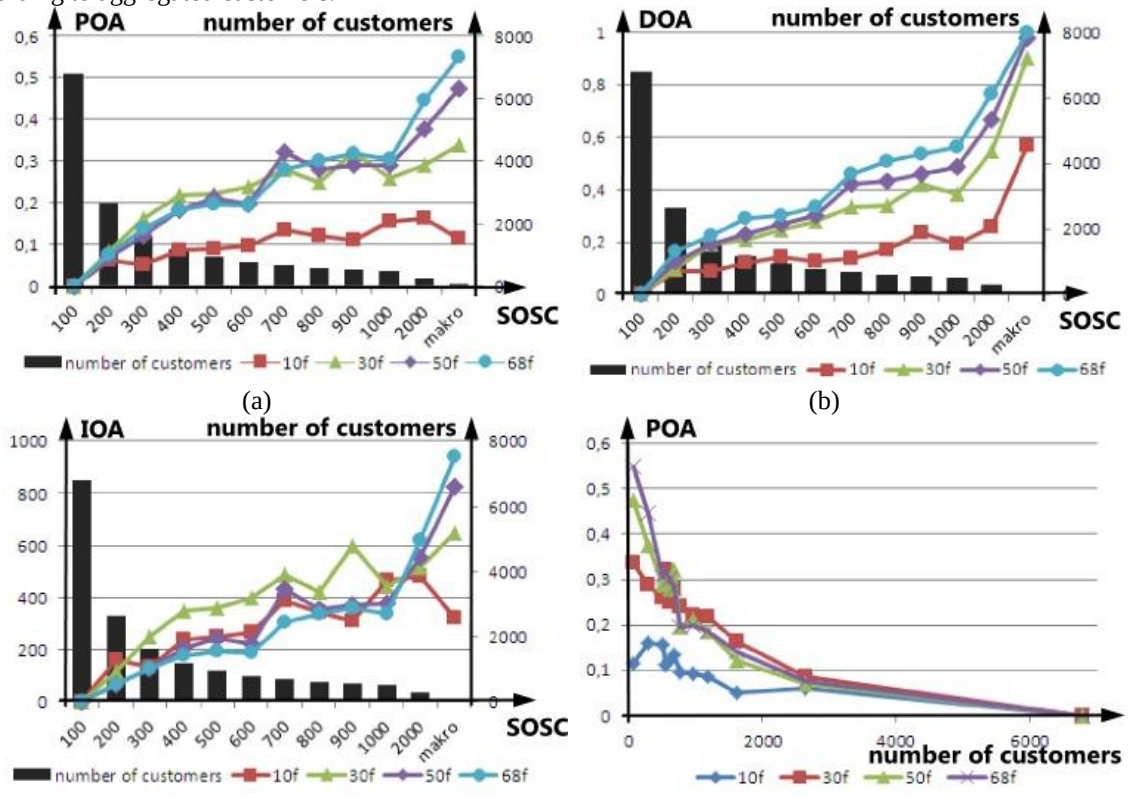


Figure 5 The results of the experiments on network MT_TT. The histograms in graphs represent the number of aggregated customers in each SOSC. (a) The POA, (b) DOA and (c) IOA according to SOSC and the (d) POA according to aggregated customers.

4 Conclusions

This paper was focused on the aggregation impact on the facility layout precision and its overall system effectiveness. We have created three different realistic microscopic models, on which the GEOP method was applied. This method aggregates the population into squares of defined dimension. With different value of the square dimension *SOSC*(100, 200, 300, 400, 500, 600, 700, 800, 900, 1000, 2000) we have created different levels of aggregated customers. Received number of aggregated customers, as well as their layout, depends on the geographical space and gained data for *GEOP*. For this reason it may occur, that for example SOSC(200) does not necessary offer better solution regarding to the SOSC(100), that for example SOSC(500) is giving. This may have impact at the worse solution finding comparing to macroscopic customers aggregation at low numbers of located facilities. The reason of this is depicted in the way of cutting plane into grid using the *GEOP*, where SOSC(200) does not have the same coordinates for aggregated customers as aggregated customers of SOSC(100), but SOSC(500) has. Anyway it is very small degradation according to the SOSC(100).

On the data models generated this way we placed different number of facilities. We observed two errors that can be encroached according to overall passed distance of all customers to their closest facilities. This summed distance represents the task solution. First error is linked to location of facilities, which we have called *price of aggregation*. This error reflects the consequence of the selection from the smaller set of candidates for placement, that represents bigger area compared to the SOSC(100). The price of fairness gets values from 0 to 1, the closer to 0, the closer is to the SOSC(100) solution. We have called the second monitored error *distortion of aggregation* that represents the error of distance measurement. This represents the result distortion that represents the solution of the aggregated task proportionally to the SOSC(100) solution. This error can gain also negative values, when reaching solution worse than SOSC(100) up to the value 1, when it offers the solution, in which is everyone served without costs or traveling no distance. Value 0 of distortion of aggregation stands for no distortion and the solution gained in the aggregated task is equal to the SOSC(100) solution.

Experiments results are depicted in figure 3.-5. From the results presented in the charts one can see, that monitored properties nonlinearly decrease when the number of aggregated customers raises. The decrease is bigger for lower number of aggregated customers. The expected assumption has been confirmed. This fact says, if the bigger number of centers is being placed that the bigger aggregation impact on the errors in the placing and distance measuring will be. On the contrary with the lower number of centers, the aggregation influence is falling. Even some values, the example can be seen in figure 5(d) when placing 10 centers, have found worse solutions than the macroscopic level. Globally we can assume, that on all three considered data models there is exponential decrease stabilizing on the 25% limit of aggregated customers from SOSC(100). Of course, it depends on how big precision has to be provided. But it is possible, that this 25% is a sufficient amount to achieve relatively good precision. We want to improve this estimation in next research using larger experiments for bigger number of different networks of different dimensions and topologies. Also we want to focus on the price of aggregation and distortion of aggregation, but not from the view of the whole system and average distance, as it is in this paper, but from the point of individual customers and how the aggregation influences these distances.

The majority of the Operational Research researchers work on the better and faster algorithms with the goal to solve larger tasks. We want to solve very large tasks by the heuristic based on the results presented in this paper. We assume that the solution by the heuristic on the aggregation better describing the reality can be better solution than the optimal solution obtained on the macroscopic level. In the next work we will also focus on the research of this option and the given heuristic design.

Acknowledgements

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Real-time versus revised Czech data: a DSGE analysis

Jan Čapek[1]

Abstract. The paper investigates the effects of using real-time data instead of revised final data within Dynamic Stochastic General Equilibrium (DSGE) framework. Theoretically, using most recent, updated historical data for ex-post analyses for a historical time-sample may be misleading, because revised data were not available at that time.

The contribution uses a small-scale monetary macroeconomic DSGE model to analyze the importance of real-time data compared to most-recent revised data with a focus on the differences in decision-making of the monetary authority. The analysis proceeds from Bayesian estimation of model parameters in a model with real-time data and in a model with the most recent revised data.

Keywords: real-time data, data revision, recursive estimate, DSGE model

JEL classification: C11, C32, C52, C54, E52, F41
AMS classification: 91B51, 91B64, 91B84, 62P20

1 Introduction

The paper investigates the effects of using real-time data instead of revised final data within Dynamic Stochastic General Equilibrium (DSGE) framework. Theoretically, using most recent, updated historical data for ex-post analyses for a historical time-sample may be misleading, because revised data were not available at that time. In case of e.g. monetary macroeconomic models with the usage of revised data, the monetary authority would be expected to make decisions based on then-unknown data. The contribution uses a small-scale monetary macroeconomic DSGE model to analyze the importance of real-time data compared to most-recent revised data with a focus on the differences in decision-making of the monetary authority. The analysis proceeds from Bayesian estimation of model parameters in a model with real-time data and in a model with the most recent revised data. Statistical significance of the differences in the Bayesian estimates is presented.

Following section introduces the reader to real-time data literature. Literature review focuses on papers that are relevant for this paper's aim. Section 3 presents analysis of the data itself with descriptive statistic. Section 4 presents results of recursive estimation on a SOE DSGE model with real-time and revised data. Final section concludes.

The term "real-time data" addresses data that become available right after collection. In accordance with the literature on similar topics, this paper understands "real-time data" as data that are available 3-4 months after the end of a quarter and are typically the first estimates published for that quarter.

A "vintage" is a quarter at which the data becomes available, or, the time of publishing. For example, if Czech statistical office releases during April of 2013 an estimate of GDP growth for last quarter of 2012, it is said that the data for fourth quarter of 2012 are in vintage of April of 2013.

2 Literature review

The issue of data revising and properties of real-time versus revised data has been investigated in the literature for a long time and until now, it is still a topic for scientific discussion.

The orientation of various literature and methodology differs. Some articles focus on the data itself and investigate properties of real-time data in comparison with revised data. There is also a group of articles that focus on the influence of real-time data on model results, e.g. for forecasting or reactions of monetary policy. This contribution follows both of these lines of research.

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Croushore and Stark [2] investigate properties of a real-time dataset and show that data vintages matter in comparison to revised data. Taylor [8] focuses on forecast quality when real-time data are used for its construction. He shows that forecasting models that use timely data have higher forecasting power. Business cycle analysis in real-time data environment is done e.g. by Lee et al. [4], who construct real-time macroeconomic database for Australia. The database exposes the difficulties in drawing inferences and decision-making based on macroeconomic data that is subsequently revised. Molodtsova et al [6] estimate Taylor rule with revised data and real-time data. There is a difference between the two estimates in Germany but not so much in the USA. Orphanides and van Norden [7] examine the reliability of alternative output detrending methods with special attention to the accuracy of real-time estimates of the output gap. They find out that the revision of published data is not the primary source of revisions in measured output gaps; the bulk of the problem is due to the pervasive unreliability of end-of-sample estimates of the trend in output. Boivin [1] searches for the differences in drifting parameters with real-time data. The findings suggest important but gradual changes in the Taylor rule coefficients, not adequately captured by the usual split-sample estimation.

This contribution follows the trends in literature and it presents analysis on the data itself by descriptive statistics in section 3 and it also calculates model results with recursive estimates in section 4.

3 Data analysis

There is a number of sources of uncertainty concerning the data and its model implementation. This contribution addresses three of them: (i) the differences in the data by source, (ii) the magnitude of data revisions and (iii) the differences in detrended data due to data availability at time of detrending.

3.1 Differences in the data by source

Figure 1 depicts the differences between the data from two widely used databases: OECD and Eurostat. The data should be the same with the exception of inflation, which is only available as CPI in OECD real-time database and HICP in Eurostat database. In most cases, the differences are small or none at all, which is the case of both domestic and foreign interest rates with no error and EuroArea growth with only 0.02 percentage points of average absolute error. On the other hand, average absolute difference in data series for Czech inflation differs by a half of a percentage point, which is not negligible. As the errorbars display, there are no systematic patterns in differences between Eurostat and OECD data.

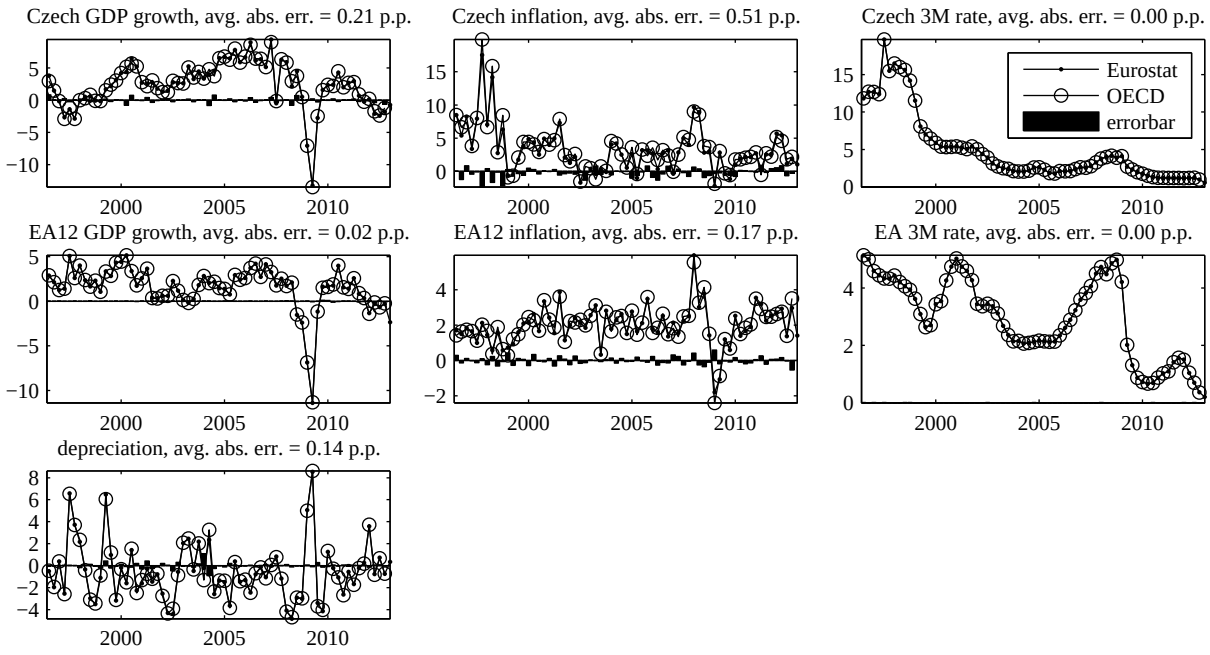

Figure 1 Eurostat and OECD data, *Note*: (error)bar plot around zero = difference between Eurostat and OECD data, "avg. abs. err." = average absolute error in percentage points

3.2 Magnitude of data revisions

Figure 2 displays the series of real-time data and most-recent updated data, all from OECD database. The differences in the series are not insignificant and the errorbars in cases of GDP growths show that the deviations are also non-random. Real-time data tend to underestimate the actual development. For example, in periods of expansion, real-time data underestimate positive growth in the Czech Republic in 2004/2005 and also in EuroArea in 2007/2008. As for recessions, both real-time values for domestic and EuroArea economy underestimate the severity of the crisis of 2009. The average absolute error tops at 1.34 percentage poins (quarterly, per annum) for Czech GDP growth, which means that the average value of the error is approximately 56% of average GDP growth over the sample. *Average error-to-average value* is also very high for EuroArea growth with 39%.

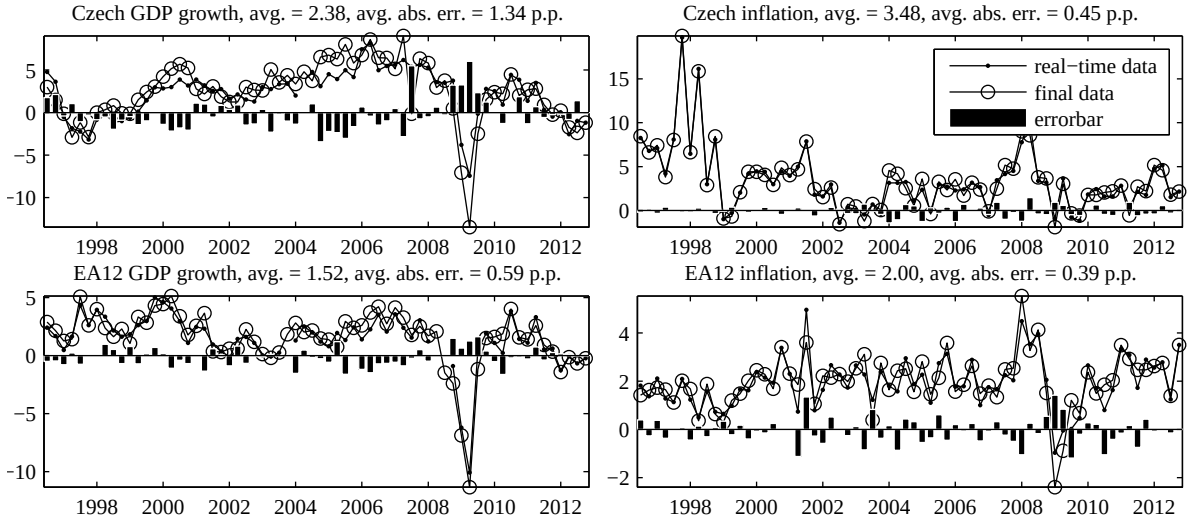

Figure 2 Real-time and final (revised) data, *Note*: (error)bar plot around zero = difference between final and revised data, "avg." = simple average of final data over the sample, "avg. abs. err." = average absolute error in percentage points

3.3 Differences in detrended data due to data availability at time of detrending

This section offers a depiction of a source of uncertainty which is not in the data itself but stems from the need to detrend the time series prior to modeling. In the model used, EuroArea interest rate is detrended with a linear trend and since Czech interest rates display even more curvature, Hodrick Prescott trend is used.

This type of analysis uses only revised data series. At first, the trend is computed on time series truncated to 30 observations. Then, the truncation moves one-by-one to more observations and the trend is recomputed each time. This procedure results in a number of time series of different length, each one detrended on its sample. Such results depict changes in the time series stemming only from the recomputation of the trend.

Figure 3 displays all vintages of detrended data with a solid line and final data with a circled line for comparison. The last values of each vintage form a real-time data in the second row panels. Time series formed this way will be addressed "detrend real-time" data in the remainder of the paper. Note that the difference between "detrend real-time" data and final revised detrended data stems only from the fact that detrending is calculated repeatedly for each length of the series. The third row in Figure 3 displays errorbars that are expectedly systematic.

4 Estimation

4.1 Model and Identification

This paper uses a New Keynesian (NK) Dynamic Stochastic General Equilibrium (DSGE) model. The model is derived from microeconomic behavior of particular economic agents. These include domestic and foreign households, domestic and foreign producers, domestic importers and domestic and foreign monetary authority. The

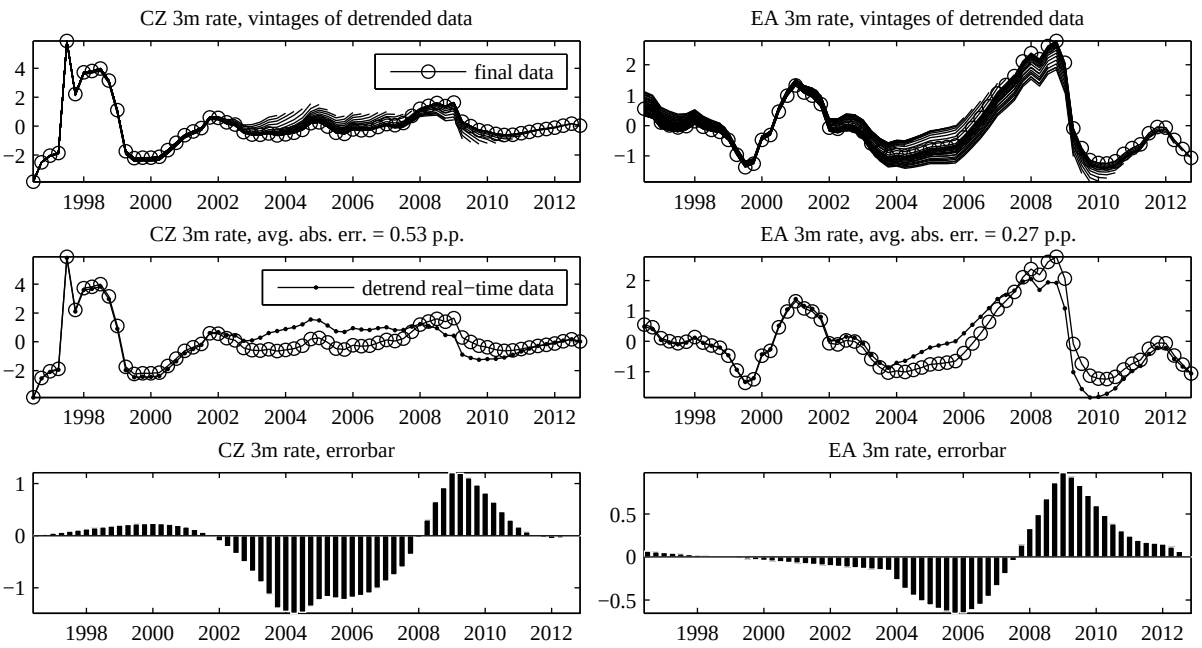

Figure 3 Differences in detrended data due to data availability

model is in small open economy (SOE) setting, so that it presumes two countries - small open economy that is influenced by a big closed economy. The small open economy is the home Czech economy, the big large economy is the foreign EuroArea economy. Most of the model assumptions are adopted from Lubik and Schorfheide [5].

The model is estimated by Bayesian methods in Dynare package[1] on a dataset with 7 variables. All the estimates share the same prior setting. Convergence of each estimate is checked by Brooks and Gelman convergence diagnostics.

4.2 Recursive estimates

Recursive estimates are constructed so that as a new data point is added to the existing set, the model is re-estimated. New values of parameter estimates are then reported. One can argue that if parameters estimates change, it reflects the information in the newly-added data.

Recursive analysis is regularly done by policy institutions if newly-data becomes available to actualize the model for a new round of predictions or forecasting. Recursive analysis is also done on historical sample to analyze the evolution of policy. The former inevitably uses real-time data, the latter mostly uses updated final data, because the estimation is easier to implement and older vintages of real-time data may not be available. This sections shows two examples where the choice of real-time versus final data does matter.

Figures in this section present four estimates to compare. Estimates based on Eurostat and OECD data are standard recursive estimates – only the time series of most updated final data is used, detrended on the whole sample and the detrended series is then truncated to mimic shorter sample. Note that the detrended series do not change at all in this type of estimation, there is just one new data point added in each longer time-frame.

Another approach is a genuine real-time estimate. This sequence of estimates use only data available at the vintage in question. Each time series may undergo revisions and therefore may be different. Also, detrending is made on these potentially different series with different number of observations. Although this approach mimics the historical possibilities of analysis best, it has a drawback that the influence of the data revisions and the influence of potentially changing trends cannot be distinguished.

To address this problem, another recursive estimate is introduced. The original data series for estimation are revised final series, but the truncation of the series is done prior to detrending. This estimation therefore does not reflect any issues of data revisions but it captures the influence of re-detrending in each quarter. It is then possible

[1] `www.dynare.org`

to infer the relevance of data revisions by comparing the genuine real-time estimate and the estimate that covers the problems arising from consecutive detrending. Note that this type of results was earlier addressed to as based on "detrend real-time" data. In order to easily see the significance of the results, a 90 % probability band of the "detrend real-time" estimate is drawn.

The estimates start in second quarter of 1996 and have at least 30 observations. The last observation is third or fourth quarter of 2012 which makes 66 or 67 observations in the most-recent estimates.

Estimates based on Eurostat and OECD data usually behave similarly, although there may be some vertical difference between the series of the estimates. However, on the two presented cases, the "detrend real-time" estimate that covers only the problem of consecutive detrending is different than the former two. This means that knowledge of the trend is significant and not knowing future trend leads to different estimates. Also, the real-time estimates differs from "detrend real-time" estimates, which again means that also the nature of the changes in the data itself due to revisions are significant.

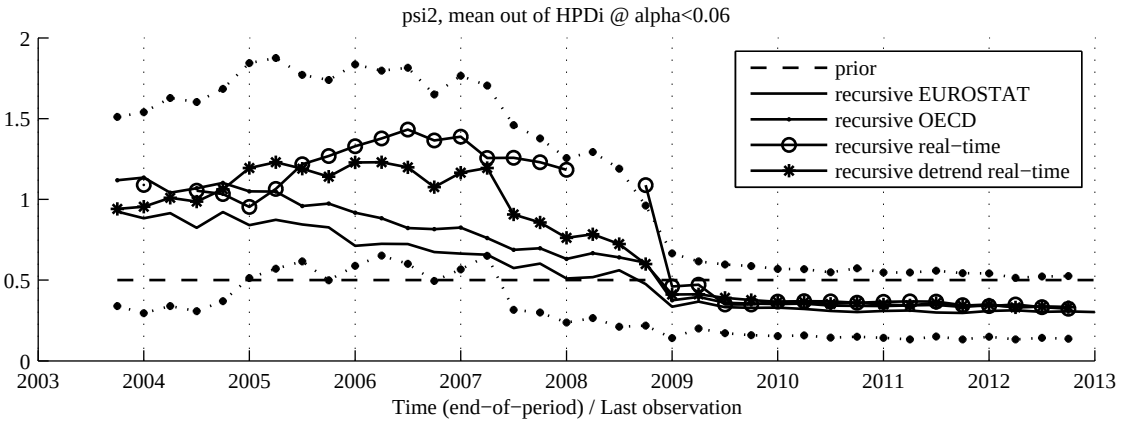

Figure 4 Recursive estimates of the weight on output growth in a domestic Talyor rule

Figure 4 displays the evolution of recursive estimates of a parameter that is the weight on output growth in a domestic Talyor rule. It is therefore a preference parameter of domestic monetary authority. The higher it is, the more important is output growth for the central bank when it set its policy interest rates.

All of the estimates display a tendency of a decline. The tendency is gradual in standard recursive estimates. On the other hand, the real-time estimate shows no gradual tendency but there is a great drop in the estimate in fourth quarter of 2008 (denoted as 2009 in the figure).[2] Since the evolution of the real-time estimate is outside of 90 % probability bands of "detrend real-time" estimate, the difference is statistically significant. Such result may be interpreted intuitivelly – from the point of view of real-time data, the oncoming crisis was a shock that is captured by a major change in the estimate in the quarter in question. Other estimates display gradual decline, because the information may have been included in revised data and the future trend was known.

Figure 5 displays estimation results for the persistence of domestic interest rate and it displays rather different development. Both standard recursive estimates calculated on final revised data tend upwards from 0.6 to final 0.7. However, both remaining estimates also show a decline in years 2006 and 2007 before converging upwards. This results may mean that 2006/2007 period is consistent with lower interest rate smoothing that would seem from the estimates on revised data. Since the evolution of real-time and "detrend real-time" data is similar, the differences are probably mainly due to re-estimation of the trend rather than in data revision itself. Note also that the results are not significant at conventional levels so that we can only discuss tendencies.

5 Conclusion

Data from OECD and Eurostat are very similar and the use in a DSGE model does not make much difference. Comparing revised and real-time data shows that the differences are not insignificant and also with systematic patterns. Also, if the preparation of the data for a model use means detrending (other than demeaning), this makes rise to a problem of trend recomputation, which can also be significant.

[2] Note that missing values of the estimates are due to missing real-time data at respective quarters.

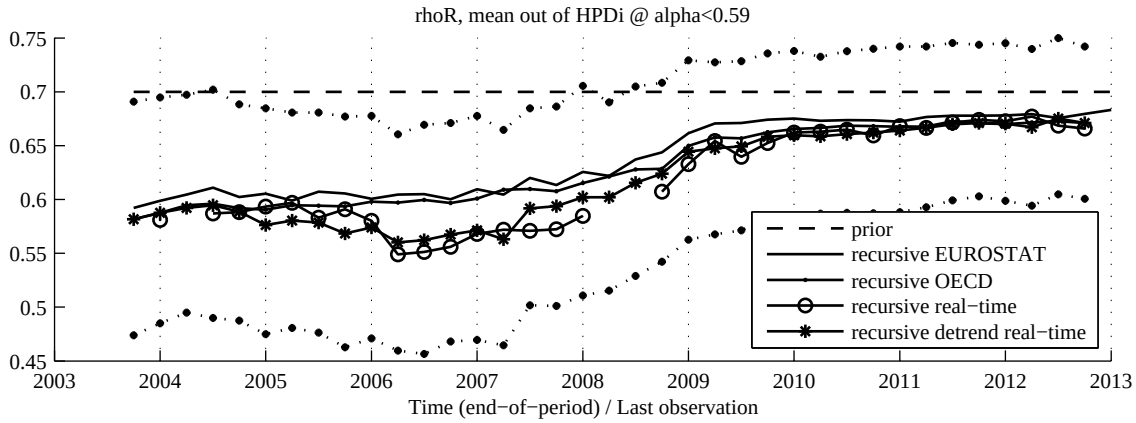

Figure 5 Recursive estimates of the persistence of domestic interest rate

Analysis on a DSGE model shows results where the re-detrending is significant and also where the nature of real-time data in comparison to revised data is significant.

Paper's results are in line with existing literature that finds the use of real-time data significant in various estimation or forecasting exercises. This contribution confirms the results on Czech data.

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Partially Joint Transport on Networks

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Abstract. The paper deals with a road network represented by a graph. It is supposed that there are several objects of the same type (containers, persons etc.) to be transported on the graph from their origins to their destinations at the same time. Any object can pass its route alone or can join some other(s) for passing some segments of the graph together. The cost of joint passing a segment is smaller than the sum of individual passing. In an extreme case, the cost of the joint passing equals to the cost of the individual one. Moreover, the limit of joined objects for transportation together is given.
The goal is to find such routes for all objects that minimize the total cost.
In the paper, several particular cases are studied where all objects have the same destination or the same origin.
Two main approaches to the solution are analyzed. First, several models based on flows on graphs with costs are discussed and afterwards linear programming models are presented.
These problems can be met in practice e.g. in travelling of pupils to schools, of employees to work or of football referees to matches.

Keywords: digraph, routing, joint travelling, costs, optimization.

JEL Classification: C65, L62, O18, R42
MS Classification: 90C15

1 Introduction

In practice, one can meet a situation where some objects are transported partially alone and partially together by the same type of transport. The second is advantageous since the joint transport is more efficient. For instance, two employees of the same employer live in different but not distant places *A* and *B*. Each morning they can drive their cars to their workplace *C* separately. However, they prefer that, in turns, one day the first takes the car, drives from *A* to *B*, picks up the colleague and continues to *C*. The next day, on the contrary, the second colleague takes the car for the trip $B - A - C$. Of course, such a type of partially common transport may relate to more than two persons to more than one destination. The present paper will deal with this type of problems.

1.1 General Problem

Given a network, represented by a graph $G = (V, E)$. A set of objects Q is transported on the network G, an object $q \in Q$ is transported from its origin o_q to its destination d_q. Any object can pass its route alone or it can join some other(s) for passing some edges of the graph together. The cost of joint passing an edge is smaller than the sum of costs of individual passing, i.e. on each edge $e \in E$, besides the price of "solo" passage $c_1(e)$ it is specified the price of "joint" passage of $k > 1$ objects $c_k(e) < kc_1(e)$. The difference between these numbers is significant. Sometimes there is even value $c_k(e)$ so close to the value of $c_1(e)$ that in the simplified model it is expected to be equal. The problem is to find a path p_q from o_q to d_q for each $q \in Q$ in such a manner that the total cost of transport is minimal.

Remark 1. It may seem strange that no requirement concerning time is introduced into the formulation. The authors are convinced that it is not necessary to introduce time-space network in this particular case, where one can suppose that all transportation is going to take place around the same time (e.g. travel to work which starts, say, at 8 a.m. for all employees).

1.2 Cost of Passing an Edge

The cost $c_k(e)$ of passing k objects through the edge e can be defined in several ways, having the common feature that $c_0(e) = 0$ and $c_1(e) > 0$:

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- if no upper bound for the number of together transported objects is given then it may be:
  - constant $c_k(e) = c_1(e)$ for all $k = 2, 3, \ldots$;
  - linear $c_k(e) = c_1(e) + a_e(k-1)$ for all $k = 1, 2, 3, \ldots$ and $e \in E$ where $a_e \geq 0$ is a given constant;
  - nonlinear $c_k(e) = c_1(e) + f_e(k)$ for all $k = 1, 2, 3, \ldots$, where $f_e > 0$ is a given (nonlinear) function;
- if an upper bound b for the number of together transported objects is given then it may be:
  - piecewise constant $c_k(e) = (\text{int}((k-1)/b) + 1)\, c_1(e)$ for all $k = 1, 2, 3, \ldots$;
  - piecewise linear $c_k(e) = (\text{int}((k-1)/b) + 1)c_1(e) + a_e(k - \text{int}(k/b))$ for all $k = 1, 2, 3, \ldots$ and $e \in E$ where $a_e > 0$ is a given constant having the property $a_e b < c_1(e)$;
  - nonlinear $c_k(e) = c_1(e) + f_e(k)$ for all $k = 1, 2, 3, \ldots$ $a > 0$, where $f_e > 0$ is a given function wih "jumps" after each b values of k.

It is necessary to note that saying "linear" means de facto piecewise linear, since for $k \leq 0$ it is $c_k(e) = 0$ and for $k > 0$ it is $c_k(e) = c_1(e) + a_e(k-1)$.

1.3 Variations of the Problem 1.1

From 1.1 it follows that there are at least six variants of the problem 1.1. However, their number is much greater, since it is necessary to distinguish between cases with one or several destinations and, similarly, with one or more origins of trips.

Moreover, as concerns approaches to solution, one should distinguish between different types of nonlinearities. E.g. convex cost is studied in [1], the concave one in [3] the general nonlinear in[6].

One can say that more than ten variants of the problem 1.1 ought to be examined separately. The present paper is focused on one of them.

2 One-to-Many Problem with Linear Cost and Unlimited Joining

In this section it is supposed that the problem 1.1 is specified by the following requirements:

- A vertex $o \in V$ is given such that $o_q = o$ for all $q \in Q$;
- Cost of passing is linear $c_k(e) = c_1(e) + a_e(k-1)$ for all $k = 1, 2, 3, \ldots$ and $e \in E$ where $a_e \geq 0$ is a given constant.

It is obvious that the same mathematical model can be used for the symmetric case "Many-to-One".

First, the case of constant costs, i.e. $a_e \equiv 0$, is tackled.

2.1 Problem 1: Constant Cost

Now, a graph $G = (V, E, c)$ is given where $c(e)$ is the cost of transport of any positive number of objects from the set Q through the edge $e \in E$. The **Problem 1** is to find a path p_q from o to d_q for each $q \in Q$ in such a manner that the total cost of transport is minimal, i.e. the number

$$c(P) = \sum_{e \in P} c(e) \to \min$$

for $P = \{e \in E: e \in p_q \text{ for some } q \in Q\}$.

The solution of the Problem 1 is based on the following propositions:

Proposition 1. Let $G = (V, E, c)$ be a graph. Let $Q \neq \varnothing$, and let $D = \{d_q \in V: q \in Q\}$. Let $G' = (D \cup \{o\}, P)$ be the shortest Steiner tree for $D \cup \{o\} \subset V$ in G. Let p_q be the (uniquely determined) path connecting the vertices o and d_q in G'. Then the paths p_q, $q \in Q$ represent the solution of the Problem 1.

Proof. Indirectly: Let other paths p_q', $q \in Q$ represent the solution of Problem 1, let $P' = \{e \in E: e \in p_q' \text{ for some } q \in Q\}$ and let $c(P') < c(P)$. That is in contradiction with the fact that the shortest Steiner tree represents the cheapest connected sub-graph of the graph G.

Proposition 2. The Problem 1 is NP-hard.

Proof is a consequence of Proposition 1 and the fact (see [2]) that the problem of the shortest Steiner tree on a graph G is NP-complete.

Remark 2. It follows from Proposition 2 that there is no sense in looking for a polynomial algorithm for an exact solution of the Problem 1. Therefore the exact solution can be reached by integer linear programming as seen e.g. in [5], or by some heuristics.

2.2 Problem 2: Linear Cost of Passing

Let a non-empty set of objects Q be given. Let a graph $G = (V, E)$ be given together with a cost function

$$c_k(e) = c_1(e) + a_e(k-1) \text{ for all } k = 1, 2, 3, \ldots \text{ and } e \in E$$

where $c_1(e) > 0$, $a_e \geq 0$.

The **Problem 2** is to find a path p_q from o to d_q for each $q \in Q$ in such a manner that the total cost of transport is minimal, i.e. the number

$$c(P) = \sum_{e \in P} c_1(e) + a_e\left(k_P(e) - 1\right) \to \min$$

for $P = \{e \in E: e \in p_q \text{ for some } q \in Q\}$ and $k_P(e) = \text{card}\{q. \in Q: e \in p_q\}$

Remark 3. Since the Problem 1 as a particular case of the Problem 2 is NP-hard, the same is true for the Problem 2. Therefore, there is no sense in looking for a polynomial algorithm for an exact solution of the Problem 2. The exact solution can be reached by integer linear programming or by some heuristics. A possible approach to linear programming is the following:

2.3 Problem 3: Transformation of the Problem 2 to the Cheapest Flow through a Digraph

Assume that the Problem 2 is fully determined as in 2.2.

Let us denote $V^* = V \cup \{s\}$ where s denotes the sink of the digraph $G^* = (V^*, H^*)$. Let $H^* = \{(o, v) \in E\} \cup \{(u, v) \in E: u \neq o \neq v\} \cup \{(d_q, s): q \in Q\}$. Let each $h \in H^*$ have infinite capacity. Let the cost $c^*(h) = 0$ for $h \in \{(d_q, s): q \in Q\}$ and let $c^*(u, v) = c(u, v) = c_1(u, v) + a_{(u, v)}(f(u, v) - 1)$ for all $(u, v) \in E$, where $f(u, v)$ is the flow through (u, v). The **Problem 3** is to find the flow $f(h) \in \{0, 1, 2, \ldots, \text{card}Q\}$ through the digraph $G^* = (V^*, A^*, c^*)$ meeting the constraints

$$f(d_q, s) = 1 \tag{1}$$

$$\sum_{(o,v) \in H^*} f(o, v) = \text{card}Q \tag{2}$$

$$\sum_{(u,v) \in H^*} f(u, v) = \sum_{(v,w) \in H^*} f(v, w) \quad \text{for each } v \in V^* - \{o, s\} \tag{3}$$

$$\sum_{h \in H^*: f(h) > 0} c_1(h) + a_h\left(f(h) - 1\right) \to \min \tag{4}$$

2.4 Problem 4: Linear Programming Solution of the Problem 3

Assume that the Problem 3 is fully defined. Let us define a binary variable x_h and an integer variable y_h for each $h \in H^*$, where $y_h = f(h)$ is the flow through h and $x_h = 1$ means that $f(h) > 0$. The **Problem 4** is to find values of variables x_h and y_h that meet the following constraints where M is a "very big number" as a substitute for ∞, e.g. $M = (\text{card}Q)^3$:

$$y_h \leq M x_h \tag{5}$$

$$\sum_{h \in \{(o,v) \in H\}} y_h = \text{card}Q \tag{6}$$

$$y_{(d_q, s)} = 1 \quad \text{for each } q \in Q \tag{7}$$

$$\sum_{(u,v) \in H^*} y_{(u,v)} = \sum_{(v,w) \in H^*} y_{(v,w)} \quad \text{for each } u \neq o, v \neq s \tag{8}$$

$$\sum_{h=(u,v) \in H^*, v \neq s} \left(c_1(h) x_h + a_h(y_h - 1)\right) \to \min \tag{9}$$

3 Notes to Practical Application

The problem of partially joint transport of goods or passengers may be found in cases of pupils travelling to schools, employees travelling to work or football referees travelling to matches. In the transport of goods, one can mention the collection of containers dispersed for a "single shot" collection of some material.

By looking at the paper [4] one could argue that application of partially joint transportation problems could be found also in freight train routing.

4 Conclusion

At the beginning a general problem of partially joint transportation was formulated and it was shown that there are many possible variations that differ in the type of costs, in number of origins and destinations.

Afterwards two types of "One-to-many" problem with linear cost are studied. The first one has constant cost, the second one works with linear cost depending on the transported quantity. It was shown that the first problem leads to the Steiner tree problem on graphs, which is known as NP-complete. Since it is a sub-problem of the second one, the later should be NP-hard.

For the sake of solvability, the second problem was reformulated to a minimum cost flow problem and a linear programming model to its solution was presented.

Since the general problem can be applied in wide area of transportation of goods and persons, the authors hope that many not yet solved variant problems will find their solvers in the near future.

Moreover, the set of constraints may be extended. E.g. an incompatibility function g on $Q \times Q$ could be defined such that $g(q, q') = 0$ means that the objects q and q' are fully compatible for joint transportation, whereas $g(q, q') = 1$ means that the common transportation is absolutely impossible and $g(q, q') \in (0, 1)$ means something in between. Adding a member containing g may be an object of further research as well.

Acknowledgements

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Interval data and linear regression: some properties and examples of the possibilistic approach

Michal Černý[1]

Abstract. We consider the linear regression model where we have partial information on its data. Only intervals are available, containing the unobservable values. We deal with the possibilistic approach: we study the ranges of all possible values which a given statistic can attain. In particular, we focus on the OLS-estimator, estimator of the variance of error terms and t-ratios. We also study complexity-theoretic classification of computation of the lower and upper bounds of the ranges. We illustrate the theory by an example motivated by the analysis of production functions, showing that even a small perturbation in data of a regression model can lead to quite wide intervals for some statistics.

Keywords: linear regression, possibilistic regression, interval data

JEL classification: C46
AMS classification: 62J86

1 Introduction

Consider the linear regression model

$$y = X\beta + \varepsilon, \tag{1}$$

where y is the vector of (observations of) the dependent variable, X is the design matrix, β is the vector of regression parameters to be estimated and ε is the vector of disturbances. From now on, n and p will denote the number of observations and the number of regression parameters, respectively. The tuple (X, y) is called *data* for the model (1). In this text we treat the data as fixed constants (i.e., as observations) rather than random variables.

An *interval matrix* $\boldsymbol{X} = [\underline{X}, \overline{X}]$ of the dimension $n \times p$ is a family of matrices $\{X \in \mathbb{R}^{n\times p} : \underline{X} \leq X \leq \overline{X}\}$, where the relation $\leq$ is understood entrywise. An *interval vector* is a one-column interval matrix. Interval matrices and vectors are denoted in boldface.

We are interested in the following problem. Assume that the data (X, y) for the model (1) are unobservable. The only information available to us is a pair $(\boldsymbol{X}, \boldsymbol{y})$ such that we are guaranteed that $X \in \boldsymbol{X}$ and $y \in \boldsymbol{y}$. Then we lose some information: for example, we cannot compute the value of the Ordinary Least Squares (OLS) estimator $\widehat{\beta} = (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}y$, since we do not know the values (X, y).

There exist various approaches to this problem. Following [2], we mention two of them here. First assume that the process, which generated intervals $(\boldsymbol{X}, \boldsymbol{y})$ from the real-valued data (X, y), is known. An example is *rounding*: then we have

$$\overline{X} = \lceil X \rceil, \quad \underline{X} = \lfloor X \rfloor, \quad \overline{y} = \lceil y \rceil, \quad \underline{y} = \lfloor y \rfloor,$$

where the operations $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ are understood entrywise. Another example is *additional random error*, that is,

$$\overline{X}_{ij} = X_{ij} + \gamma_{ij}^{+}, \quad \underline{X}_{ij} = X_{ij} - \gamma_{ij}^{-}, \quad \overline{y}_i = y_i + \delta_i^{+}, \quad \underline{y}_i = y_i - \delta_i^{-}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, p,$$

where $\gamma_{ij}^{+}, \gamma_{ij}^{-}, \delta_i^{+}, \delta_i^{-}$ are nonnegative random variables.

Of course, we are interested in computation of various statistics, such as the OLS-estimator $\widehat{\beta}$, Residual Sum of Squares, an estimate of the covariance matrix of $\widehat{\beta}$ etc. In general, such a statistic is a function

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of data, say $S(X, y)$. Since the data X, y are not observable, the natural approach is to replace the statistic $S(X, y)$ by another statistic $S^*(\underline{X}, \overline{X}, \underline{y}, \overline{y})$, giving us similar information like $S(X, y)$, and study its properties. For example, we can replace the OLS-estimator $\widehat{\beta}(X, y) = (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}y$ by the central estimator

$$\widehat{\beta}^*(\underline{X}, \overline{X}, \underline{y}, \overline{y}) = (X_C^{\mathrm{T}}X_C)^{-1}X_C^{\mathrm{T}}y_C, \tag{2}$$

where $X_C = \frac{1}{2}(\underline{X} + \overline{X})$ and $y_C = \frac{1}{2}(\underline{y} + \overline{y})$ are the *centers* of $\boldsymbol{X}$ and $\boldsymbol{y}$, respectively. Similarly we can compute e.g. RSS using the centers. Of course, important properties of the statistic S^*, such as consistency, efficiency etc., will depend on the nature of the interval-generating process. This is an interesting direction of study — an example of an important result from this area is Sheppard's Correction [9].

Now we turn to the second approach, which is applicable in case when the interval-generating process is unknown. Then it is natural to consider *all possible values* which the statistic $S(X, y)$ can attain when X ranges over $\boldsymbol{X}$ and y ranges over $\boldsymbol{y}$. Taking the OLS estimator as an example, we are interested in the range of possible values of $\widehat{\beta}_i$ $(i = 1, \ldots, p)$, that is,

$$\overline{\widehat{\beta}_i} = \sup\{b_i : X^{\mathrm{T}}Xb = X^{\mathrm{T}}y \text{ for some } X \in \boldsymbol{X}, y \in \boldsymbol{y}\},$$
$$\underline{\widehat{\beta}_i} = \inf\{b_i : X^{\mathrm{T}}Xb = X^{\mathrm{T}}y \text{ for some } X \in \boldsymbol{X}, y \in \boldsymbol{y}\}.$$

The second approach, called *possibilistic approach*, is complementary to the first one: first we can use e.g. the central estimator (2), and then the range $[\underline{\widehat{\beta}_i}, \overline{\widehat{\beta}_i}]$ quantifies the worst-case error, i.e. the worst possible deviation of the central estimator from the true value $\widehat{\beta} = (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}y$.

However, this approach often leads to serious computational problems. One of them is the following.

Proposition 1 (a consequence of Theorem 4 from [1]). *Computation of the ranges* $[\underline{\widehat{\beta}_1}, \overline{\widehat{\beta}_1}], \ldots, [\underline{\widehat{\beta}_p}, \overline{\widehat{\beta}_p}]$ *is an NP-hard problem.* □

It follows that we can expect only exponential-time algorithms for the problem — and this is intractable, especially when the number of observations is large.

In general, it is also interesting to study not only the ranges of possible values of the statistic, but also the simultaneous regions of possible values for two or more dependent statistics. A motivation for this approach can be found in [10], where simultaneous regions for expectation and variance, median and interquartile ratio, mean and Gini coefficient and many other combinations of parameters, estimated from one-dimensional data, are studied. In our setting, it is interesting to consider the simultaneous region for OLS-estimates of all regression coefficients. The region is called OLS-set [4, 1]:

$$B := \{b \in \mathbb{R}^p : \ X^{\mathrm{T}}Xb = X^{\mathrm{T}}y \text{ for some } X \in \boldsymbol{X} \text{ and } y \in \boldsymbol{y}\}.$$

With regard to Proposition 1, we cannot expect nice structural properties (or, at least, computationally testable properties). The set B need not be bounded, it need not be convex. In fact, just testing boundedness is a co-NP-hard problem [1]. Taking an example from [3], the OLS-set B with data

$$\boldsymbol{X} = \begin{pmatrix} 1 & 1 \\ 1 & [0;5] \\ 1 & [2;4] \\ 1 & 4 \end{pmatrix}, \qquad y = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

is plotted in Fig. 1.

2 Interval dependent variable

The general model described in the previous section, where both the design matrix $\boldsymbol{X}$ and the dependent variable $\boldsymbol{y}$ is interval, suffers from serious computational problems illustrated by Proposition 1. Hence it makes sense to restrict the problem to less general cases, where more optimistic results could be expected. In this paper we restrict ourselves to the case when the design matrix X is real-valued (i.e., we have $\underline{X} = \overline{X} = X$) and only the output variable is interval. From now on we assume that the matrix X has full column rank.

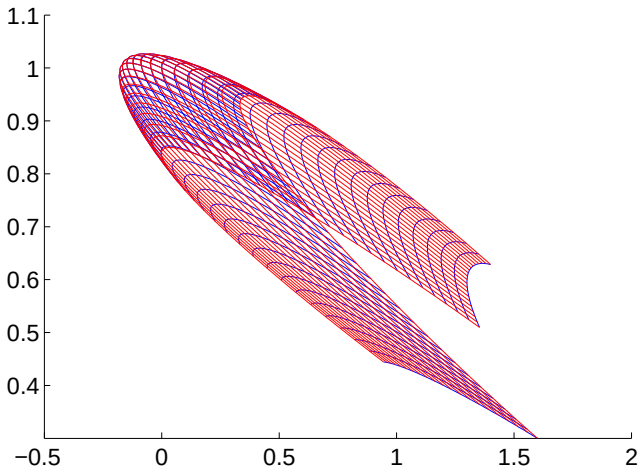

Figure 1 Example of the OLS-set.

Now the bounds for OLS-estimates are easily computable: we have

$$\widehat{\boldsymbol{\beta}} = (X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}\boldsymbol{y}, \tag{3}$$

where $\widehat{\boldsymbol{\beta}}_i = [\underline{\widehat{\beta}_i}, \overline{\widehat{\beta}_i}]$, and the expression (3) is evaluated using the *interval arithmetic* [7, 8], defined for two intervals $\boldsymbol{u} = [\underline{u}, \overline{u}]$ and $\boldsymbol{v} = [\underline{v}, \overline{v}]$ as

$$\boldsymbol{u} + \boldsymbol{v} = [\underline{u} + \underline{v}, \overline{u} + \overline{v}], \tag{4}$$

$$\boldsymbol{u} \cdot \boldsymbol{v} = [\min\{\underline{u}\cdot\underline{v}, \underline{u}\cdot\overline{v}, \overline{u}\cdot\underline{v}, \overline{u}\cdot\overline{v}\}, \max\{\underline{u}\cdot\underline{v}, \underline{u}\cdot\overline{v}, \overline{u}\cdot\underline{v}, \overline{u}\cdot\overline{v}\}]. \tag{5}$$

Moreover, the simultaneous region of possible values of the OLS-estimator has the form

$$B = \{(X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}y : y \in \boldsymbol{y}\}$$

and it is apparent that it is a convex polytope in the parameter space. Moreover, we get a more precise geometric characterization of B.

Proposition 2 ([4, 1]). *The set B is a zonotope centered in the point $\widehat{\beta}_C = \frac{1}{2}(X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}(\underline{y} + \overline{y})$ with generators $g_i := q_i(\overline{y}_i - \underline{y}_i)$ with $i = 1, \ldots, n$, where q_i is ith column of the matrix $(X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}$.* □

Now we turn our attention to another important statistic: the estimator of the standard error of the error terms ε, which has the form

$$\widehat{\sigma} = \sqrt{\frac{1}{n-p}\cdot y^{\mathrm{T}}(I-H)y},$$

where $H = X(X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}$ is the hat matrix. We would like to obtain the range of its possible values, i.e.

$$\overline{\widehat{\sigma}} = \sup\left\{\sqrt{\frac{1}{n-p}\cdot y^{\mathrm{T}}(I-H)y} : y \in \boldsymbol{y}\right\},$$

$$\underline{\widehat{\sigma}} = \inf\left\{\sqrt{\frac{1}{n-p}\cdot y^{\mathrm{T}}(I-H)y} : y \in \boldsymbol{y}\right\}.$$

The value $\underline{\widehat{\sigma}}$ can be computed efficiently: indeed, it suffices to solve the convex quadratic program

$$\min y^{\mathrm{T}}(I-H)y \text{ s.t. } \underline{y} \le y \le \overline{y}. \tag{6}$$

On the other hand, we have the following disappointing result.

Proposition 3 ([11, 5, 2]). *Computation of $\overline{\widehat{\sigma}}$ is an NP-hard problem.* □

t	Y_t	K_t	L_t	t	Y_t	K_t	L_t	t	Y_t	K_t	L_t
1	173.5	125	78	6	193.0	110	115	11	182.2	133	112
2	203.4	120	80	7	190.3	90	145	12	224.9	161	110
3	166.4	115	92	8	208.6	101	143	13	175.3	141	85
4	202.2	135	102	9	172.5	133	99	14	207.4	132	92
5	195.9	132	90	10	190.1	127	89	15	153.3	105	79

Table 1 Data for the Cobb-Douglas function.

Intuitively, computation of $\overline{\widehat{\sigma}}$ is a convex maximization over the n-dimensional cube $\boldsymbol{y}$, and it is easy to see that the maximum is attained in one of its vertices. Therefore, we have the following exponential-time algorithm (and better cannot be expected due to the result of Proposition 3):

$$\overline{\widehat{\sigma}} = \max\left\{\sqrt{\frac{1}{n-p}\cdot(y_C+\operatorname{diag}(s)y_\Delta)^{\mathrm{T}}(I-H)(y_C+\operatorname{diag}(s)y_\Delta)} : s\in\{-1,1\}^n\right\}, \tag{7}$$

where $y_C = \frac{1}{2}(\overline{y}+\underline{y})$ is the center of $\boldsymbol{y}$ and $y_\Delta = \frac{1}{2}(\overline{y}-\underline{y})$ is the *radius* of $\boldsymbol{y}$. The algorithm (7) can be used for small n only; in the next section, we will use it with $n = 15$, which is a situation when the number $2^n = 32768$ is still tractable.

3 Example

As an example we consider the Cobb-Douglas production function of the form

$$\ln Y_t = \beta_0 + \beta_1 \ln K_t + \beta_2 \ln L_t + \varepsilon_t,$$

where Y_t, K_t, L_t stand for output, capital and labor, respectively, and the error terms ε_t are assumed iid with zero mean and finite variance. Then, OLS estimator is applicable. We will use data from Table 1. We have $n = 15$ and $p = 3$. We will study the possible impact of a change in the dependent variable $\ln Y_t$ on the OLS estimates $\widehat{\beta}_1$ and $\widehat{\beta}_2$ and the estimate $\widehat{\sigma}$ (which has further consequences, since it affects the estimate of the covariance matrix of the estimator $\widehat{\beta}$, t-ratios etc.).

We will replace the vector $\ln Y_t$ by the interval vector

$$\boldsymbol{z} := [-\delta + \ln Y_t, \delta + \ln Y_t],$$

where $\delta \in [0, 0.1]$. For example, the choice $\delta = 0.05$ shows us, how the estimates $\widehat{\beta}$ and $\widehat{\sigma}$ can change in the worst case, if the output variable $\ln Y_t$ is measured with an error at most ± 0.05. This can happen, for example, when we round non-integer numbers to one decimal place. Figure 2 shows the resulting intervals $[\underline{\widehat{\beta}_1}, \overline{\widehat{\beta}_1}]$, $[\underline{\widehat{\beta}_2}, \overline{\widehat{\beta}_2}]$, $[\underline{\widehat{\sigma}}, \overline{\widehat{\sigma}}]$. The intervals $[\underline{\widehat{\beta}_1}, \overline{\widehat{\beta}_1}]$ and $[\underline{\widehat{\beta}_2}, \overline{\widehat{\beta}_2}]$ were computed using (3), the values $\underline{\widehat{\sigma}}$ were computed using (6) and the values $\overline{\widehat{\sigma}}$ were computed using (7).

We can see that even with a small δ (i.e. with a narrow interval vector $\boldsymbol{z}$) the ranges of possible values of $\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\sigma}$ are quite large.

Assuming normality of error terms, it is also tempting to have a look at the t-ratio for testing the null hypothesis $c^{\mathrm{T}}\beta = \gamma_0$, where c is a vector of parameters and γ_0 is a constant. The test statistic has the form

$$t = \frac{c^{\mathrm{T}}\widehat{\beta} - \gamma_0}{\sqrt{\widehat{\sigma}^2 c^{\mathrm{T}}(X^{\mathrm{T}}X)^{-1}c}}.$$

In fact, it is not easy to compute the upper bound $\overline{t}$ and the lower bound $\underline{t}$ exactly; it amounts to solving the optimization problems

$$\overline{t} = \max\left\{\frac{c^{\mathrm{T}}(X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}y - \gamma_0}{\sqrt{\frac{y^{\mathrm{T}}(I-X(X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}})y}{n-p}c^{\mathrm{T}}(X^{\mathrm{T}}X)^{-1}c}} : \underline{y}\le y\le\overline{y}\right\}, \tag{8}$$

$$\underline{t} = \min\left\{\frac{c^{\mathrm{T}}(X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}}y - \gamma_0}{\sqrt{\frac{y^{\mathrm{T}}(I-X(X^{\mathrm{T}}X)^{-1}X^{\mathrm{T}})y}{n-p}c^{\mathrm{T}}(X^{\mathrm{T}}X)^{-1}c}} : \underline{y}\le y\le\overline{y}\right\}, \tag{9}$$

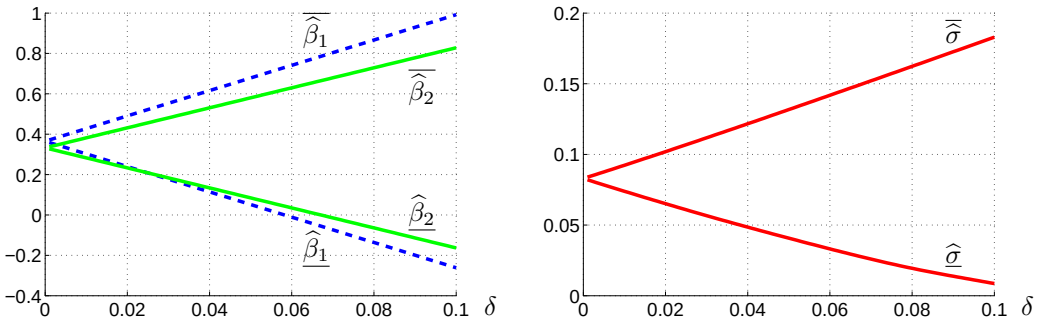

Figure 2 Ranges for $\widehat{\beta}_1$, $\widehat{\beta}_2$ and $\widehat{\sigma}$ with $\delta \in [0, 0.1]$.

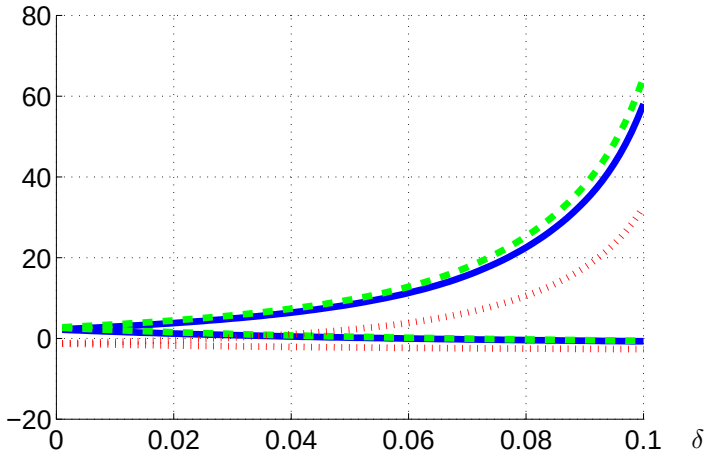

Figure 3 Approximate ranges for t-ratios for testing the hypotheses $\beta_1 = 0$ (solid), $\beta_2 = 0$ (dashed), $\beta_1 + \beta_2 = 1$ (dotted).

where X, c, γ_0, n, p are data and y are variables. The optimization problems seem to be difficult due to the occurrence of y both in the numerator and the denominator. And indeed, the bad news is that (8) is an NP-hard problem [2]. (The complexity-theoretic classification of (9) is not known at the moment.)

However, we will use the following approximation: we set

$$\overline{t}^* = \frac{c^{\mathrm{T}}\overline{\widehat{\beta}} - \gamma_0}{\sqrt{\underline{\widehat{\sigma}}^2 c^{\mathrm{T}}(X^{\mathrm{T}}X)^{-1}c}}, \qquad \underline{t}^* = \frac{c^{\mathrm{T}}\underline{\widehat{\beta}} - \gamma_0}{\sqrt{\overline{\widehat{\sigma}}^2 c^{\mathrm{T}}(X^{\mathrm{T}}X)^{-1}c}},$$

since the values $\underline{\sigma}$ and $\overline{\sigma}$ are available from Figure 2. We make three choices of c:

$c^{\mathrm{T}} = (0, 1, 0),\ \gamma_0 = 0$ for testing the hypothesis $\beta_1 = 0$,
$c^{\mathrm{T}} = (0, 0, 1),\ \gamma_0 = 0$ for testing the hypothesis $\beta_2 = 0$,
$c^{\mathrm{T}} = (0, 1, 1),\ \gamma_0 = 1$ for testing the hypothesis that returns to scale are constant.

The resulting intervals $[\underline{t}^*, \overline{t}^*]$ are plotted in Figure 3. Observe that even with small values of δ, the interval is quite large. It shows that even with a narrow interval $\boldsymbol{z}$, the conclusions of the t-test must be read carefully — it might happen that the value of the t-statistic can attain values from a quite wide interval.

4 Conclusions

We have studied how a replacement of real-valued data by interval data in a regression model can affect the values of some usual statistics, such as the OLS-estimate of regression parameters, estimate of variance

of error terms or t-ratios. Complexity-theoretic results show that these questions are generally very hard (usually NP-hard or co-NP-hard) in the general setting. The situation is slightly better when we restrict ourselves to the case when only the observations of the dependent variable are affected by interval uncertainty. We have presented an application in modeling the Cobb-Douglas production function.

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Modeling Effects of Economic Crisis on Potential Output: Application to the Eurozone

Ondřej Čížek[1]

Abstract. Economic cycles and long run relations are traditionally studied separately, which is not appropriate in the context of the current economic crisis. The goal of this paper is to propose a framework, which links short run fluctuations with long run dynamics within unobserved components methodology. The formulated model is nonlinear and is put in a conditionally Gaussian type of state space form. The model will be analyzed with respect to identifiability and estimated by maximum likelihood method. Finally, estimates and forecasts of potential output of the Euro zone will be calculated for the purpose of illustration and the obtained results will be interpreted.

Keywords: unobserved components method, Kalman filter, maximum likelihood, output gap, potential output, economic crisis.

JEL Classification: C51
AMS Classification: 90C15

1 Introduction

Nowadays, there is a debate among economists and politics whether or not governments should implement active economic policy to stimulate the economy or whether they should take low-budget measures to reduce indebtedness. There is no doubt that indebtedness is a serious economic problem of the Eurozone, but even despite this fact, there is a widespread consensus that now is not a time to tackle this problem. The argument is that contractionary economic policy would cause serious damages to economic activity in a situation of the current economic recession characterized by high unemployment rate. Lots of companies would definitely go bankrupt and many people would loose their change of ever finding a job. A short run transitory recession would thus overgrow into a permanent long run fall in economic activity which describes the relation between economic cycles and long run trends. Traditionally, economic cycles and long run relations are studied separately, which is not appropriate in the context of the current economic crisis. The goal of this paper is to propose a framework, which describes the above mentioned effects of short run fluctuations on long run dynamics.

2 Model

I will use unobserved components methodology, which was pioneered by Watson [6], who separated output into a trend and a cycle. The (natural logarithm of) output y_t is specified as:

$$y_t = \bar{y}_t + \hat{y}_t, \tag{1}$$

where $\bar{y}_t$ is potential output and $\hat{y}_t$ represents output gap.

Watson [6] assumed that trend component follow a random walk with drift and a cyclical component was assumed to follow an AR(2) process. I will differ from this popular specification. Linear trend of the potential output will not be affected by random errors, but by the output gap, which will describe the effect of the business cycle on the long run growth of the potential:

$$\bar{y}_t = \bar{y}_{t-1} + \mu + \alpha_{t-1} \cdot \hat{y}_{t-1}, \tag{2}$$

where

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$$\alpha_{t-1} = \begin{cases} 0, & \hat{y}_{t-1} \geq 0 \\ \alpha, & \hat{y}_{t-1} < 0 \end{cases} \tag{3}$$

Time-varying parameter α_{t-1} becomes zero if $\hat{y}_{t-1} \geq 0$, which describes the fact that increasing output above potential does not have any effects on the dynamics of potential output. On the contrary, $\alpha_{t-1} = \alpha > 0$ when $\hat{y}_{t-1} < 0$.

The output gap is assumed to follow AR(1) process, which is, however, modified by an inertial variable ν_t representing external shock to the European economy caused by the economic crisis in the United States:

$$\hat{y}_t = \lambda \cdot \hat{y}_{t-1} + \nu_t + u_t, \tag{4}$$

$$\nu_t = \lambda \cdot \nu_{t-1} + \varepsilon_t, \tag{5}$$

where $u_t \sim N\left(0, \sigma^u\right)$ is i.i.d. random error and

$$\varepsilon_t = \begin{cases} 0, & t \neq t_0 \\ \varepsilon, & t = t_0 \end{cases} \tag{6}$$

is a time-varying parameter, which is equal to $\varepsilon > 0$ only in time $t = t_0$, where t_0 is the second quarter of the year 2008, when the economic crisis from the United States hit the European economy.

The goal is to estimate the parameters μ, α, λ, ε and σ^u as well as the unobserved variables $\bar{y}_t$, $\hat{y}_t$, which is both done simultaneously by the applied methods.

3 Discussion of the concept of the output gap

The univariate approach can be extended to include other observable variables into the model. This was firstly done by Kuttner [4], who specified potential output as the level of output at which inflation is constant. The advantage of this multiple variable approach is that it uses additional information (e.g. from inflation) to estimate the parameters and especially to decompose output to its components. The other advantage is that the potential output has a precise economic meaning. Disadvantage of the multiple approach is that it is vulnerable to the specification errors of the structural economic equations (e.g. Phillips curve).

The idea of using (observable) inflation as an indicator for the (unobservable) potential output stems from the historical experience with high inflation caused by expansionary economic policy trying to stimulate output. It turned out that high inflation was the only effect of the expansion in the long run. From this point of view, it is then natural to say that the level of output causing rise in inflation is not sustainable in the long run and treat this excess of the output as a short run business cycle fluctuations.

There are, however, problems with defining potential output as the level of output at which inflation is constant. Typical example is the other historical experience with the current economic crisis, which showed that stable inflation is not a sufficient condition for the output to be sustainable in the long run. Typical reaction to this experience was to redefine potential output and to include other observable indicators of the potential output (e.g. indebtedness, prices of other assets) into the structural model (Borio et. al [2]). It is clear that including indebtedness as an indicator will lower the estimated potential output, because indebtedness is very high. It can be then argued that if the potential output had been defined using the indebtedness as an indicator before the economic crisis, it would gave us a signal of output above potential, which caused "overheating" of the economy.

Nevertheless, I do not consider the approach of adding additional observables into the structural model to be helpful as future experience will probably show that lots of other indicators would be needed. Especially the indebtedness is not an appropriate indicator of the level of output, which is sustainable in the long run. In my view, overheating of the US economy was not caused by high output, but by the problems in the financial system, which needed reforms of its institutional framework. In this context, Sims [5] argues that there are gaps in the institutional structure of the Euro area, which will need to be tackled, in order to solve the problem of indebtedness of some countries of the Euro zone. Sims, therefore, proposes institutional reforms to solve high indebt-

edness of the Eurozone and not contractionary economic policy, which would now only cause high damages in the economic activity.

4 State space form

The formulated model is nonlinear because of the time-varying parameter α_{t-1}. It can, however, be put in a conditionally Gaussian type of state space form, to which the linear Kalman filter algorithm can be applied to form a likelihood function.

The transition and the measurement equation of the conditionally Gaussian state space form is as follows:

$$\mathbf{x_t} = \mathbf{A_t}\left(\mathbf{Z_{t-1}}\right)\cdot\mathbf{x_{t-1}} + \mathbf{R_t}\left(\mathbf{Z_{t-1}}\right)\cdot\mathbf{u_t}, \tag{7}$$

$$\mathbf{z_t} = \mathbf{D_t}\left(\mathbf{Z_{t-1}}\right)\cdot\mathbf{x_t} + \boldsymbol{\upsilon}_\mathbf{t}, \tag{8}$$

where $\mathbf{A_t}\left(\mathbf{Z_{t-1}}\right)$ represents that the matrix $\mathbf{A_t}$ depends on the values of the observed variables till the time $t-1$, i.e. on the vector $\mathbf{Z_{t-1}} \equiv \left(\mathbf{z_1}', ..., \mathbf{z_{t-1}}'\right)'$.

In the case of the formulated model, the state vector is defined as $\mathbf{x_t} = \left(\bar{y}_t \quad \hat{y}_t \quad \nu_t \quad 1_t\right)'$ and the matrices from the transition equation (7) take the form:

$$\mathbf{A_t}\left(\mathbf{Z_{t-1}}\right) = \begin{pmatrix} \mathbf{e_1} + \mu\cdot\mathbf{e_4} + \alpha_{t-1}\cdot\mathbf{e_2} \\ \lambda\cdot\mathbf{e_2} - \left(\lambda\cdot\mathbf{e_3} + \varepsilon_t\cdot\mathbf{e_4}\right) \\ \lambda\cdot\mathbf{e_3} + \varepsilon_t\cdot\mathbf{e_4} \\ \mathbf{e_4} \end{pmatrix} \qquad \mathbf{R_t}\left(\mathbf{Z_{t-1}}\right) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

where $\mathbf{e_k}$ is a $\left(1 \text{ x } 4\right)$ vector, which has a number 1 at the k-th position and zeros elsewhere.

The matrix from the measurement equation (8) is as follows:

$$\mathbf{D_t}\left(\mathbf{Z_{t-1}}\right) = \left(\mathbf{e_1} + \mathbf{e_2}\right).$$

Vector of random errors in the transition equation is $\mathbf{u_t} = u_t$ and the vector of measurement errors $\boldsymbol{\upsilon}_\mathbf{t}$ is a vector of zeros.

Because the output gap $\hat{y}_t$ is unobservable, the time-varying parameter α_{t-1} had to be slightly redefined as:

$$\alpha_{t-1} = \begin{cases} 0, & \hat{y}_{t-1|t-1} \geq 0 \\ \alpha, & \hat{y}_{t-1|t-1} < 0 \end{cases}$$

where $\hat{y}_{t-1|t-1}$ represents the estimate of the unobserved variable $\hat{y}_{t-1}$, which was formed by the Kalman filter algorithm on the basis of the information available in $\mathbf{Z_{t-1}} \equiv \left(\mathbf{z_1}', ..., \mathbf{z_{t-1}}'\right)'$.

5 Econometric methodology

Full Information Maximum Likelihood method was used to estimate the parameters. Using the assumption that random error u_t is normally distributed, Kalman filter algorithm was applied to the model written in state space form. The outcomes of this algorithm were then used to construct likelihood function, which was then maximized by standard numerical procedures in Matlab. The construction of the likelihood function is described in Harvey [3]. The square-root version of the Kalman algorithm was used, in order to attain grater numerical precision. Description of this algorithm can be found in Anderson, Moore [1]. Parameters of the model were appropriately transformed, in order to ensure that their values lie within economically reasonable intervals. Because of

the problem with local extremes, lots of initial points were randomly generated, from which the Matlab procedure called fminunc was initialized.

6 Calibration of selected parameters

The applied econometric methodology enables to estimate all the parameters of the model. It is, however, necessary to calibrate some of them because of the limited number of observations in the data set, which is in not large enough to guarantee precise estimates of the parameters characterizing long run relationships. It will be assumed that $\mu = 0.0025$, which means that potential output would grow 1% per year if $\hat{y}_{t-1} \geq 0$. I will also assume that $\bar{y}_0 = 14.4943$, which was obtained as an ordinary least square estimate from a linear regression model $y_t = y_0 + 0.0025 \cdot t$, $t = 1,...,t_0 - 1$.

The econometric estimate of the parameter α was unrealistically high. Potential output closely followed the output, because only small changes in the output gap sufficed for that. Almost all variability in the output was caused by the variability in the potential. It will be, therefore, more reasonable to calibrate the coefficient α. I will base the calibration on the fact that very small value of α will mean almost linear trend for the output gap. It is clear that increasing α will cause higher variability in the growth rate of the potential output. From this point of view, the parameter α is an analogy to the smoothing parameter ω in the generally known Hodrick-Prescott filter defined by the optimization problem:

$$\underset{\{\bar{y}_t\}_{t=1}^{T}}{Min} \sum_{t=1}^{T}\left(y_t - \bar{y}_t\right)^2 + \omega \cdot \sum_{t=2}^{T-1}\left[\left(\bar{y}_{t+1} - \bar{y}_t\right) - \left(\bar{y}_{t+1} - \bar{y}_t\right)\right]^2 .$$

I will calibrate the parameter α in such a way so as to ensure the required (economically sensible) variability of the growth rate of the potential output. Combining the following facts:

$$\mathrm{var}\left(\bar{y}_t - \bar{y}_{t-1}\right) = \mathrm{var}\left(\alpha_{t-1} \cdot \hat{y}_{t-1}\right),$$

$$\frac{\mathrm{var}\left(\alpha_{t-1} \cdot \hat{y}_{t-1}\right)}{\mathrm{var}\left(\alpha \cdot \hat{y}_{t-1}\right)} \approx 0.34,$$

will yield

$$\mathrm{var}\left(\bar{y}_t - \bar{y}_{t-1}\right) = 0.34 \cdot \alpha^2 \cdot \mathrm{var}\left(\hat{y}_{t-1}\right),$$

or

$$\alpha = \frac{\sigma^{\Delta\bar{y}}}{\sigma^{\hat{y}}} \cdot \sqrt{\frac{1}{0.34}},$$

where $\sigma^{\Delta\bar{y}} = \sqrt{\mathrm{var}\left(\bar{y}_t - \bar{y}_{t-1}\right)}$,
$\sigma^{\hat{y}} = \sqrt{\mathrm{var}\left(\hat{y}_{t-1}\right)}$.

The value of the parameter α is determined by the required ratio of the standard error of the growth rate of potential output $\sigma^{\Delta\bar{y}}$ to the standard error of the output gap $\sigma^{\hat{y}}$. I will assume that:

$$\frac{\sigma^{\Delta\bar{y}}}{\sigma^{\hat{y}}} = \frac{0.0625\ \%}{5\ \%} = \frac{1}{80},$$

which gives the value for the parameter $\alpha = 0.0216$.

7 Data

I used the data on quarterly nominal GDP (in millions of Euro) for the Euro zone 17 in fixed composition. The dataset begins at the first quarter of 1995 and ends at the fourth quarter of 2012. The data were already working day and seasonally adjusted. GDP deflator was then used to calculate real GDP. The complete datasets can be found at the following internet addresses:

http://sdw.ecb.europa.eu/quickview.do?node=9484571&SERIES_KEY=119.ESA.Q.I6.Y.0000.B1QG00.1000.TTTT.L.U.A

http://sdw.ecb.europa.eu/browseTable.do?ADJUSTMENT=Y&saf8=1&REF_AREA=566&ESA95_ACCOUNT=B1QG00&node=2120780&FREQ=Q&saf3=3&sfl2=3&saf4=1&sfl1=3&saf5=1&saf6=1&sfl3=4&DATASET=0&saf7=1&advFil=y

8 Interpretation of the parameter estimates

Econometric estimates of the parameters, which were not calibrated is $\lambda = 0.9619$, $\varepsilon = 0.0125$, $\sigma^u = 0.0059$. Values of these parameters are reasonable from an economic point of view. High value of the coefficient λ is in line with the common view that output gap is highly persistent. The value of the parameter ε can be interpreted in such a way that output fell below potential by more than 1 % at time $t = t_0$ at the beginning of the economic crisis in the European union as a consequence of the economic crisis in the USA. This negative economic shock was however persistent due to the high value of λ and caused additional falls in the output below potential in subsequent periods by $1.25 \cdot 0.9619^{t-t_0}$ %.

9 Estimating and forecasting potential output

Estimate of the potential output is depicted at the graph 1. It can be seen that the growth rate of the potential output was zero since the beginning of the economic crisis in 2008. Forecasts 10 years ahead also show that potential output will not rise during this long period. Dynamics of the output is also depicted at this graph. It predicts that output will rise quite rapidly, but this will happen only because the diminishing output gap. The model also estimates that output is now approximately 14 % below potential and predicts that output in 2023 will still be approximately 8 % below potential even after the its growth.

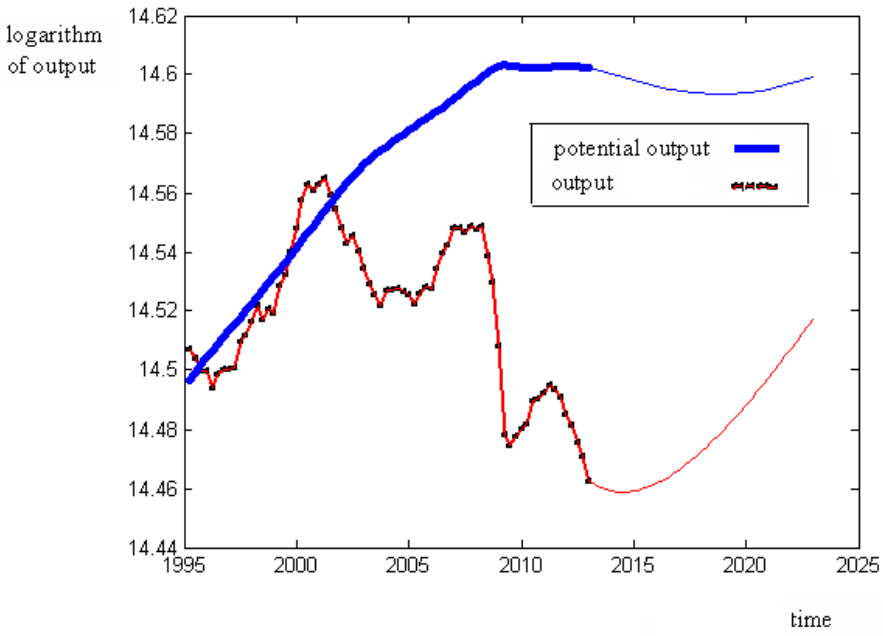

Figure 1 Estimates and forecasts of the (potential) output

10 Conclusion

Economic cycles and long run relations are traditionally studied separately, which is not appropriate in the context of the current economic crisis. In the presented paper, I formulated a model, which links short run fluctuations with long run dynamics within unobserved components methodology. Some of the parameters had to be calibrated, while the others were econometrically estimated by the method of maximum likelihood. Estimation of the parameters was in line with stylized facts. The model was also used to forecast (potential) output of the Eurozone with a forecasting period of 10 years, which showed that potential output will not rise and output will be below potential by 8 % in 2023. The model also predicts that the output in 2023 will still be at a lower level than in pre-crisis periods.

Acknowledgements

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Subsidy Allocation as a Coalition Game

Martin Dlouhý[1]

Abstract. The City of Prague allocates a part of its budget to the city districts in the form of subsidies. The objective of the City of Prague is to find such a set of criteria and their weights that estimate in the best way the expected cost of public administration. The objective of city district is subsidy maximization by influencing the criteria and their weights. The change of allocation rules requires that the city district is able to form a coalition of city districts with majority of votes. The strength of each city districts is evaluated by the number of members of the Prague City Assembly coming from the given city district. The conflict between the city districts is studied as a coalition game. The illustrative calculations for the 2013 subsidy allocation are presented.

Keywords: City of Prague, subsidy allocation, game theory, coalition game.
JEL Classification: H72, C71
AMS Classification: 91A12

1 Introduction

The City of Prague, the capital of the Czech Republic, has an administrative structure of three tiers. The highest tier is the City of Prague itself, represented by the Prague City Assembly and the Magistrate of the City of Prague. At the lowest level, the City of Prague is divided into 57 self-governing city districts. The city districts differ in the size of population and territory. The city districts Prague 1–22 are the so-called "large" city districts with high numbers of inhabitants (from 6 627 to 127 723) and are mostly located in the inner city. The districts Prague 23–57 are the so-called "small" city districts with much lower numbers of inhabitants (from 277 to 10 071) and are mostly located in suburban areas. In total, less than 10% of population live in the city districts Prague 23–57. In 2001, the middle administrative level was introduced. 57 city districts have been grouped into 22 administrative districts for state-administration purposes. One "large" city district in each administrative district has responsibility for providing services of state administration for the entire administrative district. One has to distinguish whether by Prague 4 is meant one of the self-governing city districts Prague 1–57 or one of the administrative districts Prague 1–22.

The City of Prague allocates a part of its budget to the city districts in the form of subsidies. The objective of the City of Prague is to find such a set of criteria and such a set of weights that estimate in the best way the expected cost of public administration. The objective of city district is subsidy maximization by manipulating the criteria and their weights. The decision about the subsidy allocation is made by the Prague City Assembly (63 elected members). During the election period 2010 - 2014, four political parties are represented in the Prague City Assembly: TOP 09 (26 members), ODS (20 members), ČSSD (14 members) and KSČM (3 members). The strength of each city districts is evaluated by the number of members of the Prague City Assembly coming from the given city district. This means that members of the assembly form also coalitions based on the city districts they come from, irrespectively of their political membership and the current governing coalition (TOP 09 and ODS). Not surprisingly, each year the subsidy allocation to city districts is a subject of political struggle among and within the political parties and among the city districts. This process is an interesting topic for the theory of coalition games (Osborne, 2004, Dlouhý, Fiala, 2009).

In brief, we summarize the historical development of the subsidy allocation process for the large city districts Prague 1–22 between years 2001 and 2013. In 2001 and 2002, the city districts received three types of subsidies: 30% of income tax paid by physical persons-entrepreneurs and the so-called "direct subsidy" and "criteria subsidy". The criteria subsidy was based on five criteria: population (50%), the number of pupils at primary schools and kindergartens (35%), the area of urban green (10%), the road maintenance measured as area of roads in square meters (5%). In 2002, the same model as in the year 2001 was used with some minor modifications. In 2003, the special solidarity model was used due to floods in August 2002.

In 2004 - 2006, the index method was used, which was derived from the annual growth of taxes collected by the City of Prague. The minimum average subsidy per inhabitant was introduced in 2004 at the level of 1900 Czech korunas (CZK). This minimum average subsidy was increased to 2000 CZK in 2005 and to 2100 CZK in 2006. In 2007 - 2009, the subsidy allocation processes for the large city districts (Prague 1–22) and the small city districts (Prague 23–57) were separated and varied indicators were used for these two groups of city districts. This separation was used in the following years. The allocation for the large city districts Prague 1–22 was based

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on three criteria: income tax paid by physical persons-entrepreneurs, population and territory. The minimum average subsidy per inhabitant was gradually increased: 2200 CZK in 2007, 2300 CZK in 2008, and 2400 CZK in 2009. The allocation for the small city districts Prague 23–57 was based on six criteria: population (40%), territory (15%), the number of pupils at primary schools and kindergartens (20%), the area urban green (10%), the road maintenance measured as area of roads in square meters (10%), and units of volunteer fire fighters (5%). In year 2010, due to the worsened economic situation in the country, the subsidy for each city districts was indexed by 93.5% of the 2009 subsidy level; however the minimum subsidy per inhabitant 2400 CZK was preserved. In 2011, the 2009 criteria were used again and the minimum average subsidy 2400 CZK per inhabitant was used.

In 2012 - 2013, six allocation criteria were used to calculate subsidies for the large city districts Prague 1–22 (City of Prague, 2011, 2012). The city districts received 30% of income tax paid by physical persons-entrepreneurs (average 2008–2010), which made 9.5% of the total subsidy. The rest of the total subsidy was distributed according to five criteria and their weights: population (30%), territory in square kilometres (10%), the number of pupils at primary schools and kindergartens owned by the city district (30%), the area of urban green served by the city district in hectares (20%), and the road maintenance measured as the area of the roads served by the city district in square meters (10%). It may look as if the income tax has some special position in the allocation. However, it just means that the real weights of criteria in 2013 were as follows: income tax paid by physical persons–entrepreneurs (6.8%), population (28.0%), territory in square kilometres (9.3%), the number of pupils and students at the schools owned by the city district in 2011 (28.0%), the area of urban green served by the city district (18.6%), and the road maintenance measured as the area of the roads served by the city district in square meters (9.3%). For the small city districts Prague 23–57, a different six-criterion subsidy allocation model (see criteria and weights in period 2007-2009). The minimum average subsidy 2400 CZK per inhabitant was again used. Moreover, two city districts received additional direct subsidy for public dumps.

2 Data

In this study, we will focus on the large city districts Prague 1–22 that are responsible for both the self-governing public administration and state administration. The input data on six criteria used for the 2013 subsidy allocation among city districts Prague 1–22 are presented in Table 1 (City of Prague, 2012).

District	Population	Territory (km^2)	Pupils and Students	Urban Green (ha)	Roads (m^2)	Income Tax
Prague 1	29 857	5.51	2 977	43.0	13 413	66 901
Prague 2	49 237	4.18	4 367	58.5	8 862	56 863
Prague 3	71 140	6.48	5 272	68.7	34 289	39 433
Prague 4	127 723	24.20	11 447	248.6	261 144	87 048
Prague 5	81 161	27.50	6 366	87.0	204 290	58 129
Prague 6	97 951	41.56	9 694	209.0	501 522	99 305
Prague 7	41 416	7.12	3 037	7.0	30 630	23 826
Prague 8	103 757	21.80	9 091	207.0	148 652	58 036
Prague 9	53 382	13.31	3 486	120.0	140 641	17 062
Prague 10	108 998	18.60	7 641	145.0	208 085	50 748
Prague 11	77 170	9.79	7 316	227.4	313 640	23 021
Prague 12	54 426	23.32	5 191	136.0	633 960	18 442
Prague 13	59 874	13.23	6 781	166.4	502 047	23 057
Prague 14	45 822	13.53	4 189	91.4	400 229	9 825
Prague 15	30 954	10.25	2 805	53.9	168 983	10 237
Prague 16	8 141	9.30	935	14.2	253 430	4 999
Prague 17	24 973	3.25	2 174	38.9	136 475	7 106
Prague 18	17 323	5.61	1 739	24.0	201 770	1 934
Prague 19	6 627	6.00	763	22.6	137 619	1 988
Prague 20	15 028	16.94	2 094	46.6	663 611	7 898
Prague 21	10 334	10.15	1 144	33.5	313 008	6 562
Prague 22	9 025	15.62	1 159	20.4	256 425	4 035

Table 1: Input Data 2013, Prague 1 – Prague 22

3 Methods

Let us have m city districts and n allocation criteria. In this case, the values are m=22 and n=6. The real values for city district i in each criteria j are y_{ij}. The relative proportion of city district i in dimension j on the total sum for all city districts is denoted r_{ij}, and is calculated as

$$r_{ij} = y_{ij} / \sum_{i=1}^{m} y_{ij} . \tag{1}$$

The total subsidy is denoted B. The subsidy to city district i is denoted s_i. Let us further suppose that each criterion will have the minimum weight at least 0.05. The maximum value of weight is (1-0.05(n-1)). The best case scenario, an upper limit of the subsidy for the ith city district u_i, is that subsidy is calculated as

$$u_i = ((1-0.05n)\max_j r_{ij} + 0.05\sum_{j=1}^{n} r_{ij})B. \tag{2}$$

On the other hand, the worst case scenario, a lower limit of the subsidy l_i, is calculated as

$$l_i = ((1-0.05n)\min_j r_{ij} + 0.05\sum_{j=1}^{n} r_{ij})B. \tag{3}$$

It is certainly good if the city district knows what is upper subsidy limit and optimal weights, however the city district needs to persuade a sufficient number of members of the Prague City Assembly for adoption of its proposal. We assume that the members from other city districts will agree only if the proposed subsidy allocation will also be profitable for the city districts they come from. This is a basic assumption about the players to cooperate in coalition games (Dlouhý, Fiala, 2009). In this case a coalition of city districts C with minimally 32 out of 63 votes in the Prague City Assembly has to be formed. The subsidy from the proposed allocation s_i has to be higher than the existing subsidy allocation s^0_i for the coalition members:

$$s_i \geq s_i^0, \quad i \in C. \tag{4}$$

If the allocation related to the upper limit u_q is profitable also for other city districts with majority of votes, the proposal will adopted. It is however likely that such proposal will not achieve the majority of votes. In such case, the city district has to look for the best available compromise. City district q can achieve the maximum level of subsidy s_q by changing the weights and at the same time having coalition with minimally 32 votes, which can be obtained as a solution of the following program:

$$\begin{aligned}
&\max s_q = \sum_{j=1}^{n} v_{qj} r_{qj} \\
&\text{subject to} \\
&0.05 \leq v_{qj} \qquad \text{for } j = 1, 2, \ldots 6, \\
&\sum_{j=1}^{n} v_{qj} = 1, \\
&\sum_{j=1}^{n} v_{qj} r_{ij} \leq s_i^0 - Mh_i \quad \text{for } i = 1, 2, \ldots, 22, \\
&\sum_{i=1}^{m} w_i h_i \geq 32,
\end{aligned} \tag{5}$$

where h_i is a binary variable representing the vote of each city district, w_i is number of members of the Prague City Assembly from city district i, M is sufficiently high constant, and v_{qj} are optimal weights for the city district q. This program has to have a feasible solution, because the existing subsidy allocation with weights (0.280, 0.093, 0.280, 0.186, 0.093, 0.068), which represent the case of the grand coalition, is included in the feasible set.

Alternatively, we can formulate a model which assumes that the condition of 32 votes is necessary for adoption of the subsidy allocation proposal, however this number of votes has to be achieved by the governing coalition TOP 09 and ODS, not by a random group of members of the Prague City Assembly. The city district i

is a member of coalition *C* if there is at least one member of the Prague City Assembly from the governing coalition (TOP 09 and ODS) and comes the given city district. The city district *i* is not a member of coalition *C* if no member of the Prague City Assembly from the given district is member of the governing coalition or if the given city district is not represented in the Prague City Assembly at all (as of March 2013). The allocation model can be formulated as follows:

$$\begin{aligned}
&\max \sum_{i \in C} s_i = \sum_{i \in C} \sum_{j=1}^{n} v_j r_{ij} \\
&\text{subject to} \\
&0.05 \le v_j \qquad \text{for } j = 1, 2, \ldots, 6; \\
&\sum_{j=1}^{n} v_j = 1; \\
&\sum_{j=1}^{n} v_j r_{ij} \le s_i^0 - M h_i \quad \text{for } i = 1, 2, \ldots, 22; \\
&\sum_{i=1}^{m} w_i h_i \ge 32.
\end{aligned} \tag{6}$$

The similar problem of dual structure can be observed in other situations. For example, Turnovec, Mercik, and Mazurkiewicz (2008) studied the case of the European Parliament. European Parliament has a dual structure because its members represent their own countries and at the same time they are clustered in European political parties, forming clubs in the Parliament.

The performance of the city district in the allocation process can be measured by the ratio of achieved subsidy and existing value of subsidy:

$$s_i / s_i^0 \times 100, \tag{7}$$

or by formula (8), which takes into account that the city district can benefit, but it can also loose, from any new allocation rules:

$$\frac{s_i - l_i}{u_i - l_i} \times 100. \tag{8}$$

4 An Illustrative Example

Firstly, we calculate the lower and upper subsidy limits u_i and l_i with formulas (2) and (3). The difference between the lower and upper limits shows how sensitive the district subsidy is to a change in the weights (Table 2). The last column in Table 2 shows the allocation criterion that is preferred by the given district, i.e. the criterion with the weight that equals to 0.75. Third criterion, the number of pupils and students, is not preferred by any city district. There are six extreme combinations of weights that can be theoretically optimal for the city districts: one of six weights with a value 0.75 and the other five weights with values 0.05. We denote them as *Proposal 1* to *Proposal 6*. *Proposal 1* has weights (0.75, 0.25, 0.25, 0.25, 0.25, 0.25), *Proposal 2* has weights (0.25, 0.75, 0.25, 0.25, 0.25, 0.25), etc.

Secondly, we investigate above mentioned six subsidy allocation proposals by checking the total number of votes for each proposal (Table 3). Two proposals (proposals 3 and 4) can achieve the majority of votes in the Prague City Assembly. As an illustrative example, the subsidy allocation related to *Proposal 3* and its impact on city districts is presented in Table 4. The relative performance of the city district is measured by performance criteria suggested in formulas (7) and (8).

District	s^0_i	l_i	u_i	$u_i - l_i$	Criterion
Prague 1	2.86	1.15	7.91	6.75	6
Prague 2	3.71	1.19	6.96	5.77	6
Prague 3	4.52	1.61	5.60	4.00	1
Prague 4	10.69	6.32	12.02	5.70	6
Prague 5	6.36	4.54	8.22	3.68	2
Prague 6	10.16	9.39	13.57	4.18	6
Prague 7	2.46	0.91	3.25	2.34	1
Prague 8	8.49	4.22	9.34	5.12	4
Prague 9	4.19	2.94	5.23	2.29	4
Prague 10	7.58	4.72	8.87	4.15	1
Prague 11	7.06	4.10	9.57	5.46	4
Prague 12	5.98	3.83	9.94	6.11	5
Prague 13	6.35	4.23	8.20	3.97	5
Prague 14	4.31	2.31	6.35	4.05	5
Prague 15	2.73	1.86	3.14	1.28	2
Prague 16	1.35	1.02	3.74	2.73	5
Prague 17	1.98	1.28	2.27	0.99	5
Prague 18	1.66	0.71	3.06	2.35	5
Prague 19	1.01	0.56	2.10	1.54	5
Prague 20	3.08	2.04	9.61	7.58	5
Prague 21	1.78	1.32	4.64	3.32	5
Prague 22	1.67	1.08	4.22	3.14	2

Table 2: Real, Minimum and Maximum Subsidies as a Percentage of Total Budget

District	Proposal 1	Proposal 2	Proposal 3	Proposal 4	Proposal 5	Proposal 6
Prague 1	0	0	4	0	0	4
Prague 2	3	0	3	0	0	3
Prague 3	6	0	6	0	0	6
Prague 4	5	0	5	5	0	5
Prague 5	2	2	2	0	0	2
Prague 6	0	8	0	8	0	8
Prague 7	1	0	1	0	0	1
Prague 8	6	0	6	6	0	0
Prague 9	1	1	0	1	0	0
Prague 10	7	0	0	0	0	0
Prague 11	0	0	0	2	0	0
Prague 12	0	4	0	4	4	0
Prague 13	0	0	5	5	5	0
Prague 14	0	0	0	0	0	0
Prague 15	0	4	4	0	4	0
Prague 16	0	0	0	0	0	0
Prague 17	0	0	0	0	0	0
Prague 18	0	1	1	0	1	0
Prague 19	0	1	0	1	1	0
Prague 20	0	1	0	0	1	0
Prague 21	0	2	0	2	2	0
Prague 22	0	0	0	0	0	0
Votes	31	24	37	34	18	29

Table 3: Number of Votes for Proposals by City District

District	Real value s^0_i	Proposal 3 s_i	Performance (formula (7))	Performance (formula (8))
Prague 1	2.86	3.07	107.4	28.5
Prague 2	3.71	4.14	111.6	51.2
Prague 3	4.52	4.88	107.8	81.8
Prague 4	10.69	11.06	103.5	83.0
Prague 5	6.36	6.42	101.0	51.3
Prague 6	10.16	10.10	99.4	17.0
Prague 7	2.46	2.81	114.2	81.0
Prague 8	8.49	8.72	102.7	87.9
Prague 9	4.19	3.62	86.4	29.8
Prague 10	7.58	7.45	98.2	65.8
Prague 11	7.06	7.01	99.3	53.2
Prague 12	5.98	5.57	93.1	28.4
Prague 13	6.35	6.61	104.0	59.9
Prague 14	4.31	4.23	98.2	47.6
Prague 15	2.73	2.77	101.4	71.4
Prague 16	1.35	1.19	88.3	6.5
Prague 17	1.98	2.07	104.7	79.8
Prague 18	1.66	1.73	104.5	43.4
Prague 19	1.01	0.89	88.4	21.5
Prague 20	3.08	2.69	87.2	8.6
Prague 21	1.78	1.48	83.5	4.8
Prague 22	1.67	1.48	88.2	12.6

Table 4: Impact of Proposal 3 on Subsidy Allocation

5 Conclusion

We have described the subsidy allocation that is used by the City of Prague in 2013. We have shown that the allocation problem can be seen as a coalition game with a dual structure (city districts and political parties). In the analysis, we do not consider the possibilities that some criterion is abolished (its weight is zero) or that some new criterion was introduced. All the calculations we have made have to be seen as illustrative examples. We are not able to offer any definitive answers about which subsidy allocation model is the best one; however an analysis of the allocation process can help us in understanding the nature of the allocation problem, its advantages and disadvantages (Dlouhý, 2013).

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RAROI or IRR?

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Abstract. In this study we deal with estimating profitability of consumer finance loans on a single loan level. We compare two different theoretical approaches to measuring profitability: RAROI (Risk Adjusted Return on Investment) and IRR (Internal Rate of Return). We discuss the usage of these two approaches in portfolio management and we particularly focus on the conditions under which one of the measures prevails over the other in case of profit maximization with constrained or unconstrained capital.

Keywords: Return on Investment, Internal Rate of Return, Consumer finance, Profitability.

JEL classification: G23
AMS classification: 90A09

1 Introduction

Assume we are in the position of a financial institution providing loans to its customers. Our aim is to estimate profitability of a signle loan at the date it enters the approval proces in order to be able to decide whether it is worth of approval. Profitability stands for some measure of net profit that is made on a given exposure. In reality there are few ways how to measure this profit. For example we can take into account time value of money or not, it can be expressed as a percentage of the original amount, as an absolute value or as a profitability rate p.a. We will deal with two approaches:

- RAROI (Risk Adjusted Return on Investment), which stands for the sum of discounted cash flows of the investment divided by the original principal
- IRR (Internal Rate of Return), which stands for the interest rate p.a. that equals the rate we would have to have on a risk-free investment in order to gain the same profit in the same period. For further details on IRR see the articles [4] and [1].

We will assume 90 days default on an exposure level, i.e. application is said to be in default, when it is or ever was at least 90 days past due. This also means, that 90 days default is an absorbing state. As the date of default we will consider the date on which the exposure is 90 days past due the first time. Moreover we assume the German standard 30E/360 for computing number of days between distinct payments and interest being imposed monthly. As a discount rate we use the cost of equity e p.a. as it expresses the best alternative investment. We assume only annuity exposures.

To be formaly correct, we also assume the set of exposures[3] $\mathcal{K}$ and we denote the cardinality of this set K and introduce a bijective (one-to-one) mapping ρ from the set $\mathcal{K}$ to the set of integers $\tilde{K} = \{1, \ldots, K\}$. Each particular exposure κ from the set $\mathcal{K}$ we will refer to as to the integer $k := \rho(\kappa)$ form the set $\tilde{K}$.

We assume that the credit demand in a period Δ is given by the number n of clients applying for a loan during the period Δ. This number of applicants is exogeneous for the risk management. It can be at most boosted by marketing or sales actions, but not by risk department. We wish to compare RAROI with IRR under four different conditions. These conditions reflect four distinct states:

1. We wish to maximize short-term profit while we have a given set of customers, bounded capital and no possibility to reinvest the repaid money.

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[3]We will speak of exposures or loans. These two notions are freely substitutable throughout this text.

2. We wish to maximize short-term profit while we have a given set of customers, unbounded capital and no possibility to reinvest the repaid money.
3. We wish to maximize long-term profit while we have bounded capital and we can always reinvest the repaid money.
4. We wish to maximize long-term profit while we have unbounded capital and we can always reinvest the repaid money.

We wish to find out, which of the two proposed measures of profitability are better in which of these situations.

2 Cash flows

Assume the loan k is an annuity with term T_k months, where $T_k \in \mathbb{N}$. Assume in addition we have estimated the probabilities of default on each monthly payment for the loan k. We denote the probability of default on the m-th payment[4] $P_k(m)$. In addition we assume i_k is the interest rate p.a. imposed on the loan k. The original nominal value (original principal) of the loan k we will denote $X_k(0)$. There can be a fee paid monthly which is not part of the annuity. We denote this monthly fee of the loan k f_k.

2.1 Annuity payment

The present value of all the annuity payments discounted by the interest rate i_k (when the payments are performed monthly) must sum up to the nominal value of the loan, i.e.

$$S_k \left(\sum_{t=1}^{T_k} \left(\frac{1}{1+\frac{i_k}{12}} \right)^t \right) = X_k(0), \tag{1}$$

where S_k stands for the annuity payment of the loan k. From (1) we easily derive the equation for annuity payment:

$$S_k = \frac{X_k(0)\,\frac{i_k}{12}}{1-\left(\frac{1}{1+\frac{i_k}{12}}\right)^{T_k}}. \tag{2}$$

2.2 Outstanding balance

We will denote $X_k(t)$ the outstanding balance of the loan k after t-th payment was paid (when repaid on time). It is easy to derive the formula for the computation of $X_k(t)$ as the recursive equality 3 must hold for any $s \in \{0, \ldots, T_{k-1}\}$.

$$X_k(s)\left(1+\frac{i_k}{12}\right) - S_k = X_k(s+1) \tag{3}$$

The equation derived for the outstanding balance is thus

$$X_k(t) = X_k(0)\left(\frac{1-\left(1+\frac{i_k}{12}\right)^{t-T}}{1-\left(1+\frac{i_k}{12}\right)^{-T}}\right). \tag{4}$$

2.3 Cost of money

We assumee that we repay the borrowed loan as a revolving with the same interest rate ϕ p.a. This revolving is set so as to pay just the principal that is to be repaid by the client in each month according to the instalment schedule. Moreover at the date of 90 days default we repay the whole remaining

[4]Probability that the m-th ($m > 1$) payment will not be repaid before 90 days past due while the $(m-1)$-th payment will. For $m = 1$ it is just the probability that no payment will be repaid before 90 days past due.

principal and so the interest after default does not have to be paid. We repay the non-equity $(1 - EQ)$ part of the borrowed loan principal provided by a third party as a revolving payments with interest rate ϕ p.a. The t-th payment equals the t-th scheduled instalment of the loan k, except for the case of default. So the cost of money is the interest paid on this loan before the default occurs

$$NM_k^t = (1 - EQ)\left(1 - \sum_{j=1}^{t} P_k(j)\right)\left(\frac{\phi}{12} \cdot X_k(t-1)\right) + 3(1 - EQ)\, P_k(t)\left(\frac{\phi}{12} \cdot X_k(t-1)\right). \tag{5}$$

2.4 Loss

The expected loss EL_k^t from the period between the scheduled date of $(t-1)$-th and the scheduled date of the t-th payment[5] (including this date) can be computed for the exposure k when the loan is financed through revolving as

$$EL_k^t = P_k(t)\, X_k(t-1). \tag{6}$$

If it is financed through an annuity, the expected loss can be computed as

$$EL_k^t = P_k(t)\, X_k(t-1). \tag{7}$$

2.5 Interest revenue

The interest revenue RE_k from the period between the scheduled date of $(t-1)$-th and the scheduled date of the t-th payment[6] including this date can be computed for the exposure k simply as

$$RE_k^t = \left(1 - \sum_{j=1}^{t} P_k(j)\right) X_k(t-1)\, \frac{i_k}{12}. \tag{8}$$

2.6 Recoveries

For the loan k a recovery rate $r_k(t)$ for a loan defaulted on t-th payment is estimated based on the loan's characteristics. The recovery RC_k^t from exposure defaulted on the t-th payment can be computed for the exposure k simply as

$$RC_k^t = P_k(t)\, \frac{X_k(t-1)\, r_k(t)}{\left(1 + \frac{e}{12}\right)^3}. \tag{9}$$

The recovery rate $r_k(t)$ stands for all the recovered payments after the date of default, no matter whether they are allocated as a principal payment, penalty payment, interest payment or late fee payment. The estimation is always made using the payment number on which the default occured, but it can generaly be estimated using other loan or client characteristics.

2.7 Monthly fees

The loan fees are to be paid monthly together with the annuity payment. The total expected fee paid on exposure k between the scheduled $t-1$-th and t-th payment we denote FE_k^t and it can be computed simply as

$$FE_k^t = \left(1 - \sum_{j=1}^{t} P_k(j)\right) f_k. \tag{10}$$

2.8 Commissions

Let's denote c_k the commissions paid to dealers for the acquisition of the loan k.

[5] For $t = 1$ it is the period before the scheduled date of first payment.
[6] For $t = 1$ it is the period before the scheduled date of first payment.

2.9 Variable operational cost

The variable operational cost should be estimated based on financial data. It can vary accross product types, clients' risk profiles and other characteristics. The variable operational costs (OPEXes) can be generaly divided in at least three different types of costs: origination, servicing and termination. We will not account for termination costs, as the regular termination is more or less costless and the non-regular termination we deal separately as a cost of collections. To be rigorous, we should also include the cost of pre-termination, which is usualy not negligible, but for this, we would have to also estimate the probabilities of pre-termination, which would make the computation too complicated and the value added would be very small. So we denote $v1_k$ the variable operational cost of origination and $v2_k^t$ the monthly variable operational cost of servicing for the month between the scheduled date of $(t-1)$-th and the scheduled date of the t-th payment[7] including this date. The origination cost should depend on the rejection rate, as there are variable costs rather on each application, than just on each contract. The higher is rejection rate, the higher should be the variable costs of origination on the given loan.

2.10 Cost of equity

The cost of equity stands for the interest payments on shareholders' funds which is used in order to provide the given loan. We denote e the cost of equity expressed as a percentage p.a. The sum of equity costs over the life of the loan can be computed as

$$EQ_k^t = EQ\left(1 - \sum_{j=1}^{t} P_k\left(j\right)\right) X_k\left(t-1\right)\frac{e}{12}, \tag{11}$$

where EQ stands for the equity to productive assets ratio. It can be considered to be a constant.

2.11 Costs of collections

The costs of collections should be estimated based on financial data. It can vary accross product types and clients' risk profiles. We denote g_k the costs of collections. The expected total cost of collections is thus

$$CC_k^t = P_k\left(t\right)\frac{g_k}{\left(1+\frac{e}{12}\right)^3} \tag{12}$$

2.12 Insurance

Assume, there is a life insurance and an unemployment insurance offered to the customer. The insurance fee is treated as part of the loan, i.e. it highers the principal. Denote the fee for life insurance considering the exposure k IL_k and the unemployment insurance IU_k. Assume, that the commissions from the insurance are c_{IL} for the life insurance and c_{IU} for the unemployment insurance. When we denote CA the original credit amount that the client would apply for if there was no insurance, then we get $X_k\left(0\right) = CA + c_{IL}IL_k + c_{IU}IU_k$.

2.13 Gross margin

The expected gross margin of loan k from period between the scheduled date of $(t-1)$-th and the scheduled date of the t-th payment[8] including this date we denote GM_k^t and it can be computed as

$$GM_k^t = RE_k^t + RC_k^t + FE_k^t - EL_k^t. \tag{13}$$

[7] For $t = 1$ it is the month before the scheduled date of first payment.
[8] For $t = 1$ it is the period before the scheduled date of first payment.

3 RAROI estimation

RAROI stands for the sum of discounted cash inflows minus the sum of discounted cash outflows divided by the original investment (i.e. the original principal borrowed). Therefore it can be expresed as a percentage of the original principal.

The risk adjusted return on investment $RAROI_k$ for loan k can be computed as

$$RAROI_k = \frac{\sum_{t=1}^{T_k} \frac{GM_k^t - EQ_k^t - NM_k^t - CC_k^t - v2_k^t}{\left(1+\frac{e}{12}\right)^t} + c_{IL} IL_k + c_{IU} IU_k - v1_k - c_k}{X_k(0)} - 1. \tag{14}$$

It is important to note, that this RAROI already contains the cost of equity and so whenever it is positive the loan brings higher interest rate on equity than e p.a. When $RAROI < 0$, it can still be profitable for shareholder, but it is not profitable for the company assuming the cost of capital from shareholder is e p.a.

RAROI can also be considered as profitability index, which stands for profit divided by the original investment.

4 IRR estimation

Estimating IRR requires the same inputs as estimation of RAROI, but the concept of the final profitability measure is quite different. Instead of dividing the cash flows by the original principal, we compute the interest rate which corresponds to the expected cash flows from the investments. Then IRR is the solution for r to the equation $\Pi(r) = 0$, where $\Pi(r)$ is given by

$$\Pi(r) = \sum_{t=1}^{T_k} \frac{GM_k^t - EQ_k^t - NM_k^t - CC_k^t - v2_k^t}{\left(1+\frac{r}{12}\right)^t} - X_k(0) + c_{IL} IL_k + c_{IU} IU_k - v1_k - c_k \tag{15}$$

Finding a solution to the equation $\Pi(r) = 0$ is based on iterative numerical approaches as there is generally no analytical solution to express a root of high degree polynoma. Newton method can be used as well as the method of half intervals. In our case there is just one positive real solution, see [2].

IRR assumes each repaid money can immediately be reinvested for the same rate of return. If this is not true, modified IRR (MIRR) is proposed, see for example [3]. MIRR assumes that positive cash-flows can be reinvested only at the rate of cost of equity e. We can then rewrite the equation for IRR as follows:

$$MIRR = \sqrt[T]{\frac{\sum_{t=1}^{T_k} PCF_k(t)\left(1+\frac{e}{12}\right)^{T-t}}{\sum_{t=1}^{T_k} NCF_k(t)\left(1+\frac{e}{12}\right)^{-t}}} - 1. \tag{16}$$

5 Profitability maximization

Now we show, in which of the four situations described above prevails RAROI and in which prevails IRR or MIRR. Scoring via RAROI is based on rejecting applications with RAROI below some cut-off and scoring via IRR on rejecting applications with IRR below some cut-off.

It can be shown that

$$IRR_k(T_k, RAROI_k) = 12\left(\sqrt[T_k]{RAROI_k\left(1+\frac{e}{12}\right)^{T_k} + 1}\right) - 1. \tag{17}$$

This can be rewritten for $RAROI_k$ as

$$RAROI_k(T_k, IRR_k) = \frac{\left(1+\frac{IRR_k}{12}\right)^{T_k} - 1}{\left(1+\frac{e}{12}\right)^{T_k}}. \tag{18}$$

From (18) we know, that $RAROI_k$ is increasing in the term T_k while IRR_k is fixed and it is also increasing in IRR_k while term T_k is fixed. We can conclude, that IRR is an interest rate p.a. and so it does not directly depend on the term, while RAROI depends on term and is higher for higher terms.

If we are in the first situation of short-term profit maximization and bounded capital it is optimal strategy to score the applicants via RAROI. We do not care about the terms of the provided loans as we have no chance to reinvest the repaid money. Moreover because the capital is bounded, we will approve only $\tilde{k}$ applications with the highest RAROI such that $\forall k > \tilde{k} \sum_{i=1}^{k} X_i(0) > K$, and $\sum_{i=1}^{\tilde{k}} X_i(0) \leq K$, where K is the available capital and X_i is sorted descendingly by $RAROI_i$. The only difference from the second situation is in the cut-off. If we maximize short-term profit, have no chance to reinvest the repaid money, but the capital is unbounded, we will approve all the loans with positive RAROI.

The third situation stands for long-term profit maximization with bounded capital, where we always reinvest the repaid money by providing another loan. In this case, we need to take into account terms, as long term with a given RAROI must be less profitable than short term with the same RAROI, as we make the same money faster and so can faster reinvest them. This is the reason why IRR prevails over RAROI. IRR enables to compare profitability of two loans with different terms, while RAROI does not. Because we have a bounded capital, we set the cut-off such that we approve only $\hat{k}$ applications with the highest IRR such that $\forall k > \hat{k} \sum_{i=1}^{k} X_i(0) > K$, and $\sum_{i=1}^{\hat{k}} X_i(0) \leq K$, where K is the available capital and X_i is sorted descendingly by IRR_i. Again the difference from the fourth situation is just in the cut-off. If we maximize long-term profit, can reinvest the repaid money freely by providing another loan, but the capital is unbounded, we will approve all the loans with positive IRR.

6 Discussion and conclusion

We have provided two distinct ways of measuring profitability of a single loan, including the cost of capital (cost of money representing the debt (emitted bonds, loans provided by third parties, etc.) and cost of equity representing the cost of shareholders' capital (equity)). The first measure, Risk Adjusted Return On Investment was shown to be a measure better reflecting short-term profit maximization in an environment without possibility of reinvesting in the same asset. The second measure, Internal Rate of Return was shown to be a measure better reflecting long-term profit maximization in an environment with possibility of reinvesting in the same asset. The provided results would deserve exact derivation, but for this, there is not enough space in this short text. We have also shown a modified version of Internal Rate of Return, which could be used in the environemnt where it is possible to reinvest the repaid money in the same asset, but it does not necessarily have to be without a waiting period during which the repaid money can not be reinvested in similar loans. This situation would account for relatively low demand for loans.

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On finite single-server queue subject to non-preemptive breakdowns

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Abstract. The paper deals with modelling and simulation of a finite single-server queueing system with a server subject to breakdowns. We consider that customers come to the queueing system in the Poisson stream. Customers incoming to the system are served according to the FCFS discipline, service times are considered to follow the Erlang distribution defined by the shape parameter and the scale parameter. Customers can wait for the service in the queue which length is limited by (m-1) places, that means the total capacity of the queueing system is equal to m places. Further, we assume that the server can break down. Breakdowns of the server are considered to be non-preemptive that means when a breakdown occurs during customer servicing it is possible to finish it before server repair is started. Times between breakdowns and repair times are assumed to follow the exponential distribution. We model the queue as a quasi-birth death process for which we present steady-state diagram and equation system describing the system behaviour in the steady-state. Solving the equation system in Matlab we get stationary probabilities which are used for computing basic performance measures. The mathematical model is supported by a simulation model in order to validate the outcomes of the mathematical model.

Keywords: $M/E_n/1/m$, queue, method of stages, non-preemptive failures

JEL Classification: C44
AMS Classification: 60K25

1 Introduction

Queuing theory enables us to study the behaviour of such called queueing systems. In general, each queueing system represents a system which serves customers coming in the system. Such types of systems can be met in many sectors, for example in informatics, telecommunications, transport and, of course, economics.

One of the possible ways how queueing system can be sorted it is sorting according to the fact whether the model of the system has incorporated server failures. Failures of the server have an obvious impact on the performance measures of the studied queueing system. Therefore, in many cases it is necessary to incorporate them into the model in order to get relevant outcomes.

Queueing systems subject to server failures, in which inter-arrival or service times are considered to be Erlang distributed, have already been studied in the past. But in comparison with queues assuming exponentially or general distributed inter-arrival and service times the models of queueing systems under assumption of the Erlang distribution are not so common.

Some authors presented queueing models subject to failures under the assumption of the Erlang distribution which was most often applied to model service times. An $M/E_k/1$ queue with server vacation was studied by Jain and Agrawal [2]. The authors assume that the server takes vacations when the system is empty and when it is busy than can break down. It is further considered that the Poisson arrival rate is state dependent. An $M/E_k/1$ queueing system under similar assumptions but with so called multi optional repair was presented in paper [3]. Wang and Kuo [5] solved a finite source $M/E_k/1$- a group of identical machines which is operated under the care of an unreliable service station (so called machine repair problem). The authors employed matrix geometric method to derive the steady-state probabilities and developed the steady-state profit function to find out the optimum number of machines which are repaired by the server. In paper [4] an $M^X/E_k/1$ two-phase queueing system with a single removable server and with gating, server start-up and unpredictable breakdowns is considered.

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In paper [1] a finite queueing system with Erlang distributed inter-arrival and service times and with the service station subject to disastrous failures was studied.

The paper is devoted to a modification of a finite single-server queueing system with a server subject to breakdowns which are considered to be non-preemptive. The paper is organized as follows. In Section 2 we will make all necessary assumptions and the mathematical model is presented. In Section 3 we will introduce a simulation model which was created to validate the mathematical model. In Section 4 outcomes of executed experiments are shown.

2 Mathematical model

In this section we will present a mathematical model of a finite queueing system with a server subject to non-preemptive breakdowns. The capacity of the queueing system is equal to m, where $m \geq 2$ - a place in the service and (m-1) places in the queue. Let the arrival process of customers be the Poisson process with the parameter λ. For the Poisson arrival process it holds that customer inter-arrival times are exponentially distributed with the same parameter λ. Costumer service times are supposed to be an Erlang random variable with the shape parameter $n \geq 2$ and the scale parameter $n\mu$. The Erlang distribution with the parameters n and $n\mu$ arises from n mutually independent exponential distribution with the same parameter equal to $n\mu$; thus the mean service time is equal to $\frac{n}{n\mu} = \frac{1}{\mu}$. This property of the Erlang distribution is exploited by method of stages which models Erlang distributed service times as n exponentially distributed service times. Each customer successively goes through n individual phases of service and leaves the system only when the last phase of service has been finished.

Let us assume that server failures also occur according to the Poisson input process with the parameter η. If there already is a failure in the system, then the parameter η is equal to zero. Times to repair are exponentially distributed with the parameter ζ. In the case that a failure arises when a customer is being served, the service of customer is not interrupted (the service continues without any influence of the failure on the server) and after finishing it we begin to repair the server - we can say that server failures are nonpreemptive. Such type of failures can be also interpreted as a special case of server vacations, where the server takes vacation randomly with no dependency on the number of customers finding in the system.

Let us have three discrete random variables denoted as K, P and F. The variable K describes the number of customers finding in the system, where the variable can take the values from the set {0,1,...,m}. The variable P expresses the phase of service which has already been finished, where the variable can take the values from the set {0,1,...,n-1}. The last variable F can take 3 values from the set {0,1,2}, where the meaning of it is as follows:

- If the variable F is equal to 0, there is no failure in the system.
- If the variable F is equal to 1, the failure of the server is waiting for repair because we have to finish servicing of the customer being in the service.
- If the variable F is equal to 2, the failure of the server is being repaired.

It is obvious that the individual states of the system can be described by triplets (k,p,f). The state space of the system is the union of three states subsets:

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3,$$

where:

- The states in the subset $\Omega_1 = \{(k, p, f): k \in \{1,\ldots,m\}, p \in \{0,\ldots,n-1\}, f = 0\}$ are the states in which there are k customers in the system and the p phases of customer service have already been finished and there is no failure in the system. To these states we have to add the state (0,0,0) which corresponds to the state in which the system is idle (and empty).
- The states in the subset $\Omega_2 = \{(k, p, f), k \in \{1,\ldots,m\}, p \in \{0,\ldots,n-1\}, f = 1\}$ are the states in which there are k customers in the system and the p phases of customer service have already been finished and there is a failure of the server in the system waiting for repair.
- The states in the subset $\Omega_3 = \{(k, p, f), k \in \{0,\ldots,m-1\}, p = 0, f = 2\}$ are the states in which there are k waiting customers in the system and the server is being repaired.

To simplify the formation of equations describing the system state probabilities we created a state transition diagram which depicts transitions between individual system states – see Figure 1. Please notice that vertices represent the states of the system and oriented edges indicate the possible transitions with the corresponding rate.

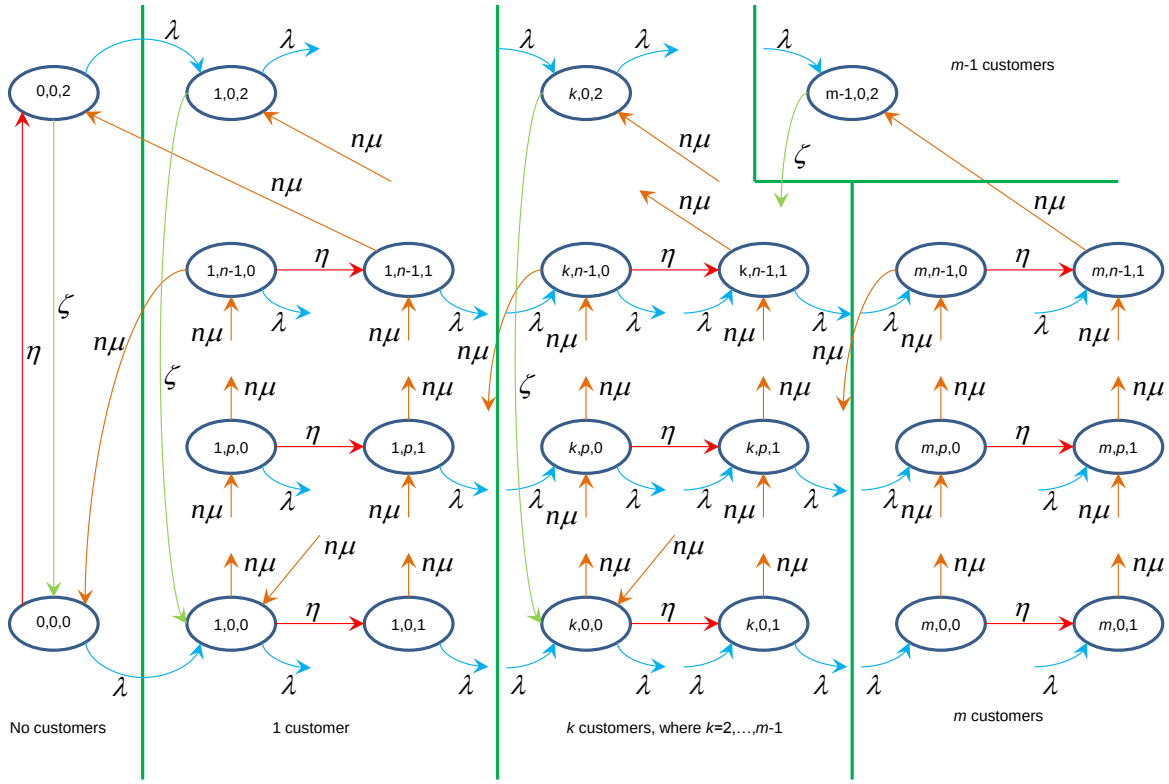

Figure 1 The state transition diagram of the system.

No we are ready to write equations for individual state probabilities. Applying generally known global balance principle we get the following linear equation system which describes the behaviour of the process in steady state:

$$(\lambda+\eta)\cdot P_{(0,0,0)} = n\mu\cdot P_{(1,n-1,0)} + \zeta\cdot P_{(0,0,2)}, \tag{1}$$

$$(\lambda+n\mu+\eta)\cdot P_{(k,0,0)} = \lambda\cdot P_{(k-1,0,0)} + n\mu\cdot P_{(k+1,n-1,0)} + \zeta\cdot P_{(k,0,2)} \text{ for } k=1,\ldots,m-1, \tag{2}$$

$$(n\mu+\eta)\cdot P_{(m,0,0)} = \lambda\cdot P_{(m-1,0,0)}, \tag{3}$$

$$(\lambda+n\mu)\cdot P_{(1,0,1)} = \eta\cdot P_{(1,0,0)}, \tag{4}$$

$$(\lambda+n\mu)\cdot P_{(k,0,1)} = \eta\cdot P_{(k,0,0)} + \lambda\cdot P_{(k-1,0,1)} \text{ for } k=2,\ldots,m-1, \tag{5}$$

$$n\mu\cdot P_{(m,0,1)} = \eta\cdot P_{(m,0,0)} + \lambda\cdot P_{(m-1,0,1)}, \tag{6}$$

$$(\lambda+n\mu+\eta)\cdot P_{(1,p,0)} = n\mu\cdot P_{(1,p-1,0)} \text{ for } p=1,\ldots,n-1, \tag{7}$$

$$(\lambda+n\mu+\eta)\cdot P_{(k,p,0)} = \lambda\cdot P_{(k-1,p,0)} + n\mu\cdot P_{(k,p-1,0)} \text{ for } k=2,\ldots,m-1 \text{ and } p=1,\ldots,n-1, \tag{8}$$

$$(n\mu+\eta)\cdot P_{(m,p,0)} = \lambda\cdot P_{(m-1,p,0)} + n\mu\cdot P_{(m,p-1,0)} \text{ for } p=1,\ldots,n-1, \tag{9}$$

$$(\lambda+n\mu)\cdot P_{(1,p,1)} = \eta\cdot P_{(1,p,0)} + n\mu\cdot P_{(1,p-1,1)} \text{ for } p=1,\ldots,n-1, \tag{10}$$

$$(\lambda+n\mu)\cdot P_{(k,p,1)} = \eta\cdot P_{(k,p,0)} + \lambda\cdot P_{(k-1,p,1)} + n\mu\cdot P_{(k,p-1,1)} \text{ for } k=2,\ldots,m-1 \text{ and } p=1,\ldots,n-1, \tag{11}$$

$$n\mu\cdot P_{(m,p,1)} = \eta\cdot P_{(m,p,0)} + \lambda\cdot P_{(m-1,p,1)} + n\mu\cdot P_{(m,p-1,1)} \text{ for } p=1,\ldots,n-1, \tag{12}$$

$$(\lambda+\zeta)\cdot P_{(0,0,2)} = \eta\cdot P_{(0,0,0)} + n\mu\cdot P_{(1,n-1,1)}, \tag{13}$$

$$(\lambda+\zeta)\cdot P_{(k,0,2)} = n\mu\cdot P_{(k+1,n-1,1)} + \lambda\cdot P_{(k-1,0,2)} \text{ for } k=1,\ldots,m-2, \tag{14}$$

$$\zeta\cdot P_{(m-1,0,2)} = n\mu\cdot P_{(m,n-1,1)} + \lambda\cdot P_{(m-2,0,2)} \tag{15}$$

Because an equation, for example equation (1), is linear combination of all others, we can omit it. To solve the equation system it is necessary to replace the omitted equation by normalization equation (16):

$$P_{(0,0,0)} + \sum_{k=1}^{m}\sum_{p=0}^{n-1}\sum_{f=0}^{1} P_{(k,p,f)} + \sum_{k=0}^{m-1} P_{k,0,2} = 1. \tag{16}$$

After omitting equation (1) we have the equation system of $2mn+m+1$ linear equations formed by equations (2) up to (16) with $2mn+m+1$ unknown stationary probabilities. To solve it using Matlab it is necessary to establish an alternative states description because applied state description in the form of (*k*,*p*,*f*) is three-dimensional and is very good for formation the equation system but is absolutely unsuitable for computations in

Matlab which uses matrices. Therefore we applied an alternative one-dimensional state description in the following form:

- The states (k,p,f) for k=1,...,m, p=0,...,n-1, and f=0,1 can be denoted using a single value $f \cdot m \cdot n + p \cdot m + k$,
- The states (k,0,2) for k=0,...m-1 can be denoted using a single value $2 \cdot m \cdot n + k + 1$.
- The state (0,0,0) is labelled as $2 \cdot m \cdot n + m + 1$.

Applying the alternative one-dimensional state description we are able to transform the equation system in the form we need for using Matlab. After numerical solving of the equation system rewritten in matrix form we obtain the stationary probabilities we need in order to compute performance measures of the studied system. Let us establish three random variables - the number of the customers in the service S, where $S \in \{0,1\}$, the number of customers waiting in the queue L, where $L \in \{0,1,\ldots,m-1\}$ and the number of broken servers F, where $F \in \{0,1\}$. In the case we have computed steady-state probabilities; we are able to compute the following performance measures.

The mean number of the customers in the service *ES* is equal to:

$$ES = \sum_{k=1}^{m} \sum_{p=0}^{n-1} \sum_{f=0}^{1} P_{k,p,f} \,, \tag{17}$$

the mean number of the customers waiting in the queue *EL* is given by expression:

$$EL = \sum_{k=2}^{m} (k-1) \cdot \sum_{p=0}^{n-1} \sum_{f=0}^{1} P_{k,p,f} + \sum_{k=1}^{m-1} k \cdot \sum_{p=0}^{n-1} P_{k,p,2} \tag{18}$$

and for the mean number of the broken servers *EF* it holds

$$EF = \sum_{k=0}^{m-1} P_{k,0,2} \,. \tag{19}$$

Another important performance measures of finite queueing system is the probability that customer will not be served. In our queueing system it means that customer is rejected upon its arrival, it happens when the system is full. Therefore for the probability of rejection P_{rej} we can write:

$$P_{rej} = \sum_{p=0}^{n-1} \sum_{f=0}^{1} P_{m,p,f} + P_{m-1,0,2} \,. \tag{20}$$

3 Simulation model

To validate the proposed mathematical model we created a simulation model of the studied queueing system. The model is based on coloured Petri net, the software CPN Tools in version 3.0.4 was used to create the Petri net model. Figure 2 depicts the created Petri net in the initial marking with the following values of the model parameters: m=5, λ=10 h^{-1}, n=10, $n\mu$=100 h^{-1}, η=0.1 h^{-1}and ζ=0.2 h^{-1}.

To generate concrete values of the random variables used in model we defined a function "*fun ET(k, mi) = round(erlang(k, mi/3600.0))*;" where *k* is the shape parameter and *mi* is the scale parameter of the Erlang probability distribution expressed in [h^{-1}]. Please notice that the values which come from the exponential distribution can be also generated by this function because the exponential distribution is a special case of the Erlang distribution with the shape parameter equal to 1. We apply a second as the basic unit of time for the simulation experiments.

The simulation model consists of 10 places and 9 transitions. The model works with following tokens:

- Tokens "*C*" represent customers; these tokens are defined to be timed in order to model exponentially distributed inter-arrival times and Erlang distributed service times.
- Tokens "*F*" model failures of the server and are declared to be timed as well.
- Auxiliary tokens "*P*" model sources of the queueing system (for example free places in the queue or free places in the service).

To obtain desired simulation outcomes some monitoring functions were defined:

- The monitoring function named "*ES"*, which is bound with the place "*Servicing*", enables estimation of the mean number of the customers in the service.
- The monitoring function named "*EL"*, which is bound with the place "*Queue"*, serves for estimation of the mean number of the waiting customers.

- The monitoring function named "*EF*", which is bound with the place "*Repairing*", was create in order to estimate the mean number of the broken servers.
- To estimate the probability that the customer is rejected upon its arrival we were obliged to define two monitoring functions bound with transitions - the monitoring function named "*Number of Rejected Customers*" is bound with the transition "*Rejection*" and the function "*Number of Arriving Customers*" with the transition "*Customer initialization*". These two functions count how many times the corresponding transition was fired during each simulation run. The probability P_{rej} can be then estimated as ratio of these values - the value of the function "*Number of Rejected Customers*" divided by the value of the function "*Number of Arriving Customers*".

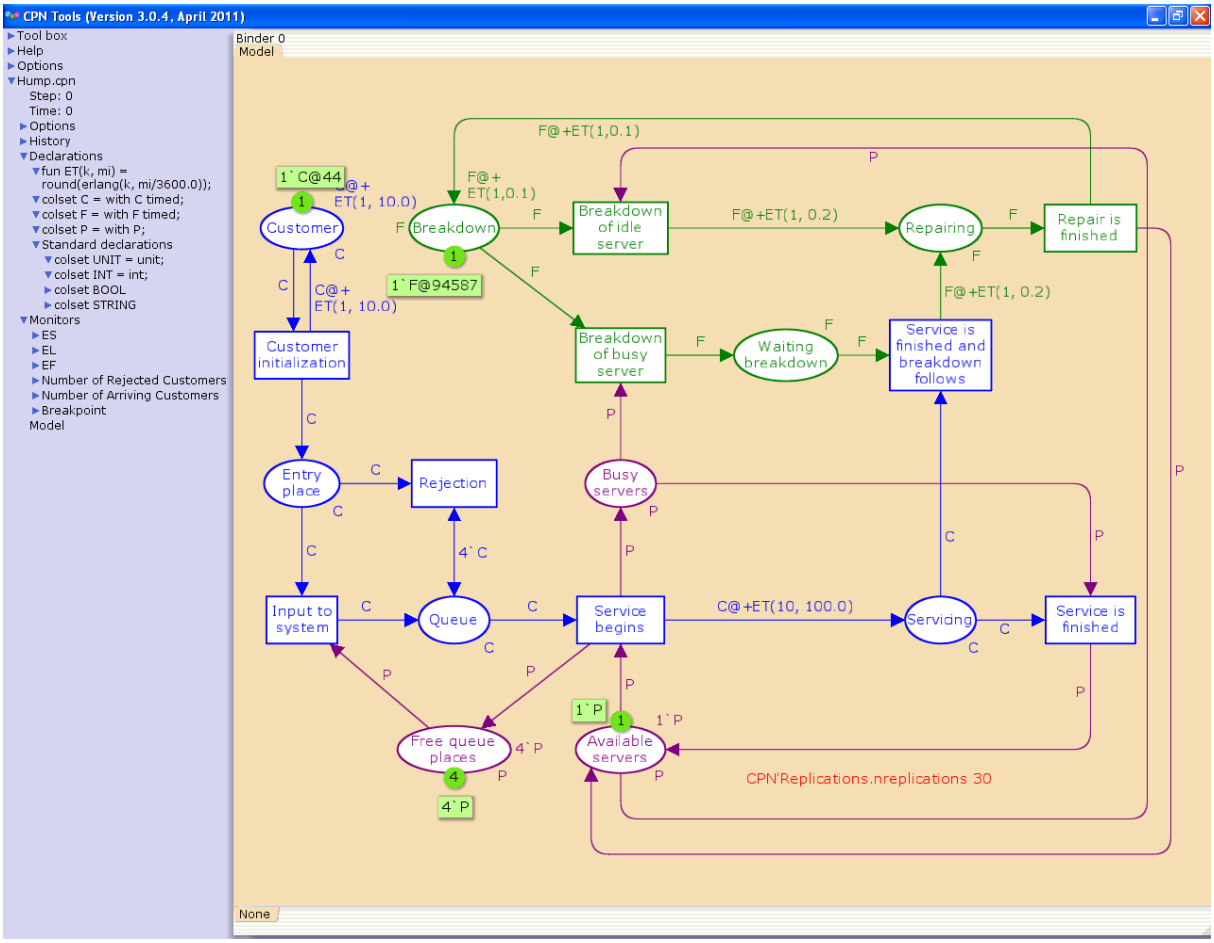

Figure 2 The simulation model created using CPN Tools - Version 3.0.4.

To enable stopping each simulation run after reaching predefined value of simulation time a breakpoint function was defined. The function stops each simulation run after reaching the simulation time equal to 31 536 000 s; that is 365 days. Because the simulation model is stochastic, it is necessary to execute several simulation runs with the model to compute confidence intervals for monitored performance measures. To ensure it we added an auxiliary text "*CPN'Replications.nreplications 30*" into the model. Evaluating the text it is automatically performed 30 simulation runs and simulation outcomes are gained.

4 Results of performed experiments

We executed several experiments with both models to compare analytical and simulation outcomes. Applied values of model parameters are summarized in table 1.

Parameter	m [-]	λ [h^{-1}]	n [-]	$n\mu$ [h^{-1}]	η [h^{-1}]	ζ [h^{-1}]
Applied value	5 up to 10	10	10	100	0.1	0.2

Table 1 Applied values of model parameters.

In table 2 we can see the comparison of analytical and simulation outcomes. Columns corresponding individual performance measures are divided into two sub-columns. Analytical outcomes were obtained using Matlab and the values of observed performance measures were computed according formulas (17) up to (20). The analytical outcomes for individual performance measures are listed in the left sub-columns. On the basis of executed simulation runs (30 runs for each value of the parameter m) we computed 95% confidence intervals for *ES*, *EL*, *EF* and P_{rej}. The bounds of the computed confidence intervals are listed in right sub-columns; the lower bound of the interval is up and the upper bound of the interval is down. We can see that for all executed experiments the analytical value of individual performance measures lies in the corresponding confidence interval. We can see that both models give very similar outcomes, we validated the mathematical model.

m	*ES*		*EL*		*EF*		P_{rej}	
5	0.599	0.596 0.601	2.460	2.454 2.473	0.331	0.330 0.337	0.401	0.399 0.405
6	0.613	0.603 0.614	3.148	3.140 3.170	0.331	0.330 0.342	0.387	0.385 0.396
7	0.622	0.620 0.627	3.851	3.827 3.860	0.331	0.327 0.334	0.378	0.373 0.380
8	0.630	0.623 0.632	4.569	4.553 4.597	0.331	0.330 0.339	0.370	0.368 0.376
9	0.636	0.632 0.641	5.304	5.281 5.335	0.332	0.326 0.336	0.364	0.359 0.368
10	0.640	0.635 0.644	6.055	6.028 6.086	0.332	0.328 0.338	0.360	0.355 0.364

Table 2 Summary of results.

5 Conclusions

In the paper we presented the mathematical and the simulation model of $M/E_n/1/m$ queueing system subject to non-preemptive service station failures. The mathematical model represented by linear equation system can be solved using suitable software, for example Matlab. The mathematical model is supported by the simulation model based on coloured Petri net which was created using software CPN Tools. Both models enable to get the values of performance measures of the studied queueing system.

As regard our future research we would like to extend the model; there are another performance measures we are interested in. The models will be applied to model shunting processes at marshalling yards. Finally we would like to try to derive closed-form formulas for computing steady-state probabilities if it is possible using known methods.

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Modelling financial flows of development projects subsidized from European funds

Stanislava Dvořáková[1], Petr Jiříček[2]

Abstract. The paper is going to deal with modelling the course of financial flows of projects which are subsidized to a various degree from European funds within the European Union Cohesion Policy. The aim of the paper is to illustrate the influence of subsidies from public financial sources on economic efficiency of investment and development projects. The behaviour of the model is demonstrated by the investment curve which is a summarizing expression of functional dependency of net present value of the project's cash-flows on the project's discount rate, represented by a polynomial function. The degree of the polynomial corresponds to the usual six year period of investment project implementation, the constant term of the polynomial will reflect the share of the subsidy of investment costs from the European funds in the project budget and the coefficients of the polynomial will correspond to the polarity and amount of cash-flows issuing from the project implementation. The model should solve the amount of subsidy from the European funds on the internal return rate of the project, represented in the model by a non-negative root of the polynomial. The sensitivity analysis of the drawn up model will examine the stability of the optimal solution of return on investment in relation to the changes of the anticipated amount of projected cash-flows. The outputs of the model programmed in MAPLE software will be analysed by graphs and complemented with explanatory economic commentary.

Keywords: financial flows, investment curve, polynomial function, sensitivity analysis, conventional projects, economical net present value, social discount rate.

JEL Classification: C20, H43
AMS Classification: 65H04, 68R10

1 Introduction

The scope and importance of the public sector in both Czech and European economy is growing and the effect of replacing private capital with capital originating from public recourses occurs. One of such public resources is also European funds which finance cohesion policy of the European Union, focused on reducing differences among both regions and member countries. Increasing the effectiveness of the cohesion policy depends on efficient financing of development projects of the public sector. For general evaluation of criteria of allocation of investment sources in the public sector we use criteria among which the so called Pareto improvement is the best known, saying that allocation of finance into a development project is optimal when it is impossible to find another allocation of financial sources which would lead to increasing the gain of the project beneficiary without making another stakeholder worse off [8]. The problem of evaluating efficiency of investment in the public sector is the measurement of this very gain, unlike the commercial sphere where the goal of investment projects is maximizing the profit or the market value of the company's equity. For measuring efficiency in the public sector, we use various methods [1], for example-cost minimum analysis (CMA), cost-effectiveness analysis (CEA), cost-utility analysis (CUA). These methods are by their output focused only on minimizing the costs, losses and damage in monetary or natural conception. For complex measurement of the difference between benefits and losses of the project in the resulting quantitative expression the generally accepted method used also for evaluating projects from European funds is the cost-benefit analysis. The aim of the paper is to present the possibilities of modelling financial flows for use of CBA in case of public development projects for determined levels of subsidy from European funds.

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2 Material and methods

In the paper, we will deal with modelling financial flows of public development projects where we will investigate the sensitivity of change of the height of allocation of the subsidy from European sources to the benefit from the project measured by the criterion of economic net present value, used in CBA according to the EU guide [9]. The behaviour of the model is illustrated by the investment curve which is a summary expression of functional dependence of net present value of the financial flows of the project on the social discount rate of the project, represented by a polynomial function [4]. For modelling financial flows, a model created by MAPLE software was applied. The model was constructed for the parameters of the investment curve of a public development project. The degree of the polynomial corresponds with the usual six year period of development projects implementation. The criteria function for evaluating public project efficiency has the form:

$$ENPV = \sum_{t=0}^{n} \frac{CF_t}{(1+k)^t}, \tag{1}$$

where CF_t is the cash flow of the public development project (i.e. the benefits) minus costs, losses of the project,

k is the common social discount rate (the required return on public investment from the perspective of the investor or donor of the project)

n is the period of economic lifespan of the project (in our case $n = 6$),

$ENPV$ is economic net present value of the public project reflecting the benefit from the project from a wider social economic view of the given investment.

The constant term of the polynomial reflects the share of investment costs subsidy from European funds in the project budget. In the Czech Republic it usually covers 100% of the project budget in non-profit making organizations, 70% in municipalities, and 30% in common projects of commercial companies and the public sector, applying for funding from European funds.

Coefficients of the polynomial correspond with the polarity and height of cash flows (CF_t) issuing from operational and liquidation stages of the project. In this phase, financial flows can generally have both negative and positive values; the positive values, however, should prevail. In the model we consider only the so called conventional projects, containing only one change of financial flow polarity [5], usually corresponding with the change from negative financial flow (from investment costs) to predicted positive financial flows from the operation of the investment.

Thus the created model shows individual stages of the investment process – the investment stage, the operation stage, and the liquidation stage. The pre-investment stage is not included in the project efficiency evaluation due to the so called sunk costs [2]. The sensitivity analysis of the model [7] should solve the influence of the height of subsidy from European funds on the $ENPV$ (using a 3D projection) and its relationship to marginal social return rate of a public project (defined as Economic Rate on Return – ERR, [9]), represented in the model by a non-negative root of the polynomial.

3 Results and discussion

3.1 The course of the $ENPV$ investment curve

The method of internal interest rate is based on the philosophy that we look for the marginal social rate of return ERR (Economic Rate of Return) from the given public investment on condition of return of the investment during the projected life span. It means that we look for the root of the $ENPV$ function (1), i.e.

$$ENPV = \sum_{t=0}^{n} \frac{CF_t}{(1+ERR)^t} = 0 .$$

When calculating the number ERR it is more suitable to convert the $ENPV$ function by means of substitution $x = \frac{1}{1+k}$ to the polynomial

$$f(x) = CF_0 + CF_1 x + \cdots + CF_n x^n .$$

Using some program (e.g. Maple or Matlab) we calculate the roots of this polynomial and by backward substitution $k = \frac{1}{x} - 1$ we convert the real roots back to the *k* variable. We do not consider imaginary roots because they do not have real interpretation in the economic practice. This procedure is more advantageous since there are more algorithms and more elaborate ones for searching for a polynomial root than algorithms for finding a general function root (see e.g. [3] and [6]). Let's also remark that it is necessary to use a calculating program because, as is well known, there are no formulas for calculating polynomial roots of a higher than the third degree.

With the help of the searched number ERR the public project efficiency can be judged according to a certain preference perspective of the investor or donor. However, this is not the aim of this paper. This issue is elaborated in more detail e.g. in [5] or [4]. Let's only say that for determining project efficiency by the method of internal rate of return there must be only one polynomial root in the positive real half-plane.

Hen creating the model for the needs of this paper, a model public project was used in which the project cash flow equals $CF = [CF_0, CF_1, CF_2, CF_3, CF_4, CF_5, CF_6] = [-6,1,1,1,1,1,1]$. It is a conventional project, it has only one change of the sign in the progression of numbers CF. In addition, it holds that $\sum_{t=0}^{n} CF_t = 0$ or also $CF_0 = -\sum_{t=1}^{n} CF_t$. Figure 1 shows the course of the investment curve $ENPV$ in dependence on the common social discount rate *k* of public projects without subsidy ($CF_0 = -6$), with 30% subsidy ($CF_0 = -4.2$), with 70% subsidy ($CF_0 = -1.8$) and with 100% subsidy ($CF_0 = 0$). Here it is clearly visible that the higher the subsidy, the higher ERR is. For 100% subsidy (the black curve) even $ERR \to \infty$, which expresses the reality of an always efficient project at the above mentioned positive financial flows from the project. Let's say about the course of the $ENPV$ curve that the curve has two asymptotes, namely a vertical one with the value of $k = -1$ and a horizontal one $ENPV = CF_0$ for $k \to \infty$. The intersection with the *y* axis, i.e. for $k = 0$, is of the value. $\sum_{t=0}^{n} CF_t$.

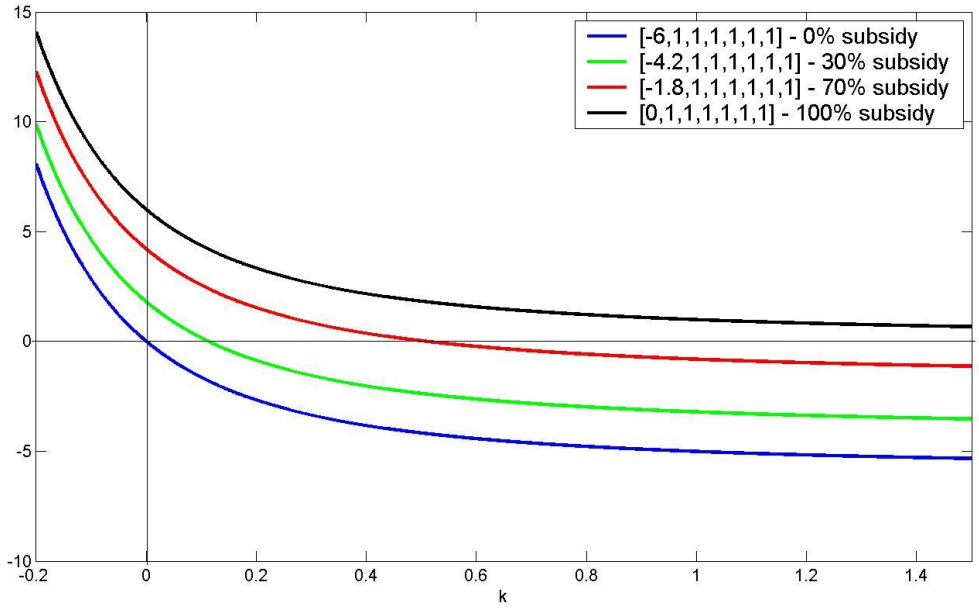

Figure 1 Graph of the course of the $ENPV$ curve

3.2 Sensitivity analysis of the model

On the above described model project, we examined the stability of the optimal solution of the return on investment depending on changes of one input quantities – the anticipated height of the projected cash-flows. We

created a 3D model of the investment curve depending on the social discount rate *k* and on the initial cash flow CF_0, see Figure 2.

For further analysis, let's divide the projection in Figure 2 into three cases. The first case is for $CF_0 \geq 0$. Here, $ENPV$ is always positive, which implies $ERR \to \infty$ and such a project is always efficient. In the real economic practice, this case only corresponds with the considered possibility of a 100% subsidy, where $CF_0 = 0$. The possibility of $CF_0 > 0$ is unreal under the conditions of financing from European funds, because it would have to be a more than 100% subsidy. The second case is for $CF_0 < -6$. Here, $ENPV$ is always negative. Thus the root ERR does not lie in positive values, the social discount rate of the project is negative and therefore the public project is not efficient. In this case it would be a project where the investment costs are higher than the discounted income from the project.

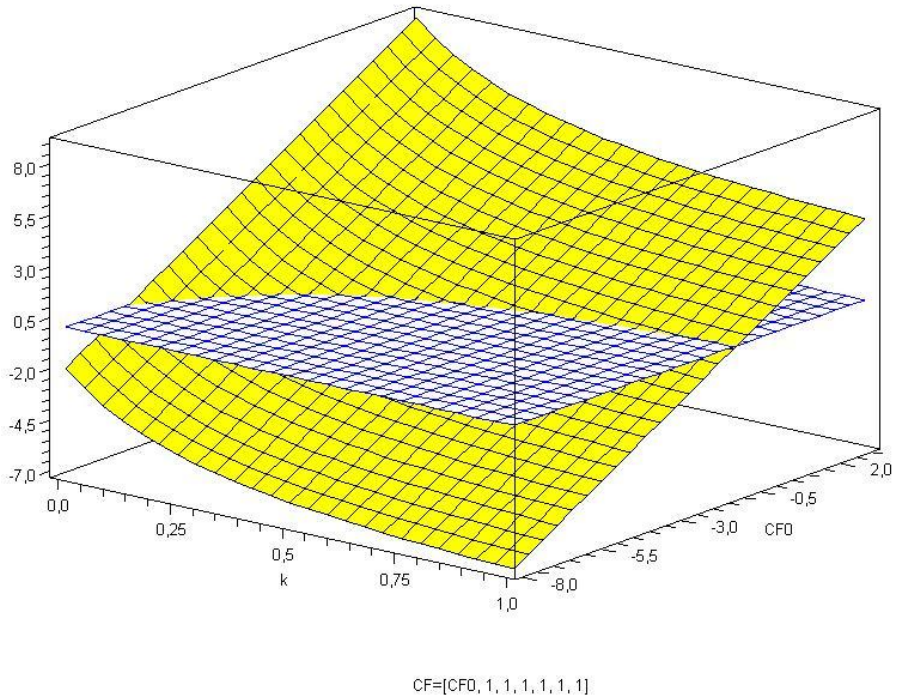

Figure 2 Three-dimensional depiction of $ENPV$

In economic practice, only the third alternative is feasible to use, that is for $CF_0 \in \langle -6, 0)$, i.e. the alternative when the investment costs are subsidized in a less than 100% share. Here, under the conditions of the model situation of positive financial flows caused by the project $ENPV$ passes from positive to negative values and therefore it is possible to find at least one positive root ERR. In Figure 3 there is the curve of the intersection of the surface $ENPV$ with the plane $ENPV = 0$ which corresponds with the roots of the investment curve. In this graph, we can deduct the marginal economic rate of return ERR (on the horizontal axis) if the initial investment is set and the amount of subsidy, i.e. the financial flow CF_0, taken into consideration. For instance, for $CF_0 = -4$, we can estimate $ERR = 0.15$. With an accurate calculation by the Maple system for this case, $ERR = 0.13$. This deviation in estimation could be further eliminated by specifying the interval of the root function display in the range $0 < k < 1$ since the social discount rate as the rate of the project return required by the investor does not excess 100% in practice (see Figure 4).

4 Conclusion

In the paper, we describe a conventional model of solving social economic efficiency of public projects where there is only one change of the cash flow polarity. The conventional model then covers the initial situation when in the submitted projects we predict positive cash flows in the operational and liquidation stages of the development project. The negative cash flow in the investment stage, i.e. $CF = [CF_0, CF_1, CF_2, CF_3, CF_4, CF_5, CF_6]$ is in our case defined by the initial conditions as $CF = [-6,1,1,1,1,1,1]$. The financial flow CF_0 refers to the difference

between investment costs (capital expenditures of a development project) and the subsidy from European funds. In the graph, this model is represented by the $ENPV$ curve which is a representation of a polynomial function of the sixth degree, which corresponds with the chosen six-year average lifetime of development projects.

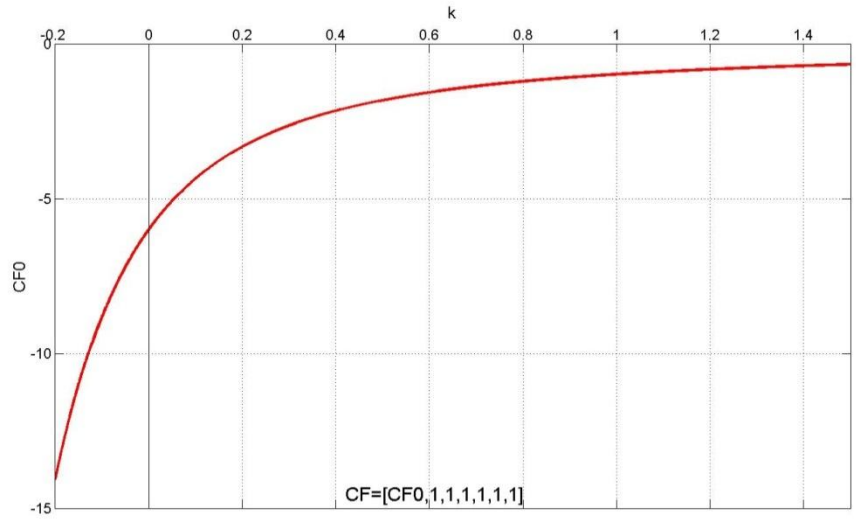

Figure 3 Dependence of the root of ERR on the CF_0 coefficient and the discount rate k

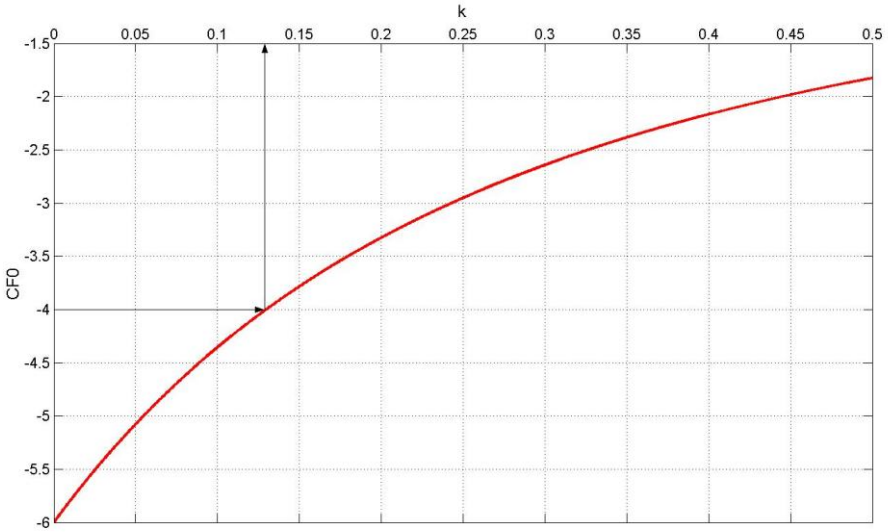

Figure 4 Detail the curve of roots ERR for $0 < k < 0.5$

Sensitivity analysis of the $ENPV$ model is carried out in a situation when the changed input quantity is the financial flow CF_0, i.e. the state when investment costs of the project are gradually subsidized from European funds up to the 100% amount. The other parameters of the project are held constant. In a two-dimensional model, this situation can be seen in the curves in Figure 1 for the states of 0%, 30%, 70%, and 100% subsidy respectively. This situation is better documented by the 3D model created with the MAPLE software (see Figure 2). The marginal situation in the model illustrates the state when the subsidy from the European funds covers 100 % of investment costs (i.e. $CF_0 = 0$). Here, it is not possible to speak rigidly about a conventional project any more since the change of the sign of financial flows does not occur and the investment curve of this marginal situation in the v 3D model does not intersect the plane $ENPV = 0$.

From the solution of the 3D model it theoretically follows that in the situation of predicted positive financial flow in the investment stage ($CF_0 > 0$) in the area to the right of the marginal investment curve the surface $ENPV$ does not intersect the plane $ENPV = 0$ any more for any chosen discount rate of the development project. In a development project constructed this way there is no marginal social discount rate (i.e. economic rate of return ERR) and so for each discount rate required by the investor or donor the project is always economically efficient in this situation. This alternative is not real under the conditions of financing from European funds because it is not possible to obtain a subsidy higher than the investment costs of the project are.

A different situation occurs in the area of real economic solutions, i.e. of negative financial flows in the investment stage of the development project and predicted positive financial flows in the operational and liquidation stages (i.e. $CF_0 < 0$). Here the surface $ENPV$ gradually intersects the plane $ENPV = 0$ along a curve which shows us in 3D depiction the set of roots of the polynomial corresponding to the marginal social discount rates of projects in the growing rate of subsidizing investment costs from European funds.

This root function can be used in 2D depiction simply for estimating ERR under various conditions of the height of investment cost subsidy (see Figure 3), namely with quite an accuracy which for practical use is comparable with a calculation based on solving a polynomial e.g. by the MAPLE program. In the graph, the confirmation of the above mentioned statements is clearly visible when in case of positive financial flow in the investment stage of the project ($CF_0 > 0$) we do not find any polynomial root, i.e. there is no economical rate of return and the project is economically efficient at any discount rate chosen by the investor or donor. In the area $CF_0 \in (-6,6)$ we will find exactly one positive solution of the root at a chosen height of subsidy. On the contrary, in case of too high investment costs of the project (i.e. $CF_0 < -6$) and on condition of zero subsidy the positive cash flows defined in the initial setting of the model will be manifested in a loss rate of the whole project, when the root passes to the negative area of ERR .

The above described model predicts a standard situation of public development project construction for an evaluation from the perspective of granting a subsidy from European funds, when the projects are designed as conventional one. However, in practice a situation which is closer to the real state in economy occurs during the project implementation, i.e. for example the first financial flow in the operational stage (project beginning) is necessarily negative or on the contrary the final financial flow in the liquidation stage of the project (e.g. environmental damage liquidation) is necessarily negative. Here the use of the created 3D model suggests itself for solving such non-predicted situations of a non-conventional project with a view of determining the resulting efficiency of the projects subsidized from European funds.

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Capacitated Postman Problem

Jan Fábry[1], Jan Pelikán[2]

Abstract. The paper deals with the Chinese Postman Problem. It has many modifications that are summarized in the first part of the paper. The topic of the work is solving the case study of urban household waste collection, i.e. a Capacitated Postman Problem. In this task, distances and requirements are given for each arc of the network and cyclical routes are searched with the objective to minimize total costs. Each route must respect the vehicle capacity. Moreover, start and finish nodes may be defined for each route. In the paper, two mathematical models of integer programming are formulated. Due to the NP-hardness of the problem, heuristic method is proposed.

Keywords: arc routing, Chinese postman problem, rural postman problem, Eulerian cycle, capacitated postman problem.

JEL Classification: C44
AMS Classification: 90C15

1 Introduction

The Arc routing problem (ARP) deals with searching for the Eulerian cycle in the graph. In 1763, Leonhard Euler defined the conditions for finding the cycle containing each arc *exactly once*. The problem is polynomially solvable in case all nodes in the connected graph have an even degree, i.e. an even number of incident arcs. In 1962, M. Guan generalized the problem to the Chinese Postman Problem (CPP), in which the objective is to find a minimal-distance cycle containing each arc *at least once*. The problem is also polynomially solvable. First, we search arcs which will be used multiple times to get *Eulerian graph*[3]; minimum-cost matching of all odd-degree nodes is used for this purpose. Then, the *Eulerian cycle* is found using Fleury's algorithm.

The CPP is formulated both for directed and for undirected graphs, and also for mixed graphs consisting of directed and undirected arcs. Solution of the CPP consists in doubling any arcs to obtain the *unicursal graph*. Graph is unicursal if:

a) in an undirected graph, all nodes have even degree,

b) in a directed graph, in-degree of each node is equal to its out-degree,

c) in a mixed graph, each subset of nodes has to satisfy the "balanced set condition".

Searching for the extended graph satisfying the condition of unicursality is polynomial in case of a) and b), and NP-hard in case of c). The special extension of the CPP with the asymmetric matrix of arcs length, i.e. where $c_{ij} = c_{ji}$ is not valid for all arcs in the graph, called the Windy Postman Problem, is also NP-hard (Brucker, 1981 and Guan, 1984).

Another NP-hard problem is the Rural Postman Problem (RPP), in which the set of all arcs contains the subset of arcs that have to be traversed (Eiselt et al., 1995). All other arcs can be also included in the final route. The paper deals with the Garbage Collection Problem that is analogous to the Vehicle Routing Problem. We search for the cyclical routes of vehicles with given capacity Q and their common depot, which has to be included in each route. Each obligatory arc has its collection requirement and therefore, has to be included at least once on any route. Although the arc can be included in multiple routes or it can be traversed several times on the same route, requirements are collected just once. Therefore, we have to consider the arc is serviced on the route or unserviced, i.e. traversed without service (so called deadheading in logistics). Each route has to satisfy the vehicle capacity and the objective is to minimize the total length of all routes. The problem is called the Capacitated Chinese Postman Problem (CCPP) and it is defined both for directed and for undirected graphs (Longo et al., 2006). Further, we consider only undirected graphs. The problem is generalization of the RPP. Because it is the NP-hard problem, heuristics and metaheuristics (Hertz et al., 2000) are used for the solving real large scale instances. In the paper, we offer heuristics very easy to apply for garbage collection problems.

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[2] Jan Pelikán, University of Economics Prague, Faculty of Informatics and Statistics, Department of Econometrics, Prague 3, W. Churchill sq.4, pelikan@vse.cz
[3] Eulerian graph is a graph containing an Eulerian cycle

2 The model of the CCPP on undirected graph – directed solution

Let $G = \{V, E\}$ be an undirected graph, where V is a set of n nodes and E is a set of arcs. Let us define $R \subseteq E$ as the set of obligatory arcs. For each arc $(i, j) \in E$, it is given its length c_{ij}, for each arc $(i, j) \in R$, it is known its collection requirement d_{ij}. Depot is situated in node 1, where m vehicles with capacity Q are prepared for garbage collection. Although graph G is undirected, solution of the problem will contain the orientation of the traverse through the arcs. Thus, we will be searching for the directed cyclical routes while in case of undirected graph the direction is not important.

Let binary variable x_{ij}^k is equal to 1 if arc $(i, j) \in E$ is traversed by vehicle k in the direction from node i to node j, 0 otherwise. Binary variable y_{ij}^k is equal to 1 if arc $(i, j) \in R$ is serviced on route k, i.e. requirement d_{ij} is collected by vehicle k, 0 otherwise. If arc (i, j) is serviced on route k it must lie on this route, therefore it is valid:

$$y_{ij}^k \leq x_{ij}^k. \tag{1}$$

If the requirement d_{ij} of arc (i, j) is collected on route k, then either $y_{ij}^k = 1$ or $y_{ji}^k = 1$. Because the arc must be serviced exactly once, i.e. on the only one route, the following equations must be respected:

$$\sum_{k=1}^{m} \left(y_{ij}^k + y_{ji}^k\right) = 1, \quad \forall (i,j) \in R. \tag{2}$$

The sum of requirements d_{ij} serviced on route k must not exceed the vehicle capacity Q:

$$\sum_{(i,j)\in R} d_{ij} y_{ij}^k \leq Q, \quad k = 1,2,\dots,m. \tag{3}$$

Necessary condition for the existence of cyclical routes is unicursality of the graph, i.e. a number of arcs entering each node must be equal to a number of arcs leaving it:

$$\sum_{\substack{i \\ (i,j)\in E}} x_{ij}^k = \sum_{\substack{i \\ (j,i)\in E}} x_{ji}^k, \quad \forall j \in V, \ k = 1,2,\dots,m. \tag{4}$$

Nevertheless, the condition of unicursality does not ensure the route is connected. Therefore, illegal cyclical subtour not containing depot can exist and it is necessary to prevent it.

If a set of arcs (including their multiplicity) for certain k satisfies the condition of unicursality and does not contain node 1, arcs represent a multigraph consisting of nodes subset $S \subset V - \{1\}$ and set of arcs $E(S)$. If a number of arcs (including their multiplicity) is less than $|S|$, then multigraph $\{S, E(S)\}$ does not represent the cyclical route. However, if $|E(S)| > |S| - 1$ then illegal cycle is possible and at least one arc connecting S and $V - S$ has to be added. The number of arcs in $E(S)$ is

$$|E(S)| = \sum_{\substack{(i,j)\in E \\ i,j\in S}} x_{ij}^k + x_{ji}^k, \quad k = 1,2,\dots,m. \tag{5}$$

Thus, avoiding of illegal cycles for tour k can be expressed as follows:

1) If $|E(S)| > |S| - 1$ then at least one arc must connect S and $V - S$, i.e. $x_{ij}^k \geq 1$ for $i \in S$ and $j \in V - S$.

2) If $|E(S)| \leq |S| - 1$ then such arc is not necessary.

These conditions are included in the following mathematical model of CCPP based on Eiselt et al. (1995):

$$\sum_{k=1}^{m} \sum_{(i,j)\in E} x_{ij}^k \, c_{ij} \to min, \tag{6}$$

$$y_{ij}^k \leq x_{ij}^k, \quad (i,j) \in E, k = 1,2,\dots,m, \tag{7}$$

$$\sum_{k=1}^{m} \left(y_{ij}^k + y_{ji}^k\right) = 1, \quad \forall (i,j) \in R, \tag{8}$$

$$\sum_{(i,j)\in R} d_{ij} y_{ij}^k \leq Q, \quad k = 1,2,\dots,m. \tag{9}$$

$$\sum_{\substack{i \\ (i,j)\in E}} x_{ij}^k = \sum_{\substack{i \\ (j,i)\in E}} x_{ji}^k, \quad j = 1,2,\dots,n, \ k = 1,2,\dots,m. \tag{10}$$

$$\sum_{\substack{(i,j)\in E \\ i,j\in S}} x_{ij}^k + x_{ji}^k \ \leq \ |S| - 1 + n^2 u_k^S, \qquad k = 1,2,\dots,m, \ S \subset V - \{1\}, \ S \neq \emptyset, \tag{11}$$

$$\sum_{\substack{i\in S \\ (i,j)\in E}} \sum_{j\notin S} x_{ij}^k \ \geq \ 1 - w_k^S, \qquad k = 1,2,\dots,m, \ S \subset V - \{1\}, \ S \neq \emptyset, \tag{12}$$

$$u_k^S + w_k^S \leq 1, \qquad k = 1,2,\dots,m, \ S \subset V - \{1\}, \ S \neq \emptyset, \tag{13}$$

$$x_{ij}^k \in \{0,1\}, \ y_{ij}^k \in \{0,1\}, \qquad (i,j) \in E, \ k = 1,2,\dots,m, \tag{14}$$

$$u_k^S \in \{0,1\}, \ w_k^S \in \{0,1\}, \qquad k = 1,2,\dots,m, \ S \subset V - \{1\}, \ S \neq \emptyset. \tag{15}$$

The objective (6) is to minimize the total cost of the travel corresponding to the total length of all routes. Illegal subtours are eliminated respecting inequalities (11) – (13). For each S and each k, there are binary variables u_k^S and w_k^S introduced. If $|E(S)| > |S| - 1$, variable u_k^S is equal to 1, and $w_k^S = 0$. Then, inequalities (12) assure subset S must be connected with subset $V - S$ which contains depot.

There are two possibilities for using the mathematical model:

1) At the beginning, all subsets S are generated and then, the mathematical model is applied. The disadvantage of this approach is computational complexity because of huge number of all subsets S.

2) The mathematical model is run without conditions eliminating illegal subtours. In case any illegal subtour is found, inequalities (11) – (13) are added for all vehicles and subset S containing all nodes included in the illegal subtour. Then, extended model is run again. This process is repeating until result is feasible in term of the elimination of illegal cycles. Although the model is not so complex (at the beginning of the process), the disadvantage of this approach is multiple run of the model, and also the gradual increase in the number of subsets S.

The use of the second approach will be demonstrated in Section 5 of the paper.

3 The model of the CCPP on undirected graph – undirected solution

In Section 2, the model is proposed for undirected graph and the optimal solution determines directed cyclical routes. In this section, we propose the model generating sets of arcs laying on each cycle; directed routes are found subsequently. Let integer variable x_{ij}^k represent a number of times arc $(i, j) \in E$ is traversed by vehicle k (in any direction). Variables x_{ij}^k and y_{ij}^k are defined only for $i < j$.

While the objective function (6), constraints (7) and (9) are unchanged, equations (8) and (10) must be modified:

$$\sum_{k=1}^{m} y_{ij}^k = 1, \quad \forall (i,j) \in R, \tag{8a}$$

$$\sum_{\substack{i \\ (i,j)\in E}} x_{ij}^k + \sum_{\substack{i \\ (j,i)\in E}} x_{ji}^k = 2 f_j^k, \quad j = 1,2,\dots,n, \ k = 1,2,\dots,m. \tag{10a}$$

Integer variable f_j^k in (10a) indicates a number of times node j is traversed by vehicle k. Constraints (11) and (13) are excluded from the model, and inequalities (12) eliminating illegal subtours are replaced by:

$$\sum_{\substack{i\in S \\ (i,j)\in E}} \sum_{j\notin S} x_{ij}^k + \sum_{\substack{j\in S \\ (j,i)\in E}} \sum_{i\notin S} x_{ji}^k \geq \frac{1}{M} \sum_{\substack{i\in S \\ (i,j)\in E}} \sum_{j\in S} x_{ij}^k, \qquad k = 1,2,\dots,m,\ S \subset V-\{1\}, S \neq \emptyset, \tag{12a}$$

where M is a high constant. The condition (12a) assures subset S must be connected with subset $V - S$ which contains depot in case there is at least one arc (i, j) with $i, j \in S$, traversed by vehicle k. Apparently, variables (15) are not relevant in the proposed model and integrality constraints for variables x_{ij}^k replace binary ones:

$$x_{ij}^k \in Z_+,\ y_{ij}^k \in \{0,1\}, \qquad (i,j) \in E,\ k = 1,2,\dots,m. \tag{14a}$$

After finding the optimum solution it is necessary to find directed cycles, e. g. with the use of Fleury's algorithm. Similarly to the model (6) – (15), two approaches can be also applied for our modification. We demonstrate the second one in Section 5 of the paper.

4 Heuristics for the CCPP on undirected graph

Because of the NP-hardness of the problem, solution of large scale instances needs the use of heuristics. In the paper, we offer the algorithm using the shortest path approach for finding cyclical routes. Let us assume notation introduced at the beginning of Section 2. Proposed algorithm will find the cyclical routes given by the sequence of nodes $v_1^k, v_2^k, \dots$ (k is the number of the route). The length of the shortest path from node i to node j is denoted p_{ij}. Total length of cyclical route k is F_k, total amount of units loaded on vehicle k is L_k and a number of nodes included in tour k is N_k. The heuristics consists of 5 steps:

Step 1:

$k = 0$.

Step 2:

if $R = \emptyset$ then goto step 5

else $k = k + 1$; $L_k = 0$; $F_k = 0$; $t = 1$; $v_t^k = 1$; $N_k = 1$.

Step 3:

if $R = \emptyset$ then

begin

$v_{t+1}^k = 1$; $N_k = N_k + 1$; $F_k = F_k + p_{v_t^k,1}$;

goto step 5

end.

Step 4:

if there is arc $(v_t^k, j) \in R$ and $L_k + d_{v_t^k,j} \leq Q$ then

begin

$F_k = F_k + c_{v_t^k,j}$; $L_k = L_k + d_{v_t^k,j}$; $N_k = N_k + 1$; $R = R - (v_t^k, j)$; $t = t + 1$; $v_t^k = j$;

if $v_t^k = 1$ then goto step 2 else goto step 3

end

else

if there is $(i, j) \in R$, $i \neq 1$ and $L_k + d_{ij} \leq Q$ then

begin

$F_k = F_k + p_{v_t^k,i} + c_{ij}$; $L_k = L_k + d_{ij}$; $N_k = N_k + 2$; $R = R - (i, j)$; $v_{t+1}^k = i$; $v_{t+2}^k = j$; $t = t + 2$;

if $v_t^k = 1$ then goto step 2 else goto step 3

end

else goto step 2.

Step 5:

End.

Because, at the end of the algorithm run, the value of k corresponds to the number of realized cyclical routes, the total length of them is given as follows:

$$G = \sum_{l=1}^{k} F_l. \tag{16}$$

In Step 4, arc $(i, j) \in R$ is selected randomly. The algorithm can by modified by finding the closest arc to node v_t^k. Therefore, the shortest paths are used for this purpose. In addition, instead of random selection of arc $(v_t^k, j) \in R$, we can use specific rules in case there are more possibilities to select, e.g. the longest arc, or the shortest one. In the demonstration of using the algorithm in the following section, the longest arc is preferred.

5 Computational experiments

Let us demonstrate operating of the mathematical models and the heuristics on the following example. In case of the mathematical models, the second approach mentioned above will be applied. For this purpose, optimization system Lingo and VBA for MS Excel are used.

Example

The following graph is defined (see Fig. 1). Each arc is evaluated by its length and requirement for collection (values in parentheses). Arc (4, 6) has no requirement, i.e. it is optional. Node 1 is the depot, in which 3 vehicles are prepared for collection. Each vehicle has the capacity 20 units. Totally, 21 units have to be collected.

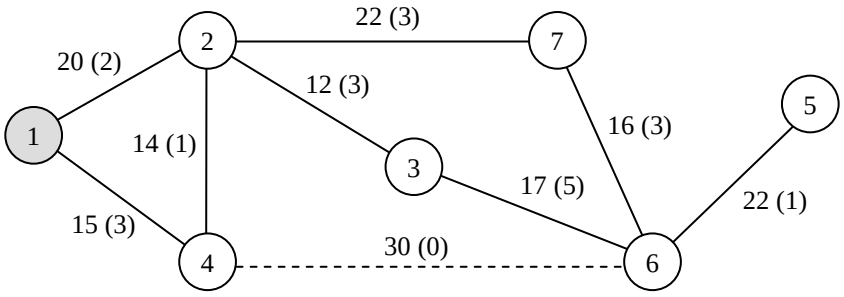

Figure 1 Garbage collection graph

Application of the model generating directed solution

In the first step, we obtained the initial solution without conditions eliminating illegal subtours. In further steps, these conditions are introduced to the model. In each step, subsets *S* of nodes, corresponding to illegal subtours generated in previous steps, are added (see Table 1).

Step	Subset *S*	Tour 1	Tour 2	Objective
1	-	1-2-4-1	2-3-6-5-6-7-2	160
2	{2,3,5,6,7}	1-2-7-6-3-2-4-1	5-6-5	160
3	{5,6}	1-4-2-3-6-5-6-7-2-1	2-3-2	184
4	{2,3}	1-2-3-6-5-6-7-2-4-1	2-4-2	188
5	{2,4}	1-2-4-1	2-4-2-7-6-5-6-3-2	188
6	{2,3,4,5,6,7}	1-4-1	1-4-2-7-6-5-6-3-2-1	190

Table 1 Application of the model – directed solution

Application of the model generating undirected solution

In each step, sets of arcs a vehicle is to traverse are found. Then, directed cycles are generated with the use of Fleury's algorithm. Conditions eliminating illegal subtours are introduced to the model until the feasible solution is obtained (see Table 2).

Step	Subset *S*	Tour 1	Tour 2	Objective
1	-	1-2-4-1	2-3-6-5-6-7-2	160
2	{2,3,5,6,7}	1-4-2-3-6-5-6-7-2-1	2-4-2	188
3	{2,4}	1-2-4-1	2-4-2-7-6-5-6-3-2	188
4	{2,3,4,5,6,7}	1-4-1	1-4-2-7-6-5-6-3-2-1	190

Table 2 Application of the model – undirected solution

Application of the heuristics

The solution obtained with the use of the proposed heuristics generates two cyclical routes: 1-2-7-6-5-6-3-2-4-1 with the length of 160, and 1-4-1 with the length of 30. Total length is 190. Thus, we obtain the alternative optimal solution.

6 Conclusions

The paper is based on the case study of the optimization of routes in garbage collection problem in Poděbrady city. The real problem contains specific conditions, e.g. separated locations of depot and dump. The instance can be classified as the Capacitated Rural Postman Problem. In the paper, two mathematical models and easy construction heuristic algorithm are introduced and demonstrated on the illustration instance. Heuristics can be extended to other steps that can make the algorithm more effective. Other conditions respecting specific requirements of the city could be included. In addition, the solution obtained with the use of the algorithm can be improved using other heuristics and metaheuristics. These are issues of the authors' future research.

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Kuhn-Tucker optimality conditions in Model of a Monopoly Production Price Differentiation

Eleonora Fendeková [1], Michal Fendek [2]

Abstract. One of the effective tools to exercise monopoly's market position or economic strength to define – in accordance with the monopoly's interests – a market price on the level guaranteeing maximum profit is a price differentiation.
In this article we discuss the general aspects of quantitative analysis of the monopoly price differentiation models and we will analyze in more detail the models of consumer's utility maximization in the conditions of price differentiation of the goods and a model of monopoly's profit maximization with price-differentiated production.
A problem of monopoly's profit optimization is built on a specific hypothesis that a monopoly gives a consumer an „all or nothing" offer, meaning that a differentiated price only applies in a case that a consumer buys the whole amount of a good offered with a differentiated price and a corresponding revenue then guarantees a monopoly's maximum profit.
We will analyze a role of consumer's behavior optimization when the consumer's willingness to spend his funds on goods with differentiated prices is related to maximization of his total utility. For the optimization problems we will formulate the Kuhn-Tucker optimality conditions and we will study their interpretation options.

Keywords: price differentiation, Lagrange function Kuhn-Tucker optimality conditions, first, second and third degree price differentiation

JEL Classification: C62, L13, L22
AMS Classification: 90C46, 49M05

1 Introduction

Due to a specific position which the subjects on a supply side have on an imperfect competition market, the producers can promote their interests without immediate danger of a competitor producing the same or similar product entering a relevant market.

We may speak of monopoly price differentiation when a monopoly uses its market position, or economic strength to define – in accordance with its interests – a market price on the level guaranteeing maximum profit. Ultimately the monopoly uses its monopolistic position to generate a monopoly profit over and above the profit achievable in the conditions of perfect competition.

A tool to achieve this goal is a market price and a monopoly has a considerable say in defining the price.

One of the effective tools to use monopoly's market position or economic power to, according to its interests, set a market price to the level which guarantees maximum profit, is a price differentiation. We can speak of a price differentiation in a situation when the identical products are being sold at different prices while this inequality is not due to different production costs. In literature price discrimination is often used as a synonym to price differentiation. We think, however, that this collocation has a slightly negativistic tone which does not correspond to its factual technical meaning, therefore following we will prefer price differentiation to indicate this microeconomic attribute.

In this article we will analyze a role of a consumer behavior optimization, whose willingness to spend his funds to purchase a good with a differentiated price is related to maximization of his total utility.

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2 Conditions for Price Differentiation

Price differentiation is of course possible only under an assumption that a consumer is willing to pay different prices for different amounts of goods. That is, for example, at a lower price he is willing to purchase more – the price-demand function is decreasing. Besides, an effective price differentiation is based on the assumption that a consumer who purchased a product at a lower, favored price would not resell it at a slightly higher price to another consumer, who would otherwise paid a higher price. When such arbitrage deals cannot be ruled out price differentiation cannot be applied.

If a price differentiation can be applied then in fact a monopoly uses its monopolistic position to gain monopolistic profit. Carlton - Perloff [2] show, that a monopoly can use price differentiation in two ways:

A tool for this is a market price while a monopoly can use price differentiation in following areas:

• to set a different price when purchasing different amounts of a same good,

• to set a different price for different consumers or consumer groups.

In a first case it is relatively simple to use a price differentiation. When meeting the conditions of purchasing volume an agreed discount is given. To successfully apply the second method of a price differentiation a mechanism must exist to identify a consumer belonging to a certain consumer group. O'Sullivan - Sheffrin - Perez [7] show, that we can find three types of price differentiation according to traditional classification:

(a) First degree price differentiation. This type of price differentiation of a monopolistic company is sometimes called a perfect price differentiation. A monopoly uses its privileged position on a market to set different prices for different volumes of a same product as well as for different consumer groups. Meaning that a seller in fact sets an individual price for each unit of a product and a price of a certain unit corresponds with a willingness to pay a maximum price by a certain consumer at a certain conditions.

(b) Second degree price differentiation represents a situation when the product prices depend on the purchasing volume of the products but do not depend on any characteristic a consumer may have. This phenomenon of pricing is also interpreted as a nonlinear pricing. Identical pricelists apply to all consumers but the pricelists vary for different purchasing volumes. A monopoly thus does not differentiate prices for particular consumers or consumer groups. The differentiation applies to varying amounts of purchased goods. An example of this approach to price differentiation is a volume rebate.

(c) Third degree price differentiation. With this type of price differentiation a monopolistic company sells any amount of goods at a same unit price. The price varies however for a specifically defined consumer groups. Third degree price differentiation represents probably the most common form of price differentiation. These are for example various types of discounts for students, child or pensioner's travel tickets, different prices on a different days (weekend discounts) and so on.

3 Model of Optimal Consumer Behavior in the Conditions of Price Differentiation

Let's now examine a simplified analytical model of price differentiation of two potential consumers S_1, S_2, who both have a price-differentiated product in their market basket. X represents its purchase in units and w is the total value of purchase of the other goods in monetary units. In [2], we can see, that a consumer can possibly decide:

- to exclusively purchase the studied product with a differentiated price and not to purchase the other goods at all,
- to purchase only the other goods from his market basket and not to purchase the studied product with a differentiated price,
- to purchase a full market basket represented by the studied product with a differentiated price as well as other goods.

Total utility expressed in monetary units, which a consumer feels when purchasing all the goods from the market basket is represented in a form of utility function:

$$f_i(x_i, w_i) = u_i(x_i) + w_i \qquad i = 1,2 \tag{1}$$

where

$u_i(x_i): R \to R$, $i = 1,2$ - is a concave, continuous and differentiable utility function for a product with differentiated price and represents a feeling of utility of a consumer in monetary units corresponding with a purchase of *x* units of the product,

w_i – expenses on other goods in the market basket,

$f_i(x_i, w_i): R^2 \to R$, $i = 1,2$ - consumer's total utility function.

To simplify, we will assume that utility for zero purchase of the studied product has a standardized zero value $u_i(0) = 0$. Maximum willingness of a consumer *i* to pay a certain price in monetary units for a purchase of x_i units of the product is represented by a function $r_i(x_i)$. This function is a solution to an equation:

$$u_i(0) + w_i = u_i(x_i) - r_i(x_i) + w_i \qquad i = 1,2 \tag{2}$$

where on the left side is utility of a purchase of zero units of the product plus a value of the other goods from the market basket and on the right side is utility of a purchase of x_i units of the product reduced by a payment for their purchase plus a value of the other goods from the market basket. When a condition of a standardized zero value of utility at a zero consumption of the product from (2) validates, we get

$$u_i(x_i) \equiv r_i(x_i) \qquad i = 1,2 \tag{3}$$

In other words, a consumer S_i is willing to pay for x_i units of the product a maximum price which correspondents to his feeling of satisfaction from the product purchase represented in monetary units. The function of utility can be thus, with a certain degree of approximation, perceived as a function representing a willingness of a consumer to pay for the studied product the maximum price $r_i(x_i)$. Function $r_i(x_i)$ has after all one more interesting economic interpretation. Its first derivation function $r_i'(x_i)$, which is a function of a marginal willingness of a consumer to pay a relevant price for a certain demand volume, in fact represents how much a consumer is willing to pay for the last unit purchased. So the value of the function of marginal willingness to pay represents the price *p* for which a consumer is willing to buy the whole amount of *x* units of goods:

$$r_i'(x_i) = p \qquad i = 1,2 \tag{4}$$

The function $r_i'(x_i)$ of marginal willingness of a consumer to pay an amount $r_i(x_i)$ de facto ultimately represents an inverse demand function of an *i*-th consumer

$$x_i = (r_i')^{-1}(p) \qquad i = 1,2 \tag{5}$$

Let's now examine a consumer's behavior optimization problem or a total utility maximization problem of a consumer who has the funds m_i at his disposal. He uses these funds to purchase a market basket (x_i, w_i), while he purchases x_i units of the studied product at a market price *p* and the variable w_i represents the total expenses related to the purchase of other goods from the market basket. The problem is to find the values of variables x_i and w_i so that the value of the utility function (1) would be maximal while respecting a budget constraint of a consumer. Mathematical programming problem for *i*-th consumer is analytically represented by:

$$f_i(x_i, w_i) = u_i(x_i) + w_i \to max$$

subject to (6)

$$px_i + w_i = m_i$$

$$x_i, w_i \geq 0$$

This optimization problem of mathematical programming represents maximization constrained extrema problem. Let's modify this problem to a standard form, which is a form of a minimization problem:

$$-f_i(x_i, w_i) = -u_i(x_i) - w_i \to min$$

subject to (7)

$$px_i + w_i = m_i$$

$$x_i, w_i \geq 0$$

Let's formulate a generalized Lagrangian function for this problem. Let us mention that a generalized Lagrangian function does not explicitly include the conditions of non-negative variables, these are accounted for implicitly in a Kuhn-Tucker optimality conditions. A generalized Lagrangian function of a mathematical programming problem (7) is:

$$\mathcal{L}_i(x_i, w_i, \lambda_i) = -u_i(x_i) - w_i + \lambda_i(px_i + w_i - m_i) \tag{8}$$

Kuhn-Tucker optimality conditions for Lagrangian function (8) of the *i*-th consumer S_i are translated in a form:

$$\frac{\partial \mathcal{L}_i(x_i, w_i, \lambda_i)}{\partial x_i} \geq 0 \qquad \frac{\partial \mathcal{L}_i(x_i, w_i, \lambda_i)}{\partial w_i} \geq 0 \qquad \frac{\partial \mathcal{L}_i(x_i, w_i, \lambda_i)}{\partial \lambda_i} = 0$$

$$x_i \frac{\partial \mathcal{L}_i(x_i, w_i, \lambda_i)}{\partial x_i} = 0 \qquad w_i \frac{\partial \mathcal{L}_i(x_i, w_i, \lambda_i)}{\partial w_i} = 0 \tag{9}$$

$$x_i \geq 0 \qquad w_i \geq 0$$

After we substitute an analytical form of the Lagrangian function (8) we can restate the Kuhn-Tucker optimality conditions (9) of the total utility maximization problem as follows:

$$-u_i'(x_i) + \lambda_i p \geq 0 \quad (9.1) \qquad -1 + \lambda_i \geq 0 \quad (9.4) \qquad p x_i + w_i - m_i = 0 \quad (9.7)$$

$$x_i(-u_i'(x_i) + \lambda_i p) = 0 \quad (9.2) \qquad w_i(-1 + \lambda_i) = 0 \quad (9.5)$$

$$x_i \geq 0 \quad (9.3) \qquad w_i \geq 0 \quad (9.6)$$

In other words, if a consumer aspires to identify an optimal consumer strategy $(x_i^\star, w_i^\star)$, meaning that a consumption of $x_i^\star$ units of the product with a differentiated price p and the expenses $w_i^\star$ of consumption of the other goods in a market basket maximize his total utility $f_i(x_i, w_i) = u_i(x_i) + w_i$, then such a Lagrange multiplier λ_i^* must exist, for which the Kuhn-Tucker optimality conditions (9) are met, i.e. the variables vector $(x_i^\star, w_i^\star, \lambda_i^*)$ is a solution to the system of equations and inequalities (9.1),..., (9.7).

We can derive some interesting consequences for the optimal combination of a supply and a price of a price-differentiated product from validation of the Kuhn-Tucker optimality conditions (9) for the total utility maximization problem (6)

a) Validity of the condition (9.7) guarantees that a consumer has precisely such an optimal consumer strategy $(x_i^\star, w_i^\star)$, that means such an optimal consumption of $x_i^\star$ units of the product with a differentiated price and on optimal expenses $w_i^\star$ of consumption of the other goods in a market basket, which he can implement using available financial resources m_i. In other words the optimal consumer strategy $(x_i^\star, w_i^\star)$ meets the condition of a budget constraint.

b) Let's now analyze a structure of an optimal market basket meeting the obvious assumption that a consumer has some other goods in his market basket than the studied product. That means that a variable representing the expenses $w_i^\star$ of consumption of the other goods in the market basket is positive $w_i^\star > 0$. Validity of (9.5) results in the optimal value of Lagrange multiplier $\lambda_i^* = 1$. However if $\lambda_i^* = 1$ and (9.2) and (9.1) validate, then at the same time for the optimal positive volume of consumption $x_i^\star > 0$ stands

$$-u_i'(x_i^\star) + \lambda_i^* p = 0 \quad \wedge \quad \lambda_i^* = 1 \Rightarrow$$

$$p = u_i'(x_i^\star) \tag{10}$$

Relation (10) represents a significant phenomenon of consumer behavior in the conditions of differentiated price. Above all we realize that (10) in fact represents an inverse demand function or a price-demand function and

$$p = p_i(x_i) = u_i'(x_i) \tag{11}$$

Price-demand function of the *i*-th consumer then determines the price $p = p_i(x_i)$, for which a consumer is willing to buy x_i units of goods, while based on (10) this price is equal to marginal utility $u_i'(x_i)$ corresponding to purchase of x_i units of the good.

Therefore a consumer increases his consumption of the price-differentiated product as long as its price is lower than marginal utility $u_i'(x_i)$ corresponding to the last purchased unit of the product. Marginal utility of the last purchased unit

$$u_i'(x_i) = u_i(x_i) - u_i(x_i - 1)$$

in monetary units represents an increase of utility caused by purchase of the last, x_i-th unit of the product. Since marginal utility corresponds to the product price, a consumer spends exactly the same amount to purchase this last unit than an increase of his utility.

c) At last let's address a theoretical yet methodologically interesting hypothesis when a consumer decides to spend all his available funds to purchase solely the product with differentiated price. In this case his optimal expenses $w^\star$ of consumption of the other goods in the market basket are zero $w^\star = 0$. For $x_i^\star$ based on (9.7) stands

$$x_i^{\star} = \frac{m_i}{p} > 0 \tag{12}$$

In this situation the optimal value of Lagrange multiplier λ_i^* resulting from optimality conditions (9.4) and (9.5) belongs to the interval $\langle 1, \infty)$. The case when $\lambda_i^* = 1$ is analyzed in section (b). For its value higher than one $\lambda_i^* > 1$, and for positive consumption of the product $x_i^{\star} > 0$ resulting from (9.2), the optimality condition (9.1) is realized as an equation and

$$-u_i'(x_i^{\star}) + \lambda_i^* p = 0.$$

At the same time $\lambda^* > 1$, so the Lagrange multiplier λ^* can be represented in a form $\lambda_i^* = 1 + \varepsilon, \epsilon > 0$ and therefore

$$-u_i'(x_i^{\star}) + (1+\varepsilon)p = 0 \qquad / \frac{1}{1+\varepsilon}$$

$$p = \frac{u_i'(x_i^{\star})}{1+\varepsilon} \tag{13}$$

Assuming that $1 + \varepsilon > 0$, the relation between marginal utility of the optimal purchasing volume of the product $x_i^{\star}$ and its market price *p* is

$$p < u_i'(x_i^{\star}) \tag{14}$$

This situation shows a favorable position of a consumer on the market of the studied product when market price *p* is lower than marginal utility $u_i'(x_i^{\star})$. In other words, in this specific case the market price is lower than the willingness of a consumer to pay the sum corresponding to his marginal utility. In this situation a consumer dedicates all his funds to purchase the studied product in the volume given by (12) and he doesn't purchase the other goods at all.

4 Conclusion

Based on the formalized analytical tools we showed that if a producer has enough market power to not only accept the market price but to be able to significantly influence and create it, he can quite effectively use his knowledge of consumer behavior to optimize a combination of supply and price of his product. As a matter of fact, it is a rational use of the information complex about the behavior of a consumer with specifically structured market basket, where they separately analyze consumer's utility regarding purchase of optimal volume of a price-differentiated product and this utility is represented in monetary units. Other goods in the market basket are being studied without any further specification of their volumes or range as one "aggregated good" and utility regarding purchase of these other goods is represented in monetary units as a simple sum of expenses spent on the purchase.

Significant is a fact that a company with a substantial market position in order to optimize its behavior at determining a combination of supply and differentiated price of a product, derives from a thorough analysis of consumers behavior while using analytical tools – demand functions and utility functions.

A mathematical programming problem which maximizes utility function of a consumer at budgetary restraints was examined in this article as it is a relevant tool for consumer behavior analysis. We showed that Kuhn-Tucker optimality conditions formulated for this optimization problem confirm the validity of consumer decision making schemes at optimization of his demand in the conditions of differentiated prices.

In a similar way we examined a monopoly profit maximization problem in the conditions of differentiated prices and we showed the fundamental schemes of price differentiation which monopoly can effectively use as a result of its market position to maximize its revenues as well as its profits from selling the products with differentiated price.

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Equilibrium analyses in allocation games

Petr Fiala[1]

Abstract. The paper considers a situation where a number of agents are connected in some network relationship. The applications are quite wide and varied. Game theory is a very powerful framework for studying decision making problems, involving a group of agents. Allocation games examine the allocation of value among agents connected by a network. Agents can be seen acting independently and willing to selfishly maximize their utility. Allocation games model the problem of maximizing network utility from the perspective of distributed non-cooperative agents. Many challenging questions arise when one seeks to design a network in such a way so as to optimize a given utility measure. Approaches for searching Nash equilibrium are presented. We provide results with respect to equilibrium existence, computation, convergence, efficiency and quality. Quality results are analyzed by the price of anarchy and the price of stability.

Keywords: networks, game theory, allocation, Nash equilibrium

JEL Classification: C44
AMS Classification: 90B50

1 Introduction

The allocation of tasks in a network of freely accessible shared resources often leads to congestion of single resources and in the worst case to a devaluation of the entire resource network utility. Two main solution approaches suggest themselves in this context:

- To achieve a load-balancing effect the unselfish distribution of tasks between the resources can be targeted by the agents in the sense of cooperative problem solving strategies (see [1]).
- To avoid excessive use of the resources, one can introduce self-interestedly acting economic agents that manage and own individual resources in the network (see [7]).

The use of cooperative agents in resource allocation games for distributed resources is not widely spread. The game resource allocation schemes are often subsumed under the description networking games. Cooperative games for resource sharing often employ Nash bargaining approach, where the bargainers negotiate for a fair contract point from the set of all feasible solutions (see [5]). The outcome is chosen based on a-priori defined fairness criteria: symmetry, Pareto optimality, and invariance with respect to utility transformations.

The concept of using self-interested agents to formulate allocation mechanisms in a game theoretical setting is closer to the classical market concept than solutions employing cooperative strategies. Network resource management is often carried out by using congestion-based pricing and routing capacity allocation. Most non-cooperative allocation strategies in distributed systems consist of following step:

- The formulation of utility functions for the system participants.
- The formulation of best response strategies.
- The existence of Nash equilibrium is proved in the system of multiple agents
- Efficiency is measured compared to achievable welfare.
- Fairness of the simulation result is checked.

In the majority of network allocation games, usually users were modeled as players. In the paper players are associated only with links in the network. It is assumed that network routers may be considered as autonomous entities, which operate independently. Only limited coordinating communication is allowed between them.

The central network resource allocation problem is the network utility maximization problem. Game theory is a very powerful framework for studying decision making problems. In the recent years a subfield known as algorithmic game theory has emerged, combining game theory and algorithms design. The problems deal with the use of algorithmic game theory include establishing the existence of Nash equilibria, designing computation-

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ally efficient procedures for determining players' strategies, and computing the price of anarchy and price of stability.

The rest of the paper is organized as follows. In Section 2, the problem formulation is provided. Approaches for computing strategies are presented in Section 3. Solution properties are discussed in Section 4. Section 5 presents conclusions.

2 Problem formulation

In the considered problem the network consists of a set of m links, each with capacity $c_i > 0$, $i = 1, 2, \ldots, m$. The state of the network is described by a vector $\mathbf{c} = (c_1, c_2, \ldots, c_m)$ of resource capacities. There are n flows (transmissions), defined by a routing matrix $\mathbf{A} = [a_{ij}]$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, where

$a_{ij} = 1$, if flow j traverses link i, and

$a_{ij} = 0$, otherwise.

The decision vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ represent partitioned allocation of capacity for each of the m flows. Each flow j is characterized by the transmission rate $x_j \geq 0$, and an associated utility measure $u_j(x_j)$, which is assumed to be strictly increasing concave and twice-differentiable function of transmission rate.

The network resource allocation problem is formulated as follows (see [3]):

$$u(\mathbf{x}) = \sum_{j=1}^{n} u_j(x_j) \tag{1}$$

subject to

$$\mathbf{Ax} \leq \mathbf{c}, \tag{2}$$

$$\mathbf{x} \geq \mathbf{0}, \tag{3}$$

In this problem the so-called isoelastic utility functions are used in the form

$$u_j(x_j) = w_j \ln x_j. \tag{4}$$

It was shown (see [4]) that such class of functions leads to proportionally fair allocations of transmission rates thus is typically employed in the analysis of network resource allocation problems.

The interaction between concurrent decision-making agents can be modeled as a network game. Each link in the network is associated with one player. Players must decide how to allocate their total capacities $c_i > 0$, $i = 1, 2, \ldots, m$, among the set of flows traversing their corresponding link. Each player makes a decision individually. The decision of player i, called player's strategy, is denoted $\mathbf{s}_i = (s_{i1}, s_{i2}, \ldots, s_{in})$, where s_{ij} is the fraction of link's capacity i allocated for flow j. The player's choice is restricted only to feasible decisions, satisfying

$$\sum_{j=1}^{n} a_{ij} s_{ij} \leq c_i \tag{5}$$

The transmission rate of a single flow is limited by the minimal allocation of some link along the path of that flow. The path is defined by the routing matrix $\mathbf{A}$. The player's payoff is computed as the value of weighted utility of transmission rates of all the flows passing through the corresponding link. The strategy of the game is defined as $\mathbf{S} = [\mathbf{s}_1, \mathbf{s}_1, \ldots, \mathbf{s}_m]$. The player's i payoff is given by:

$$v_i(\mathbf{S}) = \sum_{j=1}^{n} t_j a_{ij} u_j(\min_{k:a_{kj}=1} s_{kj}), \tag{6}$$

where $t_j \geq 0$ is a weight assigned to the flow j.

Two types of game's payoff matrices are considered:

1. A game with uniform payoffs, that is, all weights $t_j = 1$, $j = 1, 2, \ldots, n$.
2. A game with the set of weights defined by

$$t_j = \frac{1}{\sum_{k=1}^{m} a_{kj}}, \tag{7}$$

that is, the reciprocal of length of the path associated with flow j. This weight is equal for all links.

The social welfare is defined as

$$v(\mathbf{S}) = \sum_{i=1}^{m} v_i(\mathbf{S}), \tag{8}$$

that is the sum of all payoffs.

The total utility of the network, given as the objective of problem (1), corresponds to the social welfare with weights (7).

3 Computational procedures

Typical distributed methods of solving such network problems are Lagrangian relaxation-based methods for finding saddle point, or direct interior-point method decentralization. There are also algorithms for computing certain feasible strategies (see [2]). In this section two simple algorithms are presented.

Local allocation algorithm

Each player i, $i = 1, 2, \ldots, m$, solves a local concave optimization problem in the form:

$$\mathbf{s}_i^{(1)} = \arg\max_{\mathbf{s}_i \in D_i} \sum_{j=1}^{n} t_j a_{ij} u_j(s_{ij}), \tag{9}$$

where

$$D_i = \left\{ \mathbf{s}_i \middle| \sum_{j=1}^{n} a_{ij} s_{ij} \le c_i, \forall j, a_{ij} s_{ij} = s_{ij} \right\}, \tag{10}$$

For the class of utility functions (4), the solution can be derived analytically:

$$s_{ij} = a_{ij} c_j \frac{t_j w_j}{\sum_{l=1}^{n} t_l w_{jl} a_{il}} \tag{11}$$

This algorithm is a realization of the simplest rational strategy, which can be computed without any communication between players. Due to this fact, there are no synchronization issues concerning implementation in a networked environment.

Example 1

Consider two links ($m = 2$), the first with capacity $c_1 = 20$ and the second with capacity $c_2 = 80$. There are three flows ($n = 3$); first flow passes through both links, second flow single link 1 and third flow uses link 2. Thus, the routing matrix is

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

All utility functions are assumed to be in form

$$u_j(x_j) = \ln x_j.$$

The local algorithm computes the following strategy vectors for both players (links), $\mathbf{S}^{(1)} = (\mathbf{s}_1^{(1)}, \mathbf{s}_2^{(1)})$:

$$\mathbf{s}_1^{(1)} = (10, 10, 0),$$

$$\mathbf{s}_2^{(1)} = (40, 0, 40).$$

The payoffs of players (i.e. the local utilities of their corresponding flows) are:

$$v_1\left(\mathbf{S}^{(1)}\right) = \ln 10 + \ln 10,$$

$$v_2\left(\mathbf{S}^{(1)}\right) = \ln 10 + \ln 40.$$

If $\mathbf{S}^{(1)}$ were a Nash equilibrium, no player would have an incentive to unilaterally deviate from this allocation. However, since the capacities in both links are uneven, a fair allocation is suboptimal. The high-speed link with $c_2 = 100$ should promote the flow 3, as it does not pass through the bottleneck link with $c_1 = 10$. Thus the following change of player 2 strategy:

$$\mathbf{s}_2^{(1)} = (10, 0, 70)$$

gives a better outcome

$$v_2\left(\mathbf{S}^{(1)}\right) = \ln 10 + \ln 70.$$

Iterative allocation algorithm

The local allocation algorithm can be easily improved, if we allow players to interact in the following way. Initial allocations are computed with the use of the local algorithm. After these allocations are computed, all sources start sending data and transmission rates of all flows gradually increase from zero. The rate of a flow stops accelerating as soon as some link on the path becomes a bottleneck, i.e. the rate reaches minimal allocation of some link along its path. Such flows are called saturated. This means that it is no longer possible to increase its rate. However, other links on the path of such flow may have unused capacity. Thus it is possible to assign this capacity among the non-saturated flows, increase their rates and repeat that until all the flows become saturated.

Following additional notations are used:

- q iteration,
- $\mathbf{S}(q)$ the strategy vector computed in iteration q,
- $\mathbf{S}^{(2)}$ the final strategy vector returned by the algorithm,
- R_i the set of flows traversing link i,
- $R(q)$ all the flows that are not yet saturated in iteration q,
- $L(q)$ all the links that ran out of capacity in in iteration q,
- $i_0(q)$ the smallest index of link that runs out of capacity in iteration q.

The procedure of the iterative allocation algorithm can be summarized in following steps:

Step 1 (Initialization)

$$\text{Let } q = 1.\ R(q) = \{1, 2, \ldots, n\}.$$

$$\text{Execute the local allocation algorithm (solving problem (9)-(10)), } \mathbf{S}(q) = \mathbf{S}^{(1)}.$$

$$L(q) = \left\{ i : \sum_{j=1}^{n} a_{ij} \left(\min_{k:a_{kj}=1} s_{kj}^{(1)} \right) = c_i \right\}.$$

Step 2 (Next iteration)

$$\text{If } q < m, \text{ let } q = q + 1.$$

Step 3 (Saturated flows are removed)

$$R(q) = R(q-1) - R_{i_0(q-1)}$$

Step 4 (Strategy determination)

$$\mathbf{S}(q) = [\mathbf{s}_1(q), \mathbf{s}_2(q), \ldots, \mathbf{s}_m(q)].$$

$$\mathbf{s_i}(q) = \arg \max_{\mathbf{s}_i \in D_i(q)} \sum_{j \in R(q)} t_j a_{ij} u_j(s_{ij})$$

where

$$D_i(q) = \left\{ \mathbf{s}_i \,\middle|\, \sum_{j=1}^{n} a_{ij} s_{ij} \le c_i, \forall j, 0 \le s_{ij} \le a_{ij} s_{ij}, \forall j \in \bigcup_{p=1}^{q-1} R_{i_0(p)}, s_{ij} = \min_{k:a_{kj}=1} s_{ij}(q) \right\}.$$

Step 5 (Stopping rule)

If it holds $R(q) = \emptyset$ then terminate the algorithm and $\mathbf{S}^{(2)} = \mathbf{S}(q)$. Otherwise go to Step 2.

Example 2

Consider the same data as are given in Example 1 and the game with uniform payoff matrix. At the beginning of the execution of the iterative algorithm all flows are not saturated, $R(1) = \{1, 2, 3\}$, so each link allocates capacities maximizing its own objective, and the players' strategies are the same as in Example 1, i.e.:

$$\mathbf{s}_1(1) = (10, 10, 0),$$
$$\mathbf{s}_2(1) = (40, 0, 40).$$

Once all strategies are computed, the sources may transmit data at rates:

$$\mathbf{x} = (10, 10, 40).$$

Only first link becomes filled ($L(1) = \{1\}$), and the first and the second flows are saturated. There is unused capacity in the second link, which may be used in the next step.

For iteration $q = 2$, the first link does not change its strategy since it is filled. The second link sets allocations for saturated flows equal to their current transmission rates (i.e. minimal capacity allocated for these flows on their paths) and calculates the allocations for non-saturated flows (in this case for $j = 3$) to maximize its objective:

$$\mathbf{s}_2(2) = (10, 0, 70).$$

Since both stopping conditions are met in the next iteration, i.e. $q = 3 > m = 2$ and $R(2) = \emptyset$, the iteration algorithm stops with the following players' strategies, which constitute a Nash equilibrium:

$$\mathbf{s}_1^{(2)} = (10, 10, 0),$$

$$\mathbf{s}_2^{(2)} = (10, 0, 70).$$

4 Solution properties

First, the problem of determining Nash equilibria in the formulated network game is considered. The existence of Nash equilibrium for considered game can be concluded from the Rosen's theorem (see [6]), which states that a Nash equilibrium exists for a Nash game if the payoff function for each player is concave with respect to their own strategy and continuous with respect to the strategies of all players and the strategy set for each player is convex and compact.

It is however not obvious whether a given strategy is equilibrium, or how to compute one efficiently. The solution properties are summarized in theorems and corollaries, proofs are given in [2].

In general the local allocation algorithm does not produce a state of equilibrium; a player may be better off changing its allocation without informing the other players (see Example 1).

The strategy computed by iterative allocation algorithm dominates the strategy computed by local allocation algorithm. The iterative allocation algorithm provides even Nash equilibrium.

Theorem 1. *Strategy* $\mathbf{S}^{(2)}$, *provided by the iterative allocation algorithm, constitutes a pure Nash equilibrium.*

The result can be even strengthened, according to the construction of strategy $\mathbf{S}^{(2)}$.

Definition 1. *Strategy* $\boldsymbol{S^0}$ *is* ***non-dominated*** *(strongly Pareto-optimal) if there is no strategy* $\mathbf{S} \neq \mathbf{S^0}$ *such that:*

$v_i(\mathbf{S}) \geq v_i(\mathbf{S^0})$, $i = 1, 2, \ldots, m$, *and*

there exists such k that $v_k(\mathbf{S}) > v_k(\mathbf{S^0})$.

Theorem 2. *Strategy* $\mathbf{S}^{(2)}$ *is non-dominated* (*strongly Pareto-optimal*).

Second, a reversed problem of game design is considered: given an optimal solution of the network utility maximization problem, does it constitute Nash equilibrium of some game variant?

Theorem 3. *For the game with uniform payoffs* ($t_j = 1$, j =1, 2, …, *n*), *the global optimum of the network utility maximization problem (1)–(3) is a Nash equilibrium*.

There is a concept to measuring how efficient Nash equilibrium is in a specific game. The **price of stability** (*PoS*) is the ratio between the value of the best Nash equilibrium to the value of the optimal solution. The **price of anarchy** (*PoA*) is the ratio between the value of the worst Nash equilibrium to the value of the optimal solution.

Corollary 1. *For the game with uniform payoffs the price of stability PoS* = 1.

Theorem 4. *For the game with the set of weights* $t_j = \frac{1}{\sum_{k=1}^{m} a_{kj}}$, *the global optimum of problem* (1)–(3) *in general is not a Nash equilibrium.*

Theorem 5. *If there are no local* (*single-link*) *flows, i.e. for all j, j* = 1, 2, …, *n*, $\sum_{i=1}^{m} a_{ij} > 1$, *any strategy such that*

$$\forall i_1, i_2 \, \forall j\colon a_{i_1 j} = a_{i_2 j}=1,\ s_{i_1 j} = s_{i_2 j}$$

constitutes a pure Nash equilibrium of the considered game.

Remark 1. *Although the strategies characterized in Theorem 5 are points of equilibrium of the considered game, they can be arbitrarily bad in terms of the players' outcomes; the price of anarchy PoA is unbounded.*

5 Conclusions

The network resource allocation problem was formulated. The problem is applicable in several practical fields. The interaction between concurrent decision-making agents can be modeled as network games. Two types of game's payoff matrices are considered: a game with uniform payoffs and a game with the set of weights.

Typical distributed methods of solving such network problems are Lagrangian relaxation-based methods for finding saddle point, or direct interior-point method decentralization. In the paper two simple algorithms are presented and illustrative examples are solved. The algorithms were tested on simulated examples with very good results. Properties of game solutions are discussed. Interesting results for both types of games were presented.

The approach seems to be useful and promising for next research. There are some possible extensions of the approach and some areas for further research. Other types of games can be analyzed.

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Quantitative Evaluation of the EU´s Export Potential to Russia Using the Gravity Model

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Abstract. The European Union (EU) is the largest trading power in the world. Although its share in the world exports of manufactured goods is gradually declining due to various factors, it still occupies the first place in both exporting industrial goods and commercial services. The EU's share in world merchandise exports is about 15 % and exports of commercial services almost 25 % (WTO, 2011). However, the EU not only exports but also imports some commodities from third countries and its trade balance is negative in the long term. The negative trade balance is created in the import of raw materials and energy such as oil and natural gas. These minerals are imported into the EU mainly from Russia, which therefore occupies the second position among the EU main importers. The dependence of the EU-27 on imports of energy resources from Russia varies in the individual Member States.
The aim of this paper is to evaluate the untapped export potential of the EU to Russia by the member countries in 2000-2011 and to propose appropriate measures to improve export in these countries. The quantitative assessment of the export potential of the member countries of the EU to Russia will be done through a gravity model, which is very often used for the purposes of analysis in this area. Removing the unused export potential to Russia would help to achieve trade balance throughout the EU.

Keywords: foreign trade, export potential, trade balance, gravity model

JEL Classification: C23, F13, F14, F15, F41
AMS Classification: 62M10

1 Introduction

The European Union (EU) is the largest world exporter in the area of merchandise as well as commercial services trade. The larger portion of total EU trade is carried out in the area of merchandise trade (the value of 3.2 billion euros in 2011) than in commercial services trade (1.1 billion euros in 2011) [8]. But to maintain the first position among the leading world traders in the future it is necessary to find new ways leading to the growth of the EU's competitiveness and to strengthen trade relations with fast-growing countries such as Brazil, Russia, India, China and others. The growing incomes in these countries contributed to the creation of a new middle class that is able to consume not only more domestic products, but also products that are imported from other countries, including the EU. But the EU needs not only exports but also imports. From the long term point of view, the largest share of the EU imports belongs to mineral fuels, lubricant and related materials (SITC 3). In 2012, the share of mineral fuels on the total EU imports reached 30.5 % and another 4.5 % belonged to imports of raw materials (SITC 2+4). It means that industrial production in the EU is highly dependent on imports from other countries. The dependence of the EU on energy products has a negative influence on the EU trade balance as well as on the EU's competitiveness in world trade.

The EU has had the largest volume of energy imports with Russia, which is the EU's most important near neighbor. The EU is the first customer of the main Russian export – energy. 80 % of all Russian oil exports, 70 % of all Russian gas exports and 50 % of all Russian coal exports go to the EU. While the EU is by far Russia's biggest overall trade partner, Russia is the European Union's third largest trade partner. In 2012 alone the total volume of trade between the EU and Russia reached 336 billion euros and around 75 % of foreign direct investment in Russia is of European origin [6]. But in the long run, the EU has recorded a trade deficit in merchandise trade with Russia that reached 90.1 billion euros in 2012 [6,7]. Conversely, the EU's trade in services with Russia was in surpluses in the previous years. When the EU is highly dependent on energy imports from Russia, it is necessary to find untapped possibilities in the EU exports to Russia.

The aim of this paper is to evaluate untapped export potential to Russia by the EU's member countries in 2000-2011 using the gravity model. Firstly, we will introduce the theoretical background of the gravity model

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and its application to the EU export flows to Russia. Particular attention will be paid to our empirical findings about real and untapped export potential of the individual member countries of the European Union. Afterwards, the trade balance of the individual member countries with Russia will be presented in the period of 2000-2011. There is a hypothetical assumption that countries that registered an untapped trade potential also achieved a trade deficit with Russia. Finally, the institutional framework for increasing trade cooperation between the EU and Russia will be introduced.

2 Methodology and data

Section 2 concentrates on providing the theoretical foundations of the gravity model, followed by a presentation of the methodology and the data used in the empirical analysis.

2.1 Theoretical development and model identification

In order to find potential export flows of the EU to Russia, the gravity model of trade will be used. Gravity equations are one of the most popular tools in empirical studies in the area of international trade. Gravity models utilize the gravitational force concept that is based on Newtonian physics, as an analogy to explain the volume of trade, capital flows, and migration among the countries of the world. In the area of international trade, the gravity model can be applied to four categories of issues: estimating the cost of the border, explaining trade patterns, identifying effects related to regionalism and, finally, calculating trade potential.

The model was first used by Tinbergen (1962) and Linneman (1966). Although the gravity model was empirically successful in explaining trade flows and the prediction of trade potential, the model lacks strong theoretical foundations. In response to this criticism several studies were written to explain theoretical derivation. Anderson was the first who gave a theoretical explanation based on economic theory. Anderson helped to explain the presence of income variables in the gravity model by specifying demand. According to Anderson, products traded internationally are differentiated by the country of origin (the so called "Armington assumption"). This approach was also adopted by Bergstrand, who specified the supply side of economy more thoroughly. J. H. Bergstrand [3] presents empirical evidence that the gravity equation is a reduced form of a partial equilibrium subsystem of a general equilibrium model with nationally differential products. Deardorf [4] derived a model using the Hecker-Ohlin model. This theory predicts that trade patterns will be based on relative factor abundance. Those countries with a relative abundance of one factor are expected to produce goods that require a relatively large amount of that factor in their production. While the Hecker-Ohlin model and Ricardian's comparative advantage theories are accepted trade theories, they have suffered empirical problems. An alternative theory, first proposed by Staffan Linder, predicts that patterns of trade will be determined by the aggregated preferences of goods within countries. Those countries with similar preferences are expected to develop similar industries and trade in differential but similar industries. But the empirical validity of the Linder hypothesis is unclear. Several studies have found a significant impact of the Linder effect, but others have had weaker results. The monopolistic competition model of a new trade theory is another approach that provides theoretical foundations to the gravity model. In this model, the product differentiation by the country of origin approach is replaced by product differentiation among product firms, and the gravity model serves as an explanation of intra-industry trade. Anderson and Wincoop [1] have also contributed to the theoretical development of the gravity model with the method "the border puzzle". They find that borders reduce bilateral national trade levels by plausible though substantial magnitudes. On the whole, the different theories underlying the gravity model lead to different trade policy implications.

The econometric estimation of gravity equations is connected with using different variables. Tinbergen's original gravity model has been used to estimate the volume of bilateral trade as a function of two main components – the economic size of countries and the distance between them. The traditional gravity model takes a log-linear model form (1):

$$\ln(FT_{ij}) = \beta_0 + \beta_1 ln(GDP_i) + \beta_2 \ln(GDP_j) - \beta_3 \ln(D_{ij}) + \varepsilon_{ij} \quad (1)$$

where FT_{ij} is the volume of trade from country $_i$ to country $_j$, GDP_i and GDP_j present the economic size of countries $_{i,j}$, $D_{i,j}$ denotes the distance between the two countries and $\varepsilon_{i,j}$ is the random component with an expectation equal to 1.

The following empirical studies also included other explanatory variables. Kepaptsoglou et al. [11] presents almost sixty studies that were written during a 10-year period (1999-2009), in which the dependent variables are population, area size, GDP per capita, dummy variables denoting borders, islands, former colonies, membership in a free trade area or customs unions, currency unions, etc. The main dependent variables are bilateral trade flows, exports, imports and foreign direct investments. Kovářová [10] used the gravity model to verify the

Linder hypothesis of foreign trade between the USA and the EU. She includes variables in the gravity model, such as the GDP of the considered countries, the geographical distance between the countries and the absolute value of per capita GDP differential between the two countries. The gravity model has also been recently used to assess the effect of the WTO on multilateral trade patterns. For example, Rose, Subramanian and Wei evaluated the magnitude of the WTO dummy after checking for the customary gravity effects. Lissovolik et al. [12] used the gravity model to evaluate Russia's trade determinants and patterns, with particular reference to the role of the WTO. Benedictis and Salvatici [2] encompass the trade preferences as another component of the gravity model and estimate the effect of trade preferences on the EU trade flows. Despite the fact that the trade data are easily available, they struggle on the problem of missing data when evaluating the impact of the EU's preferential trade policies on developing countries' trade flows. But in many cases, gravity models have significant explanatory power, leading Deardorff [4] to refer to them as a "fact of life".

By defining the gravity model in this way, it is possible to solve the problem cross-sectionally or by using panel data. In the case of cross-sectional analysis, the data for the individual year are used. Panel data analyses enable us to explore the development of international trade among countries in a period of several years. Panel data are used in most studies for periods of at least 5 years. There are generally three types of panel data regression analyses: independently pooled panels, fixed effect models and models with random effect.

2.2 The gravity model of the EU export potential to Russia

The calculation of export potential between the individual EU member states and Russia based on the gravity model firstly supposes the arrangement of the EU member states in alphabetical order, and bilateral export flows with Russia within the sample are then estimated. Such simulated bilateral exports will be compared with the observed ones in order to infer bilateral export potentials. This methodology will be applied at the aggregate level.

The dependent variable is the natural logarithm of the EU´s exports in current USD calculated individually for 27 member countries. The explanatory variables in the model are the natural logarithm of the gross domestic product of the EU member states and Russia measured in current USD, the natural logarithm of the distance between the capitals measured in kilometers. The distance between the EU member states with Russia is measured using the great circle formula, which takes into account the longitude and latitude of the capitals. [13] For each scheme, we estimate the following log-linear equation (2):

$$\ln(EX_{EUMSRt}) = \beta_0 + \beta_1 \ln(GDP_{EUMSt}) + \beta_2 ln(GDP_{Rt}) - \beta_3 D_{EUMSR} + \varepsilon_{EUMSRt} \quad (2)$$

where EX_{EUMSRt} are exports from the EU member state to Russia in period *t*, β_0 are specific effects associated with each bilateral flow, GDP_{EUMSt} indicates the GDP of the EU member countries in period *t*, GDP_{Rt} denotes the GDP of Russia in period *t*, D_{EUMSRt} is the great circle distance between the EU member countries (*EUMS)* and Russia (*R)* and ε_{EUMSRt} is random effect.

Panel data were used for periods 2000-2011. Elasticities were estimated by means of an Ordinary Least Squares (OLS) cross-country regression on 324 observations for 27 countries. All data were obtained from the database of the United Nation Conference on Trade and Development (UNCTAD).

3 Empirical results

The cross-country OLS regression results for the gravity equations (2) are reported in table 1. The overall performance of the model seems to be surprisingly good, with high R^2 value around 0.85. All explanatory variables are found to be highly significant, indicating that the gravity model is appropriate and effective in explaining the EU member countries export flows to Russia.

Variable	Coefficient	t-Statistic	Probability
$\ln(GDP_{EUMSt})$	1.137610	39.18736	0.0000
$\ln(GDP_{Rt})$	0.446382	13.80452	0.0000
$\ln(D_{EUMSRt})$	-2.815240	-26.12049	0.0000

Table 1 Estimated coefficients
Source: self-elaboration using EViews 7

Let us first look at the coefficient on the export structure variable to identify the underlying trade model of the EU member states export flows to Russia. The presented results confirm that bilateral export flows of the individual EU member countries to Russia positively depend on their economic size measured by GDP and

negatively depend on their distance measured by the distance between their capitals. The coefficient β_1 shows a positive value with a high statistical significance. The estimated coefficient shows that, holding other variables constant, a 1 percentage point increase in the GDP of the EU member states will result in a roughly 1.14 percentage point increase in the EU member countries export flows to Russia. The coefficient β_2 also indicates a positive value. When the GDP of Russia increases by 1 percentage point, it causes a roughly 0.45 percentage point increase of the export flows to Russia from the EU member countries. On the contrary, the coefficient β_3 shows a negative value. When the distance increases by 1 percentage point, then the export flows of the EU member countries to Russia decrease by 2.82 percentage points. The estimated coefficients β_1, β_2 and β_3 are consistent with the economic theory. The variable of GDP expresses the economic size of the two countries, in terms of both the production capacity and the size of market. Larger countries, with a greater production capacity, are more likely to achieve economies of scale and increase their exports based on their comparative advantage. They also possess larger domestic markets that are able to absorb more imports. From this point of view, an increase of the GDP exporter as well as importer has a positive impact on bilateral trade among countries. However, in most gravity models the GDP of the exporter has a higher significance for bilateral trade than the GDP of the importer. On the other hand, the distance variable has a negative influence on bilateral trade flows. A decrease in the distance variable indicates that export as well as import with geographically distant countries increases relative to trade with geographically closer countries, whereas an increase indicates that trade with closer countries increases faster than that with distant countries.

In order to find the export potential of the individual member countries, the export potential was compared with the actual export volume. The export potential is simply the predicted export volume from the gravity estimation (2). The difference between the predicted and actual export flows can be interpreted as an untapped export potential. Table 2 shows that the export potential of the EU member countries to Russia ranged from 90.3 % to 105.8 % in 2011. The fifteen EU countries reached a value of more than 100 % and the twelve EU countries recorded an export potential below 100 %. While Slovenia reached the highest use export potential (105.8 %), Malta reached the highest untapped export potential to Russia (90.3 %). Bearing in mind the fact that all the EU member countries use the Common commercial policy that is applied to non EU member countries, the cause of the untapped export potential of the twelve EU member countries cannot be explained by trade barriers, but by the economic structure and comparative advantages that these countries have in the given commodity items.

Country / Code	**Actual export flows (Ex_r)**	**Predicted export flows (Ex_p)**	**Ex_r/Ex_p (%)**
Austria / AT	22,28	22,14	100,61
Belgium / BEL	22,59	21,54	104,88
Bulgaria / BGR	20,39	19,64	103,84
Cyprus / CYP	16,69	18,02	92,63
Czech Republic / CZE	22,24	21,40	103,90
Denmark / DNK	21,41	22,07	97,01
Estonia / EST	21,33	20,66	103,20
Finland / FIN	22,71	23,40	97,05
France / FRA	23,06	23,18	99,44
Germany / DEU	24,62	24,70	99,67
Greece / GRC	20,12	20,95	96,01
Hungary / HUN	21,97	21,08	104,23
Ireland / IRL	20,38	19,97	102,01
Italy / ITA	23,28	23,04	101,01
Latvia / LVA	20,96	21,02	99,70
Lithuania / LTU	22,26	21,66	102,72
Luxembourg / LUX	19,23	19,15	100,44
Malta / MLT	14,72	16,31	90,30
Netherlands / NLD	23,01	22,23	103,51
Poland / POL	22,86	23,43	97,56
Portugal / PRT	19,62	19,12	102,58
Romania / ROU	21,07	21,54	97,82
Slovakia / SVK	21,59	20,55	105,07
Slovenia / SVN	20,44	19,32	105,76
Spain / ESP	21,97	21,55	101,96
Sweden / SWE	22,13	23,29	95,00
United Kingdom / GBR	22,75	23,02	98,85

Table2 The rate utilization of export potential of the EU member countries to Russia in 2011 (%)
Source: own calculations according to data from the UNCTAD Database [13]

4 Trade balance of the EU member states with Russia

The EU has recorded a deficit in merchandise trade with Russia in the long term. It is caused by the dependence of the EU on energy imports that represent the main commodity item of the EU imports from Russia. Table 3 records the trade balance of the individual EU member states with Russia in the period of 2000-2011. The trade balance (export minus import) was calculated from the data obtained from the UNCTAD Database (UnctadStat). The data includes international trade in goods and services and is published in U.S. dollars (USD). Most of the countries recorded a trade deficit with Russia on a different level. The Netherlands, Poland, France, Spain and the United Kingdom reached the highest level of trade deficit in the individual years. Trade surpluses reached in the given years by some member countries are displayed in the grey fields in Table 3. Ireland, Denmark, Luxembourg and Slovenia belong to the countries that reached trade surpluses with Russia most often. Positive changes in trade balance with Russia have been recorded in Austria since 2007. The development of trade balance with Russia was variable especially in Germany. While in 2011 Germany recorded the largest trade surplus among the other EU countries in the amount of 7 950.1 million dollars, in 2001 it reached the largest trade deficit with Russia in the amount of 7 272.4 million dollars. On the whole, the variable level of the trade deficit of EU-27 with Russia in the reported period was influenced by currency fluctuations and changes in commodity prices, especially energy products on the world market. The analysis of the causes of the trade deficit in the individual member countries of the EU was not the object of this paper and presents another area for empirical research.

The hypothesis that has been set in the introduction of this paper was not completely confirmed. Contrary to the results of the trade analysis shown on the untapped export potential in Denmark and Germany in 2011, these countries recorded trade surpluses in merchandise trade with Russia in the same year. On the other hand, the hypothesis was confirmed in Cyprus, Finland, France, Greece, Latvia, Malta, Poland, Romania, Sweden and the United Kingdom. These countries recorded a trade deficit with Russia as well as an untapped export potential to Russia in 2011.

	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
AT	-0.548	-0.184	-0.070	-0.213	0.017	-0.697	-0.208	1.008	0.693	0.528	0.304	0.729
BEL	-0.540	-0.615	-0.559	-1.076	-1.880	-2.680	-2.409	-2.185	-2.608	-2.409	-3.685	-5.883
BGR	-1.062	-0.936	-0.831	-0.957	-1.363	-2.316	-2.536	-3.842	-5.007	-2.450	-3.280	-4.036
CYP	-0.104	-0.069	-0.131	-0.129	-0.064	-0.035	-0.044	-0.041	-0.026	-0.016	-0.047	-0.021
CZE	-1.575	-1.365	-1.564	-1.565	-1.234	-2.974	-3.374	-2.053	-4.229	-2.004	-2.466	-1.506
DNK	0.97	0.288	0.208	0.139	-0.060	-0.230	0.002	0.402	0.057	0.642	0.467	0.111
EST	-0.852	-0.775	-0.947	-0.828	-0.926	-1.164	-1.706	-0.867	-0.003	-2.081	-0.051	-0.157
FIN	-1.227	-0.600	-0.477	-1.053	-1.272	-1.054	-2.030	-2.317	-3.769	-4.179	-6.029	-7.950
FRA	-2.714	-2.490	-2.733	-3.772	-5.211	-5.885	-6.666	-7.387	-9.795	-5.869	-7.817	-8.996
DEU	-7.272	0.318	2.043	-1.411	-1.597	-6.265	-8.217	-0.822	-5.261	-6.216	-7.179	7.628
GRC	-0.836	-1.293	-2.004	-2.344	-2.537	-3.878	-3.939	-3.802	-5.939	-3.124	-5.934	-5.227
HUN	-2.171	-1.926	-1.807	-2.329	-2.535	-3.704	-4.165	-3.621	-6.226	-2.780	-3.471	-5.375
IRL	0.152	0.194	0.210	0.234	0.190	0.201	0.212	0.341	0.309	0.224	0.283	0.542
ITA	-2.521	-1.476	-1.141	-1.447	-2.028	-2.253	-0.443	0.294	2.208	-7.969	-8.935	-12.162
LVA	-0.293	-0.205	-0.222	-0.300	-0.395	0.341	-0.372	-0.521	-0.753	-0.373	-0.178	-0.046
LTU	-1.222	-1.102	-0.972	-1.435	-1.990	-2.948	-2.893	-1.822	-5.607	-3.293	-4.397	-5.692
LUX	-0.011	7.920	-1.184	0.020	-0.004	0.018	2.569	0.033	0.209	0.161	0.158	0.212
MLT	-0.035	-0.023	-0.064	-0.041	-0.044	-0.051	-0.139	-0.156	-0.298	-0.119	-0.366	-1.563
NLD	-2.242	-2.191	-3.853	-4.989	-7.852	-14.979	-21.433	-23.826	-32.764	-20.935	-32.232	-25.867
POL	-3.774	-3.382	-3.094	-3.722	-3.548	-5.025	-7.434	-7.921	-11.628	-7.819	-11.583	-16.985
PRT	-0.117	-0.277	-0.200	-0.300	-0.554	-0.883	-0.931	-0.946	-0.096	-0.257	-0.107	-0.138
ROU	-1.032	-1.101	-1.246	-1.929	-2.108	-3.108	-3.648	-3.834	-4.053	-1.378	-1.605	-1.510
SVK	-2.069	-2.050	-1.940	-2.166	-2.354	-3.158	-4.306	-4.008	-4.787	-2.867	-3.780	-6.226
SVN	-0.038	-0.156	0.051	0.026	0.119	0.143	0.268	0.225	0.595	0.401	0.257	0.144
ESP	-1.711	-1.180	-1.815	-2.350	-3.416	-5.086	-7.424	-7.960	-6.885	-4.326	-5.474	-7.845
SWE	0.006	0.491	0.124	0.105	-0.339	-1.163	-1.942	-1.061	-2.391	-2.269	-4.438	-5.630
GBR	-1.884	-2.151	-1.917	-1.909	-4.294	-5.538	-6.121	-4.930	-4.638	-3.038	-2.066	-3.453
EU27	-35.594	-24.092	-24.953	-35.740	-47.280	-75.053	-91.893	-81.617	- 112.690	-81.735	- 113.652	- 115.341

Table 3 Trade balance of the EU´s member states with Russia in 2000-2011 (bil. USD)
Source: own calculations according to data from the UNCTAD Database [13]

5 Conclusion

Taking into account the results of the empirical analysis, there is untapped export potential to Russia in 12 member states of the European Union. Ten of these countries also recorded a trade deficit with Russia in 2011. On the

whole, the EU recorded a trade deficit in the area of merchandise trade throughout the period (2000-2011). Modernization is a strategic objective of today's Russia and it should be a challenge for the EU member countries to use this opportunity as Russian's main import partners. A key step for increasing the EU export growth to Russia is the creation of a proper institutional framework. Political and economic relations between the EU and Russia have been carried out under the Partnership and Cooperation Agreement since 1997. Now, a new EU-Russia Agreement is being negotiated. An ambitious and comprehensive new agreement, which includes a developed regulatory framework with common standards and norms, trade and energy provisions, would help to create wider cooperative approaches with win-win situations. Russia, who joined the WTO in 2012, should also help to lower the tariffs, to remove non-tariff obstacles of trade and make access to the Russian market easier for the EU exporters. Russia participates in EU research and development programmes. The proposal of Jose M. Barosso, which was brought up at the Moscow conference in 2013, to establish a European Union-Russia Strategic Partnership in Research and Innovation is a very important step forward in deepening the EU-Russia trade relations, too. The long-term vision of the EU is a common economic and human space from Lisbon to Vladivostok with free travel of people, free exchange of goods and services and very close overall cooperation [5].

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Output gap and structural shock estimates for Czech Republic and other selected EU members

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Abstract. Our paper is focused on different theoretical and practical aspects of estimation of key unobservable macroeconomic variables: the output gap, potential output and supply and demand shocks for Czech Republic and other selected EU countries. We use, evaluate and compare different methods for output gap estimation. Hodrick-Prescott (HP) filtering is used for baseline estimates. We compare those baseline results with improved alternative estimates from other sources: a modified Phillips-curve model, a simple Beveridge-Nelson unobserved components method and the Blanchard-Quah decomposition (based on macroeconomic structural VAR models). Results obtained for data from Czech Republic are compared to other selected EU economies and the overall inter-country homogeneity of macroeconomic behavior and internal (unobservable) dynamics of individual economies are evaluated and compared to each other. Overall, we conclude that prominent heterogeneities exist in the macroeconomic internal dynamics of individual countries as measured and evaluated in this contribution, with potentially negative impacts on the formation and evaluation of economic policy actions. Our results are relatively robust, as they are based on diverse econometric models and estimation methods.

Keywords: output gap, potential product, structural shocks, VAR model.

JEL Classification: C32, C52, E23
AMS Classification: 91B55

1 Introduction

In this paper, we present multiple approaches to consistent estimation of selected key unobservable macroeconomic variables across different economies, with appropriately devised subsequent evaluation of homogeneity across individual countries. The expected (estimated) behavior of both observable and unobservable variables is a key econometric tool for macroeconomic analysis. Namely, this approach may be conveniently used for the evaluation of expected costs and benefits of potential monetary policy actions under consideration by the European Central Bank (ECB). Also, should there be any EU-wide consensus on coordinating individual fiscal policies of the member states, the consistent inter-country comparison techniques presented in our contribution would only gain importance. Although we focus on the estimation and comparison of unobservable data series, our approach is relatively conservative as the estimates of "unobservables" are mainly based on observed data with only a few relatively simple and theoretically well-established (identifying) restrictions. Specifically, in this contribution we do not adopt the DSGE methodology as shown and referenced to in [8].

This article is structured as follows: Next section describes key aspects and general methodology used for estimation of the unobservable variables and the basic principles and constraints for subsequent inter-country analysis of homogeneity, as well as some basic data handling topics. The third chapter contains detailed description of the models used for estimation along with their theoretical background and the empirical results of our contribution. Those results are organized in tables with pair wise correlation coefficients among the estimated unobservable variables for different countries. Evaluation of the results and complementary comments are also provided. Last section and the list of references conclude our contribution.

2 Methodology and data

We aim to estimate and compare a set of relatively small, yet theoretically sound and practical macroeconometric models with intuitively comprehensible estimation outputs, suitable for subsequent comparison of the results among all economies included in our research. This approach has the advantage of providing the reader with multiple shots at some of the most important directly unobservable macroeconomic variables. Individual models contained in this contribution are based on diverse theoretical backgrounds, approaches and estimation methods.

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Although some of the estimation methods are compatible and generate consistent results, others differ in theoretical framework aspects that may limit our ability to directly compare the results. On the other hand, subsequent homogeneity evaluation based on pair wise correlation analyses may be easily compared as described throughout next chapter. Moreover, many theoretical pitfalls and practical errors may be avoided if fiscal and monetary policy decisions are exercised upon complex evaluation involving *multiple perspectives* rather than based on a single criterion, regardless of its potential sophistication and complexity.

We use quarterly data (2004:Q1 – 2012:Q4) for the following countries: Austria, Czech Republic, Denmark, Estonia, Germany, Hungary, Poland, Slovakia and Slovenia for estimation of all our models, except for specification (3) where we took advantage of available monthly series for the period 2004 to 2012. Our data originate from the International Monetary Fund's IFS database (http://elibrary-data.imf.org/). We use real GDP, inflation (CPI and PPI) and unemployment. All variables were de-seasoned by the authors, unless a series was published by the IMF as already seasonally adjusted. Due to data availability and model significance issues, we had to drop some individual economies from certain model estimations (details may be observed by comparing tables 1, 2 and 3). Fortunately, such individual drop-outs do not hamper significantly with our subsequent comparison and evaluation. EViews6 software was used for all estimations and most of the data handling.

The output gap definition (1) used throughout this paper follows e.g. from [3]. Although this is not a sole specification, it's fairly common to define output gap y_{gap} as the relative deviation of output y from its estimated potential value y^* expressed in percentage points:

$$y_{gap,t} = 100(y_t - y_t^*)/y_t^*. \tag{1}$$

In our paper, we use real GDP base index (2005 = 100%) to measure output. Inflation and unemployment gaps are also constructed and interpreted using the concept defined by equation (1), with the exception of an Expectations Augmented Phillips Curve (EAPC) model described by equation (3) where we closely adhere to the textbook specification used in [10]. Data series based on specification (1) are directly comparable among different economies, avoiding the need for currency translation or additional normalization for the size of individual economies. Apart from model (4), we approximate all potential values y^* using Hodrick and Prescott [5] HP-filter, which is a relatively flexible and widely adopted approach. Technically, we associate the HP-trend component of y with y^*, using (minimizing) the HP-filter expression (where λ determines the smoothness of the trend component):

$$\left(\sum_{t=0}^{T} (y_t - y_t^*)^2 + \lambda \sum_{t=2}^{T-1} \left[(y_{t+1}^* - y_t^*) - (y_t^* - y_{t-1}^*) \right]^2 \right) \rightarrow \min \tag{2}$$

Due to space limitations, in this contribution we only follow-up on data obtained by using the usual λ values (λ=1,600 for quarterly data and λ=14,400 for monthly data). However, we have been able to simulate significantly diverse levels of smoothness in potential product time series by arbitrarily varying the λ parameter, with significant impact on subsequent comparison among individual economies. Generally, higher values of λ put additional weight on smoothing of the trend component and therefore they increase fluctuations in gap values calculated from equations (1) and (2). On the other hand, low λ values may result in spurious cycles in the trend series. While focusing only on the economy of Sweden, Cerra and Saxena [3] provide examples for potential output and output gap using the HP-filter with λ set to 100, 200 and 1,000.

We adopt the approach described by equations (1) and (2) to generate basic output gap estimates, as shown in figure 1. Those estimates may be compared with alternative and arguably more sophisticated results from other models. Again, due to space limitations we only show results obtained from three models, chosen for this contribution so that different theoretical bases and estimation (econometric) approaches may be demonstrated. Alternative methods and models are provided and referenced to in [12], among others. Our first model is a single equation EAPC model based on theory detailed in [10], where the estimated residuals for a specific economy are interpreted as supply shocks. As a second approach, the Beveridge and Nelson [1] BN-decomposition is used to estimate (calculate) the "trend and cycle" of a series, i.e. to decompose an integrated time series into a permanent component (trend) and transitory shocks (cycles). Last but not least, two-variable real GDP and PPI inflation VAR(2) models are estimated for individual economies. Upon applying long-term structural identifying restrictions, the Blanchard and Quah [2] BQ-decomposition is performed. This relatively complex procedure described in detail by [2], [4] and [7] allows us to decompose the estimated VAR model residuals into supply and demand shocks that may be used for further analysis and comparison. Other econometric models for output-gap or supply and demand shock estimation assembled by the authors are either omitted from this contribution or just briefly mentioned. Also, we only have space to show the core of our results. Underlying estimates, econometric verification outputs and generated unobservable macroeconomic series are available from the authors upon request.

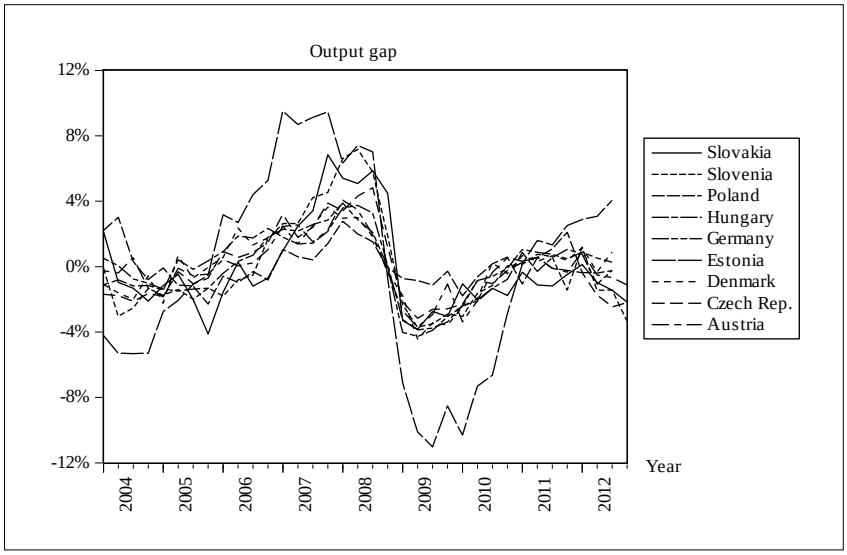

Figure 1 Output gap

Figure 1 shows the baseline HP-filter generated output gap, which is used for most of the following estimations. A prominent 2008-2009 dip in figure 1 reflects the onset of the 2008 economic crisis on different economies. Data for Estonia make for the biggest decline in output gap: from +7.4% in 2008:Q2 to -11.0% in 2009:Q3 (output gap is negative when the actual output lies below potential output). This is in compliance with the fact that Baltic states were among those EU members that were most significantly affected by the crisis onset. Pair wise correlation coefficients for data shown in figure 1 are relatively high and range from approximately +0.75 to +0.92. Surprisingly, Poland is an exception with *corr*(*POL*,*c*) values ranging from +0.35 to +0.60. This might be also observed from figure 1, where output gap for Poland is nearly zero for most of the 2008-2009 period. Whether the 2008 negative shock may have helped synchronizing the business cycles throughout EU economies would be a core question for monetary policy decision making by the ECB and for the potentially upcoming fiscal policy coordination within EU. However, we see no evidence for such synchronizing effect and this question remains open for discussion; see [6] and [7] for details on this topic.

3 Models, empirical results and interpretation

Expectations Augmented Phillips Curve (EAPC) model

Our first model, the EAPC, is a relatively simple modification to the Phillips curve (see [10] for theoretical inference). The corresponding linear regression model (LRM) used in this contribution has been amended by controlling for the autoregressive nature of variables and may be defined as

$$\left(\pi_{ct} - \pi^{e}_{ct}\right) = \beta_0 + \beta_1\left(\pi_{c,t-1} - \pi^{e}_{c,t-1}\right) + \beta_2(u_{ct} - u^{*}_{ct}) + \beta_3(u_{c,t-1} - u^{*}_{c,t-1}) + \varepsilon_{ct} \qquad (3)$$

where π_{ct} and u_{ct} represent CPI inflation and unemployment rate observations for country c at time period t. We use HP-filter generated trends as proxies for the unobserved variables π^{e}_{ct} and u^{*}_{ct}: the expected rate of inflation and the natural rate of unemployment. Therefore, the expressions in brackets may be interpreted as *unanticipated inflation* and *cyclical unemployment* respectively. The estimated residual part $\hat{\varepsilon}_{ct}$ is now defined as the supply shock, according to [10].

	Austria	**Czech Rep.**	**Denmark**	**Germany**	**Hungary**	**Poland**	**Slovenia**	**Slovakia**
Austria	1.000	0.945	0.491	0.329	0.051	0.221	0.601	0.399
Czech Rep.		1.000	0.491	0.335	-0.014	0.234	0.577	0.446
Denmark			1.000	0.342	0.048	0.234	0.381	0.188
Germany				1.000	-0.084	0.322	0.424	0.236
Hungary					1.000	0.144	0.063	-0.002
Poland						1.000	0.395	0.198
Slovenia							1.000	0.261
Slovakia								1.000

Table 1 EAPC-model supply shock correlations

We estimate equation (3) for individual economies and compare the supply shocks obtained. The estimation process may be briefly described as follows: data for all estimated countries were non stationary and cointegrated. Serial correlation of residuals was addressed for all economies by estimating (3) using the Cochrane-Orcutt generalized least squares (GLS) method as described and referenced to in [6]. The R^2 values are satisfactory (above 0.80) and further econometric verifications confirm models' significance and specification. Except for Hungary, every estimated model (3) meets the theoretical constraint $\beta_2 + \beta_3 < 0$ which is a core feature of the Phillips curve and reflects the tradeoff between unanticipated inflation and cyclical unemployment.

Limited theoretical justification of the supply shocks' definition derived from [10] and the relative simplicity of the single equation specification (3) effectively make this model inferior to the following two methods. Yet, the overall heterogeneity in supply shocks as shown in table 1 conforms to the low correlations of the arguably more sophisticated supply shocks as calculated using BQ-decomposition and described in table 3. Also, supply shocks from the EAPC model (3) keep an economically interpretable scale of the dependent variable, whereas shocks from table 3 lose their scale due to orthogonalization.

Beveridge-Nelson (BN) decomposition

This approach belongs to a wider class of *unobserved component* methods. Based on the reasoning performed in [11], we use and interpret the BN-decomposition as a tool providing estimates of unobservable variables instead of treating it as a way of defining a specific type of trend: the BN-trend (as originally suggested in [1]). Lütkepohl [9] and many others provide detailed explanation for the terminology and methods used to describe and estimate model (4). Assuming output y_{ct} has a known ARIMA(p,1,q) structure, we may construct a suitable state-space representation by defining the output as a sum of two unobservable macroeconomic variables: a permanent component (trend) τ_{ct} and a transitory (cycle) component γ_{ct}. For a country c, the state-space representation may be written as

$$y_{ct} = \tau_{ct} + \gamma_{ct}, \tag{4.a}$$

$$\tau_{ct} = \mu_c + \tau_{c,t-1} + \eta_{ct}, \quad \eta_{ct} \sim \mathrm{N}(0, \sigma^2_{c\eta}), \text{ i.i.d.}, \tag{4.b}$$

$$\phi(\mathrm{L})\gamma_{ct} = \theta(\mathrm{L})\omega_{ct}, \quad \omega_{ct} \sim \mathrm{N}(0, \sigma^2_{c\omega}), \text{ i.i.d.}, \tag{4.c}$$

$$corr(\eta_{ct}, \omega_{ct}) = \rho^{(c)}_{\eta\omega}. \tag{4.d}$$

The unobservable permanent component follows a random walk with a drift, while the transitory component is an ARMA (stationary) process with zero mean. Diffuse prior specification (4.d) leaves the correlation between permanent and transitory shocks unconstrained, although various BN-type estimates have been published for both orthogonal ($\rho^{(c)}_{\eta\omega} = 0$) and perfectly correlated ($\rho^{(c)}_{\eta\omega} = 1$) innovations. By enforcing a "smooth trend" prior, we choose ARMA(0,1) structure for all Δy_c series in order to estimate (4) through a slightly modified BNDecomp script, a publicly available EViews Add-in (downloadable from http://www.eviews.com) that calculates the BN-trend as the conditional optimal long-horizon (100 periods ahead) forecast of y_c (with future drift removed). Then, the transitory component may be simply calculated as $\gamma_{ct} = y_{ct} - \tau_{ct}$. Further theoretical background and estimation examples for the state-space model as defined by (4) may be found in [3] and [11].

	Austria	**Czech R.**	**Denmark**	**Estonia**	**Germany**	**Hungary**	**Poland**	**Slovenia**	**Slovakia**
Austria	1.000	0.885′	-0.054	0.159	0.764′	0.989″	0.999″	0.527	0.967″
Czech R.	0.531	1.000	0.362	0.453	0.779′	0.932″	0.870′	0.843′	0.967″
Denmark	0.339	0.807′	1.000	0.903″	0.372	0.026	-0.079	0.729′	0.136
Estonia	0.203	0.710′	0.763′	1.000	0.626	0.190	0.141	0.724′	0.280
Germany	-0.053	-0.207	-0.272	-0.132	1.000	0.741′	0.760′	0.677	0.760′
Hungary	0.916″	0.572	0.315	0.125	-0.021	1.000	0.984″	0.611	0.991″
Poland	0.788′	0.281	0.012	-0.050	0.056	0.771′	1.000	0.500	0.958″
Slovenia	0.342	0.726′	0.594	0.796′	0.034	0.302	0.115	1.000	0.704′
Slovakia	0.731′	0.854′	0.636	0.493	-0.116	0.752′	0.502	0.618	1.000

Table 2 BN-type trend and transitory components: country-wise correlations

Table 2 shows correlations of BN-trends among different countries (upper triangle) as well as the corresponding correlations of transitory components (lower triangle). Hence, for Czech Republic and Slovakia, the coefficient 0.843 refers to $corr(\tau_{CZE}, \tau_{SVK})$ and $corr(\gamma_{CZE}, \gamma_{SVK}) = 0.854$. We use Fisher's z-transformation to evaluate the homogeneity (similarity) among unobservable components for individual economies. Correlations

significantly exceeding arbitrary thresholds of similarity set to $corr(c1,c2) > +0.5$ and $corr(c1,c2) > +0.8$ at the 5% significance level are marked by ′ and ″ respectively. Although correlations in table 2 are significantly higher than correlations for the supply and demand shocks, numerous and significant instances of heterogeneous behavior in table 2 must not be overlooked. At the same time, we need to bear in mind that BN-trend and cycle data may not be *directly* compared to supply and demand shocks from tables 1 and 3 and we only assess the inter-country homogeneity levels obtained through different specification and estimation approaches.

Blanchard-Quah (BQ) structural decomposition, supply and demand shocks

Neoclassical economy provides theoretical grounds for the Blanchard and Quah [2] BQ-decomposition, which is based on *long term identifying restrictions imposed on impulse-response functions* (IRFs) calculated from estimated VAR(*p*) models. It is not a generally applicable procedure and may be utilized only with VAR models composed of certain sets of macroeconomic variables where specific long term economic restrictions are justifiable. As shown in [4] and [7], BQ-decomposition may be also used to untangle the *orthogonal unobservable* neoclassical *supply and demand shocks* (sometimes referred to as structural shocks) from the residuals of an estimated VAR model for each country *c*. BQ-decomposition is performed through implementing additional (identifying) zero-value restriction on the cumulative functions of IRFs: provided demand shocks affect only inflation in the long run, the cumulative response of output to a demand shock may be set to equal zero after an ad-hoc established number of periods representing the long term, whereas reactions to a supply shock are considered permanent (technically, no prior mathematical restriction is imposed on cumulative responses to supply shocks). Lütkepohl [9] provides theoretical description of different aspects of the BQ-decomposition technique (VARs, orthogonalization of residuals, Choleski decomposition and BQ additional long-term identifying conditions).

We have evaluated various specifications of the output and inflation VAR model but neither substituting inflation for unemployment (as suggested by [3]) nor VARX generalization by incorporating dummies (covering the 2008 crisis onset) would yield convincing improvement over the relatively simple specification (5). Also, PPI inflation leads to statistically superior VAR model estimations when compared to our first specification option: the CPI inflation. Finally, a reduced-form second order VAR model was chosen with respect to consistency requirements on VAR estimation (as an input to BQ-decomposition and subsequent comparison among countries) as well as by using the Akaike information criterion. Our model may be written in a simple matrix form as

$$\boldsymbol{y}_{ct} = \boldsymbol{A}_{c1}\boldsymbol{y}_{ct-1} + \boldsymbol{A}_{c2}\boldsymbol{y}_{ct-2} + \boldsymbol{u}_{ct}, \tag{5.a}$$

$$\boldsymbol{\varepsilon}_{ct} = \boldsymbol{G}_c^{-1}\boldsymbol{u}_{ct}, \tag{5.b}$$

where $\boldsymbol{y}_{ct}$ is a 2×1 vector of endogenous variables: output and PPI inflation, both expressed in terms of equation (1). $\boldsymbol{A}_{c1}$ and $\boldsymbol{A}_{c2}$ are 2×2 coefficient matrices for the lagged $\boldsymbol{y}_{ct}$ vectors, $\boldsymbol{u}_{ct}$ is the 2×1 random-element vector and $\boldsymbol{\varepsilon}_{ct}$ is a 2×1 vector containing the estimated supply and demand shocks $\varepsilon_{ct}^{supply}$ and $\varepsilon_{ct}^{demand}$. Details on BQ identification and estimation of the 2×2 decomposition matrix $\boldsymbol{G}_c$ from equation (5.b) are provided in [7]. The main purpose of estimating this model is to generate comparable residual series for subsequent BQ decomposition. Hence, we omit individual tables containing VAR model estimates for individual countries from this contribution. However, proper econometrical verification involving ADF tests for stationarity of endogenous series (only up to the 2008:Q3 shock in some cases) and stationarity of the residuals was performed. As a result of this verification, we had to drop specification (5.a) for Poland from further analysis, as the estimated model would not meet significance criteria (even after relaxing α=5% to α=10%). Such outcome for Poland does seem rather unexpected, yet it may be traced back to the properties of Poland's output gap as in figure 1 and the accompanying description. Further empirically oriented discussion on specification, ordering of variables, model estimation and potential caveats for this type of BQ-decomposition based inter-country analysis is provided in [4] and [7].

	Austria	**Czech R.**	**Denmark**	**Estonia**	**Germany**	**Hungary**	**Slovenia**	**Slovakia**
Austria	1.000	0.572	0.779′	0.264	0.446	0.321	0.494	0.082
Czech R.	0.241	1.000	0.680	0.387	0.466	0.287	0.544	0.172
Denmark	0.224	0.063	1.000	0.361	0.642	0.017	0.457	-0.037
Estonia	-0.001	0.175	0.167	1.000	0.410	-0.268	0.568	0.070
Germany	0.214	0.177	-0.019	-0.259	1.000	-0.049	0.259	0.363
Hungary	0.164	0.322	0.141	0.276	0.311	1.000	-0.095	0.155
Slovenia	0.148	0.369	0.086	0.190	0.195	0.407	1.000	0.154
Slovakia	0.042	0.587	-0.045	-0.030	0.282	0.519	0.072	1.000

Table 3 Supply and demand shocks: country-wise correlations

Table 3 shows the correlations among directly unobservable (decomposed) supply shocks in the upper triangle and demand shocks in the lower triangle as estimated from the residuals for period 2004:Q3 to 2012:Q3. Generally speaking, the level of homogeneity observed among individual economies as in table 3 is significantly lower when compared to the results obtained through Beveridge-Nelson approach. Specifically, only the correlation between supply shocks of Austria and Denmark exceeds our +0.5 similarity threshold at the 5% significance level. Although both unobservable shocks exhibit significant heterogeneity, the lack of homogeneity in supply shocks is more important given the permanent (unconstrained in time) nature of their effects on observable variables. Our results provide evidence against the possibility of reliable EU/EMU-wide unified evaluation of the expected costs and benefits of monetary policy measures under consideration by the ECB and the same unfavorable conclusions must be drawn towards any prospective centralized fiscal policy actions by the EU. Moreover, it should be stressed out that the heterogeneous behavior identified here is not an isolated result, as similar conclusions arise from [7] and from other studies referenced therein.

4 Conclusions

Focusing on some of the core tools and indicators for monetary and fiscal policy decision making process, our contribution finds significant heterogeneity among the potential output, output gap, supply and demand shocks and other unobservable macroeconomic variables that we have consistently estimated and compared across different EU countries. We use diverse models, theoretical backgrounds and disparate estimation methods in order to provide the reader with multiple perspectives on the topic under consideration. Given this approach, the results obtained from different models are not directly nor simply comparable to each other. However, the homogeneity (or the lack of it) that is always evaluated upon consistently estimated and generated data sets may in fact be considered and evaluated across different models.

Our results provide evidence against the synchronization of real business cycles among the EU countries considered. The observed heterogeneity of supply and demand shocks has a strong potential for decreasing the accuracy of consistently evaluated expected impacts of monetary and fiscal policy actions (as opposed to ad-hoc evaluations for individual countries based on diverse and potentially incompatible models). Therefore, we may conclude that prevalent heterogeneity exists between the macroeconomic internal dynamics of Czech Republic and other individual countries as measured and evaluated in this contribution, with potentially negative impacts on the centralized formation of economic policy actions and their unified and consistent evaluation.

Acknowledgements

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A NEW SINGLE-CLASS DEMAND UNCONSTRAINING METHOD

Gregory Fridman[1], Maria Lapina[2]

Abstract.

The authors propose a new statistical unconstraining method which is based on the construction of the distribution function for the censored demand and application of the maximum likelihood approach to estimate distribution parameters. Numerical results are presented of comparative analysis of existing unconstraining methods and the method advocated in the paper. It is demonstrated that the new method has proven to be more efficient in the case of a high percentage of observed censored elements of sample data. Yet another important advantage of the method connected to the fact that it enables one to process the situation of censoring information incompleteness when some elements of the observed sample data are known to be censored or not and for the others this information is not available. Mathematical computer environment Wolfram *Mathematica* has been used for obtaining all the results presented in the paper.

Keywords: demand forecast, unconstraining, statistical distribution, *Mathematica*

JEL classification: C44
AMS classification: 62N

1 Introduction

The theoretical basis and practical applications of unconstraining methods are discussed in many papers, reference to which can be found, for instance, in [1], [2], [3]. It is reported that Projection Detruncation (PD) and Expectation Maximization (EM) methods are the most effective ones. PD was developed at Boeing by Hopperstad [4]. Salch [5] was the first who applied EM approach to the censored data of airline passenger demand. In [6] both iterative statistical methods were compared to four other ones through extensive numerical simulations to demonstrate their advantages and to analyse their impact on forecasting of Revenue Management and therefore on revenue itself.

In the present paper authors advocate a new statistical unconstraining method. It is based on deriving a new statistical distribution which describes a random variable from censored sample data. Unknown parameters of this distribution are then estimated by the maximum likelihood algorithm. Results of numerical calculations are produced for comparative analysis of the proposed method with another three methods.

2 Mathematical formulation and solution to the problem

Consider a normally distributed random variable $X \sim N(\mu_1, \sigma_1)$ with cumulative distribution function $\Phi\Big(\frac{z-\mu_1}{\sigma_1}\Big)$, where

$$\Phi(x) = \frac{1}{2}\left(1 + \operatorname{erf}\Big(\frac{x}{\sqrt{2}}\Big)\right), \quad \varphi(x) = \frac{\mathrm{d}\Phi}{\mathrm{d}x} = \frac{1}{\sqrt{2\,\pi}}\,\mathrm{e}^{-x^2/2}, \quad \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int\limits_0^x \mathrm{e}^{-t^2}\,\mathrm{d}t$$

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are cumulative distribution function and the probability density function of a normal standard distribution (with zero mean and unit standard deviation) and the error function. Assume that this variable describes a true (unconstrained) demand for a given airticket fare class. Denote $\bar{x} = (x_1, \ldots, x_n)$ to be a sample data for n observations of random variable X.

Let a normally distributed random variable $Y \sim N(\mu_2, \sigma_2)$ characterize restrictions (i.e. booking limits) for this fare class. Its cumulative distribution function $\Phi\left(\frac{z-\mu_2}{\sigma_2}\right)$ depends on parameters (μ_2, σ_2).

Finally, consider a random variable $Z = \min(X, Y)$ which describes the number of bookings for the fare class. Then $z_1 = \min(x_1, y_1)\,, \quad \ldots \quad z_n = \min(x_n, y_n)\,,$ where $\bar{z} = (z_1, \ldots, z_n)$ – a *censored* sample data of the random variable X in which the value x_k is observed only if $x_k \leqslant y_k$. If $y_k < x_k$ then $z_k = y_k$ and in this case the observation x_k is censored:

$$z_k = \begin{cases} x_k\,, & \text{if } \; x_k \leqslant y_k\,; \\ y_k\,, & \text{if } \; x_k > y_k\,, \end{cases} \qquad k = 1, \ldots, n \tag{$*$}$$

From mathematical viewpoint the problem is to estimate distribution parameters μ_1 and σ_1 of random variable X (or both sets of distribution parameters (μ_1, σ_1) and (μ_2, σ_2) of random variables X and Y simultaneously) using censored sample data $\bar{z} = (z_1, \ldots, z_n)$ under assumption that distributions laws of X and Y are known. In general case an additional information is given as well, namely, for some values of $k = 1, \ldots, n$ it is known either an element z_k of the observed sample data $\bar{z}$ represents censored value of the corresponding x_k (i.e. $z_k = y_k$) or not (i.e. $z_k = x_k$).

A straightforward approach to solution to the problem consists of two steps: first, one has to construct a probability density function $f_Z\,(x; \mu_1, \sigma_1; \mu_2, \sigma_2)$ of the random variable $Z = \min(X, Y)$ and then, second, apply maximum likelihood estimation (MLE) technique, originally developed by R.A. Fisher in the 1920s, to estimate all the unknown parameters of derived distribution.

The MLE methods involves the maximization procedure for so-called likelihood function $\mathcal{L}$ which in the considered problem is a joint probability distribution of the n observed values of the censored sample data z_k, $k = 1, \ldots, n$. In the standard case of independent and identically distributed observations z_k, $k = 1, \ldots, n$ the likelihood function $\mathcal{L}$ is given by the product of the individual densities $f_Z\,(z_k; \mu_1, \sigma_1; \mu_2, \sigma_2)$ and takes the form

$$\mathcal{L}(\mu_1, \sigma_1; \mu_2, \sigma_2) = \prod_{k=1}^{n} f_Z\,(z_k; \mu_1, \sigma_1; \mu_2, \sigma_2)\;. \tag{1}$$

The maximum likelihood estimators of the parameters (μ_1, σ_1) and (μ_2, σ_2) are defined as those values $\hat{\mu}_1, \hat{\sigma}_1; \hat{\mu}_2, \hat{\sigma}_2$ which globally maximize function $\mathcal{L}$. For technical and also for theoretical reasons it is easier to work with the logarithm (a monotonically increasing function of its argument) of the likelihood function $\mathcal{L}$.

$$\log \mathcal{L}(\mu_1, \sigma_1; \mu_2, \sigma_2) = \sum_{i=1}^{n} \log f_Z\,(z_k; \mu_1, \sigma_1; \mu_2, \sigma_2) \to \max\;. \tag{2}$$

Consider three cases of the problem formulated above:

• there is no information on which elements of sample data $\bar{z} = (z_1, \ldots, z_n)$ reflects censored or unconstrained values of $\bar{x}$;

• such information can be used for all elements of sample data $\bar{z}$;

• generalized case when such information is partially accessible, i.e. some elements of $\bar{z}$ are known to be censored or not and there is no such information about the others.

It is to be underlined that an assumption as to how X and Y are distributed makes no difference for solution's algorithm.

There is no information on which elements of sample data $\bar{z}$ has been censored

To derive a cumulative distribution function F_Z of random variable $Z = \min\{X, Y\}$ one has to calculate the probability $P(Z > z)$ of the event that random variable Z takes the value greater then z. As a result

one arrives at the formula

$$F_Z(z;\mu_1,\sigma_1,\mu_2,\sigma_2) = 1 - P(Z > z) = 1 - \left(1 - \Phi\Big(\frac{z-\mu_1}{\sigma_1}\Big)\right)\left(1 - \Phi\Big(\frac{z-\mu_2}{\sigma_2}\Big)\right), \tag{3}$$

what readily yields the probability density function $f_Z = \mathrm{d}F_Z/\mathrm{d}Z$ of Z:

$$f_Z(z;\mu_1,\sigma_1,\mu_2,\sigma_2) = \frac{1}{\sigma_1}\,\varphi\Big(\frac{z-\mu_1}{\sigma_1}\Big)\left(1 - \Phi\Big(\frac{z-\mu_2}{\sigma_2}\Big)\right) + \frac{1}{\sigma_2}\,\varphi\Big(\frac{z-\mu_2}{\sigma_2}\Big)\left(1 - \Phi\Big(\frac{z-\mu_1}{\sigma_1}\Big)\right) \tag{4}$$

and the log-likelihood function $\log\mathcal{L}(\mu_1,\sigma_1,\mu_2,\sigma_2)$ takes the form

$$\log\mathcal{L}(\mu_1,\sigma_1,\mu_2,\sigma_2) = \sum_{k=1}^{n}\log\left\{\frac{1}{\sigma_1}\,\varphi\Big(\frac{z_k-\mu_1}{\sigma_1}\Big)\left(1 - \Phi\Big(\frac{z_k-\mu_2}{\sigma_2}\Big)\right) + \right.$$
$$\left. + \frac{1}{\sigma_2}\,\varphi\Big(\frac{z_k-\mu_2}{\sigma_2}\Big)\left(1 - \Phi\Big(\frac{z_k-\mu_1}{\sigma_1}\Big)\right)\right\}. \tag{5}$$

Censoring information can be used for all elements of sample data $\bar{z}$

Following the same analytical procedure, the likelihood function $\mathcal{L}$ can be obtained as follows

$$\mathcal{L}(\mu_1,\sigma_1,\mu_2,\sigma_2) = \prod_{k=1}^{m}\frac{1}{\sigma_1}\varphi\Big(\frac{x_k-\mu_1}{\sigma_1}\Big)\left(1 - \Phi\Big(\frac{x_k-\mu_2}{\sigma_2}\Big)\right)\prod_{j=1}^{r}\frac{1}{\sigma_2}\varphi\Big(\frac{y_j-\mu_2}{\sigma_2}\Big)\left(1 - \Phi\Big(\frac{y_j-\mu_1}{\sigma_1}\Big)\right), \tag{6}$$

where m denotes the number of unconstrained elements and r is the number of censored elements in the sample data $\bar{z}$ and, naturally, $m + r = n$. Introduction of log-likelihood function is effective as well from the viewpoint of the global maximization procedure $\log\mathcal{L}(\mu_1,\sigma_1,\mu_2,\sigma_2) \to \max$.

Generalized case when censoring information is incomplete

Assume that the censoring information for the observed sample data $\bar{z}$ is incomplete, i.e. for some elements z_k it is unknown if they represent x_k or y_k. In this case the likelihood function is constructed as a product of expression for unconstrained and censored elements of $\bar{z}$ and expression for those z_k of which the censoring information is absent:

$$\mathcal{L}(\mu_1,\sigma_1,\mu_2,\sigma_2) = \prod_{k=1}^{m}\frac{1}{\sigma_1}\,\varphi\Big(\frac{x_k-\mu_1}{\sigma_1}\Big)\left(1 - \Phi\Big(\frac{x_k-\mu_2}{\sigma_2}\Big)\right)\times\prod_{j=1}^{r}\frac{1}{\sigma_2}\,\varphi\Big(\frac{y_j-\mu_2}{\sigma_2}\Big)\left(1 - \Phi\Big(\frac{y_j-\mu_1}{\sigma_1}\Big)\right)\times$$
$$\times\prod_{l=1}^{n-m-r}\left\{\frac{1}{\sigma_1}\,\varphi\Big(\frac{z_l-\mu_1}{\sigma_1}\Big)\left(1 - \Phi\Big(\frac{z_l-\mu_2}{\sigma_2}\Big)\right) + \frac{1}{\sigma_2}\,\varphi\Big(\frac{z_l-\mu_2}{\sigma_2}\Big)\left(1 - \Phi\Big(\frac{z_l-\mu_1}{\sigma_1}\Big)\right)\right\}. \tag{7}$$

where, again, m denotes the number of unconstrained elements in observed sample data $\bar{z}$ and r is the number of censored elements in observed sample data $\bar{z}$.

Note that the problem of finding the estimations of parameters of the probability density function f_Z in all three cases can be considered either under the assumption that parameters μ_2 and σ_2 for random variable Y are known or without it. Even if the problem is to find estimations to all four parameters (μ_1,σ_1) and (μ_2,σ_2) it can be subdivided into two independent ones due to the fact that partial derivatives $\log\mathcal{L}$ with respect to μ_1 and σ_1 do not depend on μ_2 and σ_2 and vise versa. The global optimization can be implemented basing on one of the well-known algorithms.

3 Numerical results and discussions

The results of computer numerical simulation are presented in this section to analyse advantages and drawbacks of the proposed unconstraining method and to compare its effectiveness (first of all, accuracy and performance) with those of PD and EM approaches. The simulation methodology was chosen to be partly similar to that used in [6]. Some important alterations were introduced to the process of

unconstrained demand and booking limit generation and additional numerical analysis was performed taking into account the effect of censoring information incompleteness degree in observed sample data on accuracy of the proposed method. Two simulated data sets (i.e. observed sample data $\bar{z}$) were generated. The first one is obtained using initial parameters $\mu_1 = 20$, $\sigma_1 = 4$ and $\sigma_2 = 4$ and the second one using $\mu_1 = 0$, $\sigma_1 = 1$ and $\sigma_2 = 1$. The percentage of unconstraining observations was chosen to be 50%, 75%, 85%, 90%, and 98%.

Figure 1 presents comparative numerical results for unconstrained estimations of mean (left) and standard deviation (right) obtained using N3, PD, EM, and Z1 methods for the first simulated data set in the form of box-and-whisker chart. It is assumed that censoring information can be used for all elements of sample data $\bar{z}$.

Calculations demonstrate that method Z1 gives identical results with EM for moderate and high values of percentage of censored observations and better results for very high censored sample data (more than 96%). Mean absolute errors are calculated in Table 1 for the unconstrained mean and the unconstrained standard deviation.

	N3	PD	EM	Z1
50%	(1.59634, 1.25069)	(0.614932, 0.948666)	(0.229581, 0.207031)	(0.229581, 0.207031)
75%	(3.10603, 1.93417)	(1.66712, 1.8206)	(0.515712, 0.528677)	(0.515687, 0.528612)
85%	(3.91338, 2.34192)	(2.34452, 2.36615)	(0.591423, 0.53379)	(0.591017, 0.533552)
90%	(4.59367, 2.57692)	(2.91175, 2.68328)	(0.855289, 0.713381)	(0.855284, 0.713378)
98%	(6.63266, 3.17802)	(4.5955, 3.46528)	(2.12046, 1.38938)	(1.91494, 1.21637)

Table 1 Mean Absolute Error (MAE) for the first data set, $\mu_1 = 20$, $\sigma_1 = 4$. The first number in each pair gives MAE for the unconstrained mean and the second – for the unconstrained standard deviation.

Note that box-and-whisker charts demonstrated in the paper for presentation of the numerical results are much more informative than simple histograms that usually used, see, for instance [2], [6].

Figure 2 as well as Table 2 illustrate numerical results for the second data set. Again, it is seen that Z1 method is more effective especially for very high values of percentage of observed censored elements.

	N3	PD	EM	Z1
50%	(0.406433, 0.313069)	(0.155855, 0.236075)	(0.053293, 0.052207)	(0.053293, 0.052207)
75%	(0.759536, 0.499272)	(0.406477, 0.471294)	(0.129863, 0.128624)	(0.128468, 0.12785)
85%	(1.01825, 0.574064)	(0.599651, 0.579925)	(0.152037, 0.135692)	(0.152034, 0.135692)
90%	(1.1424, 0.637409)	(0.697426, 0.666965)	(0.181933, 0.160816)	(0.181937, 0.160817)
98%	(1.63486, 0.8068)	(1.14214, 0.882418)	(0.525655, 0.389394)	(0.421494, 0.264834)

Table 2 Mean Absolute Error (MAE) for the second data set, $\mu_1 = 0$, $\sigma_1 = 1$. The first number in each pair gives MAE for the unconstrained mean and the second – for the unconstrained standard deviation.

Finally, figures 3 and 4 contain numerical results of the unconstraining procedure for the first and the second data set correspondingly in the case when some elements of the observed sample data are known to be censored or not and for the others this information is not available. Note again that Z1 method is alone can be used in such a situation of all the unconstraining methods.

Conclusions

This paper presents a new unconstraining approach based on construction of probability density function of the censored random variable and estimation procedure for the distribution parameters using maximum likelihood method. The results of numerical simulations demonstrated that the proposed method has better accuracy than well-known PD and EM approaches at a high percentage of the censoring. Another important advantage of new method is connected to the fact that it enables one to process the situation of censoring information incompleteness when some elements of the observed sample data are known to be censored or not and for the others this information is not available. Other methods cannot be used to work with such kind of data. All the calculations are produced in computer mathematical environment Wolfram *Mathematica*.

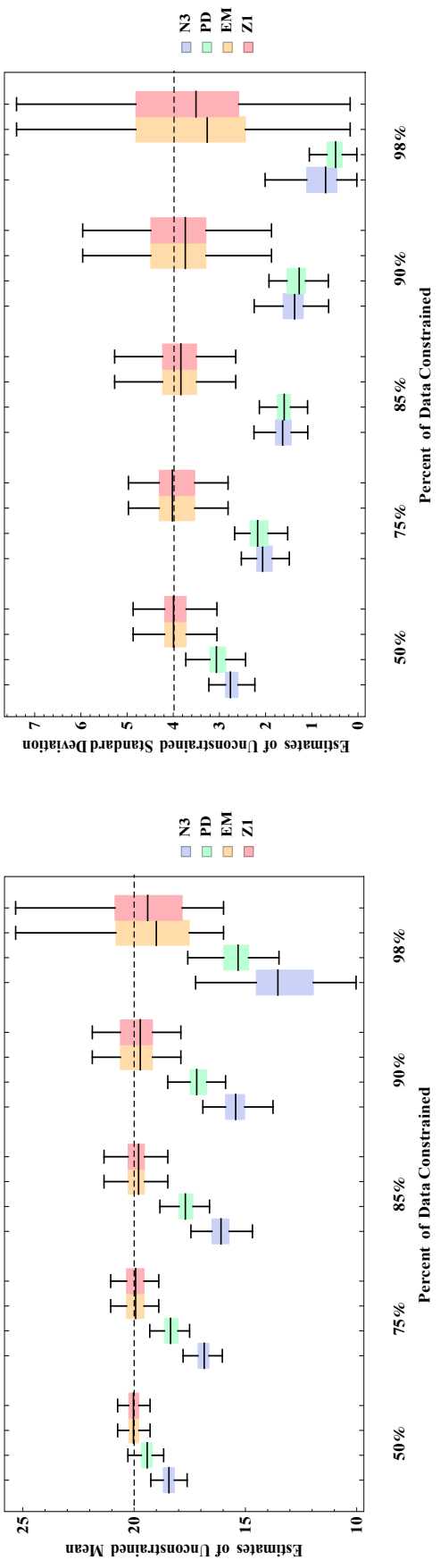

Figure 1 Unconstrained estimations of mean (left) and standard deviation (right) obtained using N3, PD, EM, and Z1 methods for the first simulated data set ($\mu_1 = 20$, $\sigma_1 = 4$ and $\sigma_2 = 4$) for different percentage of censored observations.

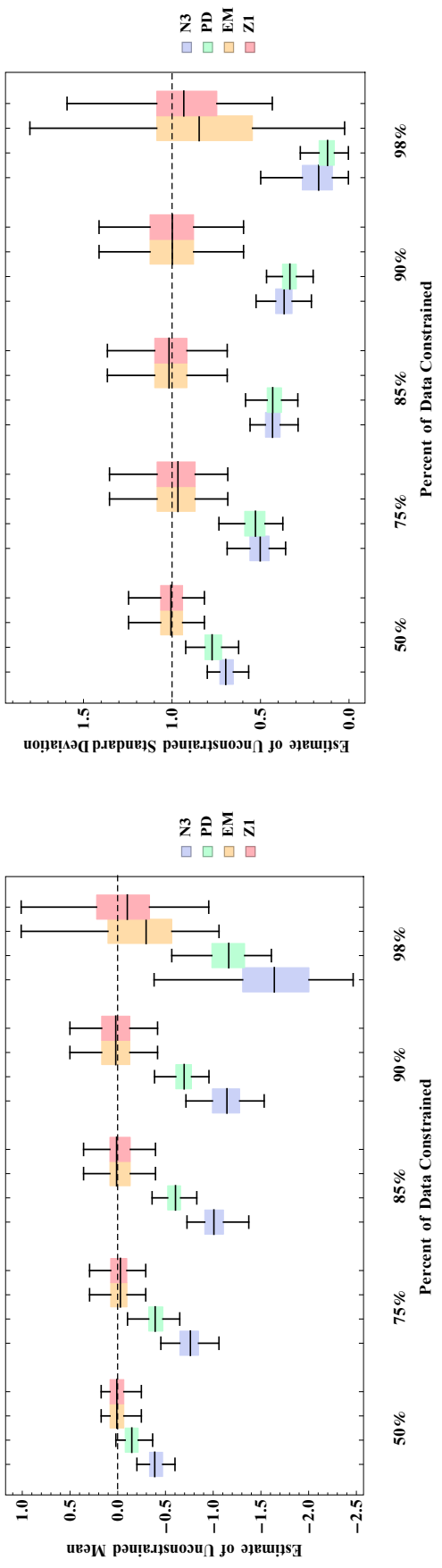

Figure 2 Unconstrained estimations of mean (left) and standard deviation (right) obtained using N3, PD, EM, and Z1 methods for the second simulated data set ($\mu_1 = 0$, $\sigma_1 = 1$ and $\sigma_2 = 1$) for different percentage of censored observations.

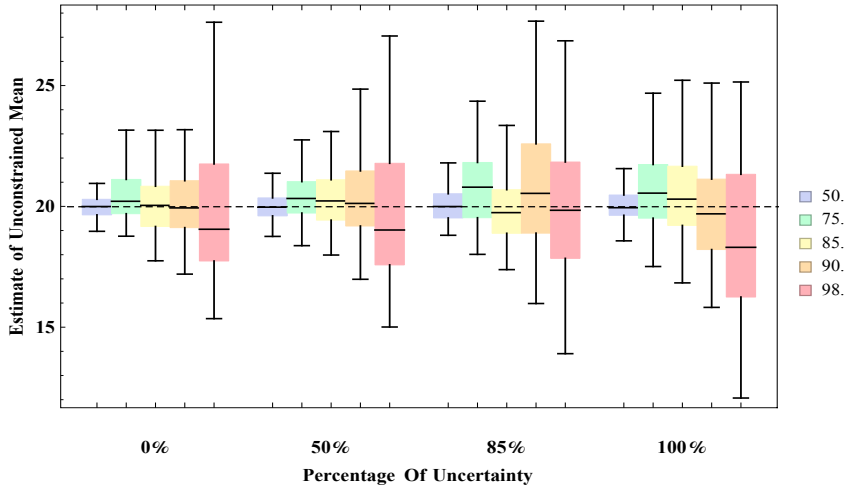

Figure 3 Unconstrained estimations of mean obtained using Z1 method for the first simulated data set ($\mu_1 = 20$, $\sigma_1 = 4$ and $\sigma_2 = 4$) for different percentage of censored observations and information incompleteness.

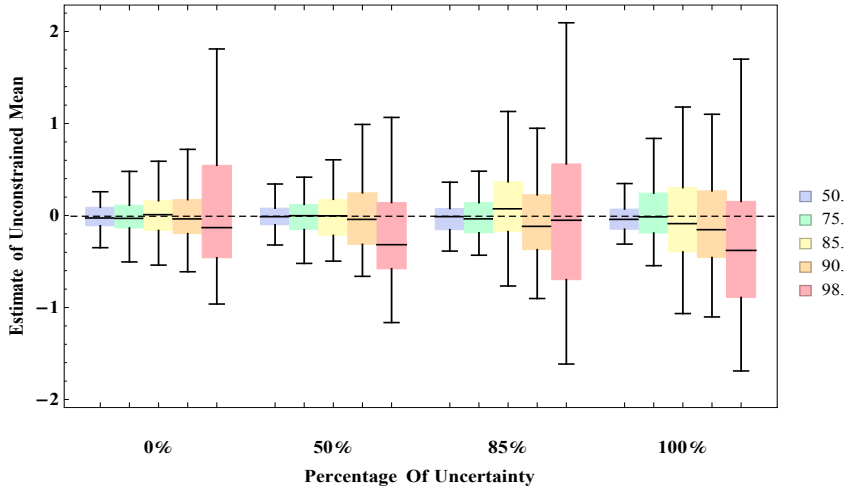

Figure 4 Unconstrained estimations of mean obtained using Z1 method for the second simulated data set ($\mu_1 = 0$, $\sigma_1 = 1$ and $\sigma_2 = 1$) for different percentage of censored observations and information incompleteness.

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Stochastic analysis of costs during the pig breeding process

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Abstract: This contribution deals with the economy of agricultural production, specifically in animal farming. The main goal of this paper is to find and apply an efficient simulation model of pig breeding at a particular company in order to improve its economic results. The partial goal is to identify a pig breed that is the most profitable using current technologies. Costs at particular parts of the breeding process were used as the main simulation inputs, together with frequency of litters, length of meantime, number of weaned piglets depending on a sequence of the litter, length of breast-feeding time, and the length of time of pregnancy of sows. Additional inputs were price of produced piglets, price of insemination doses etc. Statistical distributions for the simulations of the inputs were obtained with theoretical curve fitting of sample data, which were provided from research projects focused on piglet production efficiency. There also was obtained the empirical the curve of sow´s profitability depending on number of litters which can help to decide about the culling of sows from the herd. A simulation was performed using the program @RISK. The obtained results and recommendations are further discussed in the paper.

Key words: simulation; pig breeding; probability distribution; weaning of piglets; culling of sows;

JEL Classification: C13, C15, P4
AMS Classification: 60H30

1 Introduction

Many authors have dealt with the economy of piglet production farms, e.g. Pulkrábek [9], Brož [1]. The key factors of economy of piglet production system can be divided into two groups – exogenous and endogenous. Exogenous factors, such as the price of feeding, the price of energy and the price of pork meat, can be influenced negatively. In contrast, the endogenous factors can be managed with success. The optimal replacement policy of sows is described by Rodriguez et al in [10]. Plà et al. have developed a production model using Markow [8] and Semi-Markow chains [7]. A review of mathematical models for sow herd management is described by Plà [6]. Usage of stochastic process models in sow herd management is dealt with by Marín [5].

In this contribution, we focus on the distribution of time spent by farrowing batches at different parts of the production process during meantime. Because of the stochastic character of animal reproduction, we have developed a stochastic simulation model [3] covering costs and incomes arising during the whole reproductive cycle.

2 Methodology

Stochastic analysis of the pig breeding process had to work in the circumstances of a model farm. The pig breeding process is depicted in Figure 1. At the top of Figure 1, we can see the time axis.

Because the duration of production cycles is mandatory (biologically) set, we can only change the duration of the particular stage of meantime of the sows and sows with piglets. However, the possible change of the above mentioned stages have to be ethologically admissible.

The second parameter of the pig breeding process which can be changed with success and which has great economic impact is the culling policy. In Czech farms, it is typical that the culling of sows is induced by veterinary conditions (in our reference farm 68%) and is not regulated statistically as in the other countries.

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Placing the inefficient sows into the brewing process from the point of view gestation and fertility has an undesirable impact on the economy of the whole farm.

On the model farm, two breeds of pig were entered into the breeding process. An additional aim of our contribution is to determine which of them is more efficient from the economic point of view.

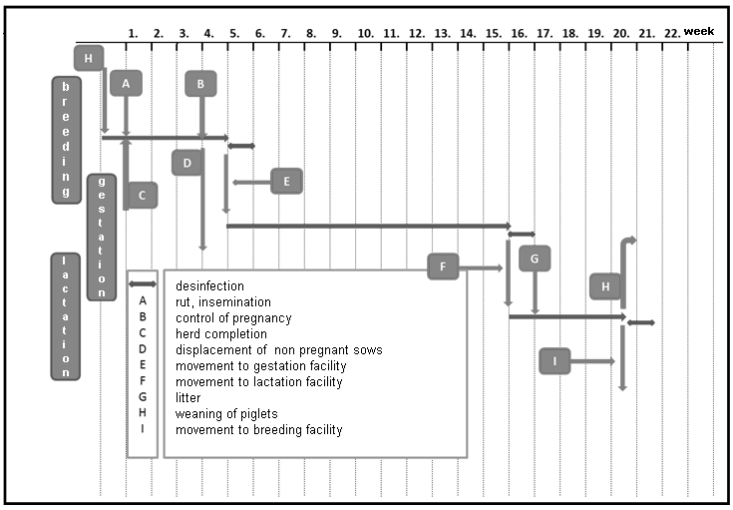

Figure 1: Production diagram

2.1. Data

The data file included 1431 records of Czech Landrase sows (CL) and 3145 of Large White pig (LW) sows. Each record included the birth date, date of culling, reason of culling, date of litter, number of piglets weaned in each litter, number of insemination, litter weight at birth, weaning weight etc. The data had to be processed in order to gain necessary simulation inputs.

2.2. Structure of the herd

The herd structure is a crucial parameter which influences the total efficiency of piglet production. This indicator is primarily given by the veterinary conditions on the farm. The secondary impact is the culling policy depending on sow efficiency. The contemporary structure of the herd is depicted in Table 1.

Litter count	CL-non culled	LW-non culled
0	43	141
1	45	127
2	30	78
3	28	21
4	9	31
5	5	12
6	3	7
7	0	1
Total	163	418

Table 1: Structure of herd depending on frequency of litters

2.3. Costs and incomes

Total costs do not include the purchase of gilts and insemination doses. The highest costs occur in the lactation facility. The level of costs there is 200% higher than elsewhere.

Feeding costs, water costs, wage costs, veterinary costs, energy, purchase of gilts, and insemination doses are quantified in Table 2. For a detailed analysis of costs see http://home.ef.jcu.cz/~ludva/farm1.

Litter count	CZK/head
Lactation	59,09
Gestation	26,53
Breeding	25,88
Gilts	28,34
Contact animals	24,75
Teaser male	29,16
Weaner pen	10,03
Piglets	1,14

Table 2: Total daily costs

The main income of a pig breeding company is from piglets sold. The average price is 72 CZK per kilogram. This price was computed as an average within the last year. The average weight of a weaned piglet is 7.6 kg, the weight of a piglet sold is 30 kg.

2.4. Simulated inputs

Number of inseminations

The price of an insemination dose is one of the important costs. The number of inseminations strongly depends on the number of ruts before becoming pregnant. In order to create this simulation input we used the Distribution Fitting function in the program @RISK. The most suitable were the Binomial and Poisson distributions [4]. Because the probability of becoming pregnant differs from the increasing number of litters, we had to suggest a particular probability distribution function for each litter. In Figure 2, we can see the PDF of becoming pregnant for LW and for the first litter. The number of inseminations was monitored only up to the 15th litter for the same reasons as for the litter size.

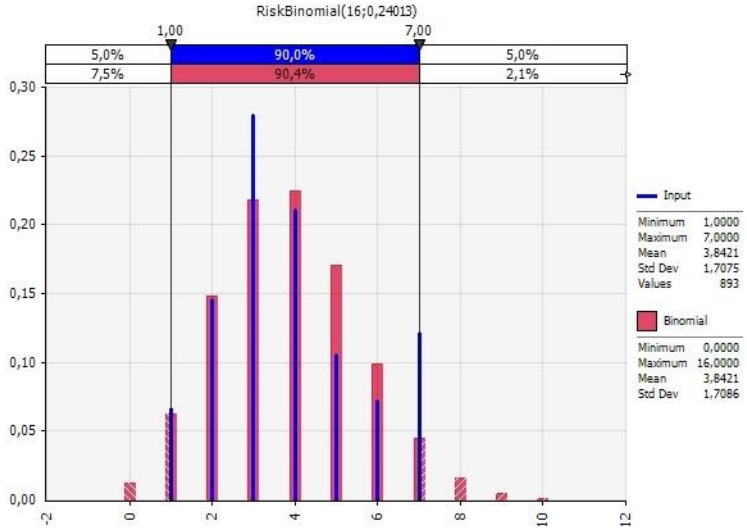

Figure 2: Fitting of number of inseminations

Litter size - number of piglets in a particular litter

The litter size depends on the number of litters. For CL sows, this size increases up to the fourth litter and thereafter decreases. A similar situation exists for LW-sows. In order to create this simulation input we used the Distribution Fitting function in the program @RISK [2]. The most suitable was the Binomial distribution. Because the probability of becoming pregnant differs from litter to litter, we had to suggest a particular PDF for each one. The litter size was monitored only up to 15th litter, thereafter the efficiency of the sow slopes down mainly because of death losses of baby pigs. In Figure 3, we can see the PDF of litter size of CL sows and of the first litter. Because of completeness, we used for the creation of this simulation the input data about culled sows of both breeds.

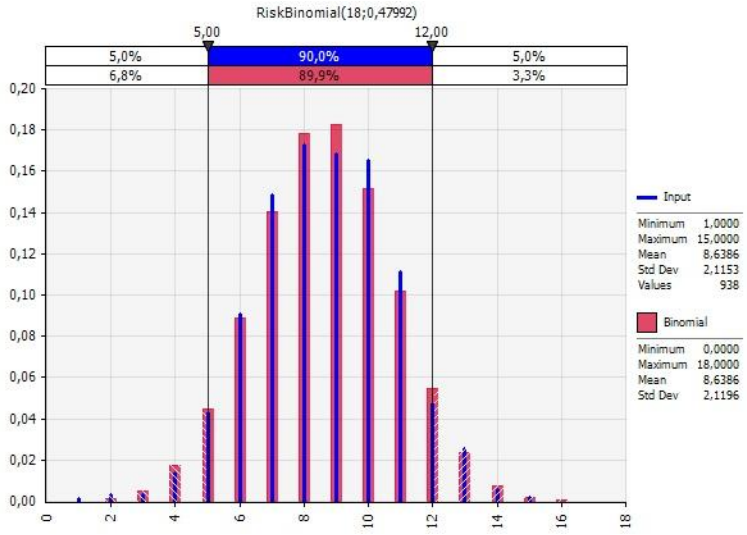

Figure 3: Number of piglets

2.5. Building of a simulation model

Simulation inputs

- Total cost in particular facility;
- Price of weaned piglets;
- Price of insemination dose;
- Litter frequency;
- Meantime (interval between two litters);
- Litter size;
- Number of inseminations before becoming pregnant.

This model was built for each breed separately in order to compare their efficiency. We formed a model batch of sows in accordance with the structure of the total herd. For this model batch we performed a relevant number of iterations with different parameters. The model was moreover divided into models of cost and incomes (see Table 3 and 4).

b	p	v	TB	CB	TG	CG	TL	CLS	PW	CPL	t	TC	CP
1	1	1	98	3 600.74	77		29		9	225.72	204.00	7 532.83	4 062.15
1	2	2	56	1 986.08	77		29		10	250.80	162.00	5 943.25	4 513.50
1	3	3	35	1 178.75	77		29		10	250.80	141.00	5 135.92	4 513.50
1	4	4	35	1 178.75	77		29		10	250.80	141.00	5 135.92	4 513.50
1	5	5	56	1 986.08	77	1 992.76	29	1 713.61	10	250.80	162.00	5 943.25	4 513.50
1	6	6	35	1 178.75	77		29		9	225.72	141.00	5 110.84	4 062.15
1	7	7	35	1 178.75	77		29		9	225.72	141.00	5 110.84	4 062.15
1	8	8	35	1 178.75	77		29		9	225.72	141.00	5 110.84	4 062.15
1	9	9	35	1 178.75	77		29		9	225.72	141.00	5 110.84	4 062.15

b - batch, p - pig, v - number of litter, TB - time spent in breeding facility [day], CB - costs in breeding facility [CZK], TG - time spent in gestation facility [day], CG - cost in gestation facility [CZK], TL - time spent in lactation facility [day], CLS - cost on sow in lactation facility [CZK], PW - number of weaned piglets, CPL - costs on piglets in lactation facility [CZK], t - reproductive cycle [day], TC - Total costs for a sow from piglets to weaning [CZK], CP - cost in weaner pen [CZK]

Table 3: Simulation model costs

PP	PR	SL	TO	PS	NS
14 445.00	2850.02	0.3357	1.7893	1 712.00	5.41
16 050.00	5593.25	0.3046	2.2531	3 838.04	6.86
16 050.00	6400.58	0.1871	2.5887	3 099.21	4.84
16 050.00	6400.58	0.0504	2.5887	834.40	1.30
16 050.00	5593.25	0.0743	2.2531	936.84	1.67
14 445.00	5272.01	0.0288	2.5887	392.73	0.67
14 445.00	5272.01	0.0168	2.5887	229.09	0.39
14 445.00	5272.01	0.0024	2.5887	32.73	0.06
14 445.00	5272.01	0	2.5887	0.00	0.00
			AS	11075.05	
				AP	21.21

PP - Price for piglets sold per litter [CZK], PR - profit per sow during reproductive cycle [CZK], SL - share of particular number of litter in a batch, TO - herd turnover, PS - Profit per sow for a year [CZK], NS - number of piglets, AS - average profit per sow and year [CZK], AP - Average number of piglets per sow and year.

Table 4: simulation model profitability

3 Results

We performed 1000 iterations for different weaning periods. We tested seven possible periods of weaning for 22–28 days for both breeds of pigs and we monitored the total costs and incomes. The total profitability for both breeds is given in Tables 5 and 6. It is evident that the CL pig is more efficient with a weaning time of 22 days, but with a lower risk quantified in the standard deviation. The optimal time to wean is identical for both breeds. Histograms for LW and CL sows are depicted in Figures 5 and 6. It is evident that stochastic CL sows achieve more balanced economic results, the double sided 90% quantile is narrower than for LW stochastic sows. For complete assessment we have to carefully monitor the impact of the time to wean on the weaning weight.

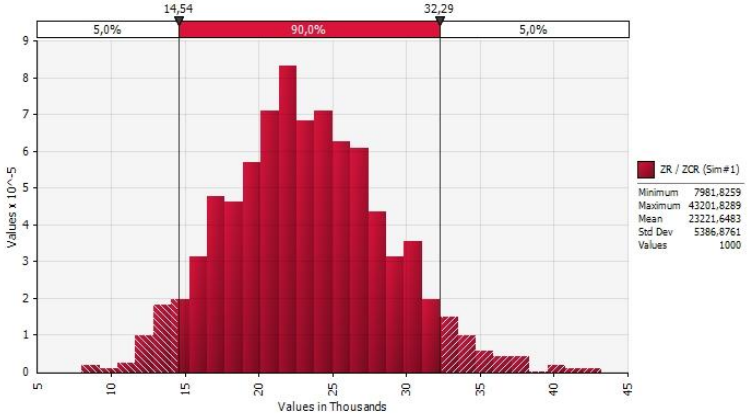

Figure 4: probability distribution function of profit per sow of LW for weaning in 22 days

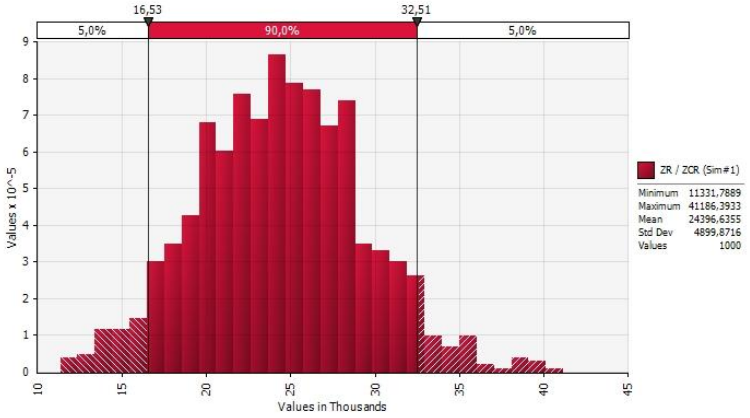

Figure 5: probability distribution function of profit per sow of CL for weaning in 22 days

An interesting fact is that the mean value of piglets weaned per sow for the LW is greater than for the CL although the CL is profitable. This paradox is caused by a greater meantime for the LW, which give birth to more piglets per litter, but the litters occur less frequently.

Days to weaning	**22**	**23**	**24**	**25**	**26**	**27**	**28**
Mean	23222	23124	23027	22931	22837	22745	22653
Std Dev	5387	5367	5347	5328	5309	5290	5272

Table 5: Profitability LW (Profit per sow and year)

Days to weaning	**22**	**23**	**24**	**25**	**26**	**27**	**28**
Mean	24397	24287	24179	24072	23967	23864	23762
Std Dev	4900	4879	4858	4837	4817	4798	4778

Table 6: Profitability CL (Profit per sow and year)

In the presented model, we supposed a uniform distribution of costs during the whole stay for particular parts of production. In further research we will focus on a detailed analysis of the costs during the stay of sows on particular facilities. For example, consumption of feeding there increases during the first 13 days by more than three times, see http://home.ef.jcu.cz/~ludva/farm1. The most expensive feeding occurs at the weaning pen where the situation is very similar.

As was mentioned in the methodology part, one of the most important questions of the pig breeding process with a great economic impact is the culling policy. This policy is closely connected with purchasing gilts and herd replenishing. Revenues for braked sows are economically inconsiderable.

Besides other things, the proposed model can be used for developing pig farms, including dimensioning particular production facilities.

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On Active Set Method and Second-Order Inclusion Probabilities

Wojciech Gamrot [1]

Abstract. Under various sampling schemes the exact calculation of inclusion probabilities is prohibitively complex even for modest population sizes. Such a difficulty may be tackled by replacing unknown first-order inclusion probabilities in the Horvitz-Thompson expansion estimator of population total with estimates computed in a simulation study. However, one may also desire to estimate the variance of this statistic using simulated second-order inclusion probabilities. To improve the accuracy of their estimates, the available auxiliary information may be employed. Such an information often takes form of partial (hierarchic) ordering inequalities that may be incorporated into estimation using isotonic regresson methods. In this paper such an approach is adopted for a fixed-cost sequential sampling scheme. The active set optimization algorithm is applied to estimate inclusion probabilities and the variance of population total estimates.

Keywords: inclusion probability, restricted estimation, population total.

JEL classification: C83
AMS classification: 62D05

1 Introduction

Design-based estimation of finite population parameters is based on the assumption that population values of the study variable are fixed and the only source of estimator's variability is the sampling scheme. Inclusion probabilities characterizing the sampling scheme play important role in the construction of estimators and assesment of their properties. Sometimes these probabilities are hard to calculate. This problem arises for various sequential sampling and rejective sampling strategies considered among others in [5], [15], [2], [16]. In such circumstances one may focus on estimators that do not utilize inclusion probabilities at all, such as sufficiency-based estimator of [10] or model-based estimators such as those in [12], [17] or [14]. Another possibility is to replace unknown first-order inclusion probabilities with estimates obtained from the simulation experiment. Such a technique has been considered in [6], [13] and [7] for estimation of population totals. It will now be extended to estimating second order inclusion probabilities and the variance of population total estimates. Let the finite population of size N be represented by a set of unit indices $U = \{1, ..., N\}$. Fixed values of the study variable are denoted by $y_1, ..., y_N$. The objective of the survey is to estimate the population total $t = \sum_{i \in U} y_i$ from the sample s. When first-order inclusion probabilities $\pi_1, ..., \pi_N$, $\pi_i = Pr(i \in U)$ for $i \in U$ are known, it is estimated without bias by the well-known expansion estimator:

$$\hat{t} = \sum_{i \in s} \frac{y_i}{\pi_i} \tag{1}$$

Its variance depends on second-order probabilities $\pi_{ij} = Pr(i, j \in s)$ for $i, j \in U$ and it is estimated by:

$$\hat{V}(\hat{t}) = \sum_{i,j \in s} \check{y}_i \check{y}_j \check{\Delta}_{ij} \tag{2}$$

where $\check{y}_i = y_i/\pi_i$, $\check{\Delta}_{ij} = \Delta_{ij}/\pi_{ij}$ and $\Delta_{ij} = \pi_{ij} - \pi_i \pi_j$ for $i, j \in U$. The statistic $\hat{V}(\hat{t})$ may occasionally become negative which is unwelcome but usually unlikely to happen. When inclusion probabilities are

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unknown, Fattorini [6] proposes to carry out a simulation to compute estimates $\hat{\pi}_1, ..., \hat{\pi}_N$ of first order inclusion probabilities and to plug them into (1). This results in a simulation-based estimator:

$$\hat{t}_* = \sum_{i \in s} \frac{y_i}{\hat{\pi}_i} \tag{3}$$

To preserve the finiteness of expansion estimator, estimates of first order inclusion probabilities should remain strictly positive. This may be achieved by using Fattorini's formula:

$$\hat{\pi}_{iF} = \frac{k_i + 1}{R + 1} \tag{4}$$

where R is the number of simulated sample replications and k_i is the number of times the i-th unit is drawn for $i \in U$. Other likelihood-based non-negative estimators are also considered in [9]. When suitable auxiliary information in the form of ordering constraints is available, Gamrot [8] proposes to apply isotonic regression procedures to compute improved estimates of first order inclusion probabilities, and studies properties of resulting population total estimator. When variance estimates for (3) are desired, they may be constructed by replacing unknown true second-order probabilities in (2) with Fattorini's estimates:

$$\hat{\pi}_{ijF} = \frac{k_{ij} + 1}{R + 1} \tag{5}$$

where k_{ij} is the number of times the i-th and j-th unit is jointly drawn for $i \neq j \in U$. However, true second-order inclusion probabilities tend to be much lower than first-order ones. Consequently, sampling weights computed as reciprocals of $\hat{\pi}_{ijF}$'s become very unstable. As a result, variance estimates are also highly variable and the probability of obtaining a negative variance estimate increases as well. To mitigate this effect, one may again consider using auxiliary information to improve precision of second-order inclusion probability estimates in the spirit of Gamrot [8]. In the next section this approach is adopted for a certain sampling scheme. An alternative variance estimator is then constructed.

2 Fixed-cost sampling

The fixed-cost sequential sampling scheme of Pathak [11] is characterized by varying inclusion probabilities which are computationally extremely demanding even for very low population sizes. Although sufficiency-based design-unbiased estimates of population totals do exist for this scheme and may be calculated without knowing inclusion probabilities, the simulation approach may be of interest when nonresponse adjustments need to be incorporated or when the original scheme is modified. Overall the scheme constitutes a convenient model for presenting the proposed approach.

Let $c(1), ..., c(N)$ be individual costs of surveying respective population units, known in advance and let L be the fixed survey budget. Budget excesses are unacceptable. The procedure draws units to the sample one by one without replacement and with equal probabilities, until the cumulative cost of the sample becomes greater than or equal L. The element for which it occurs is not included in the sample. Hence the sample selection under such a scheme may be treated as happening in two phases. In the first phase all population units are sorted randomly into a sequence $(A_1, ..., A_N)$ of size N in such a way that all permutations of U are equally likely to occur. In the second phase the cumulative cost of succeeding units in the obtained sequence is computed and first M units are sampled, where M is chosen in such a way that $c(A_1) + ... + c(A_M) < L$ and $c(A_1) + ... + c(A_{M+1}) \geq L$. The sample size is random.

Let $\Psi = \{\psi_1, ..., \psi_{N!}\}$ represent all permutations of U where the k-th permutation is denoted by $\psi_k = (a_1^{(k)}, ..., a_N^{(k)})$ for $k \in \{1, ..., N!\}$. The sequence of individual costs corresponding to elements of ψ_k is $(c(a_1^{(k)}), ..., c(a_N^{(k)}))$ for $k \in \{1, ..., N!\}$. Let

$$M(\psi_k) = \max \left\{ M \in U : \sum_{j=1}^{M} c(a_j^{(k)}) < L \right\} \tag{6}$$

and let $\tau_0(\cdot)$ be a function that assigns to each permutation ψ_i a sequence of its $M(\psi_i)$ first units. Hence $c(a_1^{(k)}) + ... + c(a_{M(\psi_k)}^{(k)}) < L$ and $c(a_1^{(k)}) + ... + c(a_{M(\psi_k)+1}^{(k)}) \geq L$. Moreover, let $\tau_1(\cdot)$ be a reduction function that assigns to any ordered sequence $O = (o_1, ..., o_z)$ an unorderet set $\tau_1(O) = \{o_1, ..., o_z\}$

of its elements. Let $\tau(\cdot) = \tau_1(\tau_0(\cdot))$ represent the composition of τ_1 and τ_0. Hence for any $\psi \in \Psi$, $\tau(\psi)$ represents an unordered sample corresponding to ψ. Let $\Psi_i = \{\psi \in \Psi : a_i \in \tau(\psi)\}$ and let $\Psi_{ij} = \{\psi \in \Psi : a_i, a_j \in \tau(\psi)\}$ for any $i \neq j \in U$. Inclusion probabilities of first two orders may be expressed as

$$\pi_i = \frac{\#\Psi_i}{\#\Psi} \tag{7}$$

and

$$\pi_{ij} = \frac{\#\Psi_{ij}}{\#\Psi} \tag{8}$$

for $i \neq j \in U$ while $\#\Psi = N!$. Calculation of inclusion probabilities through (7) and (8) requires enumerating all possible permutations of U and in most cases is not feasible. However several useful properties of these probabilities may be proven. They are formally stated in following four propositions.

Proposition 1. *If the sample is drawn from a finite population using the Pathak scheme with individual costs $c(1), ..., c(N)$ and if costs of some i-th and i'-th unit $(i, i' \in U)$ satisfy $c(i) \geq c(i')$ then $\pi_i \leq \pi_{i'}$.*

Proof. Let $\psi_k = (a_1^{(k)}, ..., a_N^{(k)}) \in \Psi_i$ and let $a_g^{(k)} = i$ and $a_{g'}^{(k)} = i'$. Let $\psi_k^* = (a_1^{*(k)}, ..., a_N^{*(k)})$ be a permutation of elements in U obtained by exchanging the g-th element and g'-th element in ψ_k, so that $a_g^{*(k)} = i'$ and $a_{g'}^{*(k)} = i$ while $a_z^{*(k)} = a_z^{(k)}$ for $z \in \{1, ..., N\} - \{g, g'\}$. Let $\Psi_i^* = \{\psi_k^* : \psi_k \in \Psi_i\}$. For any $\psi_k \in \Psi_i$, recalling that $c(i) \geq c(i')$ we have: $c(a_1^{*(k)}) + ... + c(a_{M(\psi_k)}^{*(k)}) \leq c(a_1^{(k)}) + ... + c(a_{M(\psi_k)}^{(k)})$ and consequently $M(\psi_k^*) \geq M(\psi_k)$. Meanwhile, among first $M(\psi_k)$ elements in ψ_k^*, one is certainly equal to i' and hence $i' \in \tau(\psi_k^*)$. Consequently $\Psi_i^* \subseteq \Psi_{i'}$ and $\#\Psi_i = \#\Psi_i^* \leq \#\Psi_{i'}$. So from (7) we have $\pi_i \leq \pi_{i'}$. □

Proposition 2. *If the sample is drawn from a finite population using the Pathak scheme with individual costs $c(1), ..., c(N)$ and if costs of some i-th, j-th, i'-th and j'-th unit $(i \neq j, i' \neq j' \in U)$ satisfy $c(i) \geq c(i')$ and $c(j) \geq c(j')$ then $\pi_{ij} \leq \pi_{i'j'}$.*

Proof. Let $\psi_k = (a_1^{(k)}, ..., a_N^{(k)}) \in \Psi_{ij}$ and let $a_g^{(k)} = i$, $a_{g'}^{(k)} = i'$, $a_h^{(k)} = j$, $a_{h'}^{(k)} = j'$. Let $\psi_k^* = (a_1^{*(k)}, ..., a_N^{*(k)})$ be a permutation of elements in U obtained by exchanging the g-th and g'-th element, then exchanging h-th and h'-th element in ψ_k, so that $a_g^{*(k)} = i'$, $a_{g'}^{*(k)} = i$, $a_h^{*(k)} = j'$, $a_{h'}^{*(k)} = j$ while $a_z^{*(k)} = a_z^{(k)}$ for $z \in \{1, ..., N\} - \{g, h, g', h'\}$. Let $\Psi_{ij}^* = \{\psi_k^* : \psi_k \in \Psi_{ij}\}$. For any $\psi_k \in \Psi_{ij}$, recalling that $c(i) \geq c(i')$ and $c(j) \geq c(j')$ we have: $c(a_1^{*(k)}) + ... + c(a_{M(\psi_k)}^{*(k)}) \leq c(a_1^{(k)}) + ... + c(a_{M(\psi_k)}^{(k)})$ and consequently $M(\psi_k^*) \geq M(\psi_k)$. Meanwhile, among first $M(\psi_k)$ elements in ψ_k^*, one element is certainly equal to i' and one element is certainly equal to j' and hence $i', j' \in \tau(\psi_k^*)$. Consequently, $\Psi_{ij}^* \subseteq \Psi_{i'j'}$ and $\#\Psi_{ij} = \#\Psi_{ij}^* \leq \#\Psi_{i'j'}$. From (8) we have $\pi_{ij} \leq \pi_{i'j'}$. □

Proposition 3. *If the sample is drawn from a finite population using the Pathak scheme with individual costs $c(1), ..., c(N)$ and if costs of some i-th and i'-th unit $(i, i' \in U)$ satisfy $c(i) = c(i')$ then $\pi_i = \pi_{i'}$.*

Proof. If $c(i) = c(i')$ then it is true that $c(i) \geq c(i')$ and $c(i) \leq c(i')$. Applying twice proposition 1 we conclude that $\pi_i \leq \pi_{i'}$ and at the same time $\pi_i \geq \pi_{i'}$. Both conditions may simultaneously be true only when $\pi_i = \pi_{i'}$ □

Proposition 4. *If the sample is drawn from a finite population using the Pathak scheme with individual costs $c(1), ..., c(N)$ and if costs of some i-th, j-th, i'-th and j'-th unit $(i \neq j, i' \neq j' \in U)$ satisfy $c(i) = c(i')$ and $c(j) = c(j')$ then $\pi_{ij} = \pi_{i'j'}$.*

Proof. If $c(i) = c(i')$ and $c(j) = c(j')$ then it is also true that $c(i) \geq c(i')$ and $c(j) \geq c(j')$. Applying proposition 2 we get $\pi_{ij} \leq \pi_{i'j'}$. However from assumptions $c(i) = c(i')$ and $c(j) = c(j')$ we may also infer that $c(i) \leq c(i')$ and $c(j) \leq c(j')$, and applying again proposition 2 we get $\pi_{ij} \geq \pi_{i'j'}$. Conditions $\pi_{ij} \leq \pi_{i'j'}$ and $\pi_{ij} \geq \pi_{i'j'}$ are simultaneously satisfied only when $\pi_{ij} = \pi_{i'j'}$. □

Without a loss of generality, let us assume that the finite population is ordered in such a way that $c(1) \leq c(2) \leq ... \leq c(N)$. Proposition 1 leads to a conclusion that first-order inclusion probabilities must satisfy a simple order: $\pi_1 \geq \pi_2 \geq ... \geq \pi_N$. Moreover, according to Proposition 2 second-order inclusion probabilities must satisfy $\pi_{ij} \geq \pi_{i'j'}$ whenever $i \leq i'$ and $j \leq j'$ for $i \neq j, i' \neq j' \in U$. The resulting

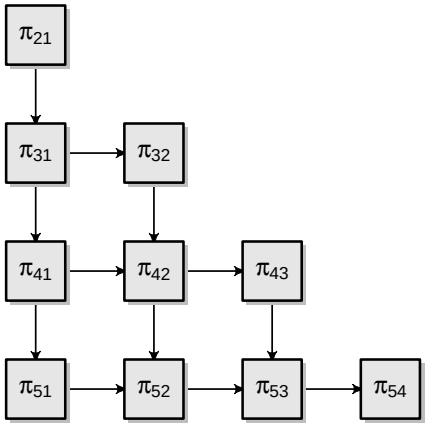

Figure 1 Cover graph for a population of $N = 5$ elements

system of inequalities is illustrated by the Hasse diagram (cover graph) shown in the figure 1. There is no guarantee that Fattorini's estimators of inclusion probabilities satisfy the whole system of inequalities dictated by propositions 1 and 2. Forcing these constraints to be satisfied may improve the accuracy of population total estimates. To preserve constraints one may consider calculating estimates of first-order inclusion probabilities as solutions to the quadratic program

$$\begin{cases} f_1(\hat{\pi}_1, ..., \hat{\pi}_N) \to min \\ \hat{\pi}_1 \geq ... \geq \hat{\pi}_N \\ \hat{\pi}_i \in <0, 1>, i \in U \end{cases} \tag{9}$$

where

$$f_1(\hat{\pi}_1, ..., \hat{\pi}_N) = \sum_{i \in U} R_i (\hat{\hat{\pi}}_{iF} - \hat{\pi}_i)^2 \tag{10}$$

while weights $R_1, ..., R_N$ represent numbers of replications observed for individual units. They will take the same value R when all individual unit costs values are unique, but according to the proposition 3 they may also equal multiples of R in groups of units with exactly the same costs. Let $a^* = a + 1$ for $a = i, j$. Estimates of second-order probabilities are obtained by solving a separate quadratic program:

$$\begin{cases} f_2(\hat{\pi}_{ij}; j < i) \to min \\ \hat{\pi}_{ij} \geq \hat{\pi}_{ij^*}; \;\; j, j^* < i \in U \\ \hat{\pi}_{ij} \geq \hat{\pi}_{i^* j}; \;\; j < i, i^* \in U \\ \hat{\pi}_{ij} \in <0, 1>; \;\; j < i \in U \end{cases} \tag{11}$$

where

$$f_2(\hat{\pi}_{ij}, j < i) = \sum_{j < i \in U} R_{ij} (\hat{\pi}_{ijF} - \hat{\pi}_{ij})^2 \tag{12}$$

while R_{ij} for $i > j \in U$ represent numbers of replications observed for i-th and j-th unit. They will all be equal to R when all unit costs are unique and may be multiples of R in groups of units featuring exactly the same individual costs, on the basis of the proposition 4. Programs (9) and (11) are formulated in terms of inclusion probability estimates $\hat{\pi}_{ij}$ for $i = j$ (first order) and $i < j$ (second order), but knowing that $\pi_{ij} = \pi_{ji}$ for $i, j \in U$ one in fact estimates all inclusion probabilities of first two orders in the whole population. Simulation results applying to units both included and not included in the sample s may then be applied to improve accuracy of finite population totals. It should also be noted that one might also formulate additional constraints involving both first and second inclusion probabilities. Such constraints would however be easier to satisfy and hence less useful to constuct estimates. Both programs may be solved using the active set method discussed in [4]. Plugging obtained estimates $\hat{\pi}_i$ and $\hat{\pi}_{ij}$ into (2) instead of π_i and π_{ij} (for $i, j \in U$) one may construct the variance estimator for (1) in the form:

$$\hat{V}(\hat{t}_*) = \sum_{i,j \in s} \hat{\hat{y}}_i \hat{\hat{y}}_j \hat{\hat{\Delta}}_{ij} \tag{13}$$

where $\hat{\hat{y}}_i = y_i / \hat{\pi}_i$, $\hat{\hat{\Delta}}_{ij} = \hat{\Delta}_{ij} / \hat{\pi}_{ij}$ and $\hat{\Delta}_{ij} = \hat{\pi}_{ij} - \hat{\pi}_i \hat{\pi}_j$ for $i, j \in U$.

3 Numerical illustration

Assume that the finite population consists of $N = 24$ units. Individual costs are given by a vector $\mathbf{c} = [c(1), ..., c(N)]' = [1, 2, ..., 24]'$ and the budget constraint is $L = 90$ which constitutes 30% of the census cost. A simulation involving $R = 500$ sample replications was carried out to estimate unknown inclusion probabilities under Pathak sampling. Estimates of the $N \times N$ inclusion probability matrix $\Pi = [\pi_{ij}]$ (where $\pi_{ii} = \pi_i$ for $i \in U$) obtained using Fattorini's formulae (4), (5), and obtained by solving (9) and (11) through active set method are shown in figure 2. Visual inspection reveals that the introduction of hierarchical constraints reduces volatility of inclusion probability estimates.

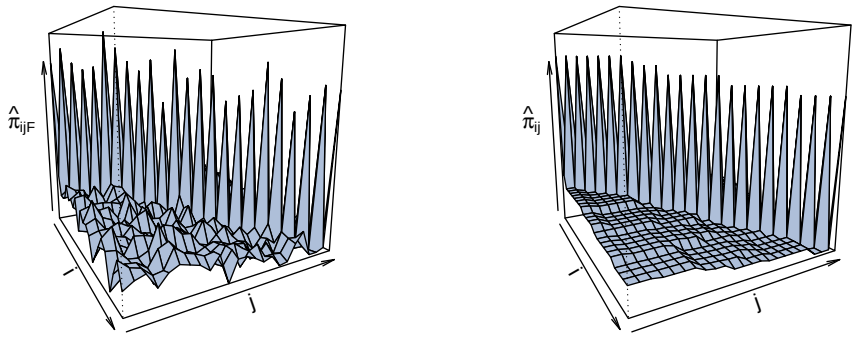

Figure 2 Estimated matrices of inclusion probabilities

To sched some light on properties of the proposed variance estimator a simulation was carried out for values of $\mathbf{c}$ and L introduced above. A total of 5000 realizations of the sample s were drawn with $R = 500$ replications generated for each drawn sample. Two variants of the estimator (3) were computed. The first one denoted by the letter F utilized raw Fattorini's estimates of inclusion probabilities given by (4) and (5). The second one denoted by the letter A utilized inclusion probability estimates obtained by solving (9) and (11) through the active set method. Table 1 shows percentages P_F and P_A of negative variance estimates, their ratio P_A/P_F as well as ratios of variances V_A/V_F and mean square errors M_A/M_F of both estimators, for several value sets $\mathbf{y} = [y_1, ..., y_N]'$ of the study variable.

$\mathbf{y}$	P_F	P_A	P_A/P_F,	V_A/V_F	M_A/M_F
$[1, 2, ..., 24]'$	0.3695	0.2134	0.5776	0.4329	0.5028
$[1, ..., 12, 12, ..., 1]'$	0.0447	0.0026	0.0581	0.6647	0.7269
$[12, ..., 1, 1, ..., 12]'$	0.1365	0.0544	0.3985	0.7076	0.7537
$[24, 23, ..., 1]'$	0.0054	0.0003	0.0645	0.9853	1.0008

Table 1 Selected characteristics of variance estimators' empirical distributions

The results suggest that the proposed estimation method may lead to a substantial reduction in the number of unwelcome negative estimates as compared to Fattorini's approach. Moreover, in three out four investigated cases the variance was considerably reduced and the bias introduced through constraints did not overwhelm the variance reduction effect, which led to a reduced mean squre error as well. In the fourth case both variance and mean square error were comparable.

4 Conclusion

Due to very small expected sample size in the presented example the bias and standard deviation of variance estimates for the estimator $\hat{t}_*$ were large in comparison to its true variance. This had to result in high frequencies of negative variance estimates. In practice samples would be much larger and these frequencies would be very small. Nevertheless, incorporation of hierarchical constraints in the estimation process seems to mitigate risks of such unwelcome developments. The variance reduction is also a benefit. After modifications the proposed approach might potentially be adopted to assess the variance of estimates for population parameters other than the total, such as covariances, ratios, or price indices of [1]. The

main practical challenge seems to be associated with computational complexity of the method as the number of constraints grows with the square of the population size. This might be tackled by solving proposed quadratic programs only for some subset of population units (in particular by collapsing unit groups on the basis of propositions 3 and 4), using parallel computing and/or adopting scalable quadratic programming techniques such as those in [3].

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An Alternative Approach to the Structure Determination of Hierarchical Archimedean Copulas

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Abstract. Copulas offer very a flexible tool for a stochastic dependence modeling. One of the most popular classes of copulas is the class of hierarchical Archimedean copulas, which gained its popularity due to the fact that the models from the class are able to model the stochastic dependencies conveniently even in high dimensions. One critical issue when estimating a hierarchical Archimedean copula is to correctly determine its structure. The paper describes an approach to the problem of the structure determination of a hierarchical Archimedean copula, which is based on the close relationship of the copula structure and the values of measure of concordance computed on all its bivariate margins. The presented approach is conveniently summarized as a simple algorithm.

Keywords: hierarchical Archimedean copula, structure determination, measure of concordance, bivariate margins, nesting condition

JEL classification: C51, C46
AMS classification: 62H99

1 Introduction

Hierarchical Archimedean copulas (HACs), which generalize Archimedean copulas (ACs), overcome some limitations and bring some advantages compared to the most popular class of Gaussian copulas [2]. There already emerged successful applications of HACs in finance, e.g., in collateral debt obligation pricing, see [2, 5]. One critical issue when estimating HAC is to properly determine its structure. Despite the popularity of HACs, there exists only one paper [9] addressing generally the structure determination. The method presented in that paper mainly focus on maximum likelihood estimation (MLE) for the estimation of HAC's parameters, which are later used for the structure determination. The MLE used in the method involves the computation of the density of a HAC that needs up to d derivatives, where d is the data dimension. The authors claim, that the approach is feasible in high dimensions when using numerical method for the density computation and present two examples for $d = 5$, which involves only homogeneous HAC and which incorporates ACs belonging to one Archimedean family. Our approach provides an alternative way to the problem, which completely avoids the need of the HAC's density computation for some Archimedean families, hence is feasible even in very high dimensions.

The paper is structured as follows. The second section recalls some necessary theoretical concepts concerning copulas, the third section presents the proposed approach to the structure determination of HAC and the fourth section concludes the paper.

2 Preliminaries

2.1 Copulas

Definition 1. For every $d \geq 2$, a *d-dimensional copula* (shortly, *d-copula*) is a d-variate distribution function on $\mathbb{I}^d$ ($\mathbb{I}$ is unit interval), whose univariate margins are uniformly distributed on $\mathbb{I}$.

At the first look, copulas (denote the set of all copulas as $\mathcal{C}$) form one of many classes of joint distribution functions (shortly, joint d.f.s). What makes copulas interesting is that they establish a

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connection between general joint d.f. and its univariate margins (in text below we use only *margin* for term *univariate margin*).

Theorem 1. ***(Sklar's Theorem)*** *[10] Let H be a d-dimensional d.f. with margins $F_1, ..., F_d$. Let A_j denote the range of F_j, $A_j := F_j(\overline{\mathbb{R}})(j = 1, ..., d), \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. Than there exists a copula C such for all $(x_1, ..., x_d) \in \overline{\mathbb{R}}^d$,*

$$H(x_1, ..., x_d) = C(F_1(x_1), ..., F_d(x_d)). \tag{1}$$

Such a C is uniquely determined on $A_1 \times ... \times A_d$ and, hence, it is unique if $F_1, ..., F_d$ are all continuous.

Through the Sklar's theorem, one can derive for any d-variate d.f. its copula C using (1). In case that the margins $F_1, ..., F_d$ are all continuous, copula C is given by $C(u_1, ..., u_d) = H(F_1^-(u_1), ..., F_d^-(u_d))$, where $F_i^-, i \in \{1, ..., d\}$ denotes pseudo-inverse of F_i given by $F_i^-(s) = \inf\{t|\ F_i(t) \geq s\}, s \in \mathbb{I}$. Many classes of copulas are derived in this way from popular joint d.f.s, e.g., the most popular class of Gaussian copulas is derived using H corresponding to d-variate Gaussian distribution. But, using this process often results in copula forms not representable in closed form, what can bring difficulties in some applications.

2.2 Archimedean Copulas

This drawback is overcame while using Archimedean copulas due to their different construction process. ACs are not constructed using thr Sklar's theorem, but instead of it, one starts with a given functional form and asks for properties in order to obtain a proper copula. As a result of such a construction, ACs are always expressed in closed form, which is one of the main advantages of this class of copulas [3]. To construct ACs we need a notion of an *Archimedean generator* and a *complete monotonicity*.

Definition 2. *Archimedean generator* (shortly, *generator*) is continuous, nonincreasing function $\psi : [0, \infty] \to [0, 1]$, which satisfies $\psi(0) = 1, \psi(\infty) = \lim_{t\to\infty} \psi(t) = 0$ and is strictly decreasing on $[0, \inf\{t : \psi(t) = 0\}]$.

Remark 1. We denote set of all generators as Ψ.

Definition 3. Function f is called *completely monotone* (shortly, c.m.) on $[a, b]$, if $(-1)^k f^{(k)}(x) \geq 0$ holds for every $k \in \mathbb{N}_0, x \in (a, b)$.

Definition 4. Any d-copula C is called *Archimedean copula* (we denote it d-AC), if it admits the form

$$C(\mathbf{u}) := C(\mathbf{u}; \psi) := \psi(\psi^{-1}(u_1) + ... + \psi^{-1}(u_d)), \mathbf{u} \in \mathbb{I}^d, \tag{2}$$

where $\psi \in \Psi$ and its inverse $\psi^{-1} : [0, 1] \to [0, \infty]$ is defined $\psi^{-1}(0) = \inf\{t : \psi(t) = 0\}$.

For verifying whether function C given by (2) is a proper copula, we can use the property stated in Definiton 3. A condition sufficient[1] for C to be a copula is stated as follows.

Theorem 2. *If $\psi \in \Psi$ is completely monotone, then function C given by (2) is copula.*

We can see from Definition 4 that having a random vector $\mathbf{U}$ distributed according to some AC, all its k-dimensional ($k < d$) marginal copulas have the same marginal distribution. It implies that all multivariate margins of the same dimension are equal, thus, e.g., the dependence among all pairs of components is identical. This symmetry of ACs is often considered to be a rather strong restriction, especially in high dimensional applications.

2.3 Hierarchical Archimedean Copulas

To allow for asymmetries, one may consider the class of HACs[2], recursively defined as follows.

Definition 5. A d-dimensional copula C is called *hierarchical Archimedean copula* if it is an AC with arguments possibly replaced by other hierarchical Archimedean copulas. If C is given recursively by (2) for $d = 2$ and

$$C(\mathbf{u}; \psi_0, ..., \psi_{d-2}) = \psi_0(\psi_0^{-1}(u_1) + \psi_0^{-1}(C(u_2, ..., u_d; \psi_1, ..., \psi_{d-2}))), \mathbf{u} \in \mathbb{I}^d, \tag{3}$$

[1] Necessary and sufficient condition for C to be a copula can be found in [6]

[2] often also called *nested Archimedean copulas*

for $d \geq 3$, C is called *fully-nested hierarchical Archimedean copula* with $d-1$ *nesting levels*. Otherwise C is called *partilally-nested hierarchical Archimedean copula*. [4]

Remark 2. We denote a d-dimensional HAC as d-HAC. We refer to the hierarchical ordering of $C(\cdot;\psi_0), ..., C(\cdot;\psi_{d-2})$ together with the ordering of variables $u_1, ..., u_d$ as the *structure* of a d-HAC.

From the definition, we can see that ACs are special cases of HACs. The most simple proper fully-nested HAC is copula C obtained for $d = 3$ with two nesting levels. The structure of this copula is given by

$$\begin{aligned} C(\mathbf{u};\psi_0,\psi_1) &= C(u_1, C(u_2,u_3;\psi_1);\psi_0) \\ &= \psi_0(\psi_0^{-1}(u_1) + \psi_0^{-1}(\psi_1(\psi_1^{-1}(u_2) + \psi_1^{-1}(u_3)))), \mathbf{u} \in \mathbb{I}^3. \end{aligned} \tag{4}$$

As in the case of ACs we can ask for necessary and sufficient condition for function C given by (3) to be a proper copula. Partial answer for this question in form of sufficient condition is contained in the following theorem [6].

Theorem 3. ***(McNeil (2009))***. *If $\psi_j \in \Psi_\infty, j \in \{0, ..., d-2\}$ such that $\psi_k^{-1} \circ \psi_{k+1}$ have completely monotone derivatives for all $k \in \{0, ..., d-3\}$, then $C(\boldsymbol{u};\psi_0, ..., \psi_{d-2}), \boldsymbol{u} \in \mathbb{I}^d$, given by (3) is a copula.*

If we take the most simple 3-HAC given by (4), we can see that the condition for C to be a proper copula following from McNeil's theorem is $(\psi_0^{-1} \circ \psi_1)'$ to be completely monotone. As this condition will be essential for the rest of this paper we put it in individual definition.

Definition 6. Let $\psi_a, \psi_b \in \Psi_\infty, a, b \in \{0, ..., d-2\}, a \neq b$ and $C(\cdot;\psi_a)$ corresponds to parent of $C(\cdot;\psi_b)$ in the tree structure of C. Then condition for $(\psi_a^{-1} \circ \psi_b)'$ to be compete monotone is called *nesting condition*.

As we can observe, verification of conditions in McNeil's theorem is just $d-2$ verifications of nesting condition for $d-2$ different pairs $\psi_k, \psi_{k+1}, k \in \{0, ..., d-2\}$. McNeil's theorem is stated only for fully-nested HACs, but it can be easily translated also for use with partially-nested HACs.

For the sake of simplicity, assume that each d-HAC structure corresponds to some binary tree t. Each node in t represents one 2-AC. Each 2-AC is determined just by its corresponding generator, so we identify each node in t with one generator and hence we have always nodes $\psi_0, ..., \psi_{d-2}$. For a node ψ denote as $\mathcal{D}_n(\psi)$ the set of all descendant nodes of ψ, $\mathcal{P}(\psi)$ the parent node of ψ, $\mathcal{H}_l(\psi)$ the left child of ψ and $\mathcal{H}_r(\psi)$ the right child of ψ. The leafs of t correspond to the variables $u_1, ..., u_d$.

2.4 Measure of concordance

A measure of concordance (MoC) is a measure, which reflects a degree of dependency between two random variables independently on their univariate distributions. There also exist generalizations for more than two random variables, but we present only pairwise measure of concordance. As $\mathcal{C}$ allows for partial ordering known as *concordance ordering*, a measure of concordance also reflects this ordering (see [3, 7]). One of the most popular measures of concordance is *Kendall's tau*. As we are interested in its relationship with a general bivariate copula, we use its the definition given by (as in [1])

$$\tau(C) = 4 \int_{\mathbb{I}^2} C(u_1,u_2) dC(u_1,u_2) - 1. \tag{5}$$

If C is 2-AC based on a generator ψ and ψdepends on the parameter θ, then (5) states a relationship between θ and τ. This relationship is very important for our approach and is used extensively later in Section 3.

2.5 Okhrin's algorithm for the structure determination of HAC

We recall the algorithm presented in [8] for the structure determination of HAC, which returns for some unknown HAC C its structure using only the known forms of its bivariate margins. The algorithm uses the following definition.

Definition 7. Let C be a d-HAC with generators $\psi_0, ..., \psi_{d-2}$ and $(U_1, ..., U_d) \sim C$. Then denote as $\mathcal{U}_C(\psi_k), k = 0, ..., d-2$, the set of indexes $\mathcal{U}_C(\psi_k) = \{i | (\exists U_j)(U_i, U_j) \sim C(\cdot;\psi_k) \vee (U_j, U_i) \sim C(\cdot;\psi_k), 1 \leq i < j \leq d\}, k = 0, ..., d-2$.

Proposition 4. *Defining $\mathcal{U}_C(u_i) = \{i\}$ for the leaf $i, 1 \leq i \leq d$, there is an unique disjunctive decomposition of $\mathcal{U}_C(\psi_k)$ given by*

$$\mathcal{U}_C(\psi_k) = \mathcal{U}_C(\mathcal{H}_l(\psi_k)) \cup \mathcal{U}_C(\mathcal{H}_r(\psi_k)). \tag{6}$$

Due to space limitations we do not state the proof for the proposition and we refer the reader to the Okhrin's work [8], which includes detailed description of the method and the necessary proofs.

For an unknown d-HAC C, knowing all its bivariate margins, its structure can be easily determined with Algorithm 1, which returns the unknown structure t of C. We start from the sets $\mathcal{U}_C(u_1), ..., \mathcal{U}_C(u_d)$ joining them together through (6) until we reach the node ψ for which $\mathcal{U}_C(\psi) = \{1, ..., d\}$.

Algorithm 1 The HAC structure determination

$\mathcal{I} = \{0, ..., d-2\}$
while $\mathcal{I} \neq \emptyset$ **do**
1. $k = \text{argmin}_{i \in \mathcal{I}}(\#\mathcal{U}_C(\psi_i))$, if there are more minima, then choose as k one of them arbitrarily.
2. Find the nodes ψ_l, ψ_r, for which $\mathcal{U}_C(\psi_k) = \mathcal{U}_C(\psi_l) \cup \mathcal{U}_C(\psi_r)$.
3. $\mathcal{H}_l(\psi_k) := \psi_l, \mathcal{H}_r(\psi_k) := \psi_r$.
4. Set $\mathcal{I} := \mathcal{I} \backslash \{k\}$.

end while

3 Our approach

Recalling Theorem 3, the sufficient condition for C to be a proper copula is, that the nesting condition must hold for each generator and its parent in a HAC structure. As this is the only known condition that assures that C is a proper copula, we concern in this work only the copulas, which fulfill this condition. The nesting condition results in constraints for the parameters θ_0, θ_1 of the involved generators ψ_0, ψ_1 (see [4, 3]). As $\theta_i, i = 1, 2$ is closely related to a MoC, e.g. τ and θ_i relationship established through (5), there is also an important relationship between the MoC and the HAC tree structure following from the nesting condition. This relationship is described for the fully-nested 3-HAC given by the form (4) in Remark 2.3.2 in [3]. There is stated that if the nesting condition holds for the parent-child pair (ψ_0, ψ_1), then $0 \leq \kappa(\psi_0) \leq \kappa(\psi_1)$, where κ is a MoC (as we concern only HACs with binary structures, which incorporates only 2-ACs, which are fully determined only by its generator, we use as domain of κ the set Ψ instead of the usually used set of all 2-copulas). We generalize this statement, using our notion, as follows.

Proposition 5. *Let C be a d-HAC with the structure t and the generators $\psi_0, ..., \psi_{d-2}$, where each parent-child pair satisfy the nesting condition. Let κ be a MoC. Than $\kappa(\psi_i) \leq \kappa(\psi_j)$,* where *$\psi_j \in \mathcal{D}_n(\psi_i)$, holds for each $\psi_i, i = 0, ..., d-2$.*

Proof. If $\psi_i = \mathcal{P}(\psi_j)$, then we get directly $\kappa(\psi_i) \leq \kappa(\psi_j)$ using Remark 2.3.2 from [3]. Otherwise, as $\psi_j \in \mathcal{D}(\psi_i)$, there exists a unique sequence $\psi_{k_1}, ..., \psi_{k_l}$, where $0 \leq k_m \leq d-2, m = 1, ..., l, l \leq d-1, \psi_{k_1} = \psi_i, \psi_{k_l} = \psi_j$ and $\psi_{k-1} = \mathcal{P}(\psi_k)$ for $k = 2, ..., l$. Applying the above mentioned remark for each pair $(\psi_{k-1}, \psi_k), k = 2, ..., l$, we get $\kappa(\psi_{k_1}) \leq ... \leq \kappa(\psi_{k_l})$. □

Thus, having a branch from t, all its nodes are uniquely ordered according to their value of κ assuming unequal values of κ for all parent-child pairs. This provides us an alternative algorithm for the HAC structure determination. We have to assign the generators with the highest values of κ to the lowest levels of the branches in the structure and ascending to higher levels we assign the generators with lower values of κ.

To allow for computation of MoC among m (possibly > 2) random variables (r.v.s) we state the following definition. For simplification, denote the set of pairs of r.v.s as $\mathbf{U}_{IJ} = \{(U_i, U_j) | (i, j) \in I \times J\}$, where $I, J \subset \mathbb{N}, I \neq \emptyset \neq J$.

Definition 8. Let $m \in \mathbb{N}$ and κ be a MoC. Then define an *aggregated* MoC κ^+ as

$$\kappa^+(\mathbf{U}_{IJ}) = \begin{cases} \kappa(U_i, U_j) & \text{if } I = \{i\}, J = \{j\} \\ +(\kappa(U_i, U_j)_{i \in I, j \in J}), & \text{else,} \end{cases} \tag{7}$$

where the non-empty sets $I, J \subset \{1, ..., m\}, I \cap J = \emptyset$ and $+$ denotes an aggregation function[3], for which $+(x, ..., x) = x$ for all $x \in \mathbb{I}$.

Remark 3. $\kappa(\psi_k) = \kappa^+(\mathbf{U}_{\mathcal{U}_C(\mathcal{H}_l(\psi_k))\mathcal{U}_C(\mathcal{H}_r(\psi_k))})$ for a d-HAC C and for each $k = 0, ..., d-2$.

Let us illustrate our approach to the structure determination for $d = 4$. Assume three different structures t_1, t_2, t_3 corresponding to copulas C_1, C_2, C_3. For t_1 let $\mathcal{U}_{C_1}(\psi_2) = \mathcal{U}_{C_1}(\psi_1) \cup \mathcal{U}_{C_1}(\psi_0) = \{3, 4\} \cup \{1, 2\}$. For simplification denote $\{3, 4\} \cup \{1, 2\}$ as $((34)(12))$. For t_2 let $\mathcal{U}_{C_2}(\psi_2) = \{u_4\} \cup (\mathcal{U}_{C_2}(\psi_1) \cup \mathcal{U}_{C_2}(\psi_0)) = \{u_4\} \cup (\{u_3\} \cup \{u_1, u_2\}) = (4(3(21)))$. For t_3 let $\mathcal{U}_{C_3}(\psi_0) = (3(4(12)))$. We see that t_1 is the structure of a partially-nested 4-HAC and t_2, t_3 are the structures of fully-nested 4-HACs. Also assume (without a loss of generality) $\kappa(\psi_2) = \alpha, \kappa(\psi_0) = \gamma$ and $\alpha < \kappa(\psi_1) < \gamma, \alpha, \gamma \in \mathbb{I}$ for all t_1, t_2, t_3. The case when $\alpha = \kappa(\psi_1)$ or $\kappa(\psi_1) = \gamma$ is discussed later for a 3-HAC. Denote $\beta_1 = \kappa(\psi_1)$ for t_1, $\beta_2 = \kappa(\psi_1)$ for t_2 and $\beta_3 = \kappa(\psi_1)$ for t_3. The quantities $\alpha, \beta_1, \beta_2, \beta_3, \gamma$ can be determined from corresponding bivariate distributions as for t_1 is $\alpha = \kappa(\psi_2) = \kappa(U_3, U_1) = \kappa(U_3, U_2) = \kappa(U_4, U_1) = \kappa(U_4, U_2), \beta_1 = \kappa(\psi_1) = \kappa(U_3, U_4), \gamma = \kappa(\psi_0) = \kappa(U_1, U_2)$. For t_2 we have $\alpha = \kappa(\psi_2) = \kappa(U_4, U_3) = \kappa(U_4, U_1) = \kappa(U_4, U_2), \beta_2 = \kappa(\psi_1) = \kappa(U_3, U_1) = \kappa(U_3, U_2), \gamma = \kappa(\psi_0) = \kappa(U_1, U_2)$. For t_3 similarly $\alpha = \kappa(\psi_2) = \kappa(U_4, U_3) = \kappa(U_3, U_1) = \kappa(U_3, U_2), \beta_3 = \kappa(\psi_1) = \kappa(U_4, U_1) = \kappa(U_4, U_2), \gamma = \kappa(\psi_0) = \kappa(U_1, U_2)$.

Now assume a 4-HAC C with unknown structure $t \in \{t_1, t_2, t_3\}$ and $(U_1, U_2, U_3, U_4) \sim C$. Compute κ for all pairs of the r.v.s. If follows from the assumptions that $\kappa(U_1, U_2) = \gamma$ is always (for $t = t_1, t_2, t_3$) the maximum from those values. To satisfy Proposition 5, it is necessarily $\mathcal{U}_C(\psi_0) = \{12\}$, what assures through Algorithm 1 that ψ_0 is assigned to the lowest level of a branch from t. We introduce the a new variable $Z = (U_1, U_2)$, which represents r.v.s U_1, U_2. Once again compute κ for all the pairs of the new r.v.s, which are now r.v.s (U_3, U_4, Z). As Z represents two r.v.s we use generalized κ^+. Thus we get $\beta_1 = \kappa^+(U_3, U_4) = \kappa(U_3, U_4), \beta_2 = \kappa^+(U_3, Z) = \kappa^+(\mathbf{U}_{\{3\}\{12\}})$ and $\beta_3 = \kappa^+(U_4, Z) = \kappa^+(\mathbf{U}_{\{4\}\{12\}})$. Consider that under $t = t_1$ is $\beta_1 > \beta_2 = \beta_3 = \alpha$. Under $t = t_2$ is $\beta_2 > \beta_1 = \beta_3 = \alpha$ and under $t = t_3$ is $\beta_3 > \beta_1 = \beta_2 = \alpha$. The determination of $\mathcal{U}_C(\psi_1)$ in accordance with Proposition 5 is then obvious - $\mathcal{U}_C(\psi_1) = \{3, 4\}$ if $\beta_1 = \max(\beta_1, \beta_2, \beta_3)$ or $\mathcal{U}_C(\psi_1) = \{3, 2, 1\}$ if $\beta_2 = \max(\beta_1, \beta_2, \beta_3)$ or $\mathcal{U}_C(\psi_1) = \{4, 2, 1\}$ if $\beta_3 = \max(\beta_1, \beta_2, \beta_3)$. The set $\mathcal{U}_C(\psi_2) = \{4, 3, 2, 1\}$ for all t_1, t_2, t_3.

The described process is generalized in Algorithm 2 for arbitrary $d > 2$. The algorithm returns the sets $\mathcal{U}_C(z_{d+k+1})$ corresponding to the sets $\mathcal{U}_C(\psi_k), k = 0, ..., d-2$. Passing them to Algorithm 1, we avoid their computation from Definition 7 and we get the requested d-HAC structure without a need of knowing the forms of the bivariate margins.

Algorithm 2 The HAC structure determination based on κ

Input:

1) $\mathcal{I} = \{1, ..., d\}$, 2) $(U_1, ..., U_d) \sim C$, 3) κ^+ ... an aggregated MoC, 4) $z_k = u_k, \mathcal{U}_C(z_k) = \{k\}, k = 1, ..., d$

The structure determination:

for $k = 0, ..., d-2$ **do**

1. $(i, j) := \underset{i^* < j^*, i^* \in \mathcal{I}, j^* \in \mathcal{I}}{\operatorname{argmax}} \kappa^+(\mathbf{U}_{\mathcal{U}_C(z_{i^*})\mathcal{U}_C(z_{j^*})})$
2. $\mathcal{U}_C(z_{d+k+1}) := \mathcal{U}_C(z_i) \cup \mathcal{U}_C(z_j)$
3. $\mathcal{I} := \mathcal{I} \cup \{d+k+1\} \backslash \{i, j\}$

end for

Output:

$\mathcal{U}_C(z_{d+k+1}), k = 0, ..., d-2$

Now consider a fully-nested 3-HAC with two equal generators given by the form $C(u_1, C(u_2, u_3; \psi); \psi) = \psi(\psi^{-1}(u_1) + \psi^{-1}(\psi(\psi^{-1}(u_2) + \psi^{-1}(u_3))))$. As it equals to $\psi(\psi^{-1}(u_1) + \psi^{-1}(u_2) + \psi^{-1}(u_3))$, which is the 3-AC $C(u_1, u_2, u_3; \psi)$, we get for this copula in Step 1. of the algorithm three pairs (1,2), (1,3), (2,3) corresponding to the maximal value of κ^+. This is because all bivariate margins of $C(u_1, u_2, u_3; \psi)$ are distributed equally. Choosing the first pair to be the pair (i, j) we get the result of the algorithm as $\mathcal{U}_C(\psi_0) = \{1, 2\}, \mathcal{U}_C(\psi_1) = \{1, 2, 3\}$. Passing it to Algorithm 1 we get the corresponding structure and denote it as r_1. In the same way we obtain for the second and the third pair the structures we denote as r_2, r_3. But, as $C(u_1, C(u_2, u_3; \psi); \psi) = C(u_1, u_2, u_3; \psi)$, all those structures r_1, r_2, r_3 corresponds to the same copula. Thus, in the case that there are more than one pair corresponding to the maximal value of κ^+ in Step 1., we can choose the pair arbitrarily, because it does not affect the resulting copula, i.e.

[3] like, e.g., max, min or mean

the algorithm return different structures, which however correspond to the same copula. This fact can be also easily generalized for the case when $d > 3$.

4 Conclusions

As the aggregated κ^+ depends only on the pairwise κ and the aggregation function $+$, we can easily derive its empirical version κ_n^+ just by substituting κ in κ^+ by its empirical version κ_n, e.g., by empirical version of Kendall's tau. Using κ_n^+ instead of κ^+ we can easily derive the empirical version of the structure determination process represented by Algorithms 1, 2. Conclude that in this way we base the structure determination only on the values of the pairwise MoC. This is the essential property of our approach, because if the relationship between κ and θ established through (5) is explicitly known, whole HAC, including its structure and its parameters, can be estimated just from κ_n computed on the realizations of $(U_i, U_j), 1 \leq i < j \leq d$ completely avoiding the use of the MLE.

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Quantitative modelling of consumption patterns in the CEEC

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Abstract. This paper proposes an econometric model of consumption function focusing on macroeconomic determinants under the condition of the CEE countries. Hence, the main aim of the paper is verifying selected models of consumption function and subsequently, the suitability of their application to conditions of real economic situation in the CEE countries. The attention is paid to short- and long-run patterns of behaviour applying a vector error correction model (VECM) that is suitable to study abovementioned problems. The long-run relations are estimated using real private consumption, real national disposable income per capita, real interest rates, anticipated inflation, and dummy for the CEE countries in levels and the short-run fluctuations from equilibrium using the same sample of variables are studied in differences. Hall's hypothesis on the impact of wealth on consumption is also examined including M3 aggregate as a proxy of financial wealth. The research is based on unbalanced panel analysis using selected developed countries data covering the period from 1970 to 2011. Since there is a tendency in changing consumer behaviour during 2008 crisis period it is convenient to use the dummy variable in order to prevent from biased results and to obtain interesting information about the topic.
Keywords: Consumption function, CEEC, unbalanced panel

JEL Classification: C01, C33, E21
AMS Classification: 91G70, 91B42, 91B84

1 Introduction

Private consumption represents an indispensable part of aggregate demand in most of the countries. Not only due to this fact private consumption was, is and will be unimpugnably of a considerable interest of economists or politicians. It represents circa two-thirds of the nation's GDP. For example, in the EU private consumption represents a remarkable 55–57% of GDP on average, in the CEEC the ratio of private consumption to GDP is negligibly higher ranging from 58–60% of GDP, which makes this component of aggregate demand extremely important. This is also reason why both theoretical and empirical approaches dealing with private consumption has evolved rapidly in last decades. In particular, among macroeconomic models we can mention the most influential ones as absolute income hypothesis (Keynes [15]), life-cycle hypothesis (Modigliani [18]), and permanent income hypothesis (Friedman [11]). Among fundamental empirical studies a research based on vector error correction model that models long-run relationship between consumption and income considering also short-run distortions from the equilibrium by Davidson, Hendry, Srba and Yeo [8] must be reminded.

The aim of this contribution is then to develop an econometric model of the consumption function for chosen industrial countries capturing the specifics of CEE countries that is in line with theoretical assumptions. For this purpose we will employ vector error correction model to describe the impact of both short- and long-run determinants as real national disposable income per capita, M3 aggregate, real interest rates, anticipated inflation.

The paper is organized as follows: section 2 brings a brief overview on fundamental and recent literature discussing consumption function, section 3 introduces data and methodology used to estimate private consumption in industrial countries, section 4 dissects results and section 5 concludes.

2 Literature review

Since private consumption is considered to be an important component of aggregate demand it is of a great importance in macroeconomic analysis. Most of recent studies are based on the following fundamental theories and models: absolute income hypothesis (Keynes [15]), life-cycle hypothesis (Modigliani [18]), and permanent income hypothesis (Friedman [11]).

Keynes' absolute income hypothesis (further AIH) introduces static explanation of consumption explained by current disposable income. As only current disposable income is taken into account this theory cannot explain

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the trade-off between present and future income. Nevertheless absolute income hypothesis can be regarded as a good approximation for stable economies.

Fisher's idea [10] including inter-temporal aspects into consumption issues is extended by Modigliani [18] who emphasizes the systematic changes of income during consumer life. Thus, consumer can transfer the money in periods in which the income is high to periods with low expected income. This rationale provides the basis for Modigliani's life-cycle hypothesis (further LCH). According to Modigliani a consumer tries to achieve smooth consumption through savings. That is consumption depends on current income, expected life income and wealth.

Also Friedman's permanent income hypothesis (further PIH) is built on Fisher's theory in terms that consumption does not depend solely on current income. Decisions related to consumption stem from both current income and also expected income in the future. Current income consists of permanent income and transitory income. In general, permanent income is expected to be a long-run income persistent into the future while transitory income is considered as a short-run income that fluctuates around average income.

In terms of econometric testing of consumption function is a major work written by Davidson, Hendry, Serb and Yeo [8] (further DHSY approach). This paper examines the relationship between consumption expenditure and disposable income. Data of Great Britain relating to years 1958 and 1976 were used for empirical research. Using VECM the authors try to point out the discrepancy in the average propensity to consume in the short-term and long-term period, to which economists have to face when they are trying to prove or disprove the theory of Keynesian consumption function (AIH), is not only economic, but also a econometric problem. It was found that an econometric model examining the relationship between consumption and disposable income includes a component of autocorrelation. As autocorrelation manifests itself differently in the short and long time series, which can cause a bias in estimates of the average propensity to consume, authors suggest dynamisation of the model that is in line with the assumptions of LCH and PIH. However, the authors admit that this model does not have to correctly catch the relationship between consumption and disposable income due to many important questions that remain unanswered, such as changes in the distribution of incomes or the direct effect of wealth.

This shortcut inspired Hall [13] who tested the effect of disposable income on consumption and then the effect of wealth on consumption. He links rational expectation hypothesis (further REH) with PIH tests it on time series of the United States in the postwar years. In his work, he concludes that consumption represents a process of random walk monitoring a particular trend. This is due to the fact that consumers try continually to optimize their benefits. Therefore, their wealth develops randomly. It was confirmed that real disposable income has no relation to the formation of future expectations about consumption unless unexpected event occurs. Hall's model finds its support for countries with developed financial markets allowing consumer in position of lender or borrower to maintain a certain level of consumption.

Consequently, the DSHY and Hall's approach has been adopted by many authors like Davis [9], Molana [19] and Chambers [6] who conclude that the model by Davidson, Hendry, Serb and Yeo represents the best specification for United Kingdom, or by Byrne and Davis [2] focusing on variables to be included in the consumption function of G7 countries.

Following the ideas of Hall [13] and basing the model on DSHY approach Byrne and Davis [2] claim that consumption function based on aggregate wealth is somehow instable. Their improvement can be seen in a distinction of liquid and illiquid financial wealth. Using quarterly data over 1972-1998 and SUR estimation of LCH they demonstrate that for G7 countries liquid financial wealth is less important long-term factor determining consumption than illiquid financial wealth.

Among advocates of Keynes' AIH can be listed Campbell and Mankiw [3] that approved the validity of AIH in the US, while Arlt et al. [1] find an evidence of AIH in conditions of Czech economy. In particular, Campbell and Mankiw [3] revise the PIH showing that US consumers violent this hypothesis. They use American postwar data and find that about 50% of consumers in the United States behave according to the Keynesian AIH which means that their consumption decisions depends on their current income rather than permanent income.

The aspects of the consumption function in conditions of the CEE countries were considered in the study written by Arlt et al. [1], Castiglione et al. [5] or Grochová [12] for the Czech Republic or Leszkiewicz-Kędzior and Welfe [16] for Poland. Arlt et al. [1] aimed at testing the theory of PIH. The result of his work was the statement that an excessive sensitivity of real consumption to the change in real disposable income was demonstrated which is in line with AIH, the permanent income hypothesis being rejected. The same conclusions are made by Leszkiewicz-Kędzior and Welfe [16] who base their study on annual polish data between 1970 and 2008.

In contrast to aforementioned studies Carroll [4] using US data suggests that Friedman's PIH including uncertainty matches much better than other models. A strong support of the consumption function under the PIH is found also by Manitsaris [17] who analysed selected fifteen European Union members combining adaptive ex-

pectations with partial adjustment model. A validation of the PIH is also performed by Pánková [21] using an instrumental variables approach under the condition of the Czech economy.

Aiming at the scope of the contribution together with the variables implied from the theoretical discussion, geographical and time aspects as the CEEC and crisis period will be taken into consideration when specifying an appropriate model for industrial countries' private consumption expenditure.

Even if one can claim that private consumption can be determined by country specific factors because of similar abovementioned factors and their determinants like uncertainty, credit constraint, consumption-leisure interactions, habits and durability, alternative to rational expectations, liquidity constraints, the illiquidity of some assets, financial deregulation or demographic factors (Muellbauer [20]), especially in the examined CEE countries, we expect that some common patterns related to consumption are verifiable.

3 Data and Methodology

The research is based on unbalanced panel analysis using selected developed countries data covering the period from 1970 to 2011. The dataset consists of the group of 40 industrial countries over the period 1970–2011, annual data in logarithms. In order to estimate real private consumption, following explanatory variables are used:

- real national disposable income per capita (real GDP per capita as a proxy of real disposable income in case of robustness check),
- real interest rates,
- anticipated inflation,
- M2 aggregate as a proxy of financial wealth,
- dummy for the CEE countries,
- dummy for the global crisis period starting in the 2008.

Real national disposable income per capita (either in current period, lagged or permanent) is the main determinant of consumption. The higher the income, (non-linearly) the higher the consumption, thus, we expect a positive signed coefficient of real national disposable income per capita with an elasticity ranged from 0-1. If current income is significant while lagged insignificant, Keynes' absolute income hypothesis valid and Hall's model is not the appropriate one. This results from the idea that if consumption is dependent on both current and anticipated income, then one might expect that decisions related to consumption are affected by income earned over more time periods and therefore permanent income hypothesis would be confirmed.

Wealth is considered to be one of the most important determinants of consumption from the perspective of the LCH. If the LCH is approved then monetary aggregate (M2) as a proxy of wealth should manifest a positive sign, wealth elasticity being in the interval (0, 1). If significant, the LCH is validated, while an insignificant coefficient foreshadows either AIH or PIH.

Real interest rates are connected with the possibilities deriving from financial market as lending and borrowing. Higher interest rates motivate to reduce loans and make savings more attractive. As this can lead to a reduced consumption a negative sign can be expected. However, higher interest rates increase financial wealth and so even a positively signed coefficient can result.

Anticipated inflation is proxied by 1 period lagged GDP deflator that is well described by first-order autoregressive process. The consumption that is based on anticipated inflation would be in favour of rational expectation hypothesis.

Dummy CEEC is necessary to include because of possible different patterns in consumption behaviour in the CEE countries when compared to the rest of the EU. One would expect a negative sign, since the consumption per capita variable tends to be lower in the CEE countries.

Dummy Crisis is presented in the model to control for the financial and economic crisis period during which the consumption reduced. As a consequence a negatively-signed coefficient can be expected.

Note that the long-run relations are estimated using mentioned variables in levels and the short-run fluctuations from equilibrium are estimated using the same sample of variables (excluding dummy for the CEE countries) are studied in differences. Since there can be a tendency in changing consumer behaviour during 2008 crisis period it is convenient to use the dummy variable in order to prevent from biased results and to obtain interesting information of the topical nowaday situation.

Preliminary to the estimation process, all variables have been tested for unit roots using a test developed by Im, Pesaran and Shin [14] and Fisher type unit root test modified by Choi [7]. All variables have been found to be stationary in differences, i.e. integrated of I(1). In order to estimate the real private consumption and to obtain

robust results, three different techniques could be applied: pooled estimation and fixed effect estimation. To choose appropriate model for real private consumption estimation, we apply the following panel diagnostic tests: F-test for individual effect, Breusch-Pagan LM test and Hausman specification test. Based on results of these tests we decide to use fixed effect estimation, which is also correct from the theoretical point of view due to collection of relatively homogeneous datasets.

In order to model consumption function in chosen developed countries we base the model on abovementioned theoretical and empirical studies. The following relationship is assumed:

$$C_{it} = f\left(Y_{it}, IR_{it}, Inf_{it-1}, CEEC_{it}, CRISIS_{it}\right), \tag{1}$$

where:

- C_{it} is a real private consumption of *i*-th country in time *t*,
- Y_{it} is a real national disposable income per capita (real GDP per capita in the robustness check) of *i*-th country in time *t*,
- IR_{it} is a long term interest rates of *i*-th country in time *t*,
- Inf_{it-1} is an anticipated inflation of *i*-th country in time *t+1*,
- $CEEC_{it}$ is a dummy for the CEE countries,
- $CRISIS_{it}$ is a dummy for the economic crisis period starting in the 2008.

Data source for all mentioned variables is OECD database. To obtain short-run and long-run information on real private consumption we use VECM, i.e. we estimate the following equation:

$$\Delta \ln C_{it} = \alpha_t + \beta_{1t}\Delta \ln C_{it} + \beta_{2t}\Delta \ln Y_{it} + \beta_{3t}\Delta IR_{it} + \beta_{4t}\Delta \ln Inf_{it-1} + \gamma ECT_{it-1} + \eta CEEC_{it} + \lambda CRISIS_{it} + \varepsilon_{it} \tag{2}$$

Since Hall [13] argues that consumption is not only dependent on disposable income but also on wealth that compensates low savings we reflect this including M2 aggregate as a proxy of wealth:

$$C_{it} = f\left(Y_{it}, M2_{it}, IR_{it}, Inf_{it-1}, CEEC_{it}, CRISIS_{it}\right), \tag{3}$$

where $M2_{it}$ is an M2 aggregate being a proxy of wealth of *i*-th country in time *t*. Therefore, we estimate the following equation:

$$\Delta \ln C_{it} = \alpha_t + \beta_{1t}\Delta \ln C_{it} + \beta_{2t}\Delta \ln Y_{it} + \beta_{3t}\Delta IR_{it} + \beta_{4t}\Delta \ln Inf_{it-1} + \beta_{5t}\Delta \ln M2_{it} + \gamma ECT_{it-1} + \eta CEEC_{it} + \\ + \lambda CRISIS_{it} + \varepsilon_{it} \tag{4}$$

4 Results

Firstly, we focus on unit root test. As can be seen from Tab. 1, all variables are non-stationary in levels, but stationary in differences, i.e. all variables are integrated of I(1).

Variable	Levels		Differences	
	statistics	p-value	statistics	p-value
lnC	3.408	0.999	-7.928	0.000
lnIR	-1.044	0.148	-7.984	0.000
LlnInf	-1.317	0.094	-15.506	0.000
lnM2	-0.467	0.320	-2.276	0.011
lnY (GDP variant)	4.493	1.000	-4.868	0.000
lnY (DI variant)	3.461	0.999	-5.576	0.000

Table 1 Im, Pesaran, Shin unit root test[3]

Consequently, pooled ordinary least squares (OLS) and fixed effects are applied to estimate the real private consumption for the group of 40 industrial countries over the period 1970–2011 (see models 1–8 in Tab. II). As can be seen from the mentioned table, some explanatory variables are not statistically significant among which the proxy for wealth and both dummies. The only relevant variables are then income, long-term interest rate and in model 2 monetary aggregate M2. The distortion from equilibrium is caused by lagged income, consumption

[3] Fisher type tests are not reported but are available upon request.

and long-term interest rate. The consumption seems to be unaffected by the crisis period. Even the geographic dummy has not proven the impact on consumption.

	(1) DI pooled	(2) DI,M2 pooled	(3) GDP pooled	(4) GDP,M2 poo~d	(5) DI FE	(6) DI,M2 FE	(7) GDP FE	(8) GDP,M2 FE
D.lnIR	0.0145	0.0226	0.00212	-0.00413	0.0163	0.0133	0.00299	-0.0132
	(0.00669)*	(0.0129)	(0.00641)	(0.0117)	(0.00657)*	(0.0124)	(0.00648)	(0.0127)
D.LlnInf	0.00240	0.0352	-0.00280	0.0869	-0.0730	-0.216	-0.0463	-0.236
	(0.0575)	(0.135)	(0.0547)	(0.122)	(0.0569)	(0.127)	(0.0557)	(0.137)
D.lnY	0.597	0.507	0.659	0.700	0.614	0.560	0.678	0.730
	(0.0317)***	(0.0612)***	(0.0318)***	(0.0648)***	(0.0315)***	(0.0584)***	(0.0329)***	(0.0719)***
L.lnC	-0.0328	-0.0524	-0.0241	-0.0448	-0.201	-0.578	-0.111	-0.395
	(0.00924)***	(0.0242)*	(0.00808)**	(0.0203)*	(0.0237)***	(0.0732)***	(0.0194)***	(0.0739)***
L.lnIR	-0.00694	-0.0239	-0.00690	-0.0225	0.00783	0.00690	0.000871	-0.00584
	(0.00354)	(0.0107)*	(0.00320)*	(0.00950)*	(0.00433)	(0.0142)	(0.00404)	(0.0137)
L.LlnInf	0.0589	0.126	0.00139	0.0512	-0.0249	-0.237	-0.0412	-0.523
	(0.0457)	(0.167)	(0.0436)	(0.147)	(0.0481)	(0.193)	(0.0477)	(0.200)*
L.lnY	0.0165	0.0248	0.00594	0.00880	0.186	0.552	0.0942	0.405
	(0.00929)	(0.0195)	(0.00757)	(0.0165)	(0.0239)***	(0.0650)***	(0.0193)***	(0.0733)***
CEEC	0.00196	-0.00894	-0.000616	-0.0150				
	(0.00327)	(0.0101)	(0.00317)	(0.00867)				
CRISIS	-0.00105	0.00613	0.00331	0.0132	0.000744	0.0106	0.00322	0.0104
	(0.00300)	(0.00541)	(0.00295)	(0.00488)**	(0.00294)	(0.00727)	(0.00299)	(0.00744)
D.lnM2		0.0728		0.0169		0.00623		-0.0297
		(0.0277)**		(0.0263)		(0.0277)		(0.0291)
L.lnM2		-0.00579		-0.00504		-0.0114		-0.0301
		(0.00678)		(0.00542)		(0.0212)		(0.0221)
_cons	-0.0992	-0.196	0.186	0.219	0.155	1.208	0.307	2.545
	(0.223)	(0.777)	(0.211)	(0.679)	(0.235)	(1.008)	(0.232)	(1.017)*
N	519	158	548	161	519	158	548	161
R-sq	0.645	0.600	0.666	0.668	0.651	0.695	0.653	0.692
adj. R-sq	0.639	0.569	0.660	0.644	0.630	0.623	0.632	0.618
BIC	-2505.3	-671.0	-2678.6	-715.6	-2574.0	-758.6	-2716.7	-772.3
F	102.9	19.87	118.9	27.27	113.8	28.93	121.4	28.95

Standard errors in parentheses
* p<0.05, ** p<0.01, *** p<0.001

. DI..national disposable income variants, GDP..gross domestic product variants

Table 2 Models estimation

Moreover, as showed in Tab. 3, fixed or random effects are preferred to pooled OLS (see F-test and Breusch-Pagan LM test). Therefore, based on Hausman test we favour the fixed-effect model specification to the others

	(1) DI pooled statistics	p-value	(2) DI, M3 pooled statistics	p-value	(3) GDP pooled statistics	p-value	(4) GDP, M3 pooled statistics	p-value
Durbin-Watson test	2.179	0.761	2.202	0.763	2.171	0.761	2.180	0.762
Breusch-Godfrey/Wooldridge test	13.692	0.090	3.164	0.367	15.701	0.047	6.894	0.075
Bera, Sosa-Escudero and Yoon locally robust test	5.169	0.023	2.359	0.125	4.789	0.029	1.798	0.180
F-test for individual effects	3.367	<0.001	4.453	<0.001	1.595	0.046	2.050	0.009
F-test for time effects	2.160	<0.001	3.969	0.001	1.739	0.004	2.337	0.035
LM Test – (Breusch-Pagan)	8.911	0.003	0.333	0.564	6.653	0.010	0.003	0.953
LM Test – time effects (Breusch-Pagan)	429.568	<0.001	10.462	0.001	312.346	<0.001	2.275	0.131

Table 3 Verification of pooled models

	(5) DI FE statistics	p-value	(6) DI, M2 FE statistics	p-value	(7) GDP FE statistics	p-value	(8) GDP, M2 FE statistics	p-value
Durbin-Watson test	2.078	0.785	2.163	0.838	2.129	0.923	2.192	0.880
Breusch-Godfrey/Wooldridge test	8.428	0.393	7.519	0.057	19.939	0.011	11.574	0.009
Bera, Sosa-Escudero and Yoon locally robust test	0.002	0.961	0.204	0.652	0.999	0.318	0.003	0.955
Wooldridge Test for AR(1) Errors in FE Panel Models	0.002	0.969	0.109	0.741	35.102	<0.001	0.021	0.886
Hausman test	93.072	<0.001	96.320	<0.001	49.562	<0.001	43.895	<0.001

Table 4 Verification of fixed effect models

5 Conclusions

The contribution investigates the response of real private consumption per capita to real national disposable income per capita or real GDP per capita, respectively, long-term interest rates, monetary aggregate M2 and anticipated inflation. For the purpose of the private consumption function estimates we used pooled OLS and fixed effects on unbalanced panel data sample. Our results show a strong dependence of real private consump-

tion per capita on income (disposable income per capita or real GDP per capita as proxy of disposable income) without statistical effects of monetary aggregate and anticipated inflation. The results so show strong support for the Keynes' absolute income hypothesis in contrast to permanent income hypothesis, rational expectation hypothesis, and life cycle hypothesis. Similarly, the private consumption seems to be unaffected by the crisis period and geographical country position.

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Option Pricing and Partial Hedging in the Black-Scholes Model

Haochen Guo[1]

Abstract. Option pricing techniques are often considered on the most mathematically complex of applied areas of finance. The Black-Scholes model is the most widely used models for the option pricing. This paper presents and analyses the option pricing and hedging in the Black-Scholes model. The goal of hedging is against the risk for protect the losses from the investment, that means strategically using instruments in the financial market to offset the risk of any adverse price movements. Inside of paper, it provides example of the partial hedge, which reduces the risk, that using option pricing with the Black-Scholes model.

Keywords: Option pricing, the Black-Scholes model, partial hedging

1. Introduction

This paper is analyses the option pricing and present partial hedging in the Black-Scholes model. The application of hedging strategy in the paper it will use the delta hedging in the Black-Scholes model. Delta hedging involves creating a position with zero (See [4]). There are 2 parts inside of the paper. First part is methodology of option pricing. The binominal model and the Black-Scholes model are two methods which usually to use for the option pricing. In this part, there are 2 examples for calculation both the Black-Scholes model and the binominal model. Second part is static delta partial hedging in the Black-Scholes model. Inside of this part, it will describe the delta hedging and the partial delta hedging. Then will be the conclusion.

2. Methodology of Option Pricing

Option pricing is on no arbitrage argument. If it can add other traded assets or instruments to an option so that the resulting portfolio is risk free, in other words if the option can be perfectly hedged, then the portfolio should return the risk free rate. Since it has no risk, this risk free portfolio will have the same value today for all investors, regardless of their attitude to risk. Because the hedging instruments have market prices, which are the same for all investors, the option must have the same value for investors.

2.1. The Black-Scholes Model

The Black-Scholes model assumes that stock price movements can be described by a statistical process known as geometric Brownian motion. This process is summarized by a volatility factor, σ, which is analogous to the investor's stock price forecasts in the previous models. Formally, the stock price process assumed by Black and Scholes is

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$$\Delta S / S = \mu[\Delta T] + \sigma\eta[\Delta T]^{1/2} \tag{1}$$

So that a stock's return $(\Delta S / S)$ from the present through any future period T has both an expected component $(\mu[\Delta T])$ and a "noise" component $(\sigma\eta[\Delta T]^{1/2})$, where μ is the mean return and η is the standard normally distributed random error term.

Assuming the continuously compounded risk-free rate and the stock's variance remain constant until the expiration date T, Black and Scholes used the riskless hedge intuition to derive the following formula for valuing a call option on a no dividend-paying stock

$$C_0 = SN(d_1) - X(e^{-(r)T})N(d_2) \tag{2}$$

where $e^{-(r)T}$ is the discount function for continuously compounded variables.

$$d_1 = [\ln(S / X) + (r + 0.5\sigma^2)[T]] \div \sigma[T]^{1/2} \tag{3}$$

and

$$d_2 = d_1 - \sigma[T]^{1/2} \tag{4}$$

Properties of the model reveals that the option's value is a function of five variables, there are current security price, exercise price, time to expiration, risk-free rate, security price volatility.

Functionally, the Black-Scholes model holds that $C = f(S, X, T, r, \sigma)$. The first and fourth factors are observable market prices, and the second and third variables are defined by the contract itself. Thus, the only variable an investor must provide is the volatility factor. (See [5])

Example 1. Calculation the Black-Scholes model which use the underlying asset of PetroChina Company Limited, which is the stock from China Stock Exchange Market. PetroChina Company Limited ("PetroChina") is the largest oil and gas producer and distributor, playing a dominant role in the oil and gas industry in China. It is not only one of the companies with the biggest sales revenue in China, but also one of the largest oil companies in the world. It was listed on Shanghai Stock Exchange on November 5, 2007, the stock code is 601857. The parameters for calculation are current stock price (S) is 8.65, strike price (X) is 50, stock volatility (σ) is 0.66, risk free interest rate (r) is 0.1, option time to maturity (T) is 0.5, to compute the Black-Scholes model is based on the normal distribution. After defined the normal distribution, program the Black-Scholes formula is in following table (See [2]).

Normal Distribution	$snormaB[x\_] := Erf[x / Sqrt[2]] / 2 + 0.5$ $sndist = NormalDistribution[0,1]$
Black-Scholes model	$Clear[snormal, d1, d2, BSCALL, BSPUT]$ $d1[S\_, X\_, sigma\_, T\_, r\_] := (Log[S / X] + r + sigma\text{^}2 / 2) * T) / (sigma * Sqrt[t])$ $d2[S\_, X\_, sigma\_, T\_, r\_] := d1[S, X, sigma, T, r] - sigma * Sqrt[T]$ $BSCALL[S\_, X\_, sigma\_, T\_, r\_] := S * snormal[d1[S, X, sigma, T, r]] - X * Exp[-r * T] * snormal[d2[s, x, sigma, T, r]]$ $BSPUT[S\_, X\_, sigma\_, T\_, r\_] := BSCALL[S, X, sigma, T, r] + x * Exp[-r * T] - S$

The result of Black-Scholes model	$BSCALL[8.65,50,0.66,0.5,0.1] = 0.00029176$ $BSPUT[8.65,50,0.66,0.5,0.1] = 38.9118$ $d1[8.65,50,0.66,0.5,0.1] = -3.41889$ $d2[8.65,50,0.66,0.5,0.1] = -3.88558$

Table 1 Calculation of Black-Scholes Model

The result of Black-Scholes and the graph the functions are showing in the following.

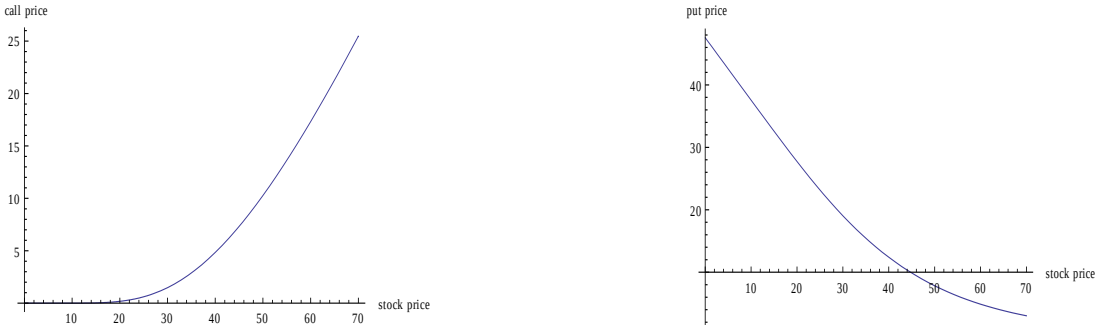

Figure 1 Call and Put Option Price in Black-Scholes Model

2.2. Binomial Model

Cox and Ross (1976) introduced a simple model for pricing an option in which the underlying asset price dynamics are governed by a binomial tree in which the price of an asset can move either upward or downward by a constant multiple, at each time step. It showed how to construct a binomial tree in which there can be no arbitrage, where the probability of an upward move is constant throughout the tree. In a simple binomial tree is assume that the transition probability, for risk neutral valuation it choose *p* so that *S* grows at the risk free rate, that is (See [1])

$$pS_0u + (1-p)S_0d = S_0e^{(r-q)T} \tag{5}$$

$$p = \frac{e^{(r-q)T} - d}{u - d} \tag{6}$$

The total return provided by the stock in a risk-neutral world must be the risk-free interest rate *r*, the dividends provide a return equal to *q*, and the return in the form of capital gains must be *(r-q)*. Because the value of the derivative is the expected payoff in a risk-neutral world discounted at the risk-free rate, that is

$$f = e^{-rT}[pf_u + (1-p)f_d] \tag{7}$$

Example 2. Calculation the binominal model which uses the underlying asset of PetroChina, the parameters data is same with example1. To define binomial option pricing for European options is in the following table (See [2]).

The Binominal Option Pricing Model	$Clear[EuropeanOption, EuropeanCall, EuropeanPut]$ $EuropeanOption[S\_, sigma\_, T\_, r\_, exercise\_Function, n\_] := Module[\{$ $u = N[Exp[Sqrt[T/n] * sigma]],$ $d = N[Exp[-Sqrt[T/n] * sigma]],$ $R = N[Exp[r * T/n]]\},$ $p = (R-d)/(R*(u-d));$ $q = (u-R)/(R*(u-d))]$

```
Sum[exercise[s*u^j*d^(n-j)]*Binomial[m,j]*p^j*q^(n-j),{j,0,n}]
```

$$\sum_{j=0}^{n} p^{j} q^{-j+n} Binomial[m, j] exercise[d^{-j+n} s u^{j}]$$

```
EuropeanCall[S_,X_,sigma_,T_,r_,n_]:=EuropeanCall[S,X,sigma,T,r,n]=EuropeanOption[S,sigma,T,r,Max[#1-x,0]&,n]
EuropeanPut[S_,X_,sigma_,T_,r_,n_]:=EuropeanOption[S,sigma,T,r,Max[S-#1,0]&,n]
```

The result of binominal model for call option	0.00024110850

Table 2 Calculation of Binominal Model

There is the graph shows option pricing in binominal model and the Black-Scholes model.

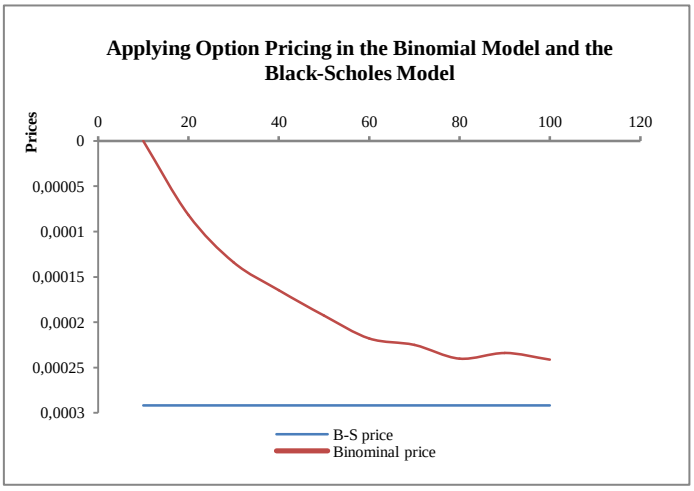

Figure 2 Applying Option Pricing in the Binomial Model and the Black-Scholes Model

3. Static Delta Partial Hedging in the Black-Scholes Model

Delta measures how volatile an option on futures premium is relative to the underlying, expressed by the change in option premium to change in futures price. The greater the extent to which the option is in the money, the greater its delta and vice versa. Delta is measured on a continuum form 0 to 1. High delta options are close to one. Delta is a metric for hedgers to determine how volatile the underlying is that they are attempting to hedge and the degree to which a hedge might be effective. In the delta hedging, it hedges the call position by purchasing $\frac{\partial C}{\partial S}$, where C is the call price. Following table shows the formulation of delta hedging for options. (See [9])

Parameters	Call Option	Put Option
Delta $\frac{\partial C}{\partial S}$	$N(d_1)$	$-N(-d_1) \equiv N(d_1) - 1$

Table 3 Formulation of Delta Hedging for Options

To hedging the position, the hedge ratio can be calculated as following table (See[3]).

$Clear[delta]$

$delta[S\_, X\_, sigma\_, T\_, r\_] := snormal[d1[S, X, sigma, T, r]]$

$delta[8.65,50,0.66,0.5,0.1] = 0.000314385$

Table 4 Calculation of Static Delta Hedging

The result of the delta hedge ratio is 0.000314385.Partial hedge is a position that has been hedged in part and not in its entirety. This will reduce, but not eliminate, adverse movements to the position being hedged. A partial hedge is undertaken if the hedger wishes to be still able to benefit from some upside but is concerned about the amount of the exposure being taken. Following table is distinguished between full hedging and partial hedging in the delta hedging strategy.

Full Hedging	$\Delta\Pi = h \cdot \Delta S - \frac{\partial C}{\partial S} \cdot \Delta S = 0$ $h = \frac{\partial C}{\partial S} = delta$
Partial Hedging	$\Delta\Pi = \Delta S \cdot (h \cdot x - \frac{\partial C}{\partial S}) = 0$ $h = \frac{\frac{\partial C}{\partial S}}{x} = \frac{delta}{x}$

Table 5 Distinguish Between Full Hedging and Parting Hedging

In the partial delta hedging which the $\prod$ is position value, S is underlying asset value, h is hedge ratio, x is determine the partial underlying asset, the $x = [0;100\%]$. When $x = 1\%$ means the full hedging,

Example 3. Calculation the partial hedging, underlying asset is PetroChina which the same computation data with previews examples. Following table is to distinguish between the full hedging and partial hedging.

x	0.01	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
h ratio	0.03143850	0.00314385	0.00157193	0.00104795	0.00078596	0.00062877	0.00052398	0.00044912	0.00039298	0.00034932	0.00031439

Table 6 Full Hedging and Partial Hedging

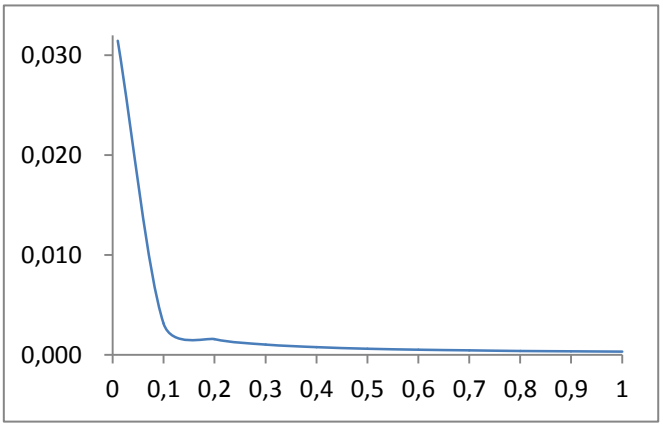

Figure 3 Partial Hedging Trends

4. Conclusion

The paper describes partial hedging strategy using delta hedging applied. It is describes about distinguish of basic static delta hedging strategy. In the paper, there are 3 examples, which are to calculation of Black-Scholes model and binominal model, static delta hedging and partial hedging in the Black-Scholes model. The Black-Scholes option-pricing model is the first successful option-pricing model, published in 1973 and based on stochastic calculus. It focuses on the pricing of European options, in which the underlying does not pay a dividend in the option period. The option is priced according to the value of the underlying, the volatility of the value of the underlying, the exercise price, the time to maturity, and the risk-free rate of interest. The model provides a general approach to option pricing and has given rise to a number of other option-pricing models. As a result, for European options, the binomial model converges on the Black-Scholes formula as the number of binomial calculation steps increases. In fact, the Black-Scholes model for European options is really a special case of the binomial model where the number of binomial steps is infinite. In other words, the binomial model provides discrete approximations to the continuous process underlying the Black-Scholes model.

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LMD1 decomposition of the changes in the industrial energy consumption in the Czech Republic

Vladimír Hajko[1]

Abstract. This article applies logarithmic mean Divisia index decomposition method to analyze the changes in the energy consumption of the industrial sector in the Czech Republic, EU-27, EU-15 and EU-12 during the period of 1997 to 2009. The results indicate several differences between the EU-12 and EU-15 countries. The major differences are: the magnitude of the activity effect; how influential was the period of the recession for the energy consumption in the respective groups; the amount of reduction in the energy consumption and the development of the energy intensity effect in the period of crisis.

Keywords: LMD1, Energy, Energy consumption, Industry

JEL classification: Q43
AMS classification: 11Y05

1 INTRODUCTION

In the last two decades (following the transition from the centrally planned economy) the industrial sector in the Czech Republic (generally in the transforming countries) has gone through significant changes. The original rather ineffective structure and technology has been gradually improving and becoming far more energy and market efficient. Given the increasing competition and general market tendency to improve efficiency, it is of little surprise that the industrial sector has been the sector that gone under most intense transformation, not only in the Czech Republic but in most European countries. For instance in the EU-12 countries[1], we can observe nearly 20 percentage points reduction in the overall industrial energy consumption share in 2011 compared to 1990. The EU-15 countries[2] recorded a similar shift from industry to transport, though with only about 5 p. p. decrease in the share of industry and somewhat less than 5 p.p. increase in the share of transport).

Despite the significant shift of the energy consumption from the industry to other sectors (as hinted above, mainly to transport), the industrial sector still represents about a quarter (in EU average) or a third (in CZ) of the total energy consumption and remains in the live interest of many government policies (including the energy conservation and CO_2 reduction scenarios such as the Energy Policy for Europe (also known as 20-20-20 plan)).

This article aims to unfold these changes, with the main focus on the energy consumption development in the industrial sector. For comparative reasons, the following results also cover, apart from the Czech Republic, the aggregates of EU-27, EU-15 (original member countries) and EU-12 (new member countries). The main tool of the analysis is the application of the so-called logarithmic mean Divisia 1 index method (LMD1), proposed by [4]. The history of application of the index decomposition analysis (IDA) to decompose an aggregate ranges back to 1970s (see [6] for comprehensive review). Nowadays, the IDA methods are widely accepted analytical tool for policy making [2], and it is indeed evidenced not only by the number of studies employing the IDA methods but also by its acceptance by the renowned international bodies such as the International Energy Agency [10].

However, rather little attention has been given to these methods not only in the Czech Republic, but also in most countries of the former Eastern Bloc. One significant burden is the data comparability across different countries. Not only for a cross-country comparison, but also for the meaningful correspondence of

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[2]Austria, Belgium, Denmark, Finland, France, Germany, Greece, Ireland, Italy, Luxembourg, Netherlands, Portugal, Spain, Sweden, United Kingdom.

the individual country analyses, it is necessary to have a unified methodological structure and collection of the data (mainly regarding the delimitation of the individual sub-sectors how the value added is measured). Eurostat databases are useful in their methodological homogeneity, but in certain cases do not provide the latest (and, given the late economic troubles the Europe and the world have been going through, probably interesting) data.

2 DATA

This section is devoted to providing a detailed information of the data used in the analysis as well as to show some of the interesting facts regarding the data. All the figures and the tables presented in this article are my own. Two main indicators necessary for this IDA application are the final energy consumption and the gross value added (GVA). The data on the energy consumption are available under Eurostat's table [nrg100a] , and are measured in thousand tonnes of oil equivalent[3]. The data on the gross value can be found under the table [nama_nace60_c] and are measured in current millions of euro (from January 1, 1999) or current millions of ECU (up to Decemeber 31, 1998). As hinted above, the reason for the selection of the data source is specifically the availability of the indicators for multiple countries in exactly the same structure and measurement methods. This should allow any further researcher to directly compare the results across different countries and allows for extensibility of the results.

The first important thing that needs to be noted is that a meaningful analysis requires that the data for individual sub-sectors are related to the same set of activities. In other words, it is of no use if we collect data that do not correspond to the identical underlying sub-sector. As the Table 1 shows, the source data structure allows for the good match in only 11 of the 13 energy subsectors, meaning that this analysis is necessarily limited (in European average) to analyze only about the 90% of the consumption that took place in all of the industrial sub-sectors.

Due to certain peculiarities in the data, one might be lead to a certain misunderstanding of the results if it is not kept in in mind that "Industry" in this analysis is (albeit unfortunately) limited to the sum of the 11 energy subsectors corresponding to the 14 NACE codes as presented in the Table 1. An example of such peculiarity can be the atypical behavior of the energy consumption in the elusive "Not elsewhere specified" sub-sector in several EU-12 countries. For instance, in the Czech Republic this sub-sector accounted for more than half of the total Czech energy consumption in the early 1990s. This however can be attributed to the somewhat imperfect statistical collection and classification of the energy consumption, especially during the first half of the 1990s. An overall picture of the all sub-sectors indicates the overall pattern is not very different from the examined part of the industry. Yet I feel it is important to distinguish the overall industry and the examined sub-sectors delimitation. Furthermore, such unhelpful statistical classification in the early 1990s also speaks for the selection of the time frame "after the dust has settled" - in case of this analysis the time period was chosen from the 1997 onwards.

Figure 1 shows the development of both the industrial (final) energy consumption and the Gross Value Added in the examined industrial sector. We can see that the average behavior in the European countries shows a downward trend in both energy consumption share and the GVA share. However, the heavier presence (compared to EU-15) of industrial sub-sector can be clearly seen in EU-12 aggregate and Czech Republic. Unlike the aggregate of EU-15, the GVA share of the industry in the Czech Republic is decreasing. Despite that the Czech Republic still has above average (compared both to EU-27 and EU-15) level of industry's contribution on the GVA.

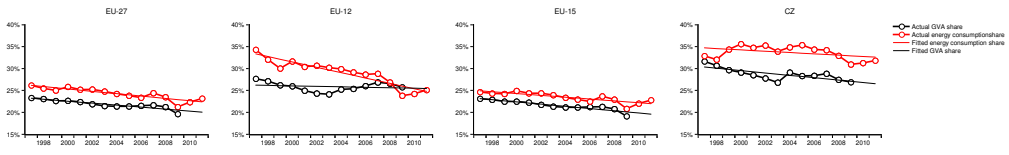


Figure 1 Gross value added and energy consumption shares, examined sub-sectors, EU-27 countries, 1990-2010

[3]The TOE unit is defined by the International Energy Agency as the amount of energy released by burning one ton of crude oil and represents 41.868 GJ.

Energy code	Energy description	NACE code	NACE description
B_101805	Iron and steel	DJ	Manufacture of basic metals and fabricated metal products
B_101810	Non-ferrous metals	#N/A	
B_101815	Chemical, including petrochemical	DF	Manufacture of coke, refined petroleum products and nuclear fuel
		DG	Manufacture of chemicals, chemical products and man-made fibres
		DH	Manufacture of rubber and plastic products
B_101820	Non-metallic mineral products	DI	Manufacture of other non-metallic mineral products
B_101825	Mining (excluding energy producing industries) and quarrying	C	Mining and quarrying
B_101830	Food processing, beverages and tobacco	DA	Manufacture of food products, beverages and tobacco
B_101835	Textile and leather	DB	Manufacture of textiles and textile products
		DC	Manufacture of leather and leather products
B_101840	Pulp, paper and printing	DE	Manufacture of pulp, paper and paper products; publishing and printing
B_101846	Transport equipment	DM	Manufacture of transport equipment
B_101847	Machinery;	DK	Manufacture of machinery and equipment n.e.c.
B_101851	Wood and wood products (other than pulp and paper);	DD	Manufacture of wood and wood products
B_101852	Construction;	F	Construction
B_101853	Not elsewhere specified.	#N/A	
#N/A		DL	Manufacture of electrical and optical equipment
#N/A		DN	Manufacturing n.e.c.

Table 1 Matching energy consumption and GVA sub-sectors

3 METHODOLOGY

The index decomposition methods are used to identify the individual contributions of several factors to the combined total change. It is especially useful method in analyzing a change[4] in an overall aggregate V (where $V = \sum_i V_i$), that is composed of multiple factors ($V_i = x_{1,i}$,$x_{2,i}, \ldots, x_{n,i}$), with non-zero changes in factors (which means the overall change is the results of multiple interacting factors). In other words, the decomposition method allows to quantify the relative contributions of the pre-defined factors to the change in the examined aggregate.

The LMD1 method applied in this article exhibits several desirable properties. First, the LMD1 fulfills the so-called Fisher's [8] tests (except the circular test[5]) - namely the time-reversal test[6] and the factor-reversal test[7]. Furthermore it is zero-value robust[8]. Detailed comparison of the properties of the decomposition methods can be found in [6].

It should be noted the additive refined Laspeyer's index decomposition methods (see e.g. [11], [7] or [1]; an example of the application of the additive refined Laspeyer's method on the cross-country energy consumption can be found in [9]) also fulfill the Fisher's factor-reversal and time-reversal tests. However, the application of the logarithmic-mean index decomposition method has advantages over the refined Laspeyer's methods. One of them is that LMD methods can be easily constructed for any number of factors, while the complexity of the interaction terms in the formulas of the refined Laspeyer's methods grows very quickly.

It is important to point out a strong advantage of the application of the decomposition methods that satisfy the factor-reversal test over the other decomposition methods. It is well known the magnitude of the unexplained residual of the actual change can be very high (even in orders of tens or hundreds of % of the actual change (though the Divisia based methods usually result in only marginal residual terms)).

[4]The change can be expressed as$\Delta V_{total} = V^T - V^0$ for the additive form, or $D_{total} = \frac{V^T}{V^0}$ for the multiplicative form

[5]The circular test has the form of $1 = D^{0S} D^{ST} D^{T0}$,where S is a point between time $t = 0$ and $t = T$ (this test can also be expressed in the (perhaps more clear) form of $D^{0T} = D^{0S} D^{ST}$). However, as shown by [8], the circular test is not met by any weighted aggregative with *changing weights*. Constant weights would, however, impose a serious questionability of the index method's usefulness.

[6]As the name suggests, to pass the time-reversal test, the method must exhibit a property, that an index number for a change from the period $t = 0$ to period $t = T$ must be a reciprocal value (for multiplicative form) of the index number for a change from period $t = T$ to period $t = 0$, i.e. $D^{0T} = \frac{1}{D^{T0}}$. For the additive form, the property is: $\Delta V^{0T} = -\Delta V^{T0}$.

[7]To pass the factor-reversal test (in the multiplicative form), the product of the individual factor contributions must equal the observed ratio of the aggregate, i.e. $D^{0T} = \frac{V^T}{V^0} = \prod_{k=1}^{n} D_k$ (for the additive form, the property is: $\Delta V^{0T} = V^T - V^0 = \sum_{k=1}^{n} \Delta V_k$). The decomposition method that satisfies this test is called *a perfect decomposition method* as it does not contain an unexplained residual.

[8]Which means the method can be used when there are 0 values in the dataset. In this article the occasional zero values in the dataset were handled in accordance with the recommendation suggested in [3], i.e. by replacing the zero values by a sufficiently small number ($\delta = 10^{-20}$). The methodological discussion of this type of zero values' handling is thoroughly discussed in [5].

As such, they leave an open question on accuracy of the results and provide a significant burden to the interpretation of the results (the more detailed discussion of the issue can be found in [2]).

Another advantage of the LMD1 method is its consistency in aggregation [4], a very desirable property that is not met by the alternative LMD2 decomposition (the difference between LMD1 and LMD2 is in the way the weights are computed) proposed by [3] (albeit LMD2 is also perfect in decomposition). Furthermore, the results from multiplicative LMD1 and additive LMD1 can be mutually linked[9], thus eliminating the need to estimate both multiplicative and additive forms, as one can be transformed into the other.

The relationship examined in the rest of the article focuses on the changes in the energy consumption E, using the relationship $E = Q \cdot EI$, where the energy consumption (measured in TOE) is defined as the product of continuous variables of the economic activity Q (measured by the value added) and the energy intensityEI (measured in TOE / 1000 Euro).

This basic relationship can be further rewritten as

$$E = Q \sum_i \frac{E_i Q_i}{Q_i Q} = Q \sum_i EI_i S_i \tag{1}$$

where $EI_i = \frac{E_i}{Q_i}$ and $S_i = \frac{Q_i}{Q}$. If we differentiate the Equation 1 by time, we can write:

$$\frac{\partial E}{\partial t} = \frac{\partial Q}{\partial t} \sum EI_i S_i + Q \sum_i \frac{\partial EI_i}{\partial t} S_i + Q \sum_i EI_i \frac{\partial S_i}{\partial t} \tag{2}$$

and if we divide this equation by $E = Q \sum_i EI_i S_i$, and rearrange:

$$\frac{\partial}{\partial t} ln\left(E\right) = \left(\frac{\partial Q}{Q \partial t}\right) Q \sum \frac{EI_i S_i}{Q \sum_i EI_i S_i} + Q \sum_i \frac{EI_i S_i}{Q \sum_i EI_i S_i} \left(\frac{\partial EI_i}{EI_i \partial t}\right) + Q \sum_i \frac{EI_i S_i}{Q \sum_i EI_i S_i} \left(\frac{\partial S_i}{S_i \partial t}\right) \tag{3}$$

The integration of Equation 3 yields:

$$ln\left(\frac{E^T}{E^0}\right) = \int_0^T \left(\sum_i w_i\left(t\right) \left(\frac{\partial}{\partial t} ln\left(Q\left(t\right)\right)\right) + \sum_i w_i\left(t\right) \left(\frac{\partial}{\partial t} ln\left(EI_i\left(t\right)\right)\right) + \sum_i w_i\left(t\right) \left(\frac{\partial}{\partial t} ln\left(S_i\left(t\right)\right)\right) \right) dt \tag{4}$$

where $w_i\left(t\right) = \frac{Q(t) EI_i(t) S_i(t)}{Q(t) \sum_i EI_i(t) S_i(t)}$. And exponentiating Equation 4 results in:

$$\frac{E^T}{E^0} = \underbrace{e^{\int_0^T \left(\sum_i w_i(t) \left(\frac{\partial}{\partial t} ln(Q(t))\right)\right) dt}}_{D_{activity}} \underbrace{e^{\int_0^T \left(\sum_i w_i(t) \left(\frac{\partial}{\partial t} ln(EI_i(t))\right)\right) dt}}_{D_{intensity}} \underbrace{e^{\int_0^T \left(\sum_i w_i(t) \left(\frac{\partial}{\partial t} ln(S_i(t))\right)\right) dt}}_{D_{structural}} \tag{5}$$

Since in practice, we can only observe discrete data, we need to work with the discrete version of the equations in question (and since the variables that constitute weights are also continuous, it motivates the use of some form of mean of two discrete values set apart by non-infinitesimal time). The discretization of Equation 5 results in:

$$\frac{E^T}{E^0} \approxeq \underbrace{e^{\left(\sum_i w_i(t^*) \left(ln\left(\frac{Q^T}{Q^0}\right)\right)\right)}}_{D_{activity}} \underbrace{e^{\left(\sum_i w_i(t^*) \left(ln\left(\frac{EI_i^T}{EI_i^0}\right)\right)\right)}}_{D_{intensity}} \underbrace{e^{\left(\sum_i w_i(t^*) \left(ln\left(\frac{S_i^T}{S_i^0}\right)\right)\right)}}_{D_{structural}} \tag{6}$$

with $t^* \in (0, T)$. A several possible options to determine the t^* can be used. The motivation to use logarithmic mean of weights was explained earlier, though one can indeed use a different scheme, such as e.g. arithmetic mean (resulting in the so called Arithmetic mean Divisia index), etc.

[9] Using the formula $\frac{\Delta V_{total}}{ln(D_{total})} = \frac{\Delta V_{effect_1}}{ln\left(D_{effect_1}\right)} = \frac{\Delta V_{effect_2}}{ln\left(D_{effect_2}\right)} \ldots = \frac{\Delta V_{effect_k}}{ln\left(D_{effect_k}\right)}$ (since $\frac{\Delta V_{x_k}}{ln\left(D_{x_k}\right)} = L\left(V^T, V^0\right)$) where ΔV terms represent the additive form of the effects (with $\Delta V_{total} = V^T - V^0 = \Delta V_{x_1} + \Delta V_{x_1} + \ldots + \Delta V_{x_n}$), D terms represent the multiplicative form of the effects (with $D_{total} = \frac{V^T}{V^0} = D_{x_1} D_{x_2} \ldots D_{x_n}$), where $L\left(V^T, V^0\right)$ is the logarithmic mean of V^T and V^0 .

However, we can recall that certain schemes (such as LMD1[10] used in this article) will result in perfect decomposition, thus:

$$\underbrace{\frac{E^T}{E^0}}_{D_{total}} = \underbrace{e^{\left(\sum_i \widetilde{w}_i(t^*)\left(ln\left(\frac{Q^T}{Q^0}\right)\right)\right)}}_{D_{activity}} \underbrace{e^{\left(\sum_i \widetilde{w}_i(t^*)\left(ln\left(\frac{EI_i^T}{EI_i^0}\right)\right)\right)}}_{D_{intensity}} \underbrace{e^{\left(\sum_i \widetilde{w}_i(t^*)\left(ln\left(\frac{S_i^T}{S_i^0}\right)\right)\right)}}_{D_{structural}} \tag{7}$$

The individual effects (activity, energy intensity, and structural) represent how the industry's energy consumption would change if the other factors did not change and therefore allow us e.g. to examine the actual energy efficiency improvements (via the energy intensity effect), in contrast to the simple computation of the energy intensity indicator.

4 RESULTS

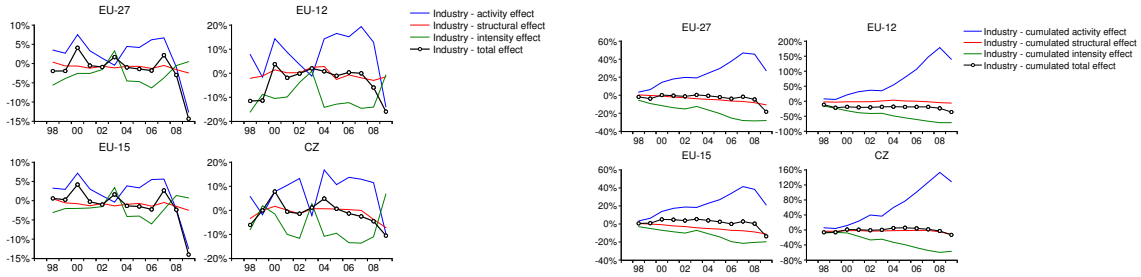
The results of decomposition (per Equation 7) are summarized in the Figure 2 (note the different scales of axes). While the relative picture might seem similar, the actual changes in the energy consumption were much more prevalent in the new member countries than in the original EU-15 countries. In 2008 (compared to the base year 1997), the overall change of the energy consumption was only +0.3% in the EU-15, but −23.7% in EU-12 (and $-2,84\%$ in CZ). We might note the cumulated impact of the period of crisis on the energy consumption in the industry. The changes in the energy consumption (compared to the level of 2007) amounted approx. to −16% in EU-15 and −21% in EU-12 (and −14.6% in CZ). An interesting point is that in the period of the crisis, unlike the both the EU-12 and CZ, only the EU-15 experienced (though rather marginal) increase in the energy intensity effect by the combined factor of 1.02.

The differences in the individual effects are also very prominent. For instance, in the peak in 2008 (comparing to the base year 1997), the change in the energy consumption due to the increase of the economic activity of the industry sector would (ceteris paribus) result in the increase by the factor of 1.38 in EU-15, but by the factor of 2.79 in EU-12 and 2.53 in CZ. The structural effect (representing the impact of changing the industry's structure, i.e. shares of individual sub-sectors) played only a minor role, as it contributed to the −8.9% change in EU-15, and −4.7% in EU-12 (−5% in CZ). The energy intensity effect, indicates that the energy efficiency improvements (not internal industrial restructuring towards inherently different sub-sectors) taking place in the new member countries were the major player in the energy intensity convergence[11]. Ceteris paribus this effect would lead to the change of the industrial energy consumption by the factor of 0.8 (≈ a reduction in consumption by 20%) in EU-15, by the factor of 0.29 in EU-12 (and by the factor of 0.40 in CZ).

As for the most influential sectors in the examined industry (due to the limited space of the article, the effects for the individual sub-sectors are not plotted here, but the figures are available from the author), it seems there are 4 commonly influential (from the energy consumption viewpoint) sub-sectors in both the original and new member countries. These sectors are (in decreasing order) Iron and steel (DJ), Petrochemical (DFGH), Non-metallic mineral products (DI) and Food processing (DA). For instance, these sectors are among top 5 sectors with both the highest activity effects (which would result in the increase of the energy consumption) and the most significant energy intensity effects (which would result in the decrease of the energy consumption) in *all* examined country groups. Overall, the major (from the industry's viewpoint) efficiency improvements took place in the Iron and steel sub-sector both in original and new member countries (though the decrease in energy consumption in 2008 and 2009 was mostly due to the decrease in the activity effect).

[10] In LMD1 the weights are given by $\widetilde{w}_i(t^*) = \frac{L(E_i^T, E_i^0)}{L(E^T, E^0)} = \frac{\frac{(E_i^T - E_i^0)}{(ln(E_i^T) - ln(E_i^0))}}{\frac{(E^T - E^0)}{(ln(E^T) - ln(E^0))}}$, where $L(a,b)$ is the logarithmic mean of a and b in the usual definition, i.e. $L(a,b) = \frac{(a-b)}{(ln(a)-ln(b))}$, with $L(a,a) = a$.

[11] Though it should be noted the energy intensity indicator (TOE / 1000 EUR in value added) is still about twice as high in EU-12 (0.19) or CZ (0.22) than in EU-15 (0.11).



(a) Individual year-by-year effects

(b) Cumulated effects

Figure 2 The examined industry's activity, structural and intensity effects

5 CONCLUSION

The results indicate the changes in energy consumption that can be attributed to the changes in economic activity, were almost 5 times as high (in terms of the relative change against the base period) in EU-12 and 4 times as high in CZ when compared to the EU-15 aggregate. The period of recession was more influential (in terms of the relative drop) in the energy consumption for the EU-12 aggregate than for the EU-15 aggregate. Despite the activity effect, the EU-12 countries managed to reduce the energy consumption by 36% (though CZ managed only 12% reduction), as opposed to the 14% reduction in EU-15. Unlike both EU-12 and CZ, the EU-15 experienced (albeit marginal) increase in the energy intensity effect by the combined factor of 1.02 during the period of crisis (2008 and 2009). In all examined groups, the major activity and intensity effects took place in the Iron and steel sub-sector.

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Individual and panel modeling of foreign business cycle transmission on small open economics

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Abstract. The purpose of this paper is to investigate a panel model of transmission of foreign business cycle on small open economics for a group of new member states in the EU (EU12) over the period 1995Q1 – 2012Q4. There are two different channels of business cycle transmission – financial markets and foreign trade. This paper will be investigated second one. We examine the relationship among the variables of domestic output growth, foreign output growth and the real exchange rate. We provide the panel unit roots tests using Levin, Lin and Chu, Breitung and Im, Pesaran, Shin tests assuming common unit root process and the ADF or PP Fisher-type tests for individual unit root process. Once the existence of a panel unit root has been established, we examine a long-run equilibrium relationship among the variables using panel cointegration test without structural breaks (seven Pedroni's tests, Kao (Engle-Granger based) tests and also combined Fisher - Johansen tests. There is evidence that both variables $\ln GDP_{it}$ and $\ln ERT_{it}$ are integrated of order one for the homogenous and the heterogeneous alternative of panel unit root tests. In our application we find evidence of cointegration between domestic output growth, foreign output growth and real exchange rate for most new EU countries. Our analysis is based on quarterly data using software Eviews 7.

Keywords: foreign trade, business cycle, DD schedule, panel unit root test, panel cointegration test.

JEL Classification: C23, C12, E32
AMS Classification: 91B26

1 Introduction

One of the most striking features of the business cycles across countries are the pattern of co-movement of output, inflation, interest rate and real equity prices. There are two different channels of business cycle transmission – financial markets and foreign trade. This paper will be investigated second one. We examine the relationship among the variables of domestic output growth, foreign output growth and the real exchange rate.

To analyze how output is determined we introduce the concept of aggregate demand for a country's output in an open economy. Aggregate demand is the amount of a country's goods and services demanded by households and firms throughout the world (Krugman, [10]). Just as the output of an individual good or services depends in part on the demand for it, a country's overall short-run output level depends on the average demand for its products. Corresponding aggregate demand for an open economy's output depends on various factors – consumption demand, investment demand, government demand, and the current account (net export demand), see [9], [14]. An important determinant of the current account is the real exchange rate, the ratio of the foreign price level measured in domestic currency to the domestic price level. We examine output market equilibrium - i.e. the relationship between output and the exchange rate (the DD schedule). We expect that any rise in the real exchange rate will cause an upward shift in the aggregate demand function and expansion of output, all else equal.

This paper examine the relationship among the variables of domestic output growth, foreign output growth and the real exchange rate (DD schedule) as transmission of foreign business cycle on small open economics for a group of new member states in the EU (EU12) over the period 1995Q1 – 2012Q4. We provide the panel unit roots tests for common or individual panel unit root process. We examine a long-run equilibrium relationship using panel cointegration tests without structural breaks. The paper is divided into four parts. Based on the introduction of the theoretical concepts of two different channels of business cycle, the second part specifies an empirical equilibrium model, panel unit roots tests and also panel cointegration tests without structural breaks. The third part deals with testing of panel unit roots and panel cointegration tests for a group of new member countries EU12 during the period 1995Q1 – 2012Q4. The final part summarizes the empirical results.

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2 Econometric methodology

One problem with results for individual countries is that they are often impaired by short data span that lowers the power of the unit root and cointegration test. In response recent studies have employed a panel unit root test combined with the panel cointegration test proposed by Pedroni [15] to exploit the extra power from combining cross-sectional and time series data.

2.1 An empirical model and data selection

As the goal to investigate the impact of foreign cycle on domestic business cyclical development we concentrate according to Juriová [7] to one transmission channel – foreign trade. The foreign business cycle has an impact on domestic economy through its exports (through changes in export demand and indirectly through changes in terms of trade). We can propose our empirical panel model as follows:

$$\ln GDP_{it} = \alpha_i + \gamma_i \cdot t + \beta_{1i} \cdot \ln GDP\_EU27_t + \beta_{2i} \ln ERT_{it} + \varepsilon_{it}, \quad (1)$$

where GDP_{it} is domestic demand (in millions of national currency, chain-linked volumes, reference year 2005, including 'euro fixed' series for euro area countries), GDP_EU27_t is real gross output EU27 countries and ERT_{it} is real effective exchange rate (deflator: consumer price indices - 27 trading partners, index, 2005=100). Subscript *t* denotes time ($t = 1,\ldots,T$) and subscript *i* is country $i = 1,\ldots, N$. We provide our analysis for a group of new member states in the EU (BG-Bulgaria, CZ-Czech Republic, EE–Estonia, CY-Cyprus, LV-Latvia, LT-Lithuania, HU-Hungary, MT-Malta, PL-Poland, RO-Romania, SK-Slovak Republic and SI-Slovenia). All variables are seasonally adjusted and transformed into natural logs. We use quarterly data over the period 1995Q1 – 2012Q4. Data are obtained from the EUROSTAT database.

2.2 Panel unit root tests

Panel unit root tests are similar, but not identical, to unit root tests carried out on a single series(see, [5]. We compute one or more of the following tests: Levin, Lin and Chu (LLC, [11]), Breitung (BR, [1]), Im, Pesaran and Shin (IPS, [6]), Fisher-type tests using ADF (FADF) and PP tests (see Maddala and Wu (FPP, [12])). Next we briefly describe the five panel unit root tests which are included in EViews 7.

For purposes of panel unit root testing, there are two natural assumptions that we can make about the ρ_i (the autoregressive coefficients in AR(1) process of y_{it} (i.e. $\ln GDP_{it}$ or $\ln ERT_{it}$). First, one can assume that the persistence parameters are common cross-sections so that $\rho_i = \rho$ for all i. The LLC, BR tests all employ this assumption. Alternatively, one can allow ρ_i to vary freely across cross-sections. The IPS, FADF and FPP tests are of this form.

Tests with common unit root process

LLC and BR employ a null hypothesis of a unit root. The LLC and BR tests consider the following basic augmented Dickey-Fuller ADF specification with deterministic variables (intercept and trend to vary across individuals) for each cross-section augmented process with the lag order s_i:

$$\Delta y_{it} = \rho_i \cdot y_{i,t-1} + \sum_{j=1}^{s_i} \omega_{ij} \Delta y_{it-j} + \alpha_i + \gamma_i \cdot t + \varepsilon_{it} \quad (2)$$

where $\rho_i = \rho$, ε_{it} are assumed to be independently distributed across *i* and *t* (*i=1, 2, ..., N; t=1, 2, ..., T*). It is worth noticing that throughout the paper we consider specification that includes a linear time trend. The null hypothesis of panel unit root is: $H_0 : \rho_1 = \rho_2 = \cdots = \rho_N = \rho = 0$ (there is unit root) against the alternative hypothesis $H_1 : \rho_1 = \rho_2 = \cdots = \rho_N = \rho < 0$ (there is no unit root). The Breitung method differs from LLC. Breitung suggests in [1] a test statistic that does not employ a bias adjustment whose power is substantially higher than that of LLC test. (First adjustment, only the autoregressive portion is removed when constructing the standardized proxies and second adjustment, the proxies are transformed and detrended).

Tests with individual unit root processes

The Im, Pesaran, and Shin (IPS), and the Fisher-ADF and Fisher-PP tests all allow for individual unit root processes so that ρ_i may vary across cross-sections. IPS test begin by specifying a separate ADF regression for

each cross section in equation (2). The null hypothesis may be written as, $H_0 : \rho_i = 0$ for all i and the alternative hypothesis is given by:

$$H_1 : \begin{cases} \rho_i = 0 & for\ i = 1,\ldots,N_1 \\ \rho_i < 0 & for\ i = N_1+1, N_1+2,\ldots,N, \end{cases} \tag{3}$$

where the i may be reordered as necessary. The IPS t-bar statistic is defined as the average of individual ADF statistics. In Monte Carlo experiments it was shown that if a large enough lag order is selected for the underlying ADF regressions, then the small sample performance of the t-bar test is reasonable and generally better than the LLC test. Fisher ADF and PP tests combine the *p*-values from individual unit root tests. This idea has been proposed by Maddala and Wu in [12]. The null and alternative hypotheses are the same as for the IPS in equation (3). In fact, the power of the Z-test is in some cases more than three times that of the IPS test. It seems that this test outperforms the other tests and is recommended.

2.3 Panel cointegration testing

There is a number of procedures for computing panel cointegration tests, see [5]. We provide a brief description of the cointegration tests supported by Eviews 7. The Pedroni and Kao tests are based on Engle-Granger two-step (residual-based) cointegration tests. The Fischer test is a combined Johansen test.

Pedroni (Engle-Granger based) cointegration tests

The Engle-Granger [2] cointegration test is based on an examination of the residuals of a spurious regression performed using I(1) variables. If the variables are cointegrated then the residuals should be I(0). On the other hand if the variables are not cointegrated then the residuals will be I(1). Pedroni [15] and Kao [8] extend the Engle-Granger framework to tests involving panel data. Pedroni proposes several tests for cointegration that allow for heterogeneous intercepts and trend coefficients across cross-sections. Consider the following regression according to [4]:

$$\ln GDP_{it} = \alpha_i + \gamma_i \cdot t + \beta_{1i} \cdot \ln GDP\_EU27_t + \beta_{2i} \ln ERT_{it} + \varepsilon_{it}, \tag{4}$$

where variables are assumed to be integrated of order one (I(1)). Parameters α_i and γ_i are individual and trend effects. Under the null hypothesis of no cointegration the residuals $\hat{\varepsilon}_{it} = e_{it}$ should be I(1). The general approach is to obtain residuals e_{it} and then to test whether they are I(1) by running the auxiliary regression

$$e_{it} = \rho_i e_{it-1} + \sum_{j=1}^{s_i} \omega_{ij} \Delta e_{it-j} + \xi_{it} \tag{5}$$

for each cross-section. Pedroni describes various methods of constructing statistics for testing for null hypothesis of no cointegration ($\rho_i = 1$). There are two alternative hypotheses: the **homogenous alternative** $(\rho_i = \rho) < 1$ for all *i*, and **the heterogeneous alternative** $(\rho_i < 1)$ for all *i*.

Kao (Engle-Granger based) cointegration tests

The Kao test follows the same basic approach as the Pedroni tests, but specifies cross-section specific intercepts and homogeneous coefficients on the first-stage regressors. In the bivariate case described in Kao [8], we have

$$\ln GDP_{it} = \alpha_i + \beta_2 \ln ERT_{it} + \varepsilon_{it} \qquad \ln GDP_{it} = \ln GDP_{it-1} + u_{it}, \qquad \ln ERT_{it} = \ln ERT_{it-1} + \nu_{it}, \tag{6}$$

More generally, we may consider running the first stage regression equation (2), requiring α_i *to be heterogeneous*, β_{2i} *to be homogeneous across cross-sections*, and setting all of the trend coefficients γ_i and β_{1i} to zero. Kao then runs either the pooled auxiliary regression

$$e_{it} = \tilde{\rho}_i e_{it-1} + \sum_{j=1}^{s} \omega_j \Delta e_{it-j} + \xi_{it}. \tag{5}$$

Under H_0: no cointegration, Kao shows defined variants statistics (Eviews 7, p. 702) converging to N(0; 1) asymptotically.

Combined individual Fisher (combined Johansen) cointegration tests

Fisher derives in [3] a combined test that uses the results of the individual independent tests. Maddala and Wu [12] use Fisher's result to propose an alternative approach to testing for cointegration in panel data by combining tests from individual cross-sections to obtain at test statistic for the full panel. If π_i is the *p*-value from an individual cointegration test for cross-section *i*, then under the null hypothesis for the panel, $-2\sum_{i=1}^{N}\log(\pi_i)\chi^2_{2N}$.

Eviews reports χ^2 value base on MacKinnon-Haug-Michelis *p*-values for Johansen's cointegration trace test and maximum eigenvalue test.

3 Empirical results

We provide the panel unit roots tests for common or individual panel unit root process. We examine a long-run equilibrium relationship among the variables $\ln GDP_{it}, \ln ERT_{it}$ and $\ln GDP\_EU27_t$ using panel cointegration tests.

3.1 Panel unit root tests

Table 1 reports the results concerning panel unit root tests for variables $\ln GDP_{it}$ and $\ln ERT_{it}$ in level or 1st difference (Δ) including deterministic components (intercept (c) or trend and intercept (c+t)) with automatic lag length selection based on Schwarz information criterion and Bartlett kernel bandwidth selection. The probabilities for Fisher tests are computed using an asymptotic Chi-square distribution. All the other tests assume asymptotic normality.

variables	**c** or **c+t**	H_0: Unit root (assumes **common** unit root)		H_0: Unit root (assumes **individual** unit root process)		
		LLC	**BRE**	**IPS**	**FADF**	**FPP**
$\ln GDP_{it}$	c+t	2.732 (0.997)	2.361 (0.991)	4.352 (1.000)	10.718 (0.991)	7.185 (0.999)
$\Delta\ln GDP_{it}$	c	-15.198*** (0.000)	X	-16.328*** (0.000)	239.929*** (0.000)	355.047*** (0.000)
$\ln ERT_{it}$	c+t	-3.154*** (0.001)	0.607 (0.728)	-3.825*** (0.0001)	54.986*** (0.0003)	51.766*** (0.0008)
$\Delta\ln ERT_{it}$	c	-18.688*** (0.000)	X	-18.445*** (0.000)	296.286*** (0.000)	331.575*** (0.000)
individual variables		**ADF**	**PP**			
$\ln GDP\_EU27_t$	c+t	-0.360 (0.987)	-0.003 (0.996)			
$\Delta\ln GDP\_EU27_t$	c	-3.393** (0.015)	-3.440** (0.013)			

Table 1 Panel unit root tests

Probabilities values are in brackets. *** or ** denotes statistical significance level at the 1% or 5%, respectively.

Our results provide evidence that we cannot reject the null hypothesis at the 5% level of significance for our variables $\ln GDP_{it}$ in level and we can reject null hypothesis for these variables in first difference i.e. *there is evidence that both variables* $\ln GDP_{it}$ *are integrated of order one for the homogenous and the heterogeneous alternative of panel unit root tests. These tests also suggest that* $\ln ERT_{it}$ *variables do not contain common or individual panel unit root after excluding deterministic linear trend.* We can also conclude that exogenous $\ln GDP\_EU27_t$ variable is I(1) using simple ADF and PP unit root tests at the 5% level of significance. We proceed on this basis to test for panel cointegration.

3.2 Panel cointegration testing

Once the existence of a panel unit root has been established, the issue arises whether there exists a long-run equilibrium relationship among the variables $\ln GDP_{it}, \ln ERT_{it}$ and $\ln GDP\_EU27_t$ for the period 1995Q1 – 2012Q4 in new member the EU countries. Given that each variable is integrated of order one, we first test for

Pedroni panel cointegration test which evaluate the null against both the homogeneous and the heterogeneous alternatives. The results are reported in table 2 for deterministic individual intercept and individual trend. None of the seven Pedroni tests reveal any evidence of common or individual cointegration. Thus, *we can conclude that the variables* $\ln GDP_{it}, \ln ERT_{it}$ *and* $\ln GDP\_EU27_t$ *do not share a long-run equilibrium relationship at 5% level of significance using both the homogeneous and the heterogeneous alternatives of seven Pedroni tests.*

Alternative hypothesis: **common** AR coefs.			Weighted Statistic		Alternative hypothesis: **individual** AR coefs.		
	Statistic	Prob.		Prob.		Statistic	Prob.
Panel v-Statistic	0.633	0.263	0.073	0.471	Group rho-Statistic	0.162	0.564
Panel rho-Statistic	-0.318	0.375	-0.806	0.210	Group PP-Statistic	-0.936	0.175
Panel PP-Statistic	-0.719	0.236	-1.552	0.060	Group ADF-Statistic	2.102	0.982
Panel ADF-Statistic	1.781	0.963	1.429	0.923			

Table 2 Pedroni's panel unit roots test

The result of the *Kao test* was tested under assumption – no deterministic trend. Kao ADF t-statistic was -2.646*** with *p*-value 0.004, which means that we reject the null hypothesis of no panel cointegration with common estimated parameter for natural logs the exchange rate. *The Kao test produces significant evidence cointegration among the variables* $\ln GDP_{it}, \ln ERT_{it}$ *and* $\ln GDP\_EU27_t$ *at 5% level of significance using the heterogeneous intercept and the homogeneous* β_{1i} and β_{2i} *across cross-sections.*

The results of the *Johansen/Fisher panel cointegration test* for the full panel presents table 3. We also assume the trend effect. To determine the number of cointegration relationship in our panel DD model we proceed sequentially from number of CE(s) =0 to 2 until we fail to reject. The Fisher trace statistics, which are reported in the second column in table 3 with p-value in the third column, recommend one cointegrating relationship for the full panel model. The second part of the output in table 3 provides the same results for the Fisher maximum eigenvalue statistic. The alternative combined *individual Fisher/Johansen test provide evidence of individual cross-section cointegration among the variables for BG, CZ, LT, HU, MT, SK and SI countries at 10% level of significance.*

Hypothesized No. of CE(s)	Fisher Stat. (from trace test)	Prob.	Fisher Stat. (from max-eigen. test)	Prob.
None	68.08	0.000	79.07	0.000
At most 1	15.57	0.903	19.48	0.726
At most 2	7.318	0.999	7.318	0.999

Table 3 Unrestricted Cointegration Rank Test (Trace and Maximum Eigenvalue)

4 Conclusions

In this paper examine output market equilibrium - i.e. the relationship between output and the exchange rate (the DD schedule in new EU12 countries over the period 1995Q1 – 2012Q4. We provide the panel unit roots tests for common or individual panel unit root process. We examine a long-run equilibrium relationship among the variables $\ln GDP_{it}, \ln ERT_{it}$ *and* $\ln GDP\_EU27_t$ using panel cointegration tests without structural breaks. The main results obtained are summarized below:

- There is evidence that *both variables* $\ln GDP_{it}$ *and* $\ln ERT_{it}$ *are integrated of order one for the homogenous and the heterogeneous alternative of panel unit root tests*. We can also conclude that exogenous $\ln GDP\_EU27_t$ variable is I(1) at the 5% level of significance. We proceed on this basis to test for panel cointegration.
- None of the seven *Pedroni tests reveal any evidence of common or individual cointegration*. Thus, we can conclude that the variables $\ln GDP_{it}, \ln ERT_{it}$ and $\ln GDP\_EU27_t$ do not share a long-run equilibrium relationship at 5% level of significance using both the homogeneous and the heterogeneous alternatives of seven Pedroni tests.

- *The Kao test produces significant evidence cointegration* among the variables $\ln GDP_{it}$, $\ln ERT_{it}$ and $\ln GDP\_EU27_t$ at 5% level of significance *using the heterogeneous intercept and the homogeneous* β_{1i} *and* β_{2i} *across cross-sections.*
- *The Fisher trace and eigenvalue statistics recommend one cointegrating relationship for the full panel model. The alternative combined individual Fisher/Johansen tests provide evidence of individual cross-section cointegration for seven countries at 10% level of significance.*

The results of panel cointegration testing support our conclusion that there is the evidence of panel long-term equilibrium in our output markets (i.e. the relationship between output and the exchange rate) with individual deterministic trend, which is probably indicator of the convergence process. Fisher/Johansen test recommended due to the verification of the results heterogeneous alternative, which can be associated with the conditional convergence process rather than absolute convergence in the most of new EU Member States in the period 1995-2012. In the next part of our research we compare results for estimation of our panel cointegration relationship in an DD model n new EU12 countries. We use estimators using Fully Modified Ordinary Least Squares (FMOLS) panel estimators. We also used Granger Causality test to perform panel data specific testing. We also include structural breaks to panel unit root tests and panel cointegration tests.

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Research of the convergence of unemployment rate of university graduates in the Czech Republic

Jana Hanclova[1], Milan Simek[2]

Abstract. The article is concerned with modeling and assessment of the process of convergence of the unemployment rate of the graduates of Bachelor's and Master's Degree study programs at the Faculties of Economics in the Czech Republic within the period of 2002 – 2012. The unemployment rate of university graduates is the ratio of the number of unemployed graduates to the difference between the total number of the graduates and the number of the graduates who continue studying at university. The theoretical concept of convergence is based on the absolute (unconditional) respectively relative (conditional) form and the method of beta convergence is used within the article. The paper empirically estimates the panel model by the means of the pooled least squares method with fixed overview effects for the Bachelor's and Master's degrees of studies separately. The results show that there are vast differences in the employment of graduates on the labor market especially after the completion of the Bachelor's degree. The results of the estimated panel models further document the (non)presence of convergence, but also the speed of the process. There are also significant differences in the convergence behavior of the unemployed graduates from individual Faculties of Economics of Czech universities.

Keywords: unemployment rate of graduates, absolute and conditional convergence, Faculties of Economics, the Czech Republic, panel unit roots tests, ADF models.

JEL Classification: C23, E24
AMS Classification: 91B40

1 Introduction

The paper focuses on the use of the concept of convergence while examining the level of unemployment of university graduates in the Czech Republic. The unemployment rate of the target group is very closely related to the issues of their chances at the labor market and to the so-called employability.

Employability may be defined as a combination of the factors that enable individuals to progress toward obtaining employment or enter into employment, continuance of employment and progress in their careers. It is a complex concept, involving not only personality traits, skills, attitudes and motivation of each individual, but also other external factors, which go beyond policy in the area of education and expert training, such as labor market regulation, demographics, economic structure and economic situation in general [6].

The European Commission considers the group of young people up to 24 years of age, including university graduates to be a vulnerable group at the labor market. The problems in employment at the labor market compared with other groups of individuals there often lead to increased levels of unemployment. Unemployment of young people is influenced by many factors [14] and [11]. For example Balcar [1] states that employers consider the reached level of soft skills to be one of the important factors of good employability of graduates. At the same time the studies of the European labor market [2] show the importance of soft skills for employment at the labor market and recommend systemic development of soft skills within the education system. The position of the graduates at the labor market and their employability in the Czech Republic is often monitored by means of indicators focused on the signs of their success in getting employment. From a quantitative point of view, it is a specific unemployment rate of university graduates (e.g. according to the area of studies, faculty and type of school).

Unemployed graduate is defined by the Ministry of Labour and Social Affairs [13] as a job seeker registered at labor offices of the Czech Republic by their permanent residence at a certain date, whose time after the successful completion of his studies did not exceed 2 years. *The unemployment rate of university graduates* is then defined as the number of unemployed university graduates to the difference between the total number of gradu-

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ates and the number of graduates who continue to study at a university. In practice, there is also used the standardized unemployment rate of graduates which is the unemployment rate of graduates without the different unemployment rate in different regional labor markets [13].

In the Czech Republic the number of unemployed graduates can be tracked always on the 30th April and 30th September. Koucký and Zelenka [13] reported that "With the possibility to split the graduates into three periods also depending on how long before the survey they completed the study, it is also possible to map in a detail the development of their employability in the transition from school to the labor market. This is the period of 0-0.5 year, 0.5 - 1 year and 1-2 years." Though in this context, the authors draw attention to the varying length of time between the individual surveys. When expressing the condition for a longer period the so-called long-term standardized unemployment rate is used, which is the average rate calculated from the standardized unemployment rates of the last 8 years [13].

Taking into account the above-mentioned facts for assessing employability of university graduates, it seems preferable to follow unemployment rate over time and based on the results obtained, and only then to consider the changes in the level of employment (employability) of graduates of a particular university, faculty, or area of studies. For example the principle of convergence may be used to monitor this. Žďárek [19] states that the concept of convergence which had been created in connection with the economic theory of growth and subsequently applied in many areas of modern economics is used for example to investigate price changes and price levels over time. The concept of convergence exists as absolute or relative one. *The absolute (unconditional)* convergence is a process where e.g. countries converge to one of the (common) values of the given parameter (the so-called steady state), regardless of the initial starting position. *The conditional (weak, relative)* convergence is a situation where the countries with different initial states of the given indicator are approaching each other, but the single common state is not reached (similar economies should converge). This approach analyzes so-called beta and sigma convergence.

This article explores the beta convergence by testing the panel stationarity of the unemployment rate of graduates of the Faculties of Economics in the Czech Republic for a group of Bachelor's as well as the following Master's Degree programs in 2002-2008 (2012). The paper is structured into four parts. Based on the introduction of the theoretical concepts of beta convergence with focus on the unemployment of university graduates, the second part specifies panel unit roots tests. The third part deals with data analysis, testing of panel unit roots and estimation of the adequate econometric ADF model including verification and economic interpretation. The final part summarizes the empirical results.

2 Panel unit roots tests for convergence

Panel unit root tests for convergence among series, or group-wise convergence, utilize Bernard and Durlauf's definition of time series convergence for long-run output movements [5], where two (or more) countries have converged when long-run forecasts of per capita output differences tend to zero as the forecasting horizon tends to infinity. In the bivariate context, tests for time series convergence require cross-country per capita output differences to be stationary. In the multivariate or panel context, a group of countries have converged if the null hypothesis that the difference between each country's output and the cross-sectional mean has a unit root can be rejected in favor of the alternative hypothesis that each difference is stationary, see Pesaran [17] and [10]. Several papers use panel methods to investigate output convergence (Ben-David [3], Evans and Karras [8], and Fleissig and Strauss [9], among others) or inflation convergence (Lee and Wu [15], Kočenda and Papell [12]).

In the panel framework, testing for (stochastic) convergence of a group of N time series requires studying the dynamic properties of the series differential with respect to the cross-sectional mean. Group-wise (stochastic) convergence implies according to Lopez and Papell [16] that:

$$\lim_{t\to\infty} E\left(y_{i,t+k} + \sum_{j=1}^{N} \frac{y_{i,t+k}}{N} \middle| I_t \right) = \eta_i \quad for\ i = 1,\ldots,N, \tag{1}$$

where I_t represents the information set available at time r. If $\eta_i = 0,$ the convergence follows Bernard and Durlauf's definition (see in [4]) of absolute convergence $\eta_i \neq 0$ the convergence is conditional or relative as defined by Durlauf and Quah [7], which implies that the series have converged toward a time-invariant equilibrium differential. We use the standard panel unit root tests to account for restriction on the intercepts when testing for group-wise convergence. Let consider the following system of ADF regressions:

$$\Delta y_{it} = \alpha_i + \beta_i y_{i,t-1} + \delta \cdot trend + \sum_{j=1}^{k_i} \phi_{ij} \Delta y_{i,t-j} + \varepsilon_{it} \quad for\ i = 1,\ldots,N, t = 1,\ldots,T, \tag{2}$$

where $\beta = \beta_i$ means the homogenous rate of convergence, k_i the lagged first differences that account for serial correlation and $\varepsilon_{it} \sim N(0, \sum)$, where $\sum$ is the non-diagonal covariance matrix. The null and alternative hypotheses tested are $\beta = 0$ and $\beta < 0$. While it would be desirable to allow for heterogeneous rates of convergence, the choices are problematic. The alternative hypothesis for these tests, however, is that $\beta_i < 0$ for at least one *i*, which is not economically relevant for investigating convergence among a group of countries.

3 The empirical results

When assessing the employability of graduates we will examine the convergence of the unemployment rate of graduates from the Faculties of Economics in the Czech Republic for the group of Bachelors and Engineers (Ing.).

Analysis of the unemployment rate of graduates

Unemployment rate of graduates of Bachelor's (URF_bc) or Master's (URF_mgr) study programs of the Faculties of Economics in the Czech Republic from 2002 to 2012 is shown in Figure 1.

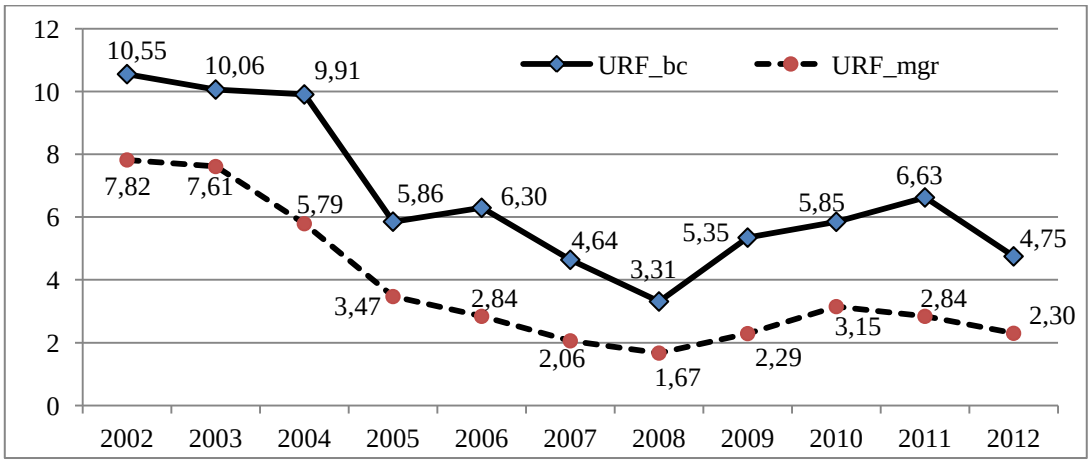

Figure 1 The development of the unemployment rate of graduates from the Faculties of Economics in the Czech Republic (%).

Source: The Education Policy Centre [18]

The level of the unemployment rate of the graduates of the surveyed faculties with economic studies was different for graduates from Bachelor's and Master's Degree programs throughout the analyzed period. In the case of the Bachelor's Degree of studies the unemployment rate was by 2-3 percentage points higher than for Master's Degree graduates. This situation has probably been caused by the reluctance of employers to employ the graduates with Bachelor's Degrees because of the lack of suitable jobs for such type of graduates. Some employers also did not consider this type of the university education to be a full-value one.

In the period under survey there have also been changes in the level of the unemployment of the graduates from the Faculties of Economics. In the period 2002-2008 there was a gradual decrease of the values of the unemployment rate of the Bachelor's Degree graduates from 10.6% to 3.3% and in the case of Master's Degree study programs from 7.8% to 1.7%. These conditions are related to the development of the Czech economy and labor market requirements as in this period the employers demanded large volume of manpower. After 2008 there was a significant reduction in the employability of these graduates, as evidenced for example by the unemployment rate 6.6% in the case of the graduates from Bachelor's Degree programs and 2.8% in the case of the graduates from Master's Degree programs. This development was influenced by the factors of the economic crisis of the Czech labor market. For the purpose of further research of convergence the time period 2002 - 2008 will be selected when there was a long-term decline in the observed unemployment rate.

Research of the beta convergence for the group of Faculties of Economics with Bachelor's programs

When estimating the panel model in equation (2) we proceed on the basis of the following indication of the variables - URF_{it} the unemployment rate of Bachelors *i* of the faculty in the year *t*, where *i=1,...,25* and *t=2002,..., 2008*; URF_{At} - the unemployment rate of Bachelors from all the Faculties of Economics in the Czech Republic in the year *t*. Furthermore, we define the difference:

$$y_{it} = \left|\ln(URF_{it}) - \ln(URF_{At})\right| \quad for\ i = 1,\ldots,N, t = 1,\ldots,T. \tag{3}$$

The Estimation of the panel model (2) has been performed alternatively by SUR (Seemingly Unrelated Regression) by the method of pooled least squares with cross-section fixed effects (PLS_FE). The verification of the estimated model (autocorrelation, heteroskedasticity, normality of residual components) highlighted the more robust estimation using the PLS_FE method. The trend component has also not been statistically significant at 5% level of significance. The resulting estimation of the modified model in the years 2002 – 2008 was the following:

$$\Delta y_{it} = \hat{\alpha}_i - 1,07\, y_{i,t-1} + \sum_{j=1}^{k_i} \hat{\phi}_{ij} \Delta y_{i,t-j} \qquad R^2 = 0,60 \quad N = 26,\ T = 7. \tag{4}$$

In the equation step of testing convergence using the ADF model (4) we test the hypothesis $H_0 : \beta = 0,$ that we reject at 5% level of Significance (prob. = 0,000) in favor of the alternative hypothesis $H_0 : \beta < 0.$ We come to the conclusion that y_{it} is stationary in terms of trends. This indicates the presence of convergence in the period examined in case of the faculties in question. In the next step, we tested the hypothesis of statistical significance of intercept. If we reject $H_0 : \alpha_i = 0,$ it is the case of conditional convergence, otherwise it is the case of absolute convergence. Conditional convergence of the unemployment rate of graduates from the Faculties of Economics with Bachelor Degree level has been proved for the majority of them - 18 faculties (VSEUJEP, ESFMU, OPFSU, FEZCU, FESUPA, FPVUT, FMEUTB, FFUVSE, FMVVSE, FPVSE, NFVSE, PEFMENDELU, VSH, VSFS, UNYP, VSEKS, VSO and VSKE) and it represents that the above-mentioned faculties converge to a various stable state in comparison with all the faculties and after reaching its stable state grow at the same pace as all the other faculties. Absolute convergence was confirmed for 8 faculties (FIMUHK, EFVSBTUO, FMKUTB, FISVSE, FMJHVSE, VSKV, SAVS and SCVI) and we may state that these faculties have the same steady state (long-term equilibrium level) of the unemployment rate of graduates as all the Faculties of Economics with Bachelor's study program (absolute variations with increasing time generate stationary process with zero medium value).

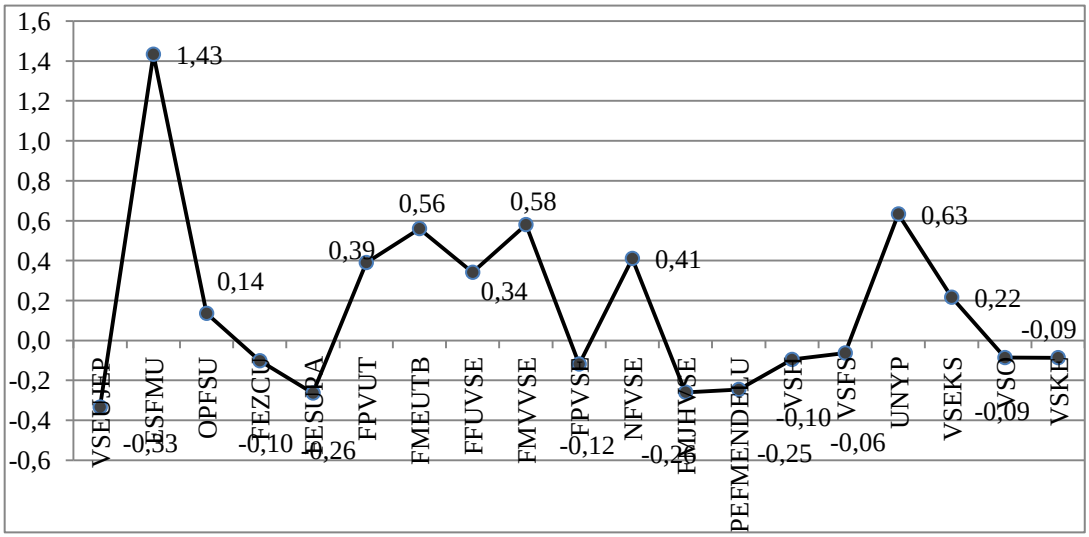

Figure 2 Conditional convergence *URF* for Bachelors in terms of the direction of approaching

The estimated level constant $\hat{\alpha}_i$ was further decomposed $\hat{\alpha}_i = \hat{\alpha}_0 + \hat{\alpha}_{1i} = \underset{sign.=0,00}{0,766} + \hat{\alpha}_{1i}.$ The development of all the estimated parameters $\hat{\alpha}_{1i}$ statistically different from zero is depicted in Figure 2 and we may observe

the convergence "from the top", i.e. the approaching of the faculty unemployment rate of Bachelors towards the national level by reducing unemployment (*positive trend*) or "from the bottom", i.e. the approximation of the faculty unemployment rate of graduates towards the national level by negative increase of the unemployment rate. We have ranked namely the faculties ESFMU, UNYP, FMVVSE and FMEUTB to the group approaching from the top. The ones converging from the bottom are namely VSEUJEP, FESUPA, FMJHVSE a PEFMENDELU.

Research of beta convergence for the group with Master's Degrees Faculties of Economics

The same procedure has been implemented for the group of the 20 Faculties of Economics with Master's Degree studies. The testing of panel unit root rejected the null hypothesis of unit root (assuming common or individual unit root process) including individual effects and linear trends. We use Levin, Lin & Chu or Breitung t-test and also Im, Pesaran and Shin W/test, ADF or PP Fisher Chi-square tests at 5% level of significance.

The estimation of the adequate ADF model according to the equations (2) by the method of pooled least squares with cross-section of fixed effects (PLS_FE) in the years 2002 – 2008 is summed up by the equation:

$$\Delta y_{it} = \hat{\alpha}_i - 0,88 y_{i,t-1} - 0,88 \cdot trend + \sum_{j=1}^{k_i} \hat{\phi}_{ij} \Delta y_{i,t-j} \qquad R^2 = 0,58 \quad N = 20,\ T = 7. \tag{5}$$

The results indicate that the trend stationary process with the inclusion of a deterministic trend, which is statistically significant at the 5% level of significance is according to the expectations negative. The unemployment rate of graduates of Master's Degree from the Faculties of Economics converges to the national level of this indicator. The estimation and decomposition of the level constant $\hat{\alpha}_i = \hat{\alpha}_0 + \hat{\alpha}_{1i} = \underset{sign.=0,00}{6,27} + \hat{\alpha}_{1i}$ indicate a higher baseline average level of the unemployment rate of Master's Degree graduates with respect to the national level. Figure 3 presents the development of the constant $\hat{\alpha}_{1i}$ for statistically significant values. From the top especially the faculties EFVSBTUO, FPVUT, OPFSU and private faculty VSH are approaching the national level. Conversely, the undesirable direction of convergence increasing the difference in the unemployment rate of Master's Degree students on the national level is shown primarily by VSEUJEP.

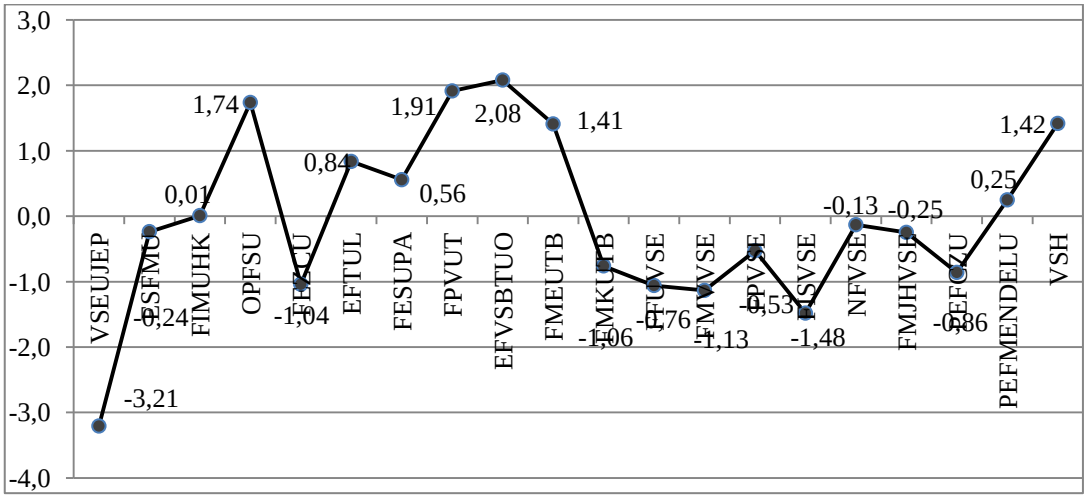

Figure 3 Conditional convergence *URF* for Master's in terms of the direction of approaching.

4 Conclusions

This paper investigates the behavior of the rate of unemployment of the graduates from the Faculties of Economics in the Czech Republic in 2002 - 2008. The main results obtained are summarized below:

- The research studies are mostly devoted to the convergence of output or prices for individual economies or panel models, but this article deals with the convergence of the unemployment rate of university graduates.
- A time series analysis showed that the unemployment rate for the entire reporting period is higher for the graduates from the Faculties of Economics with Bachelor's Degree programs, in comparison with Master's Degree programs. In the period 2002 - 2008 there was a decrease in the unemployment rate in both groups of the observed graduates. In connection with the effects of the economic crisis in the years 2009 - 2011 there had been an increase of the level of the unemployment of the graduates from the Faculties of Economics in both types of studies.

- The testing and estimating the panel unit root for the group of 26 Bachelor's programs Faculties of Economics has proved the existence of conditional convergence for the majority of 18 faculties. The other faculties converge absolutely. The development in the unemployment rate compared to the national level did not include the statistically significant negative deterministic trend.
- The results of the research of the convergence of the unemployment rate of the Master's Degree studies of 20 Faculties of Economics show that in the course of time there has been a mutual conditional convergence to the different steady states for all faculties and the differences are trend stationary with the inclusion of the deterministic linear trend.

Acknowledgements

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Microeconomic view on the inventory optimization problem

Simona Hašková[1], Pavel Kolář[2]

Abstract. The paper deals with the optimization of inventory stock. The fundamental questions of logistics are: in what quantity and when to resupply the stock. In order to answer these questions the efficient methods are derived that offer a relatively accurate and satisfactory response (including its economic interpretation) even in terms of uncertainty. The discrete case, when the unit of supply is indivisible, is studied in detail and the results are then extrapolated by the limiting transition to the continuous case, when the unit of inventory is further arbitrarily divisible. In both cases the method relies on the value of the authors own established indicator of the storage efficiency and on the fragment of the probability distribution of the demand during the delivery time, which depends on it.

Keywords: case based reasoning, logistics, daily demand, delivery time, storage costs, ordering, transferring and handling costs, reorder level, safety stock, efficiency storage indicator.
JEL Classification: C51

AMS Classification: 91B32, 91B82

1 Introduction

In practice most sellers or manufacturers, who can not manage without storing the retailed goods or components required in manufacturing, are looking for the right answer to the question "in what quantity and when to resupply their stock?". The criteria of "the correct" answer here is the minimization of the total cost associated with it. The parameters, on which the correct answer depends, are:

- JP – The variable daily demand with the jp values showing the number of removed units of the considered inventory items from storage in one working day of the current year.
- D – The variable annual consumption of the considered inventory items with values $d = \Sigma jp_i$ (summing up all the working days in the current year).
- H – The variable part of the annual costs associated with the storage of the unit of the inventory item.
- C - Costs associated with one supply of the inventory item (ordering, transferring and handling costs of one delivery that within given specific logistic terms do not depend on the number of X units of supplement delivery.
- DD – The variable delivery time of the ordered additional supply (its values dd indicate the number of working days that pass from placing the order of the additional supply to its acceptance into stock).
- Z – The loss from each considered inventory item that would not satisfy the current demand due to an empty storehouse (the loss from customer dissatisfaction in the case a retailer or the failure of production in the case of a manufacturer, e.g. the lost margin).

If we assume from the above unlimited time duration of business, it is customary to consider the possible variants within one accounting period, which is usually a calendar year. It is also suitable to split the initial question into two questions – "How much to reorder?" and "When to reorder?". If X is the answer to the question "How much to reorder?", then the values of the variable D / X are the annual numbers of orders of additional supply. Let us denote by HD (reorder level) the level of the considered inventory item ($HD \geq 0$), at which the storehouse gives the order for an additional supply. Then we can answer the question "When to reorder?" by pointing to the selected HD (reorder level).

In the following we proceed from the standard textbook model (see e.g. [1], [2], [5], [7]) and we precise the solution in terms of certainty. We give reasons for the adaptation and generalization of this model on conditions of uncertainty and then by the "case based reasoning" approach we derive a general procedure for solving the task.

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2 The model and the solution of the problem of even removal from storage in terms of certainty

Ideally, in the case of even storage removal where jp and dd are the constants (JP = jp and DD = dd), the dynamics of stocks (i.e. the course of the state of stock, depending on the time measured by the number of working days) can be approximated by the continuous model, which is the "saw" in Figure 1 (see e.g. [4], [10]). This is exactly valid, for example, for the state of stock of even flowing fluid refilled to the initial level of X at the moment of emptying the tank.

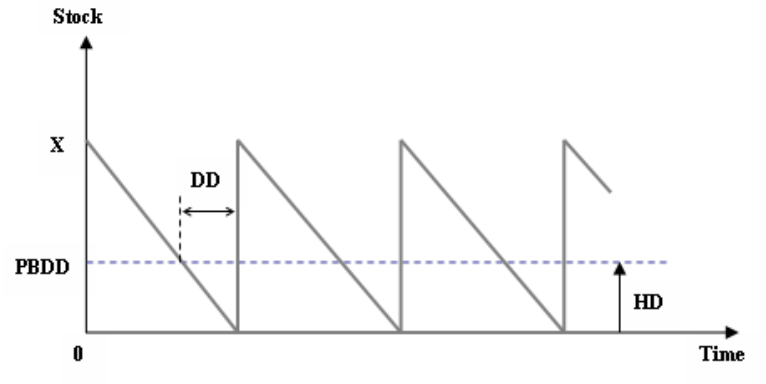

Figure 1 Dynamics of stock in the ideal case

The variable PBDD (the demand from the placing of the order to the time of delivery) is given by the product of PBDD = DD · JP, and therefore in this case has a constant value PBDD = pbdd = dd · jp. For HD_{opt} (i.e. HD minimizing the total costs associated with the storage of the inventory item and thus correctly answering the question "When to reorder?") it applies that HD_{opt} = PBDD independently of X ≥ PBDD.

It is obvious that the deviation of the HD value from the value PBDD on this or that side results in the increase of loss (costs) associated with the storage of the inventory item. Therefore HD_{opt} = PBDD. If HD = PBDD then the resulting dependence of the value of the total annual costs N(X) associated with the storage of the inventory item on the amount X of the additional supply can be described by:

$$N(X) = H \cdot X / 2 + C \cdot D / X \tag{1}$$

where H · X / 2 is the annual costs of storing and C · D / X is the annual amount of ordering, transferring and handling costs associated with replenishment of stock. Let us denote EOQ (economic order of quantity) as the volume of supplemental supply (i.e. the value of variable X), minimizing the expression (1) (for more details see [6]). Then EOQ can be obtained by solving the system $dN(X)/dX = H / 2 – C \cdot D / X^2 = 0$, $d^2N(X)/dX^2 = 2 \cdot C \cdot D / X^3 > 0$ due to the variable X:

$$EOQ = (2 \cdot C \cdot D / H)^{1/2} \tag{2}$$

By substituting EOQ for X in (1) and by the subsequent substitution of the expression $(2 \cdot C \cdot D / H)^{1/2}$ for EOQ to the right side of the equation we successively get:

$$N(EOQ) = H \cdot EOQ / 2 + C \cdot D / EOQ = (2 \cdot C \cdot D \cdot H)^{1/2} \tag{3}$$

3 The adaptation of the model to the conditions of uncertainty

In the examined ideal case (Figure 1) when HD = PBDD the last unit of the inventory item is contracted out at the time of arrival of a new supply (for more details see [9]). If HD > PBDD then the decrease of its stock would regularly stop on the level PZ = HD – PBDD. For the value PZ the teeth of the "saw" shown in Figure 1(starting from the second tooth) would move upward. The level of PZ is a safety stock, which in a case of need (an unexpected increase in the value of DD or an unexpected occurrence of jp_i > jp within the time of delivery) acts as a buffer contained in HD. Its aim is to eliminate partially or completely the threat of customer dissatisfaction due to the unexpected presence of PBDD above the usual value. In the described ideal case where this is excluded, this buffer is not needed. It would only increase the storage cost by H · PZ. Therefore, in Figure 1 PZ = 0 is selected.

The practice sometimes substantially deviates from the examined ideal. Both the daily demand (JP) and the length of delivery time (DD) may vary over time and their fluctuations make their product variable, thus the value of the variable PBDD. If values of JP and DD can, in this context, be regarded as independent random variables with known probability distributions of their values, then PBDD will be the random variable with probability distribution inferred from the distribution of variables JP and DD. Under certain assumptions, we can consider every "saw teeth" from the perspective of a long time horizon as the results of repeating of the same random process consisting of elementary random experiments that take place within individual working days. If a number of repetitions is large enough so that the law of large numbers could apply (see [3]), we can represent the variable JP, DD and PBDD as the unbiased predictions of their values, i.e. as the expected values E[JP], E[DD] a E[PBDD]. Completely analogous to Figure 1 is then Figure 2:

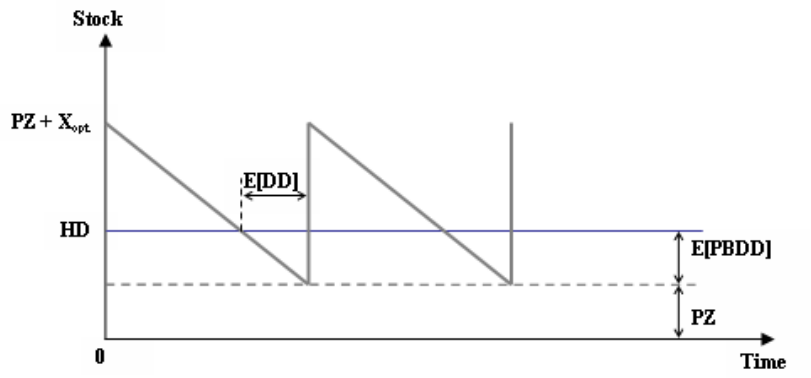

Figure 2 Model of dynamics of stock in terms of uncertainty

In it (unlike in Figure 1) the safety stock level PZ is an essential part of HD used to eliminate the unavoidable overhangs of the variable values PBDD over E[PBDD]. Therefore, it makes sense to consider PZ only within the limits $0 \leq PZ \leq (pbdd_{max} - E[PBDD])$, where for HD applies $E[PBDD] \leq HD \leq pbdd_{max}$. This model can, in a certain sense, be regarded as a statistical average of all possible random realizations of a long-term process of "development inventory stock" with the additional supply in the amount of X_{opt} units and with HD = E[PBDD] + PZ. For determination X_{opt} (the correct answer to the question "How much to reorder?") of this model it applies all that has already been mentioned in connection with the model in Figure 1. Much more difficult is finding HD_{opt}, i.e., the correct answer to the question "When to reorder?" The following part of this paper deals with the solution to this problem.

4 The general solution of the problem by the "case based reasoning" approach

Both considered models here (i.e. the model in terms of certainty - see Figure 1 and the model in terms of uncertainty - see Figure 2) answer the question "How much to reorder?" in the same way. In both cases we obtain X_{opt} by the procedure described in Article 2. The certain annual consumption of D will be replaced by the expected value $E[D] = E[\Sigma jp_i]$ in terms of uncertainty. The same thing can be said of the total annual cost (1), induced by the "saw" component of the examined fluctuations of the stock. The only difference is that while in the first case we deal with the real costs, in the second case we deal with the expected cost (see [3] and [8]).

The source of the substantial difference between the two cases is the existence of the expected annual cost E[NPZ(HD)] of the safety stock that depends on the chosen HD level under conditions of uncertainty. By analogy with equation (1), valid in terms of certainty, at $X = X_{opt}$ for the expected total cost E[N(HD)] we can write:

$$E[N(HD)] = H \cdot X_{opt} / 2 + C \cdot E[D] / X_{opt} + E[NPZ(HD)] \tag{4}$$

In the general case, E[NPZ(HD)] has two components: the expected annual storage costs of the safety stock level in the amount $H \cdot PZ = H \cdot (HD - E[PBDD])$ and the expected annual loss E[ZNZ(HD)] from a lack of stock. Therefore

$$E[NPZ(HD)] = H \cdot PZ + E[ZNZ(HD)] \tag{5}$$

To minimize E[N(HD)] according to HD means to minimize E[NPZ(HD)] according to HD. In the following we derive an algorithm to minimize E[NPZ(HD)] according to HD in the discrete case and then we generalize the idea of a solution to the continuous case.

4.1 Minimizing E [NPZ (HD)] in the discrete case

By "the discrete case" we mean the case when the unit of inventory is no more arbitrarily divisible (e.g. the washing machine in the warehouse store of washing machines, etc.). If assuming $HD_{max} = pbdd_{max}$, then $E[NPZ(HD_{max})] = H \cdot PZ_{max} = H \cdot (pbdd_{max} - E[PBDD])$. Because there is no shortage of stock, $E[ZNZ(pbdd_{max})] = 0$. A gradual reduction of HD from HD_{max} to HD – 1 gradually decreases PZ to PZ – 1 (see Figure 2). This leads to a constant marginal saving of the annual storage costs of safety stock by $H \cdot PZ - H \cdot (PZ - 1) = H$. At the same time it comes to a gradual increase of the marginal expected annual losses from the lack of stock by

$$\Delta E[ZNZ(HD)] = E[ZNZ(HD-1)] - E[ZNZ(HD)] \tag{6}$$

The idea of the algorithm of minimization E[NPZ(HD)] can be described as follows: As the first current value HD we choose the value equal to $pbdd_{max}$ and then for each current HD we count according to (6) $\Delta E[ZNZ(HD)]$. If $\Delta E[ZNZ(HD)] \geq H$, then by the transition to HD – 1 we are no better off. Thus $HD_{opt} = HD$ and the process ends. In the opposite case when $\Delta E[ZNZ(HD)] < H$ the new current HD becomes HD – 1 and the calculation according to (6) is repeated for the new current HD. The flowchart of the algorithm of this idea, in which the designation ΔE(HD) is an abbreviation for $\Delta E[ZNZ(HD)]$, is shown in Figure 3:

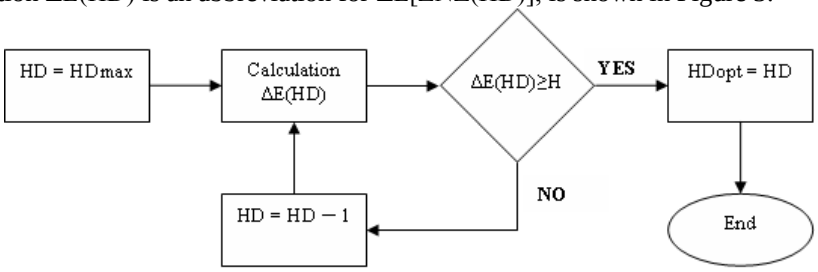

Figure 3 The flowchart of the optimization algorithm idea

The algorithm elaborates from HD_{max} to HD_{opt} from above and HD_{opt} is the one HD, to which the algorithm arrived by its recursive steps, and that as a result of (6) satisfies the condition

$$\Delta E[ZNZ(HD)] \geq H > \Delta E[ZNZ(HD+1)] \tag{7}$$

The calculation of E[ZNZ(HD)] for various HD is based on the probability distribution of the discrete random variable PBDD and on other terms, which are therefore necessary to define.

- Given a set $\Omega = \{0, 1, \ldots, pbdd_{max}\}$ of integer values of the variable PBDD and the projection $p: \Omega \rightarrow \langle 0, 1\rangle$ that for the sum of p(i) over all $i \in \Omega$ holds $\Sigma\, p(i) = 1$. The projection p defining the set of pairs $\{(i, p(i)): i \in \Omega\}$, where i is the current value of pbdd and p(i) its probability occurrence, is called probability distribution of a discrete random variable PBDD.
- The excess of demand over the HD is a discrete random variable PP(HD) with pp values dependent on the values of pbdd, for which it follows: pp = 0 for pbdd = i ≤ HD and pp = i – HD for pbdd = i > HD. Hence for the value pp = 0 we get (summing in the range from i = 0 to i = HD) its probability occurrence of Σ p(i); for pp = i – HD we get the probability occurrence of p(i). Furthermore, it is obvious that the expected value of the demand excess $E[PP(HD)] = 0 \cdot \Sigma\, p(i) + \Sigma\, (i - HD) \cdot p(i) = \Sigma\, (i - HD) \cdot p(i)$, where in the result it is summed over all i > HD.

$E[ZNZ(HD)] = E[PP(HD)] \cdot Z \cdot E[D] / X_{opt}$, from which after substitution to (6) and after adjustment we get $\Delta E[ZNZ(HD)] = (E[PP(HD-1)] - E[PP(HD)]) \cdot Z \cdot E[D] / X_{opt} = (\Sigma\, p(i)) \cdot Z \cdot E[D] / X_{opt}$, where in the result it is summed over all i ≥ HD. Analogously $\Delta E[ZNZ(HD+1)] = (\Sigma\, p(i)) \cdot Z \cdot E[D] / X_{opt}$, where in the result it is summed over all i ≥ HD + 1. Substituting both results in (7) we obtain after adjustment

$$p(HD) + \sum p(i) \geq H \cdot X_{opt} / (Z \cdot E[D]) > \sum p(i) \tag{8}$$

where in both summations it is summed over all i > HD. Sought HD_{opt} is the HD, which satisfies the condition (8). This condition can be made more transparent by designation of the constant $H \cdot X_{opt} / (Z \cdot E[D])$ by the symbol Φ, by which it takes the shape of

$$p(HD) + \sum p(i) \geq \Phi > \sum p(i) \tag{9}$$

The constant $\Phi = H \cdot X_{opt} / (Z \cdot E[D])$, sandwiched in (9) between the inequalities is the inverse value of the efficiency storage indicator.

$$\Psi = Z \cdot E[D] / (H \cdot X_{opt}) = (Z / H) \cdot (E[D] / X_{opt}) \tag{10}$$

This indicator is the product of the ratio of the unit loss from the lost margin to the annual cost of the unit storage (Z / H) and the annual number of orders ($E[D] / X_{opt}$). It measures the loss of the dissatisfaction of the annual consumption of the inventory item with the annual costs of the storing of the optimum additional supply. The condition (9) in combination with Figure 2 then says that the higher this indicator is (and therefore Φ lower), the higher the safety stock level is worth holding. In the case of the limited storage capacity it can be decided on the basis of the analysis of this indicator, which inventory items are worth storing and which are not.

4.2 Minimization of E[NPZ(HD)] in the continuous case

By the continuous case we mean the case when the unit of stock is arbitrarily divisible (as it is in the case of coal in the boiler room, the fluid reserve in the reservoir, etc.). The transition from discrete to continuous case can be reached by the "from below" unbounded physical division of the original unit. In terms of mathematical analysis it corresponds to the possibility of the limit transition from differential $\Delta x = 1$ in our discrete model to an arbitrarily small value of the differential $dx \to 0$ in the continuous model. This is related to the transition from a finite number of elements in the set of values $\Omega = \{0, 1, \dots, pbdd_{max}\}$ of the discrete random variable PBDD to the interval $\langle 0, pbdd_{max})$ of countless values of pbdd of the continuous variable PBDD.

The initial distribution $p: \Omega \to \langle 0, 1\rangle$ of the discrete PBDD in the form $\{(i, p(i)): i \in \Omega\}$, where p(i) is the probability occurrence of pbdd = i, passes in the case of the continuous PBDD in the continuous function $f: \langle 0, pbdd_{max}) \to \langle 0, 1\rangle$, in which for each $x \in \langle 0, pbdd_{max})$ and for a sufficiently short interval $dx \subset \langle 0, pbdd_{max})$ the product of $f(x) \cdot dx$ gives the probability that $x \le pbdd \le x + dx$. The function *f* is the density of probability of a continuous random variable PBDD. By assigning each value $HD \in \langle 0, pbdd_{max}\rangle$ to its corresponding value *F(HD)* obtained by integrating the density of probability *f(x)* in the range of 0 to HD, i.e. by the rule $F(HD) = \int f(x) \cdot dx$, we define the distribution function $F: \langle 0, pbdd_{max}\rangle \to \langle 0, 1\rangle$ of the continuous random variable PBDD with the extreme values of $F(0) = 0$ and $F(pbdd_{max}) = 1$. In the space between these extreme values *F(HD)* value gives the probability that pbdd < HD.

The discrete version of the continuous distribution function is the expression $F(HD) = \Sigma\, p(i)$, where is summed over all i < HD. Taking into account the fact that, if summing over the entire field Ω, Σ p(i) = 1 applies, then by substituting *F(HD)* to the equation (9) we obtain

$$1 - F(HD) \ge \Phi > 1 - F(HD) - p(HD) \tag{11}$$

The condition (11) can be achieved only when

$$\Phi = 1 - F(HD) \text{ or (which says the same thing), } F(HD) = 1 - \Phi \tag{12}$$

It is obvious that (12) holds for both the discrete and continuous case. What it means in the continuous case specifically is apparent from Figure 4:

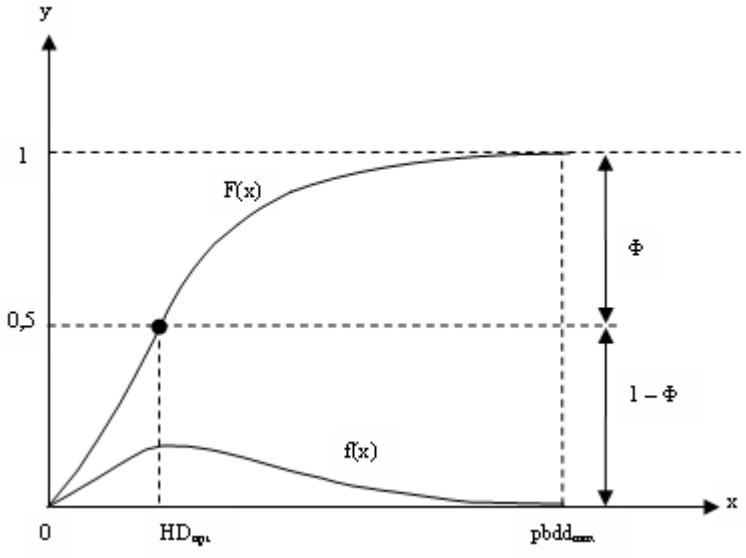

Figure 4 Identification HD_{opt} from the graph of the distribution function of a continuous random variable PBDD

We see that the sought HD_{opt} is placed under the intersection of the distribution function *F(x)* of a random variable PBDD with the level of $y = 1 - \Phi$.

5 Summary and conclusion

The paper focuses in detail on solving one of the fundamental problems of logistics, which is the question of "in what quantity?" and "when?" to resupply stock. While the literature gives a relatively accurate and satisfactory response both in terms of certainty and uncertainty to a question "how much?", the question of "when?" is often answered more or less vaguely in terms of uncertainty. The first part of the paper clarifies the standard definition of a model of the dynamics of the stock state with the steady storage removal in terms of certainty (Figure 1), draws conclusions from this, which further extends and generalizes on the conditions in terms of uncertainty.

The main contribution to the paper can be considered the derivation of a simple and effective methodology for solving this kind of task, including its economic interpretation. It turned out that for the answer to the more problematic question "when?" it suffices to know the probability distribution of the demand during the delivery time and the readily ascertainable value of, in this paper, the established indicator of the storage efficiency (10). The higher this indicator is, the higher the safety stock level is worth holding. In the case of the limited storage capacity it can be then decided on the basis of the analysis of the efficiency indicator, which inventory items are worth storing and which not.

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Comparison the Gender Wage Differences in the Selected Czech Hospitals

Veronika Hedija[1]

Abstract. The article is devoted the gender wage differences in the Czech public sector. It aims to determine and compare the amount of the gender pay gap in selected two Czech hospitals and estimate the gender pay gap which cannot be explained by different characteristics of men and women and which may be the result of wage discrimination against women.

Both hospitals are located in the same town. Data comes from the year 2010 and is administrative. Data set contains data of more than 8500 employees. We use the average treatment effect on the treated estimation to extract the part of gender pay gap, which cannot be explained by different characteristics of men and women. To reduce the potential selection bias, we supplemented this by matching as preprocessing using exact matching and coarsened exact matching.

We concluded that the gender pay gap varied in our two compared hospitals. In hospital 1, female employees took about 77.1 percent of men wage and in the hospital 2 about 88 percent. Nevertheless, unexplained gender pay gap was relative low in both hospitals and it ranged from 5 to 6 percent in favor of men. Most of the wage disparity between male and female employees could be explained by the difference in existing personal and firm characteristics of men and women working in examined hospitals.

Keywords: average treatment effect on the treated, matching, gender pay gap, wage differences, labour market

JEL Classification: J16, J24
AMS Classification: 91G70

1 Introduction

According to the Eurostat data, the Czech Republic belongs to the countries with the highest gender pay gap in the European Union. In 2010, the gender pay gap gained 25.5 percent, what was about 9 percentage points more than the gender pay gap of the whole European Union. (Eurostat [2]) In this context, there is a series of questions. Are this high wage differences between men and women due to wage discrimination and what extent? Could the gender wage differences be explained by different characteristics of men and women?

The issue of gender wage differentials in the Czech Republic and the gender wage gap decomposition is devoted a number of papers. Most of these attempts to purify the overall gender wage gap of the part that can be explained by different personal or firm characteristics of men and women and to extract the part which remains unexplained. And it is this part that can be described as potentially discriminatory.

The vast majority of papers focus on the wage differences in the private sector (for example Hedija and Musil [4], Jurajda and Paligorova [7], Mysíková [10]), but there are also studies where attention is paid to the public sector, or comparing gender pay gaps in both sectors (see Eriksson, Gottwald and Mrazek [1] or Jurajda [8]). The vast majority of these studies used data from Information System on Average Earnings (ISPV) or European Union Statistics on Income and Living Conditions (EU-SILC).

Hedija and Musil [3] and [4] presented the analysis of gender pay gap within selected firms, used data from selected Czech private and public sector companies. The conclusions of these studies cannot be generalized to the whole Czech Republic however these provide interesting information. The analysis based on the data of a particular company provides a number of advantages. It allows eliminate any bias arising from classification workers into general categories. Using data from one company enables better comparison, especially, of actual workload and performance of individual staff. This article builds on these studies and strives for the maximum micro view of gender wage differences. Unlike the above studies to calculate the unexplained gender pay gap, the average treatment effect on the treated (ATT) estimation will be used supplemented by matching procedure as preprocessing.

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The aims of the article is to determine the amount of the gender pay gap in selected two Czech hospitals and estimate and compare the gender pay gap which cannot be explained by different characteristics of men and women and which may be the result of wage discrimination against women.

The first section of the paper maps existing knowledge in the area of gender pay gap in the Czech Republic. Attention is given to the conclusions of selected studies analyzing the gender wage differences in the Czech public sector. The second section describes the methodology and data set. Unexplained portion of the pay gap is estimated by calculating the average treatment effect on the treated. In order to improve the quality of the estimation, matching procedure is used. To minimize bias arising from estimation and fundamental uncertainty, the simulation is used when estimating ATT. The final section of the article summarizes the obtained results and compares the unexplained gender pay gap in examined hospitals.

2 Gender Wage Differences: results of the selected studies

Wage differences between men and women exist not only in private but also in public sector. Empirical studies show that part of the existing gender wage gap in the Czech public sector can be explained by different characteristics of men and women. The part of wage differences remains unexplained and should be attributed to wage discrimination against women (Jurajda [8], Eriksson, Gottwald and Mrázek [1], Stupnytskyy [11]). These studies confirm that the private sector reported a higher unexplained gender pay gap compared with public sector. This finding is not surprising and can be attributed to the fact that the wage formation in public sector wages is subject to some degree of wage regulation.

Jurajda [8] used data from the Information System on Average Earnings and examined wage differences in the Czech Republic and Slovakia in 1998. He investigated the gender pay gap separately in the private and public sector using Oaxaca-Blinder Decomposition. He concluded that main source of wage differences in both sectors are various forms of employment segregation which explained over one third of the overall gender pay gap. The unexplained gender pay gaps differed dramatically between the public and private sector. In the private sector it represented more than 60 percent of the gender pay gap in comparison to the public sector, where it created just under 40 percent of the gender pay gap.

Eriksson, Gottwald and Mrázek [1] examined the gender pay gap in the public and private sector using managerial pay. They used the data set combining information from three separate sources: Information System on Average Earnings, Survey of Managerial Staff Earnings and firm register for 1998. They estimated the managerial pay function where age, education, gender, industry, region, firm size and position within firm are the explanatory variables. Their conclusions about the unexplained gender pay differences in the private and public sector are different in both periods. The wage differences due to gender were about 10.8 percent in state-owned firms and 20.1 percent in private-owned firms.

The managerial wage differences in the Czech Republic are also focused the study of Stupnytskyy [11]. Author took data from Information System on Average Earnings using Oaxaca-Blinder decomposition estimated the unexplained part of managerial gender pay gap. Gender pay gap was 0.57 in private firms and 0.26 in the stated-owned firms in the 2005. The unexplained part of gender pay gap reached 37 percent of GPG in private and only 15 percent of GPG in budgetary sector. The main factors explaining wage differences were education and position. These factors elucidated about two third of gender pay gap in budgetary sector.

3 Data

The unexplained wage difference between men and women is estimated in two selected Czech hospitals, which were willing to provide the necessary data. Both hospitals are located in the same town. The data comes from the year 2010 and is administrative.

Employees with a long-term illness, on maternity or parental leave or working on a business agreement were excluded from the sample. The final data set covers the data of 8662 employees, of which 1636 are male and 7026 female. In hospital 1, women represent about 77 percent of employees and in hospital 2 almost 83 percent.

In the hospitals the following characteristics of employees were measured: gender, age, highest level of education, occupation, number of years that the employee works in the hospital, working time (full-time means 1), department where the employee works, hours worked per year, overtime hours, days of sick leave, days of leave and annual gross wage. In the article, the hourly gross wage is used as the explained variable. The hourly gross wage for every employee is calculated as a ratio of the annual gross wage (including bonuses) and the sum of worked hours (including overtime) and hours of annual leave (days of annual leave*8*working time).

Table 1 shows the average characteristics of male and female employees in both hospitals. In the hospital 1, women earn on average 77 percent of male wage and in the hospital 2, average female wage reach 88 percent of average male wage.

	Hospital 1		Hospital 2	
Characteristic	**Men**	**Women**	**Men**	**Women**
Age (year)	43.060	41.800	41.995	38.518
Education				
primary	0.079	0.083	0.042	0.038
lower secondary vocational	0.005	0.004	0.000	0.001
secondary vocational	0.187	0.073	0.240	0.094
secondary school diploma	0.151	0.571	0.191	0.555
higher professional	0.022	0.086	0.035	0.084
bachelor's degree	0.010	0.038	0.019	0.049
tertiary	0.546	0.145	0.472	0.179
Occupation				
worker	0.190	0.038	0.276	0.038
technical and economic worker	0.128	0.115	0.105	0.084
orderly	0.130	0.097	0.103	0.111
lower medical worker	0.012	0.036	0.013	0.029
paramedical worker	0.024	0.022	0.033	0.147
medical laboratory technician	0.026	0.067	0.007	0.054
pharmaceutical assistant	0.002	0.017	0.000	0.008
midwife	0.000	0.001	0.000	0.033
nurse	0.029	0.491	0.048	0.346
another professional	0.036	0.017	0.030	0.029
pharmacist	0.000	0.003	0.003	0.006
doctor	0.423	0.096	0.383	0.115
Working time	0.896	0.959	0.910	0.934
Overtime (hour per year)	242.092	66.759	140.434	55.257
Sick leave (hour per year)	28.634	68.809	6.510	25.190
Years in firm	8.357	9.330	10.057	10.624

Table 1 Average characteristics of the employees of hospitals

A very important factor when examining the potential wage discrimination against women is the wage setting-power of managers in firm. The greater the flexibility in wage formation, the larger the effect of the different approach towards the remuneration of male and female subordinates. In 2010, wages in the Czech public health sector were regulated by Government Regulation No. 564/2006 on the salaries of employees in public service and administration. It defined the wage classes and grades and assigns workers to a wage class and to a grade level and set the base gross wage for the individual classes and grades. But the maximum amount of permitted bonuses was not regulated ant the maximum amount was regulated neither at the level of hospitals. This means that the final gross wage of an employee including bonuses depended to a large extent on the decision of the managers, who were limited by the Government Regulation, which regulates only the minimum wage, and of cause by the size of the budget of the hospital. This implies that the management of hospital have a relative flexibility in wage formation.

4 Methodology

There are several methods we can use to estimate the amount of wage differences between men and women, which cannot be explained by different characteristics of these. We estimate the unexplained part of gender pay gap using estimation of average treatment effect on the treated (ATT). Treatment literature defines the average treatment effect this way: '*The average treatment effect on the treated is the mean effect for those who actually participated in the program*'. (Wooldridge [12], p. 605) In our case the ATT is the mean effect for women in the form of a lower wage due to them being women. And this way we use it for estimating the unexplained part of gender pay gap in the two selected Czech hospitals.

We use the following equation for the calculation of the average treatment effect on the treated

$$ATT = E(y_1 - y_0 | w = 1) \quad (1)$$

Where w is the binary treatment indicator, w = 1 denotes treatment and w = 0 otherwise, y_1 is the potential

outcome with treatment and y_0 is the potential outcome without treatment. In our case, be treated means be the women. We can rewrite the ATT as

$$ATT = E(y_1|w = 1) - E(y_0|w = 1) \quad (2)$$

Where ATT represents the gender pay gap which cannot be explained different characteristics of men and women. The term $E(y_1|w = 1)$ is the sample average of logarithm of gross hourly wage of women and the term $E(y_0|w = 1)$ is the sample average of logarithm of gross hourly wage of women, if they were men. It seems to be very easy to compute the ATT bud in reality only one of the ride-side terms is known. From our sample, we can compute the first term on the right-side of the equation (2). The second term, the average of logarithm of women gross hourly wage if they were men, we have to estimate some way. There is more way to estimate this.[2] We estimate it using the regression model and simulation.

First, we estimate the coefficient of wage function of men form the regression model using least squares method.

$$y_0 = \beta_0 . X_0 + u \quad (3)$$

Where, y_0 is the logarithm of gross hourly men´s wages, β_0 is the vector of coefficients of wage function, X_0 is the vector of the chosen characteristics of women and u is a disturbance term. The explanatory variables are used: age, age squared, education, working time, years in the firm, number of overtime hours, sick leave, occupation and department.

We use the estimated coefficients of male wage function to compute the average wage of women, if they were men. We used simulation with R program to reduce the estimation and fundamental uncertainty. The simulation of estimating the parameters of the wage function and stochastic component is made, where the first step involves estimating the parameters, the second step the simulation of the random component. We use thousand simulations and receive thousand expected values of where the final expected value is calculated as the average of thousand simulations. (King et al. [9]) Than, we compute the average wage of women, if they were men using this formula

$$E(y_0|X, w = 1) = E(\beta_0 . X_{1i}). \quad (4)$$

Where $E(\beta_0 . X_{1i})$ is the mean of the expected wage of every woman in the sample after the simulation.

Finally, we estimate the average treatment effect on the treated.

$$ATT = E(y_1|w = 1) - E(\beta_0 . X_{1i}). \quad (5)$$

In this case, the ATT expresses the unexplained part of gender pay gap. But the obtained results can be biased to some extent. If the characteristics of treated and control groups are too different the coefficient of female wage function will be probably far enough the coefficients of hypothetical male wage function. To reduce bias and model dependence, we will use matching as a preprocessing procedure as proposed Ho et al. [5]. The matching is based on the idea that the simplest and least controversial way to determine the discriminatory component of the wage gap is to compare wages between men and women who show exactly the same characteristics. The aim of matching procedure is creating as much as possible homogeneous sample of men and women. There are more matching methods: one-to-one matching, exact matching, propensity score matching, monotonic imbalance bounding. The most used matching methods is exact matching which exclude women whose characteristics do not match with any man into sample and vice versa. The disadvantage of this method is that their conditions are very restrictive and leads often to selection of too small sample. This is why we use here coarsened exact matching too. Coarsened exact matching is based on exact matching but trying to overcome its shortcomings. The main idea of this method is to coarsen variables into groups, and then exact match. At the end the original values of matched data are retained (Iacus et al. [6]). Both methods offer freely available program MatchIt (Ho et al. [5]).

Application of exact matching very often caused the big loss of data and the sample of employees after matching procedure is very small. This causes the obstacles using regression to estimate potential wage of women if they are men. In the case of exact matching, we use this formula for ATT calculation

$$ATT = (\sum_{i=1}^{N} w_i)^{-1}\{\sum_{i=1}^{N} w_i \, [E(y_1|x, w = 1) - E(y_0|x, w = 0)]\} \quad (6)$$

The term $E(y_1|x, w = 1)$ is the sample average of logarithm of gross hourly wage of women in the subsample and the term $E(y_0|x, w = 0)$ is the sample average of logarithm of gross hourly wage of men in the subsample. The subsamples contain the employees with similar characteristics excluded gender.

[2] More Wooldridge [12] and Ho et al. [6].

The ATT computing without and with matching as preprocessing let as known, whether there exists the wage differences which cannot be explained by different known characteristics of men and women.

5 Empirical Results

Firstly, we estimated the ATT in both hospitals without matching. We estimated the wage function of male employees in the hospitals using equation 3 and the least square method and then calculated the average treatment effect on the treated using equation 5.

The ATT differed between examined two hospitals. In both hospitals the ATT reached negative values. The female earnings were lower than male and this difference could not be explained by known different characteristics of men and women. In the hospital 1, ATT reached -0.08945 and in the hospital 2 only -0.0293. The unexplained wage differences between men and women were higher in the hospital 1. Nevertheless, if we assess the gender wage gap we have to take into account, that not all characteristics of male and female employee in the hospital are known and quantifiable. So the part of gender wage gap can be explained for example by the higher talent or work commitment of men. The conclusion can also be partially distorted by too differences in characteristics of men and women in the sample, where the wage function of men could be far enough of hypothetical wage function of women, if they were a men.

Then we used matching procedure to get more homogenous sample of men and women and estimated ATT for this modify sample of employees. Using exact matching, we selected the subsamples of women and men with exactly identical characteristics. We worked with these characteristics: age, education, occupation, number of years in this hospital, working time, department, overtime hours and sick leave. Applying exact matching led, as expected, to the great loss of data. The resulting sample of employees was very small in the case of both hospitals and results are more informative. In hospital 1 there were chosen only 5 pairs of employees reporting exactly same characteristics, in the hospital 2 only 2 pairs. Because of very small sample of data, the ATT was calculated using equation 6. The unexplained gender pay gap reached approximately 8.4 percent in favor of men in hospital 1and 1.6 percent in favor of women in hospital 2.

Finally, we use coarsened exact matching to select subsamples of men and women with the most similar characteristics. Also in this case the sample of employees decreased significantly. After matching procedure, the sample of employees was narrowed to 255 employees in hospital 1 and to 450 employees in hospital 2. Then the wage function of men in the sample was estimated using least square method and ATT calculated. Using coarsened exact matching led to decline the ATT in hospital 1 and increase the ATT in hospital 2. However, the ATT was very similar in both hospitals after application of matching procedure it was -0.0619 in hospital 1 and -0.0531 in hospital 2.

	Without matching	**Exact matching 1)**	**Coarsened exact matching**
Hospital 1	-0.08945*	-0.08401	-0.0619*
	(0.0336)	(0.3153)	(0.0304)
N	2592	10	255
Men	584	5	103
Women	2008	5	152
Hospital 2	-0.0293	0.0158*	-0.0531*
	(0.0208)	(0.0048)	(0.0202)
N	6070	4	450
Men	1052	2	178
Women	5018	2	272

*significant at the 5% level, standard deviation in parentheses, 1) ATT is calculated using equation 6

Table 2 Average treatment effect on the treated

6 Discussion and conclusion

The main aim of this paper was analyze the amount of the gender pay gap in selected Czech public sector firms, find out and compare the unexplained part of the gender pay gap. The gender pay gap was examined in two Czech hospitals using administrative data and more methods.

The gender pay gap was different in our two compared hospitals. In hospital 1, female employees took about 77.1 percent of men wage and in the hospital 2 about 88 percent. To extract the part of the gender pay gap that could not be explained be different characteristics of male and female employee, the average treatment effect on the treated was estimated.

When we used whole sample of employees, the unexplained part of gender pay gap differed in both hospitals. The ATT reached value approximately -0.09 in hospital 1 and -0.03 in hospital 2. It means that the unexplained gender pay gap was higher in hospital 1, which reported the higher overall gender pay gap. The unexplained wage gap was approximately 9 percent in favor of men. Results for hospital 1 were very similar to Jurajda [8] who examined the unexplained part of gender pay gap in the Czech public sector in 1998. According to his conclusions, the gender pay gap in the public sector was about 0.24 and over a third of the overall gender wage gap stayed unexplained and could be due to discrimination against women. It represented the wage gap about 0.08 in favor of men.

To get more homogenous sample of men and women we applied the matching procedure as preprocessing. After using coarsen exact matching the unexplained gender pay gap was very similar in both hospitals. The ATT reached approximately -0.06 in hospital 1 and -0.05 in hospital 2.

We can conclude that identified unexplained gender pay gaps in both hospitals depend on using estimation method. Using more homogenous sample of men and women led to a convergence of values found for both hospitals. In this case, the unexplained gender pay gap was very similar in both hospitals and relative small. Most of the wage disparity between male and female employees could be explained by the difference in existing personal and firm characteristics of men and women working in examined hospitals.

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Another view on time-varying correlations: The case of stocks and bonds

Radek Hendrych [1]

Abstract. The aim of the contribution is to introduce an innovative approach to conditional covariance and correlation modelling. This can be obviously useful in multivariate financial time series analysis, e.g. in the multivariate GARCH context. The proposed method consists of two steps. The first one is based on the LDL factorization of the conditional covariance matrix, state space modelling and associated Kalman recursions. Moreover, it is able to deliver a dynamic orthogonal transformation of given stochastic vector data. The second step of the suggested technique analyses conditional covariances of transformed time series which is indeed simplified due to its simultaneously uncorrelated components. In the paper, performance of the introduced procedure is tested in an empirical financial framework. Namely, the daily correlation links between logarithmic returns on stocks and bonds are investigated and compared with other estimated dynamic correlations gained by several common methods, e.g. the moving averages, the diagonal BEKK model or the dynamic conditional correlation (DCC) models.

Keywords: conditional correlation, conditional covariance, dynamic conditional correlation, GARCH, state space modelling.

JEL classification: C32
AMS classification: 91B84

1 Introduction

Analysis of time-varying correlations is undoubtedly an important part of multivariate time series modelling. In particular, it is worth of interest from both the theoretical and the practical point of view. Correlations are crucial inputs for many tasks of financial, portfolio and risk management, e.g. an asset allocation, a construction of an optimal portfolio or a hedging problem. This issue and related topics are discussed in many academically or practically oriented publications, see e.g. the comprehensive works [1] or [4] and the references given therein.

From a general perspective, the main task is to capture time-varying behaviour of conditional covariances with special regard to modelling of conditional correlations in the given model framework. Moreover, such a class of models is indeed worthy of interest from the mathematical point of view. In particular, the conditional covariance (correlation) matrix must be symmetric and positive definite. Additionally, the conditional correlation matrix must have unit diagonal elements. Indisputably, such requirements might bring really tough constraints into estimation, especially in the case of higher dimension. Hence, it is obviously more effective to consider various representations which naturally simplify or completely eliminate these restrictions. In general, one can distinguish between two approaches: (i) *direct* - it contains an explicit modelling expression for correlations, (ii) *indirect* - time-varying correlations are simply obtained by a normalization of conditional covariances.

Section 2 introduces a general model framework (a straightforward multivariate analogy of univariate conditional heteroscedastic models). Section 3 gives a brief overview of common estimators of dynamic correlation links. Section 4 presents an innovative approach to conditional covariance and correlation modelling in details. Section 5 considers an empirical financial application which examines bivariate correlations between logarithmic returns on stocks and bonds and compares various achieved results with each other. Finally, Section 6 contains conclusions.

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2 Model framework

Consider a multivariate stochastic vector process $\{\mathbf{X}_t\}_{t\in\mathbb{Z}}$ of dimension $(n\times 1)$. Denote $\mathcal{F}_t$ the σ-algebra generated by observed time series up to and including time t, i.e. $\mathcal{F}_t = \sigma(\mathbf{X}_s, s \le t)$ is the smallest σ-algebra with respect to which $\mathbf{X}_s$ is measurable for all $s \le t$, $s,t \in \mathbb{Z}$.

In this framework, assume the following model

$$\mathbf{X}_t = \mathbf{H}_t^{1/2}\mathbf{Z}_t, \tag{1}$$

where $\mathbf{H}_t$ is the $(n\times n)$ positive definite conditional covariance matrix of $\mathbf{X}_t$ given $\mathcal{F}_{t-1}$. Furthermore, one supposes that $\{\mathbf{Z}_t\}$ is an $(n\times 1)$ i.i.d. stochastic vector process with the following first two moments: $\mathsf{E}(\mathbf{Z}_t) = \mathbf{0}$ and $\mathsf{cov}(\mathbf{Z}_t) = \mathbf{I}_n$, where $\mathbf{I}_n$ denotes the identity matrix of order n.

In the model (1), the conditional and the unconditional moments of $\mathbf{X}_t$ can be easily derived:

$$\mathsf{E}(\mathbf{X}_t|\mathcal{F}_{t-1}) = \mathbf{0}, \quad \mathsf{cov}(\mathbf{X}_t|\mathcal{F}_{t-1}) = \mathbf{H}_t^{1/2}(\mathbf{H}_t^{1/2})^\top = \mathbf{H}_t, \tag{2}$$

$$\mathsf{E}(\mathbf{X}_t) = \mathbf{0}, \quad \mathsf{cov}(\mathbf{X}_t) = \mathsf{E}(\mathbf{H}_t), \quad \mathsf{cov}(\mathbf{X}_t, \mathbf{X}_{t+h}) = \mathbf{0},\ h \neq 0. \tag{3}$$

Hence, from (2), it is evident that $\mathbf{H}_t^{1/2}$ is any $(n\times n)$ positive definite matrix such that $\mathbf{H}_t$ is the conditional covariance matrix of $\mathbf{X}_t$ given $\mathcal{F}_{t-1}$. From the theoretical point of view, $\mathbf{R}_t$ (the conditional correlation matrix of $\mathbf{X}_t$ given $\mathcal{F}_{t-1}$) can be obtained by the straightforward normalization of the conditional covariance matrix $\mathbf{H}_t$.

3 Models of conditional covariances and correlations

3.1 Basic approaches

One should start with simple multivariate moving averages (MA) in their general form, i.e.

$$\mathbf{H}_t = \frac{1}{M}\sum_{s=t-M}^{t-1}\mathbf{X}_s\mathbf{X}_s^\top, \quad M \ge 2. \tag{4}$$

Multivariate exponentially weighted moving averages (EWMA) are defined as

$$\mathbf{H}_t = (1-\lambda)\mathbf{X}_{t-1}\mathbf{X}_{t-1}^\top + \lambda\mathbf{H}_{t-1}, \quad \lambda \in (0,1). \tag{5}$$

Both models (4) and (5) are evidently simple and, therefore, they find broad practical applications. Each of them contains only one unknown parameter; e.g. $M = 100$ or $\lambda = 0.94$ are commonly set for daily data, see [4]. The given forms of (4) and (5) guarantee the positive semidefiniteness.

3.2 BEKK model

A BEKK model is a particular case of a more general VEC model, see e.g. [4]. The simplest BEKK(1,1) model can be generally represented as

$$\mathbf{H}_t = \mathbf{C}\mathbf{C}^\top + \mathbf{A}\mathbf{X}_{t-1}\mathbf{X}_{t-1}^\top\mathbf{A}^\top + \mathbf{B}\mathbf{H}_{t-1}\mathbf{B}^\top, \tag{6}$$

where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are $(n\times n)$ parameter matrices. The positive semidefiniteness is guaranteed by the construction. On the other hand, such models include lots of unknown parameters, and thus their calibration (usually via (quasi) maximum likelihood) can be challenging. Therefore, a simpler form may be used, e.g. the so-called *diagonal* BEKK(1,1), i.e. the matrices $\mathbf{A}$ and $\mathbf{B}$ are restricted to be diagonal.

3.3 Constant and dynamic conditional correlations

A *constant conditional correlation* (CCC) model by Bollerslev [2] decomposes the matrix $\mathbf{H}_t$ as

$$\mathbf{H}_t = \mathbf{C}_t \mathbf{R} \mathbf{C}_t, \tag{7}$$

where $\mathbf{C}_t$ is a diagonal matrix formed by time-varying standard deviations $\sqrt{h_{ii,t}}$ and $\mathbf{R}$ is an $(n \times n)$ constant conditional correlation matrix. The diagonal elements of $\mathbf{C}_t$ can be modelled by common (univariate) techniques for conditional variances. Moreover, the matrix $\mathbf{R}$ is usually estimated by a sample correlation matrix of standardized errors $\boldsymbol{\gamma}_t = \mathbf{C}_t^{-1}\mathbf{X}_t$. However, the assumption of constant conditional correlations may seem unrealistic and too restrictive.

Engle [5] has extended the model (7) in a natural way to a more general case of *dynamic conditional correlations* (DCC) which are defined as

$$\begin{aligned} \mathbf{H}_t &= \mathbf{C}_t \mathbf{R}_t \mathbf{C}_t, & (8)\\ \mathbf{R}_t &= \mathsf{diag}\{\mathbf{Q}_t\}^{-1/2} \mathbf{Q}_t \mathsf{diag}\{\mathbf{Q}_t\}^{-1/2}, & (9)\\ \mathbf{Q}_t &= (1-\alpha-\beta)\mathbf{S} + \alpha \boldsymbol{\gamma}_{t-1}\boldsymbol{\gamma}_{t-1}^\top + \beta \mathbf{Q}_{t-1}, & (10) \end{aligned}$$

where $\mathbf{R}_t$ is a matrix of time-varying conditional correlations with unit diagonal elements, α and β are scalars and $\mathbf{S}$ is a parameter matrix and $\mathsf{diag}\{\mathbf{Q}_t\}$ is a diagonal matrix with $q_{11,t}, \ldots, q_{nn,t}$ on the main diagonal. If $\mathbf{Q}_t$ is positive definite, then so is $\mathbf{R}_t$. To ensure that $\mathbf{Q}_t$ is positive definite, it is sufficient to suppose that $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta < 1$ and $\mathbf{S}$ is positive definite, see [5]. It is also usual to assume that $s_{ii} = 1$ in order to guarantee the unique specification of $(\alpha, \beta, \mathbf{S})$. The so-called mean-reverting DCC model (8)-(10) is an analogy of the scalar diagonal GARCH model.

There exist various alternatives to $\mathbf{Q}_t$ in (10), e.g. *integrated* DCC defined as

$$\mathbf{Q}_t = (1-\lambda)\boldsymbol{\gamma}_{t-1}\boldsymbol{\gamma}_{t-1}^\top + \lambda \mathbf{Q}_{t-1}, \quad \lambda \in (0,1). \tag{11}$$

This model is a direct analogy of the multivariate exponentially weighted moving averages, see above. The process for $\mathbf{Q}_t$ has the unit root and the covariances have no tendency to revert to a constant value. In general, the DCC models can be calibrated by two-step (quasi) maximum likelihood, i.e. the first step calibrates conditional variance terms and the second one fits conditional correlations, see [4] and [5].

3.4 Orthogonal GARCH model

An orthogonal GARCH model (OGARCH) is based on a time-invariant orthogonal transformation of multivariate time series $\{\mathbf{X}_t\}$, see [1]. Namely, assume that $\mathsf{cov}(\mathbf{X}_t) = \boldsymbol{\Sigma}$ has the spectral decomposition:

$$\boldsymbol{\Sigma} = \mathbf{P}^{-1}\boldsymbol{\Lambda}(\mathbf{P}^{-1})^\top, \tag{12}$$

where $\boldsymbol{\Lambda}$ is the diagonal matrix of eigenvalues of the unconditional covariance matrix $\boldsymbol{\Sigma}$ and $\mathbf{P}^{-1}$ is the orthonormal matrix of associated eigenvectors. Further, suppose that the conditional covariance matrix of the transformation $\mathbf{Y}_t = \mathbf{P}\mathbf{X}_t$ given $\mathcal{F}_{t-1}$, say $\mathbf{G}_t$, is diagonal and that each of its diagonal elements follows a univariate conditional variance process, e.g. a GARCH process. According to the declared assumptions, the conditional and the unconditional covariance matrix of $\mathbf{X}_t$ are

$$\mathsf{cov}(\mathbf{X}_t|\mathcal{F}_{t-1}) = \mathbf{P}^{-1}\mathbf{G}_t(\mathbf{P}^{-1})^\top, \quad \mathsf{cov}(\mathbf{X}_t) = \mathbf{P}^{-1}\mathsf{E}(\mathbf{G}_t)(\mathbf{P}^{-1})^\top. \tag{13}$$

From the practical point of view, the calibration follows a two-step procedure, see [4]. Firstly, one extracts the spectral decomposition of $\mathbf{S}$ (i.e. a sample counterpart of $\boldsymbol{\Sigma}$). Secondly, one estimates suitable univariate conditional covariance models for each element of the transformed stochastic vector process $\{\mathbf{Y}_t\}$, and thus delivers the conditional covariances (or correlations) of $\{\mathbf{X}_t\}$.

4 Conditional covariances and correlations via state space modelling

The idea of an orthogonal transformation, which has been introduced in the context of the OGARCH technique, can be further extended. In particular, one can assume a dynamic modification of this approach.

Following the algebraic theory, each real symmetric positive definite matrix has a unique LDL decomposition, see e.g. [7]. Namely, let the conditional covariance matrix $\mathbf{H}_t$ have the LDL reparametrization in the standard form, i.e.

$$\mathbf{H}_t = \mathbf{L}_t\mathbf{D}_t\mathbf{L}_t^\top \quad [= (\mathbf{L}_t\mathbf{D}_t^{1/2})(\mathbf{L}_t\mathbf{D}_t^{1/2})^\top = \mathbf{H}_t^{1/2}(\mathbf{H}_t^{1/2})^\top], \tag{14}$$

where $\mathbf{L}_t$ is a $(n \times n)$ lower triangular matrix with the unit diagonal and $\mathbf{D}_t$ is a $(n \times n)$ diagonal matrix with positive elements $d_{ii,t}$ on its diagonal. In particular, $\det(\mathbf{L}_t) = 1$, $\mathbf{L}_t$ is invertible, and the inverted matrix $\mathbf{L}_t^{-1}$ is also a $(n \times n)$ lower triangular matrix with unit diagonal elements. Point out that the decomposition (14) requires no parameter constraints for $\mathbf{H}_t$ being symmetric and positive definite since this is guaranteed by the structure.

The form of the matrix $\mathbf{L}_t$ provides a natural orthogonal transformation of $\mathbf{X}_t$:

$$\mathbf{Y}_t = \mathbf{L}_t^{-1}\mathbf{X}_t \quad [= \mathbf{L}_t^{-1}\mathbf{H}_t^{1/2}\mathbf{Z}_t = \mathbf{D}_t^{1/2}\mathbf{Z}_t]. \tag{15}$$

With respect to the declared assumptions and relations (2), (3) and (15), the transformation $\mathbf{Y}_t$ has the following conditional and unconditional moments:

$$\mathsf{E}(\mathbf{Y}_t|\mathcal{F}_{t-1}) = \mathbf{0}, \quad \mathsf{cov}(\mathbf{Y}_t|\mathcal{F}_{t-1}) = \mathbf{D}_t, \tag{16}$$

$$\mathsf{E}(\mathbf{Y}_t) = \mathbf{0}, \quad \mathsf{cov}(\mathbf{Y}_t) = \mathsf{E}(\mathbf{D}_t), \quad \mathsf{cov}(\mathbf{Y}_t, \mathbf{Y}_{t+h}) = \mathbf{0},\ h \neq 0. \tag{17}$$

For a dynamic estimation of unknown quantities in the LDL decomposition (14) for the model (1) with a given entire sample $\{\mathbf{X}_1, \ldots, \mathbf{X}_T\}$, the state space modelling seems to be truly useful. The issue of state space models and associated Kalman recursions is elaborated in various publications, see e.g. [3].

With regard to (15), assume the following dynamic discrete-time linear state space representation (a generalized analogy of the recursive OLS estimator):

$$\boldsymbol{\beta}_{t+1} = \boldsymbol{\beta}_t + \boldsymbol{\varepsilon}_t, \tag{18}$$

$$\mathbf{X}_t = \mathbf{G}_t\boldsymbol{\beta}_t + \mathbf{Y}_t,\ t = 1, \ldots, T. \tag{19}$$

Denote $\boldsymbol{\beta}_t$ the $(N \times 1)$ vector containing all unknown row elements of $\mathbf{L}_t$. The $(n \times N)$ matrix $\mathbf{G}_t$ clearly includes only zeroes and the elements of $\mathbf{X}_t$ due to (15), $N = n(n-1)/2$. The state equation (18) is a pure multivariate random walk, but one can clearly suppose some richer versions.

Moreover, recapitulate crucial assumptions of this model: $\left\{\boldsymbol{\beta}_1, ((\boldsymbol{\varepsilon}_t)^\top, \mathbf{Y}_t^\top)^\top\right\}_t$ is a sequence of uncorrelated random vectors with finite second moments and $\mathsf{E}\boldsymbol{\varepsilon}_t = \mathbf{0}$, $\mathsf{cov}(\boldsymbol{\varepsilon}_t) = \mathbf{M}_t$, $\mathsf{E}\mathbf{Y}_t = \mathbf{0}$, $\mathsf{cov}(\mathbf{Y}_t) = \mathbf{N}_t$ and also $\mathsf{cov}(\boldsymbol{\varepsilon}_t, \mathbf{Y}_t) = \mathbf{0}$. The matrix $\mathbf{N}_t$ is supposed to be diagonal due to (17). Further, the initial state vector $\boldsymbol{\beta}_1$ is assumed to be random with the expected value $\mathsf{E}(\boldsymbol{\beta}_1) = \mathbf{0}$ and the variance $\mathsf{var}(\boldsymbol{\beta}_1) = \kappa\mathbf{I}_N$, $\kappa \to \infty$, i.e. the so-called standard diffuse prior. The covariance matrices $\mathbf{M}_t$ and $\mathbf{N}_t$ could be captured essentially by constant parameter matrices which are estimated via a (quasi) maximum likelihood procedure, see [3]. In the given framework, the standard Kalman recursive formulas for predicting, filtering and smoothing can be used to obtain corresponding estimators of $\boldsymbol{\beta}_t$ and consequently also the transformed vector $\mathbf{Y}_t$ with simultaneously uncorrelated components. The conditional variances $d_{ii,t}$ of $Y_{i,t}$ can be proceed by some advanced (univariate) methods, e.g. by means of GARCH models, see [8].

5 Empirical study: Stock and bonds

To examine empirical performance of the suggested approach to conditional covariance and correlation modelling based on the LDL decomposition and the associated state space representation (see Section 4), an empirical application is considered. Namely, the daily bivariate correlations between log-returns on stocks and bonds are investigated. All mentioned estimators of conditional correlations are compared from various modelling perspectives. In general, there is no consensus about how stocks and long term bonds are related. Short-run correlations are obviously affected, e.g., by new announcements. Long-run correlations between these two types of assets should be state dependent, e.g. driven by macroeconomic factors. The way how the correlation links respond to these factors might be changed over time, see [4]. The daily logarithmic returns on the S&P 500 index and 30-year bond futures from 3 January 1990 to 30 April 2013 are observed, see [9] (get the quotes for ^GSPC and ^TYX, respectively).

The estimated conditional correlations are graphically presented in Figure 1. It indicates numerous similarities among the introduced models of conditional correlations, e.g. analogies among the state space technique (see Section 4) and both previously mentioned DCC procedures. The diagonal BEKK(1,1), the EWMA and especially the OGARCH estimators seem to be more volatile, but the main trends in the correlations are comparable with the other estimators. Note that the CCC approach is not truly competitive in this comparison due to the fact that this correlation estimator is constant, i.e. $\hat{\rho}_t = 0.029$. Generally, the time-varying correlations are mostly negative during the 90's, rather positive after the year 2000 and positive at the end of the observed period. Table 1 contains the sample correlation matrix of the estimated time-varying conditional correlations (the used abbreviations clearly follow the labels in Figure 1. In particular, one can see that the delivered dynamic correlations are strongly positively correlated with the others, which is also obvious from Figure 1.

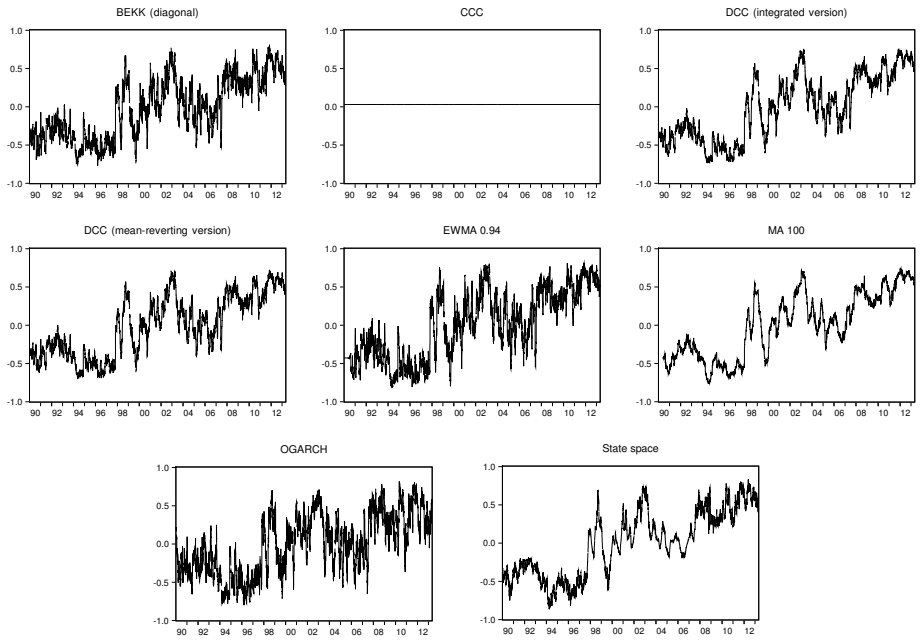

Figure 1 The estimated conditional correlations between the S&P 500 and the 30Y bond futures.

	BEKK (diag)	DCC (int)	DCC (mr)	EWMA 0.94	MA 100	OGARCH	Sspace
BEKK (diag)	1.000						
DCC (int)	0.979	1.000					
DCC (mr)	0.987	0.999	1.000				
EWMA 0.94	0.996	0.980	0.987	1.000			
MA 100	0.929	0.976	0.966	0.929	1.000		
OGARCH	0.949	0.918	0.929	0.944	0.858	1.000	
Sspace	0.947	0.956	0.957	0.948	0.937	0.906	1.000

Table 1 The sample correlations of the estimated conditional correlations.

For other performance measures, one can calculate various regression-based tests on portfolio returns, $\mathbf{w}_t\mathbf{X}_t$, where $\mathbf{w}_t$ is a vector of portfolio weights. Note that the conditional variance of $\mathbf{w}_t\mathbf{X}_t$ is $\mathbf{w}_t^\top\mathbf{H}_t\mathbf{w}_t$. Two types of portfolio weights are considered: the *equally weighted portfolio* (EWP), i.e. $\mathbf{w}_t = \mathbf{1}/n$, $\mathbf{1}$ is the $(n \times 1)$ vector of ones, and the *minimum variance portfolio* (MVP), i.e. $\mathbf{w}_t = (\mathbf{H}_t^{-1}\mathbf{1})/(\mathbf{1}^\top\mathbf{H}_t^{-1}\mathbf{1})$.

First, the Engle-Colacito regression is defined as $\{(\mathbf{w}_t\mathbf{X}_t)^2/(\mathbf{w}_t^\top\hat{\mathbf{H}}_t\mathbf{w}_t)\} - 1 = \lambda + \xi_t$, where ξ_t is an error term. The null hypothesis $\lambda = 0$ is verified (in the presence of an HAC robust estimator of the standard deviation of ξ_t), see [6]. Second, the LM test of ARCH effects is based on the property that the series $\{(\mathbf{w}_t\mathbf{X}_t)^2/(\mathbf{w}_t^\top\mathbf{H}_t\mathbf{w}_t)\}$ does not exhibit serial correlation. The null hypothesis that $\{(\mathbf{w}_t\mathbf{X}_t)^2/(\mathbf{w}_t^\top\hat{\mathbf{H}}_t\mathbf{w}_t)\}$ is serially uncorrelated is tested as in [8]. In addition, the Ljung-Box test statistics finding serial correlations in the standardized residual series are computed, see again [8].

Table 2 delivers the previously mentioned performance measures of the particular conditional correlations, i.e. the sample standard deviations $\hat{\sigma}$ and p-values corresponding to the Ljung-Box statistics Q with 9 lags, the ARCH (LM) tests with 5 lags and the Engle-Colacito (EC) tests for both portfolios.

	$Q(9)$	$\hat{\sigma}_{EWP}$	LM_{EWP}	EC_{EWP}	$\hat{\sigma}_{MVP}$	LM_{MVP}	EC_{MVP}
BEKK (diag)	0.2654	0.0090	0.0676	0.5108	0.0082	0.9865	0.1363
CCC	0.0430	0.0090	0.0000	0.2513	0.0083	0.3025	0.8774
DCC (int)	0.3911	0.0090	0.0023	0.9277	0.0082	0.7927	0.0750
DCC (mr)	0.3808	0.0090	0.0082	0.7824	0.0081	0.9035	0.2410
EWMA 0.94	0.0755	0.0090	0.2928	0.0158	0.0083	0.9993	0.0000
MA 100	0.0196	0.0091	0.0000	0.0062	0.0085	0.0030	0.0000
OGARCH	0.3821	0.0090	0.2977	0.9847	0.0091	0.3537	0.9403
Sspace	0.3382	0.0090	0.7164	0.7810	0.0080	0.9978	0.6716

Table 2 The comparison of the different conditional correlation models.

6 Conclusion

An innovative approach to conditional covariance and correlation modelling was proposed. The suggested two-step technique motivated by the principle of the OGARCH method is based on the LDL factorization of the conditional covariance matrix and the associated discrete linear state space modelling. Jointly, they can deliver a dynamic orthogonal transformation of given stochastic vector data. This indeed simplifies further conditional covariance analysis due to its simultaneously uncorrelated components. In the considered empirical financial framework, the proposed method demonstrated its capabilities. Namely, it is at least comparable with other introduced models, see e.g. Figure 1 and Table 2, and it is numerically comfortable due to the state space modelling with the associated Kalman recursions. Finally, further research will be focused on a construction of a one-step method using augmented discrete linear state space modelling.

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On alternative remedy for multicollinearity in data set

Robert Hlavatý[1]

Abstract. The paper deals with multicollinearity in terms of econometrics model. This undesirable phenomenon affects results of mathematical modelling adversely wherever there is a need for expression of dependency of endogenous and exogenous variables. Whether there is a situation with occurrence of more than one exogenous variable it is possible this unwanted dependency could occur and it is necessary to eliminate it.
Following text brings a new way of resampling for variable causing multicollinearity. An analogical approach for eliminating multicollinearity has been already known sometimes it does not bring good results though. The proposed approach offers brand new approach for multicollinearity elimination and brings various advantages compared to approaches known before. Functionality of procedures described is validated with the help of simulation model.

Keywords: multicollinearity, simulation, Markov chain, ergodicity.

JEL Classification: C15
AMS Classification: 91G70

1 Introduction

According to Hušek [5], the multicollinearity is an undesirable phenomenon found in data sets with purpose of describing dependence between the response variables and explanatory variables. It is possible to come across such data sets usually in the field of statistics and similar disciplines like econometrics, economic forecasting, stochastic modelling etc. Spanos and McGuirk [9] state that there are basically two types of problems with multicollinearity. Either it can be unwanted dependence between regressors or it can regard conditionality of correlation matrix$X^T X$. This paper will be dedicated to the first class of these two problems, particularly to multicollinearity in data set and to possibilities of its remedy while eliminating a loss of original input information. York [12] says there is main reason for multicollinearity existence – a lack of input information. If one is not able to improve data set in this regard, it is still possible to eliminate this unwanted phenomenon. There are several ways of remedy for multicollinearity in econometrics model. The utmost way is excluding a variable (variables) causing multicollinearity from the model [11]. This is only possible when the variable is not of great importance for the model. In good quality model, all of variables should be of great importance however, that is why this way of remedy is not quite advisable. The other option is excluding only some observations of the time series because it was found that some observations might affect multicollinearity to greater extent than other observations [7].

To avoid devaluation of input data by such ways there is an option of transforming the values of a variable to different values which would lead to reduction of dependency inside the model to acceptable level. By the acceptable level the low dependency is meant. From a subjective point of view it is sometimes meant as medium dependency. According to Hendl [4] low dependency is characterized by correlation coefficient $|r|$ lying in interval $|r| \in \langle 0{,}1\ \ 0{,}3\rangle$. Medium dependency is then for $|r| \in \langle 0{,}3\ \ 0{,}7\rangle$.

There are various ways of remedy for multicollinearity using data transformation. There is *ridge regression* among those oldest design by Hoerl and Kennard in 1970. Ranjit [8] proposes solution of this problem by *principal component regression*. There is also very popular approach called *bootstrapping* described by Efron and Tibshirani [2].

The purpose of following experiment is to design different way of replacing original values by n-valued sequence that will help to reduce unwanted correlation while preserving properties of original data set in best possible way. For this, the instrument of Markov chain will be used. This apparatus can be used very well for dummy variables because Markov chains are based on a sequence of independent events where every event value is dependent only on previous event value [1]. Transforming original variable to dummy variable undoubtedly carries such property. In case of dummy variable, this chain can consist of particular number of values. Creating a transition matrix of the Markov chain, it is then possible to define matrix of steady states if the transition matrix is regular [10].

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The aim of the paper is to create, upon steady states probabilities, new different n-valued vector and from this vector create back a whole new vector consisting of new values. This experiment should prove that new vector values keep properties of the original values. Simultaneously, the multicollinearity should not be present in the data set anymore.

2 Experiment design

2.1 Input data

In the beginning it is necessary to create representative experimental data set, on which the multicollinearity presence will be tested followed by eliminating the multicollinearity. The multicollinearity means correlation between two variables that is why there is no need to create large model. A pair of two variables x_1, x_2 will be sufficient enough. These variables are not real but artificially created. It is also necessary to have sufficient number of observation, so the multicollinearity could be created artificially between these two variables. Both of them will consist of 15 observations. According to Hančlová and Tvrdý [3] it is recommended to have at least 30 observations when operating with any time series. It is, however, not necessary for the simulation model. It would also bring complications for creating the artificial multicollinearity between variables. The more observations we have, the less chance of strong multicollinearity existence there is. It is then more difficult to create artificial multicollinearity. For better understanding of principles of this transformation the 15 values are enough for preserving clarity of this illustrative example.

The presence of the multicollinearity is then desired in the model, so it can be further eliminated. Creating artificial multicollinearity is based on a fact that there sure will be strong dependency between the variables if the trend functions of variables will be nearly identical in slope and different in constant. This is ensured by generating values of variables in the following way:

Let x_1 be random integer number from interval $\langle 40 \;\; 60 \rangle$.

Then x_2 is integer number defined by

$$x_2 = 2x_1 + a \,;\, a \in Z \,;\, a \in \langle -2; 2 \rangle \tag{1}$$

Variable x_2 is then integer number created as a double of x_1 with the maximum deviation of 2. Slopes of both functions will not be identical due to this deviation. This would not be desirable for it would cause so called perfect multicollineaerity (correlation coefficient would equal 1; see formula 4). That is why the values of x_2 are varied by the deviation. The number 2 was derived empirically. For lower deviation, the correlation was becoming almost too perfect, for higher deviation there was not always strong correlation between the variables. A critical value of strong correlation in this experiment is considered 0,8.

To calculate a degree of dependency the Farrar-Glauber test is used. It is based on creating a correlation matrix

$$R = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1k} \\ r_{21} & r_{22} & \cdots & r_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ r_{k1} & r_{k2} & \cdots & r_{kk} \end{pmatrix} \tag{2}$$

Its elements are calculated by multiplication of matrices $X^T X$. Elements of matrix X are calculated from input data by the formula

$$\mathrm{x_{ij}} = \frac{\mathrm{x} - \bar{\mathrm{x}}_\mathrm{j}}{\mathrm{s}.\sqrt{\mathrm{n}}} \tag{3}$$

where x represents real original values and $\bar{x}_j$ is mean of the variable, n is number of observations and s standard deviation of the variable. Using this formula we receive the correlation matrix

$$\mathrm{R} = \begin{pmatrix} 1 & \mathrm{r_{12}} \\ \mathrm{r_{21}} & 1 \end{pmatrix} \tag{4}$$

Diagonal values will always be equal 1 and the matrix is triangular so every value will be expressed twice. Test criterion here can be expressed as determinant of the matrix. However its value has no economical interpretation. That is why this criterion will be value $r_{12} = r_{21}$ from the matrix. Let us say that the degree of dependency is absolute value of $r_{12} = r_{21}$. Should this value reach over 0,8 then there is a multicollinearity between x_1 and x_2.

2.2 Bases for eliminating multicollinearity

The least appropriate solution would surely be omitting multicollinearity-causing variable(s) from a model. This solution would make impossible further examination of how omitted variable affected dependent variable. The fact of multicollinearity existence in a model does not mean that its estimated parameters are not statistically significant. On the contrary, the omitted explanatory variable could have highest degree of explanation for endogenous variable. Possible omitting the variable gives us whole new model and interpretation of this model is inappropriate because it does not consider influence of the omitted variable anymore. Therefore omitting any variable is quite irresponsible since in well-constructed model, every variable should have its importance.

The different solution is replacing original values by new values; let's say binary values 0 and 1. If the value increased compared to the previous value, the value 1 will be assigned instead. If the value decreased, the value 0 will be assigned. In this case there is a great loss of original information because reducing any data to 0 or 1 is quite vague. It is not even clear which of these two values should be assigned if there was no change compared to previous observation. It is then reasonable to use broader spectrum of values instead of two. Number of new replacing values depends on one particular trend of the variable. If there are larger fluctuations in the trend of the variable, it is then recommended to use more of new values for replacing the original. In this experiment the range of 5 values was chosen to replace the original values. The way of replacing the values is as follows:

Change compared to previous observation	New value
Significant decrease $(20\%, \infty)$	0
Slight decrease $(10\%, 20\%)$	0,25
Small change $(0\%, 10\%)$	0,5
Slight increase $(10\%, 20\%)$	0,75
Significant increase $(20\%, \infty)$	1

Table 1 Replacing original values with the new ones

Given interval boundaries of ∞ are merely theoretical. It actually means all changes greater than 20% considering the original data set was ridden of extreme values that would otherwise distort interpretation of the model. In our experimental data, let there be maximum change of 50% compared to previous observation. Still the probability of such change is very low. Probability density of change compared to previous observation is equivalent to *normal distribution* here. That is why these intervals are concentrated more to small change than to maximum possible change.

2.3 Using Markov chains

The basic assumption here says there is a state n in discrete time dependent only on state $n-1$. Knowing a whole sequence of transitions of states we are able to define transition matrix of Markov chain P. Let us first define frequencies of each type of transition from one observation to another. Number of these frequencies is 25. Those are actually all pairs of new values including the transition of one state to the same state. If any type of transition is not present, the matrix must be reduced of this type. The matrix of frequencies of transitions will have the following structure:

$$\mathrm{N} = \begin{pmatrix} 0 \to 0 & 0 \to 0{,}25 & 0 \to 0{,}5 & 0 \to 0{,}75 & 0 \to 1 \\ 0{,}25 \to 0 & 0{,}25 \to 0{,}25 & 0{,}25 \to 0{,}5 & 0{,}25 \to 0{,}75 & 0{,}25 \to 1 \\ 0{,}5 \to 0 & 0{,}5 \to 0{,}25 & 0{,}5 \to 0{,}5 & 0{,}5 \to 0{,}75 & 0{,}5 \to 1 \\ 0{,}75 \to 0 & 0{,}75 \to 0{,}25 & 0{,}75 \to 0{,}5 & 0{,}75 \to 0{,}75 & 0{,}75 \to 1 \\ 1 \to 0 & 1 \to 0{,}25 & 1 \to 0{,}5 & 1 \to 0{,}75 & 1 \to 1 \end{pmatrix} \tag{5}$$

So every element of matrix corresponds to one type of transition from one state to another. Let us define numbers of individual transition types in the following way:

$$\mathrm{N} = \begin{pmatrix} \mathrm{n}_{11} & \mathrm{n}_{12} & \cdots & \mathrm{n}_{1\mathrm{k}} \\ \mathrm{n}_{21} & \mathrm{n}_{22} & \cdots & \mathrm{n}_{2\mathrm{k}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{n}_{\mathrm{k}1} & \mathrm{n}_{\mathrm{k}2} & \cdots & \mathrm{n}_{\mathrm{kk}} \end{pmatrix} \tag{6}$$

To gain transition matrix of Markov chain it is necessary to have row sums equal 1 (100%). This can be achieved by normalizing of matrix N:

$$\mathrm{p_{kl}} = \frac{n_{kl}}{\sum_{k=1}^{l} n_{kl}} \tag{7}$$

Then the transition matrix of Markov chain is received:

$$\mathrm{P} = \begin{pmatrix} \mathrm{p_{11}} & \mathrm{p_{12}} & \cdots & \mathrm{p_{1k}} \\ \mathrm{p_{21}} & \mathrm{p_{22}} & \cdots & \mathrm{p_{2k}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{p_{k1}} & \mathrm{p_{k2}} & \cdots & \mathrm{p_{kk}} \end{pmatrix} \tag{8}$$

The transition matrix expresses probabilities of transition from one state to another. For example it is possible to say that probability of transition from state 0 to state 0 would be here p_{11}. Practically, this means if the value of variable x_1 significantly decreased in observation $n-1$, it will significantly decrease again in observation n with the probability p_{11}. These probabilities are expressed only for the following observation. It is, however, possible to calculate limiting probabilities of the states. This can be done by solving following the system of linear equations (expressed as an augmented matrix of system of linear equations using coefficients from transition matrix):

$$\left(\begin{array}{cccc|c} \mathrm{p_{11}}-1 & \mathrm{p_{21}} & \cdots & \mathrm{p_{k1}} & 0 \\ \mathrm{p_{12}} & \mathrm{p_{22}}-1 & \cdots & \mathrm{p_{k2}} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathrm{p_{1(k-1)}} & \mathrm{p_{2(k-2)}} & \cdots & \mathrm{p_{k(k-1)}}-1 & 0 \\ 1 & 1 & \cdots & 1 & 1 \end{array}\right) \tag{9}$$

Solving the system of linear equations a basic solution vector is received. This vector expresses limiting probabilities

$$p^{\infty} = (p1, p2, \ldots, pk) \tag{10}$$

where – in case of this experiment – the $k = 5$, because we have 5 different states of Markov chain. These values can be interpreted in the following way. From the limiting (long term) point of view, the variable will lie in state **0** (significant decrease) with the probability p_1, in state **0,25** (slight decrease) with probability p_2 etc. It is hereby possible to gain general structure of the variable (=time series). This structure is based on frequencies of transitions and reflects the structure of original data.

2.4 Creating new values of variable

With the help of the limiting structure **(10)** it is now possible to create new vector of the variable. Values from figure **(10)** will divide the probability spectrum into five differing sections where each of this section represents one of the values $\{0; 0{,}25; 0{,}5; 0{,}75; 1\}$. Boundaries of these sections will be created as an accumulation of values of limiting probabilities. This situation is described in figure 1:

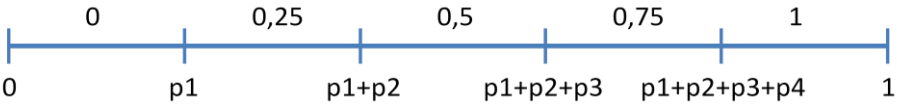

Figure 1 Probability spectrum

To get new values of the variable, a random number from $(0\,,1)$ will be generated for every observation. Generated number will always belong into one of the sections (intervals). Depending on which section it is, we will assign the appropriate value of section $\{0; 0{,}25; 0{,}5; 0{,}75; 1\}$ instead of original values. Thus we receive new variable consisting of values $\{0; 0{,}25; 0{,}5; 0{,}75; 1\}$ instead original values causing multicollinearity. Performing the test **(3)** we find out that multicollinearity is not present anymore. The new five-valued vector will be, of course, different every time we resample the original values. This is caused by randomly generated numbers. Therefore only one replacing of values cannot be considered conclusive. Performing this operation 100 000 times (i.e. 100 000 replacing of values and 100 000 multicollinearity tests) the correlation coefficient of the new variables is 74% lower than of the original variables. This eliminates multicollinearity reliably.

It is not possible to use the new vector practically for regression coefficient estimation because the new values represent original values only in terms of the limiting structure. These values, however, does not really correspond to original values in terms of their meaning. E.g. if the original values actually expressed crop yields in t/ha in a range 5-10 t/ha, then replacing these values with values $\{0; 0{,}25; 0{,}5; 0{,}75; 1\}$ would change the whole meaning of the variable and further estimation of the model parameters would be pointless.

2.5 Backwards transformation

For the obvious reasons it is necessary to transform five-valued vector into original ranges in which the original values were found. This can be done by backward approach of the first transformation of the data. This backward transformation will work with mean of the original data. For every observation, it is necessary to multiply original mean value by appropriate coefficient depending on which value of $\{0; 0{,}25; 0{,}5; 0{,}75; 1\}$ was assigned to the observation in the first transformation. E.g. in *n*-th observation the value **0,75** was assigned. This value originally expressed slight increase (see table 1). The slight increase meant change between 10% and 20%. Let us then generate random change from this interval and add it to the mean value of the original data. In other words we will multiply mean value of the original data by random coefficient generated in $\langle 1{,}1; 1{,}2\rangle$. All five ways of backwards transformation can be found in the following table:

Value assigned after first transformation	Coefficient for multiplying the mean value
0	Random number $\in \langle \mathbf{0,7}; \mathbf{0,8}\rangle$
0,25	Random number $\in \langle \mathbf{0,8}; \mathbf{0,9}\rangle$
0,5	Random number $\in \langle \mathbf{0,9}; \mathbf{1,1}\rangle$
0,75	Random number $\in \langle \mathbf{1,1}; \mathbf{1,2}\rangle$
1	Random number $\in \langle \mathbf{1,2}; \mathbf{1,3}\rangle$

Table 2 Replacing original values with the new ones

Boundary values 0,7 and 1,3 limit the new value from both left and right side to maximum change of 30% to prevent generating of extreme values.

3 Illustrative example

By illustrative example we understand only **one cycle (sample)** of creating new vector of values for **one** variable. Original values of variable where the multicollinearity is present:

Original values to be transformed (x_1)	56	60	59	44	42	50	48	59	59	55	47	40	56	48	60
Variable having unwanted correlation with variable in the first row (x_2)	111	119	116	90	86	98	98	119	117	112	92	79	114	95	121
Transforming x_1 to five-valued function	-	0,5	0,5	0	0,5	0,75	0,5	1	0	0,5	0,25	0,25	1	0,25	1

Table 3 Original variables and transformation to five-valued function

Upon frequencies of the values in the third row it is possible to create the matrix of frequencies and the transition matrix. From the transition matrix we receive liming probabilities vector (rounded) according to figure **(10)**:

$$p^{\infty} = (0{,}17; 0{,}26; 0{,}28; 0{,}06; 0{,}23) \tag{11}$$

By accumulating these limiting probabilities we gain 5 intervals (rounded): $(0; 0{,}17)$, $(0{,}17; 0{,}43)$, $(0{,}43; 0{,}71)$, $(0{,}71; 0{,}77)$, $(0{,}77; 1)$. The generated number from $(0\,; 1)$ will always belong to one of these intervals. According to this, the appropriate value $\{0; 0{,}25; 0{,}5; 0{,}75; 1\}$ will be assigned. The fact that the random number would be exactly the same as one of the interval boundaries is not taken into consideration because the probability of such phenomenon is practically zero. Row of generated random numbers for every observation and appropriate assigned values and new values after backwards transformation are in table 4 (with the original values of x_1 for comparison):

Random number (rounded)	0,58	0,02	0,94	0,57	0,15	0,28	0,84	0,74	0,13	0,46	0,65	0,47	0,77	0,44
Assigned value	0,5	0	1	0,5	0	0,25	1	0,75	0	0,5	0,5	0,5	0,75	0,5
New value	57	41	66	53	37	45	66	58	41	47	53	55	62	48
Original value (without 1st observation)	56	60	59	44	42	50	48	59	59	55	47	40	56	48

Table 4 Transformation of values

In this case the 1st original observation was omitted since there are only 14 transitions between 15 values. Performing the backwards transformation the missing value cannot be easily made up from nothing. It would be

possible to think of the 1st original value as a transition from non-existent previous value, thus the change would be marked as *no change* (0%).

Original correlation coefficient was 99% which means almost perfect multicollinearity. After resampling there is a correlation only 12%. Multicollinearity is therefore eliminated. Means of both original vector and new vector are 51,64 and 52,07 respectively. Performing 100 000 cycles of transformation leads to almost the same values of means.

4 Results

Using the backwards transformation we receive values of the variable very similar to original values. Performing 100 000 cycles of primary and backwards transformation we will receive 100 000 versions of values of new variable. All of these samples bring 77% improvement compared to original correlation coefficient. These samples alone do not have a meaning and it is necessary to work with these samples altogether. One sample alone (i.e. one resampling of 14 values to new values) is based on generating 14 random values and is not sufficiently conclusive. Limiting probabilities can be used only when the number of random experiments is large enough.

Primary objective of regression models is not eliminating multicollinearity but it is estimation of parameters (regression coefficients). It is necessary to identify which of the 100 000 versions should be used for parameter estimation. By the nature of the simulation model it is not possible to choose only one sample, but it is crucial to work with the values altogether. It is possible to use gradually all the samples instead of original values. The mean of values transformed differs from the mean of original values only by $\pm 0{,}03\%$ after 100 000 cycles.

Compared to some other approaches, this simulation brings several differences in terms of remedy for multicollinearity. It is undoubtedly better than those ways based upon reduction of input data. In the proposed solution there is also loss of original data – namely the first observation of the variable. This observation, nevertheless, affects the structure of the new vector anyway. It could be even possible to place this variable back to the vector after the transformation. Placing the one value will not cause the multicollinearity again. Proposed approach does not add any extraneous elements to the original data; it only dissolves the data and then solves them again. This process is repeated 100 000 times which makes original static data somehow dynamic.

5 Conclusion

The proposed method can be used for multicollinearity elimination while not changing original structure of the data in any harsh way. In this algorithm, only two kinds of loss can be noticed. Firstly it is loss of the first value of the variable – which can be, however, acquired again – as described in *results*. The second loss is deviation from original data caused by resampling which is actually negligible. The proposed simulation approach is opened, only the algorithm is consistent. Its parameters can be changed as one desires according to character of input data. This algorithm cannot be used for very small data sets. Such data sets are not suitable for parameters estimation either. The number of values assigned can be changed depending on the structure of the data set. In this experiment, five values were used. Future extension of this experiment is about to show, how many replacing values should be chosen in general and how much this transformation affects further estimation of parameters e.g. in econometrics model.

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DSGE model with housing sector: application to the Czech economy

Miroslav Hloušek[1]

Abstract. This paper deals with multi-sector DSGE model that is estimated on Czech economy data. The model is taken from Iacoviello and Neri (2010). There are two production sectors: consumption/investment goods sector and housing sector. These sectors contain various types of nominal and real rigidities and also different technological trends. On the demand side, there is financial friction in the form of collateral constraint which affects borrowing capacity and consumer spending. The model is estimated using Bayesian techniques. The parameters are economically interpreted, the model moments are compared to moments from data and dynamical properties of the model are studied using impulse responses and variance and shock decompositions. Results show that monetary policy has more pronounced effect on consumption and output when houses are better collateralizable. Consumption shocks, housing technology and housing preference shocks played important role in fluctuation of the real variables while inflation target shocks and cost-push shocks influenced mostly nominal variables. However, recent boom and bust in housing prices was caused primary by housing preference shocks (demand side shocks). Supply shocks were also significant but to much less extent.

Keywords: housing, DSGE model, collateral constraint, Bayesian estimation.

JEL classification: E37
AMS classification: 91B64

1 Introduction

Development in housing market in recent years attracted widespread attention, especially in U.S. where it was thought as trigger of the financial crises. Situation in the Czech Republic was not so severe but connection between housing market and macroeconomy also deserves more detailed examination. It is purpose of this paper. Multi-sector DSGE model with housing market is estimated on Czech data using Bayesian techniques, the data fit of the model is assessed and dynamical properties of the model are examined. The results from variance decomposition show that consumption shocks, shocks to housing technology and preferences and inflation target shocks were the most important driving forces for fluctuations of key macro variables. On the other hand, shock decomposition shows that high prices of houses were caused primary by housing preference shocks; housing technology shocks and consumption shocks contributed only partly. Capability of monetary policy to influence consumption and output heavily depends on loan-to-value ratio. If households have better access to credit, impact of monetary policy to above mentioned variables is substantially increased while impact to inflation is changed only slightly.

The rest of the paper is organized as follows. Section 2 describes the structure of the model, Section 3 briefly comments data and estimation technique. Results of the estimation, data fit of the model and dynamical properties are discussed in Section 4. Final section concludes.

2 Model

The model is borrowed from Iacoviello and Neri [4] and ranks among medium-scale models. Due to shortage of space I describe the model only verbally and refer the reader to the original paper for details. Figure 1 helps for better orientation. There are two types of households: patient (lenders) and impatient (borrowers). Patient households work, consume and accumulate housing. They also own capital and land and supply funds to firms and

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to impatient households. Impatient households also work, consume and accumulate housing, but they are credit constrained and their houses serve as collateral. Variations in housing values affect their borrowing and spending.

Production side of the model economy is divided into two sectors with different rates of technological progress. The non-housing sector uses capital and labor for production of wholesale goods that are subsequently used for production of consumption, business investment goods and intermediate goods. The housing sector produces new houses that are added to existing stock. This sector uses capital, labor, land and intermediate goods. There are nominal wage rigidities in both housing and non-housing sectors and price rigidity in the retail sector. The rigidities are enabled by existence of labor unions and retailers that has some market power and can influence their wages and prices. The rigidities are modeled in Calvo [3] style, the production functions have standard Cobb-Douglas form. Monetary policy follows Taylor rule with interest rate smoothing and attention to inflation and output gap. Several exogenous shocks (mostly following AR(1) processes) are added so that the model can be taken to data.

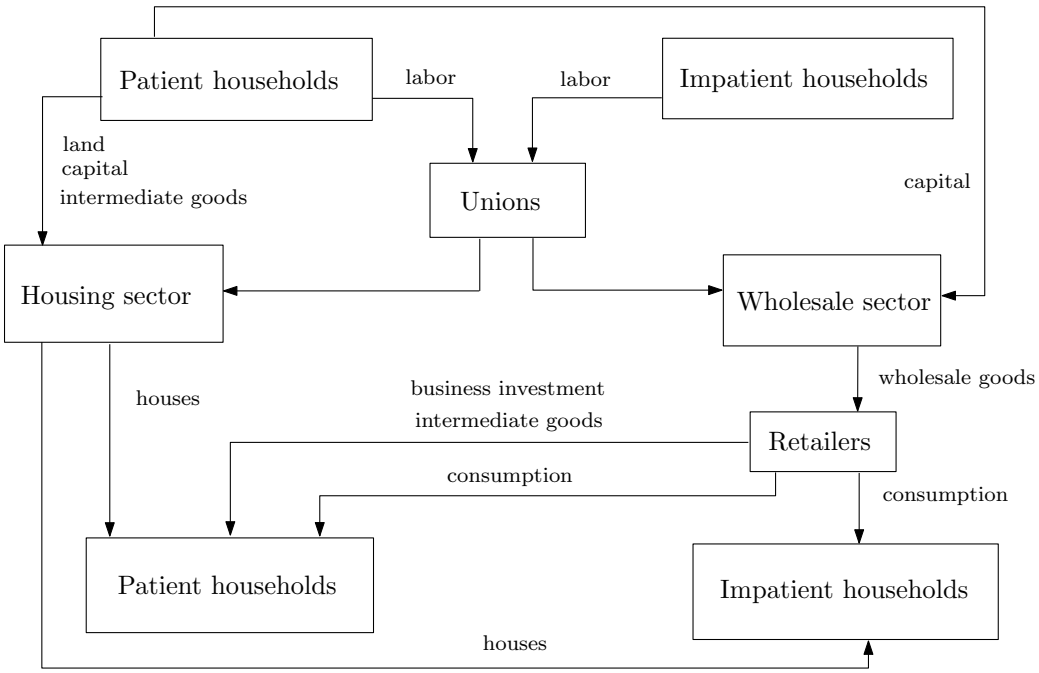

Figure 1 Model structure

3 Data and estimation

The model is estimated using data for following model variables: consumption (C_t), residential investment (IH_t), non-residential investment (IK_t), real house prices (q_t), inflation (π_t) nominal interest rate (R_t), worked hours and wage inflation in housing (NH_t, WH_t) and wholesale sector (NC_t, WC_t) respectively. Time series are quarterly, they are obtained from the Czech Statistical Office and the Czech National Bank databases and cover time period 1998:Q1 – 2012:Q4. The detailed description can be found on web appendix [7].

Some of the model parameters are calibrated according to Iacoviello and Neri [4] and data from national accounts. Specifically, discount factor of patient households was set to 0.9957 which corresponds to the real interest rate (1.7 %) calculated from data. Discount factor of impatient households was set to much lower value 0.97. Loan-to-value ratio (LTV) was calibrated to 0.75 as a combination of estimated values of LTV's for households and entrepreneurs in Hloušek [6]. Full description of calibrated parameters and their values are quoted in [7].

The rest of the model parameters was estimated using Bayesian techniques. Posterior distribution of the parameters was obtained by Random Walk Chain Metropolis-Hastings algorithm. It was generated 2,000,000 draws in two chains with 1,000,000 replications each, 90 % of replications were discarded so as to avoid influence of initial conditions. MCMC diagnostics were used for verification of the algorithm. All computations were carried out using Dynare toolbox (Adjemian et al. [1]) in Matlab software.

4 Results of estimation

This section discusses results of estimation and studies behavior of the model. Table 1 shows prior means, standard deviations and posterior means together with 95 % confidence intervals of selected estimated parameters. Rest of the estimated parameters can be found in [7].

	Prior distribution			Posterior distribution		
Parameter	Density	Mean	S.D.	Mean	2.5 %	97.5 %
Habit formation						
ε	beta	0.50	0.08	0.40	0.31	0.50
ε'	beta	0.50	0.08	0.53	0.40	0.66
Labor income share						
α	beta	0.65	0.05	0.71	0.63	0.78
Calvo parameters						
θ_π	beta	0.67	0.05	0.72	0.67	0.78
$\theta_{w,c}$	beta	0.67	0.05	0.75	0.70	0.79
$\theta_{w,h}$	beta	0.67	0.05	0.69	0.62	0.75
Taylor rule						
r_R	beta	0.75	0.10	0.91	0.89	0.93
r_π	normal	1.50	0.10	1.39	1.23	1.56
r_Y	normal	0.00	0.10	0.22	0.08	0.36
Technology growth rates						
$100\gamma_{AC}$	normal	0.50	1.00	0.42	0.37	0.47
$100\gamma_{AH}$	normal	0.50	1.00	-0.61	-1.00	-0.24
$100\gamma_{AK}$	normal	0.50	1.00	0.13	0.08	0.19

Table 1 Prior and posterior distribution of structural parameters

Parameters ε and ε' express habit consumption of patient and impatient households. The posterior mean of ε is lower than prior and the posterior mean of ε' is slightly higher than prior. Altogether, it indicates quite weak habit in consumption and does not correspond to values usually used for the Czech economy. The labor income share of constrained households $(1-\alpha)$ was estimated to 0.29. This is slightly higher than values found in empirical studies for U.S. economy (0.21) or Sweden (0.18); see Iacoviello and Neri [4] and Wallentin and Sellin [9]. Much higher estimate was obtained by Hloušek [6] for the Czech economy (0.55) and Christensen et al. [8] for Canada (0.38). However, these last two papers used different model structure. Estimated values of Calvo parameters indicate that price and wage rigidities are almost equally important. This is in contrast to empirical studies which found that wages are more rigid than prices; see e.g. Hloušek and Vašíček [5] or Andrle et.al. [2]. The reason can be again different sector structure of the model. Parameters in Taylor rule show that the Czech National Bank pays large attention to interest rate smoothing and also output gap. On the other hand, posterior mean of parameter of inflation r_π is slightly lower than mean of the prior which is usually used in calibrated models.

Estimated parameters γ's together with several model equations[2] can be used for computation of trends in the model variables. The quarterly growth rates for consumption, business and residential investment and real house prices are respectively 0.49, 0.62, -0.33 and 0.82. According to the model, the steep trend in house prices was mainly caused by negative technological progress in the housing sector.

Next step is evaluation of data fit of the model. Table 2 shows moments calculated from data and moments obtained from model simulations (with 90 % probability intervals).[3] Data for the real variables were linearly detrended so that it corresponds to treatment of the variables in the model. The outcome of the model is very good. The volatility of the variables is matched quite precisely, all volatilities fall into probability bands. The model has only minor problems to match high volatility of real house prices q. It is understandable because boom and bust in house prices during recent years was unusual. Regarding correlations the data fit is again very good. The correlations are within the probability intervals although by narrow margin in some cases. The model produces higher correlation between residential investment and output (IH, Y) and residential investment and house prices (q, IH) than the data. On the other hand, correlations of house prices with output and consumption $(q,Y$ and $q,C)$ are lower in the model than in the data. Autocorrelations from model simulation and from estimated VAR(1) model are compared in Figure 2 (again with 90 % probability intervals). The model outcome is quite satisfactory, only

[2] Equations (12) to (15) in Iacoviello and Neri [4].

[3] The model definition of output Y is the sum of consumption and business and residential investment. Corresponding definition is used for the data.

	Data	Model				Data	Model		
		Mean	5 %	95 %			Mean	5 %	95 %
Volatility					Correlations				
C	3.29	4.22	2.70	6.79	C,Y	0.94	0.89	0.73	0.96
IH	11.31	14.63	9.34	20.88	IH,Y	0.03	0.40	−0.14	0.80
IK	8.06	8.00	5.61	11.40	IK,Y	0.95	0.93	0.89	0.97
q	13.49	8.46	4.83	13.52	q,Y	0.86	0.50	−0.16	0.87
π	1.01	1.20	0.98	1.41	q,C	0.84	0.37	−0.35	0.85
R	0.29	0.38	0.23	0.65	q,IH	−0.15	0.40	−0.29	0.83
Y	4.27	5.07	3.30	7.59	q,π	−0.03	0.01	−0.37	0.47

Table 2 Moments from data and model

autocorrelations of consumption up to three lags are much higher in the model than in the data. Also empirical autocorrelations of interest rate and of inflation for one lag lies at margin of lower probability interval.

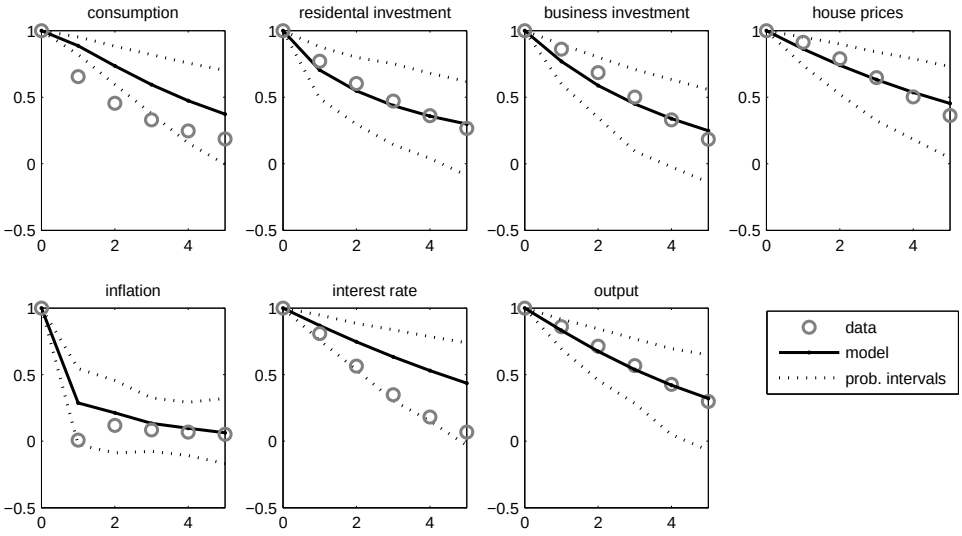

Figure 2 Autocorrelations

The dynamical properties of the model are further studied using impulse responses, variance decomposition and shock decomposition. Figure 3 depicts reaction of model variables to monetary policy shock (increase by one percentage point) for three versions of the model. In all three specifications the estimated parameters are kept at their posterior means. In the benchmark model the loan-to-value ratio (parameter m) is calibrated to value 0.75, in the "high collateral" model the value of LTV is 0.95 which means that constrained households have better position to get loan and in the "no collateral" specification the LTV is set to 0.001 which means that houses are not collateralizable and impatient households are excluded from financial markets. Reactions of the model variables for all three specifications are qualitatively same but they differ in its magnitude, especially for consumption and output. Higher LTV causes larger drop in consumption and output by 4.6 and 2.1 percentage points relative to the benchmark. On the other hand, impact for behavior of inflation is very similar across specifications.[4] It indicates that disinflation policy is more costly when collateral effect is present and is high.

Table 3 shows variance of the model variables explained by each shock. The results are quite intuitive. Consumption technology shocks explain most of the volatility of consumption and housing technology and preference shocks are important for behavior of residential investment (IH) and housing prices (q). Inflation target shocks are mainly responsible for variance of interest rate (R) and together with cost-push shocks also for variance of inflation (π). On the other hand, investment technology shocks are unimportant even for business investment (IK). Output is thus driven primarily by consumption technology shocks.

Finally, Figure 4 depicts historical decomposition of the real house prices into shocks during the estimated period. This figure shows that housing preference shocks became more important since the end of 2001; from

[4]The difference in drop of inflation between specification with high LTV ratio and the benchmark was 0.8 percentage points.

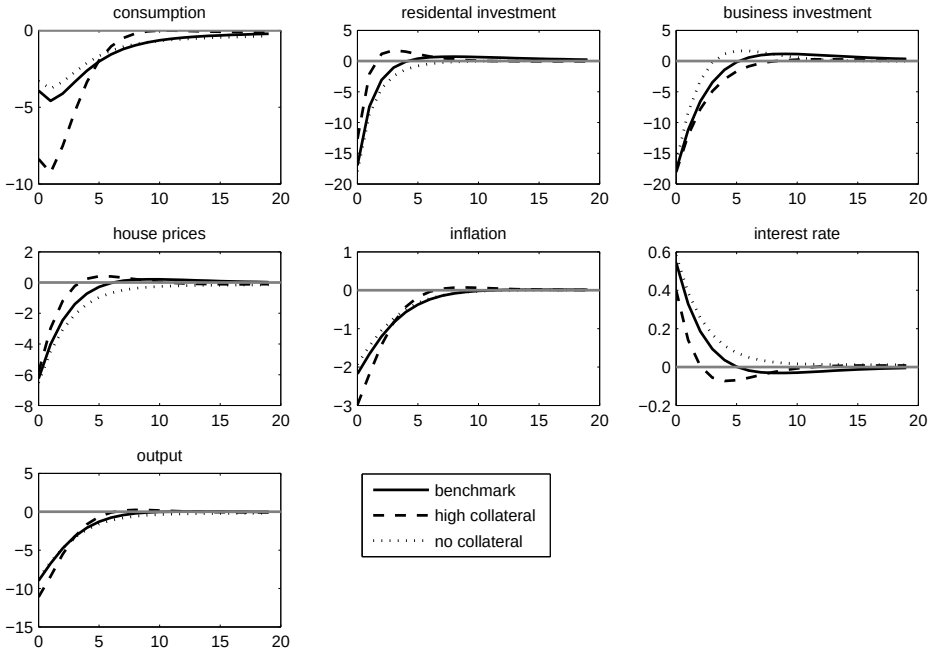

Figure 3 Impulse responses to monetary policy shock

	cons. tech.	monet.	housing tech.	housing pref.	invest. tech.	cost- -push	infl. targ.	labor supply	inter- temp.
C	79.4	4.0	0.1	0.1	0.0	1.6	3.3	4.7	6.8
IH	0.5	0.8	67.9	27.8	0.0	0.1	0.5	2.3	0.3
IK	65.6	9.8	0.6	0.5	0.2	7.3	7.2	6.6	2.3
q	8.5	0.4	38.3	51.6	0.0	0.3	0.3	0.4	0.2
π	8.2	11.5	0.1	0.2	0.0	37.0	38.2	2.5	2.4
R	14.4	2.4	0.1	0.7	0.0	2.0	76.5	1.9	2.0
Y	67.7	5.9	7.1	3.9	0.0	3.1	4.6	6.7	1.1

Table 3 Variance decomposition

this year onwards it was the main determinant of rising prices of houses. The same shock was responsible for subsequent decline during and after the crises. Housing technology shocks also contributed to the development of house prices but in more stable manner. Consumption technology shocks also increased their importance in explaining house prices behavior mainly from 2002. After the peak in 2008 consumption shocks diminish together with decline of house prices. This analysis shows that both demand and supply shocks played important role but mostly demand shocks were responsible for wild behavior of house prices.

5 Conclusion

This paper presented results of estimation of medium-scale DSGE model with housing sector on Czech data. The model fits the data in many aspects quite successfully. Detailed analysis of the model dynamics revealed which shocks explain behavior of the key macroeconomic variables. Consumption shocks, housing technology and housing preference shocks turned out as important shocks for development of the real variables in corresponding sectors. Nominal variables were primarily influenced by inflation target and cost-push shocks. If we look at behavior of house prices, shocks to housing preferences were the main driving force, especially during turbulent period in last ten years. Consumption and housing technology shocks also contributed but they were more stable. Looking at single monetary shock, reaction of consumption and output is much more pronounced if loans to constrained households are more accessible. On the other hand, impact on inflation and other variables can be considered as negligible. These results should be taken into consideration for formation of monetary policy.

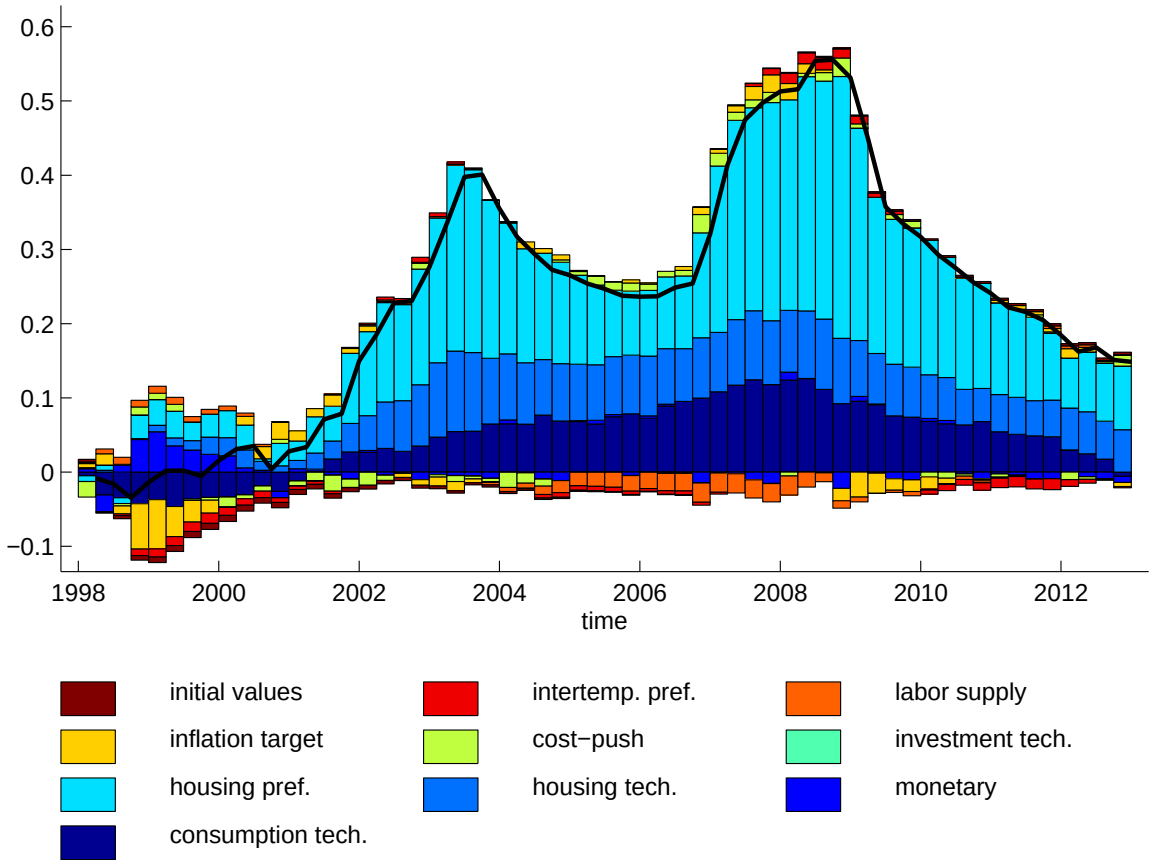

Figure 4 Shock decomposition of housing prices

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Measurement of operational complexity of supplier-customer system using entropy – a case study

Jiří Hofman [1], Ladislav Lukáš [2]

Abstract. The paper concerns with analysis of operational complexity of company supplier-customer relations. Well-known approach for measuring that operational complexity is based upon entropy. However, there are several approaches thereon. In the first part, we discuss various general measures of uncertainty of states, the power entropies in particular. In the second part, we use Shannon entropy as a base framework for the case study – supplier system of the most important commodity in brewery industry, the malted barley. We assume a problem-oriented database exists, which contains detailed records of all product orders, deliveries and forecasts both in quantity and time being scheduled and realized, within time period given. We concern ourselves with quantity perturbations since they play more crucial role as the time variations. The general procedure elaborated consists of three basic steps – preprocessing of data with consistency checks in Java, calculation of histograms and empirical distribution functions, and finally, evaluation of conditional entropy. These two last steps are realized by Mathematica modules. Illustrative results of operational complexity of malted barley supplier system focused on volume quantity variations are presented in detail.

Keywords: business economics, supplier-customer systems, firm performance, complexity measures, information and entropy, power entropy.

JEL Classification: C63, C81, L25, M21
AMS Classification: 91B42

1 Introduction

Business economics knows two types of complexity of supplier-customer systems, a structural complexity and an operational one, in principle. As usual, the structural complexity is defined as static variety of system and their design dimensions, and it describes links among various business units and their hierarchies. It has dominantly a static representation and undergoes time changes usually in long-term periods.

On the contrary, the operational complexity can be defined by uncertainties associated with dynamics of system. Hence, it reflects temporal changes in supplier-customer system, and an operational complexity measure should express behavioral uncertainties of the system during the time with respect to specified levels of control. We know that operational complexity of supplier-customer system is associated with specific data provided by inventory management. It has to record all possible types of flow variations within and across companies in detail, e.g. replenishment time disturbances, deviations of material in/out flows, etc. We refer to [3] for more details relating inventory management.

2 Theoretical background

The theoretical framework for quantification of any system complexity is provided by information theory, in general. We may refer [1] and [4] for more details about Shannon information-theoretic measure and corresponding entropy, and [5] for some generalizations, power information in particular. In [2], we can find more elaborated derivation of classic entropy, which represents the most known quantitative measure of expected amount of information required to describe the state of a system. In general, the complexity of a system increases with increasing levels of disorder and uncertainty of its states.

In [5], we can find some motivating ideas. First of all, the generalization of Shannon entropy which is called power information there, and we shall call it *α-power entropy* $H_\alpha(\pi)$, for any $\alpha>0$. It is given by expression (1).

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$$H_\alpha(\pi)=\sum_{i=1}^{N} \phi_\alpha(p_i)\,,\ \alpha>0,\ \alpha\neq 1 \quad , \quad H_1(\pi)=\sum_{i=1}^{N} \phi_1(p_i)\,,\ \alpha=1 \tag{1}$$

Where $H_1(\pi)$ is the Shannon entropy (here measured in *nats* instead of *bits*, since the binary logarithm $\log_2$ is replaced by natural one). Function $\phi_\alpha(t)$, $0 \le t \le 1$ is so called α-power information function, which is strict concave one with boundary values $\phi_\alpha(0)=\phi_\alpha(1)=0$. Finally, π stands for probability distribution of system, which states are completely described by mutually disjoint events $\{A_1,...,A_N\}$ with probabilities $(p_1,...,p_N)$. Functions $\phi_\alpha(t)$ and $\phi_1(t)$ are given by following expressions.

$$\phi_\alpha(t) = [t(1-t^{\alpha-1})]/(\alpha-1),\ \alpha>0,\ \alpha\neq 1 \quad , \quad \phi_1(t) = \lim_{\alpha\to 1}\phi_\alpha(t) = -t\ln t,\ \alpha=1 \tag{2}$$

Figure 1 shows some graphs of $\phi_\alpha(t)$, for α=0.5, 1, 2, 3, 4, which may be identified in a consequent way being depicted by plain line, thick one, dashed one, dotted one, and plain one, as well.

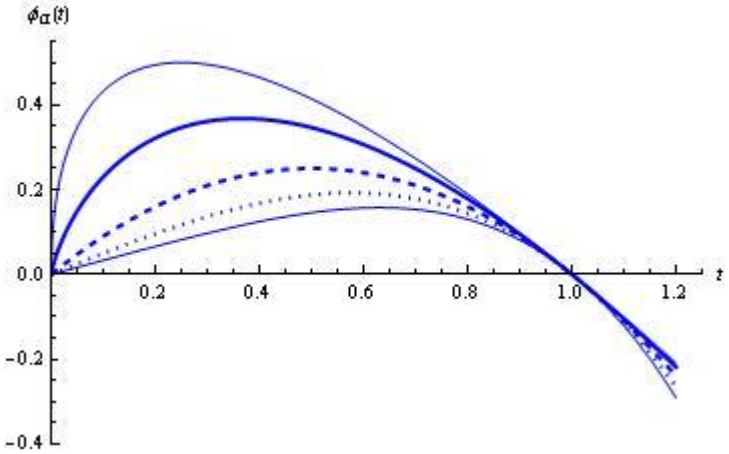

Figure 1 Power information functions, α = 0.5, 1, 2, 3, 4

The strict concavity and zero boundary conditions are preserved by passing to the adjoint α-power information functions $\psi_\alpha(t) = \phi_\alpha(1-t)$, which are plotted on Figure 2 for the same values α=0.5, 1, 2, 3, 4.

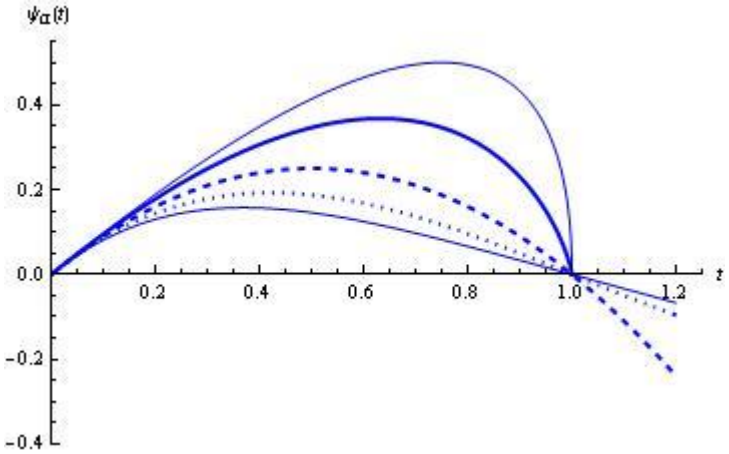

Figure 2 Adjoint power information functions, α = 0.5, 1, 2, 3, 4

As mentioned above, the complexity of a system should intuitively increase with increasing level of disorder and uncertainty of its states, and we know that $H_1(\pi)$ complies therewith. However, such characteristic property is maintained by α-power entropy $H_\alpha(\pi)$, $\alpha > 0$, as well, which is known as the information preservation law:

$$0 = H_\alpha(\pi_D) \le H_\alpha(\pi) \le H_\alpha(\pi_U) = (1-N^{1-\alpha})/(\alpha-1)\,,\ \alpha>0 \tag{3}$$

where symbols π_D and π_U stand for the singular (Dirac) and finite uniform probability distributions of system states, and $H_1(\pi_U)=\ln N$, in particular. The classic entropy, i.e. original Shannon entropy, of π_U is $H(\pi_U)=\log_2 N$, and in the sequel, we denote it simply $I_U = H(\pi_U)=\log_2 N$, too, being already measured in *bits*.

In a similar way to quantity $H_\alpha(\pi)$, one defines an *adjoint α-power entropy* $G_\alpha(\pi)$, $\alpha > 0$ by adopting adjoint α-power information functions $\psi_\alpha(t) = \phi_\alpha(1–t)$.

$$G_\alpha(\pi) = \sum_{i=1}^{N} \psi_\alpha(p_i) = \sum_{i=1}^{N} \phi_\alpha(1-p_i),\ \alpha > 0,\ \alpha \neq 1\ , \quad G_1(\pi) = \sum_{i=1}^{N} \psi_1(p_i) = \sum_{i=1}^{N} \phi_1(1-p_i),\ \alpha = 1 \tag{4}$$

Using (4) we can calculate formulas (5) and (6) for $G_\alpha(\pi_U)$ and $G_1(\pi_U)$, respectively, provided that $p_i = 1/N$, i=1,..,N. We use symbolic calculation power of Mathematica thereon, in particular

```
Sum[(1-1/N)*(1-(1-1/N)^(α-1)),{n,1,N}]//FullSimplify
Limit[(N-1-N*((N-1)/N)^α)/(α-1),α→1]//FullSimplify
```

$$\begin{aligned} G_\alpha(\pi_U) &= \sum_{i=1}^{N} \psi_\alpha(1/N) = \sum_{i=1}^{N} \phi_\alpha(1-1/N) = \left(\sum_{i=1}^{N} (1-1/N)(1-(1-1/N)^{\alpha-1})\right)/(\alpha-1) \\ &= (N-1-N)((N-1)/N)^{\alpha})/(\alpha-1)\ ,\ \alpha > 0,\ \alpha \neq 1 \end{aligned} \tag{5}$$

$$G_1(\pi_U) = \lim_{\alpha\to 1} G_\alpha(\pi_U) = \lim_{\alpha\to 1} (N-1-N)((N-1)/N)^{\alpha})/(\alpha-1) = (N-1)\ln(N/(N-1)),\ \alpha = 1 \tag{6}$$

Using (3), (5), Figures 3 and 4 show plots of values $H_\alpha(\pi_U)$ and $G_\alpha(\pi_U)$ when parameter α ranges [0.1,4.0] and for various number of system states N = 10, 50, 100, 1000. These functions are simply identifiable by plain line, thick one, dashed one, and dotted one, respectively.

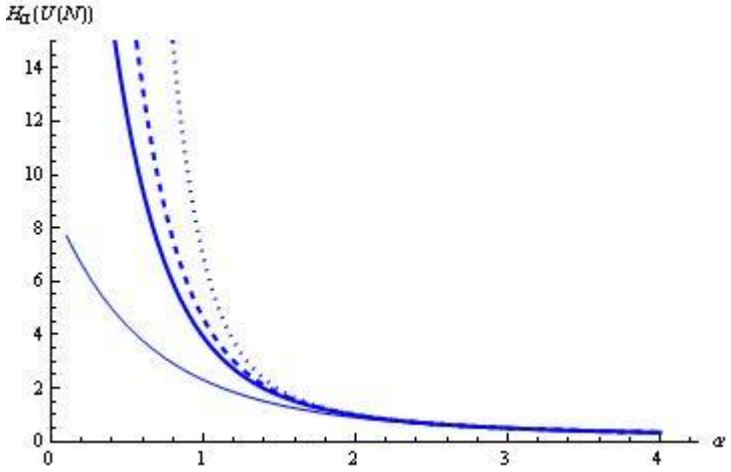

Figure 3 Power entropies of discrete uniform distributions π_U, for N = 10, 50, 100, 1000

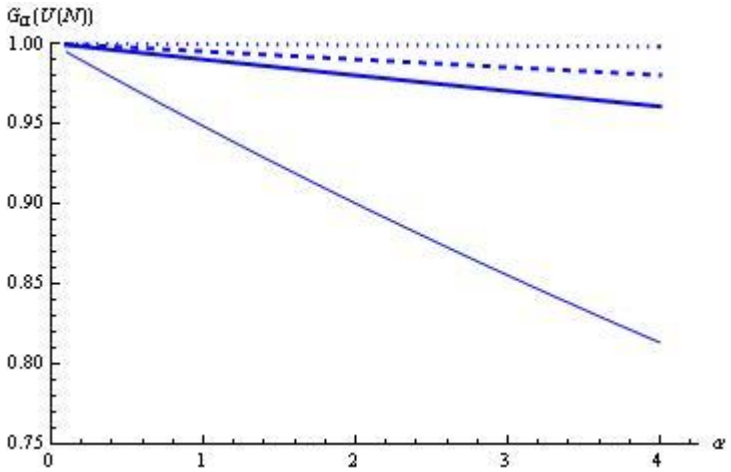

Figure 4 Adjoint power entropies of discrete uniform distributions π_U, for N = 10, 50, 100, 1000

Using Mathematica, we have also calculated several values of $H_\alpha(\pi_U)$ and $G_\alpha(\pi_U)$ for different values of parameter α and small sample space having just N = 10 different system states, see the Table 1. It shows also an

interesting property that $H_2(\pi_U) = G_2(\pi_U)$, which holds for any N. In this case the $H_2(\pi)$ is known as the *quadratic entropy*, see [5]. We can also inspect that functions $\phi_2(t)$ and $\psi_2(t)$ are equal each other, i.e. $\phi_2(t) = \psi_2(t) = t(1-t)$.

α	**0.50**	**0.99**	**1**	**1.01**	**2**	**3**	**4**
$H_\alpha(\pi_U)$	4.32456	2.3293	2.30259	2.27628	0.9	0.495	0.333
$G_\alpha(\pi_U)$	0.973666	0.948744	0.948245	0.947745	0.9	0.855	0.813

Table 1 Selected values of $H_\alpha(\pi_U)$ and $G_\alpha(\pi_U)$, for N=10

3 Operational complexity of supplier-customer system – case study

We refer to [2] and [7] for general framework and more details relating building problem-oriented database and construction of flow variation quantities denoted in general ${}_{(e,r)}Q_k$, and ${}_{(e,r)}T_k$, k=1,...,n, for set of n different products $\{P_1, ..., P_n\}$ being considered. Prefix indices e and r stand for entity and production phase, respectively, (e equals to s for supplier, i for interface, and c for customer, and r equals to s for scheduled, p for actual production, and f, o, d for forecast, order and delivery, in particular).

Our case study is focused on measuring operational complexity of malted barley supplies into known brewery in Pilsen during period 2008Q1-2011Q1. In [4], there are collected the corresponding data and building of problem-oriented database consisting of 20 main suppliers. Since we concern ourselves with volumetric variations between scheduled Q_s and actually produced and delivered Q_p volumes exclusively, we denote this quantity $\Delta Q_{p,s} = Q_p - Q_s$, and we tackle it as a random variable with empirical distribution for each supplier Di, i=1,..,20.

At present, we have two programs for operational complexity analysis. First, `EnComP1mma.java` checks consistency of input data files and generates outputs, which are fetched into the Mathematica notebook `EnComP2mma.nb` for numerical calculations and generation of graphical outputs, as well. The present version implements calculation of entropy and other related quantities by formulas (7).

$$H(\pi) = -\sum_{i=1}^{N} p_i \log_2(p_i) ,$$
$$I_u = H(\pi_U) = -\sum_{i=1}^{N} (1/N) \log_2(1/N) = \log_2(N) , \qquad h(\pi) = H(\pi)/I_u \tag{7}$$

For illustration, we present typical outputs for the suppliers D_3 and D_5 each, which are two largest ones. Figures 5 and 6 show plots of their malted barley delivery variations $\Delta Q_{p,s}$ during 2008Q1-2011Q1 and the corresponding empirical distribution functions with resized definition interval [0,1.1] because of comparison purposes with other ones in general.

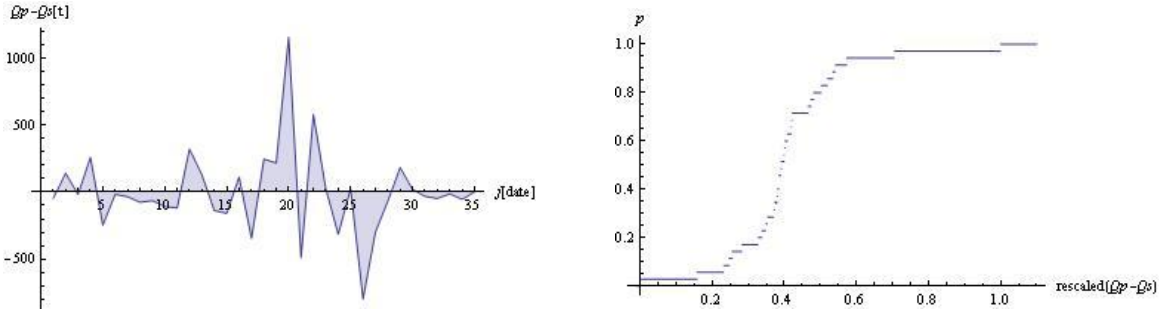

Figure 5 Supplier D_3: Q_p-Q_s – left: delivery variation $\Delta Q_{p,s}$ – right: empirical distribution function

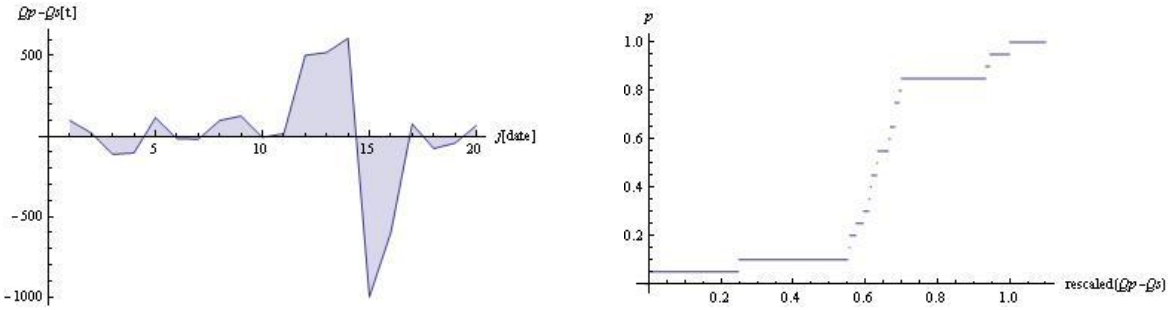

Figure 6 Supplier D_5: Q_p-Q_s – left: delivery variation $\Delta Q_{p,s}$ – right: empirical distribution function

We have selected five major suppliers of malted barley (D_1, D_3, D_5, D_{13}, D_{18}) during 2008Q1-2011Q1 period in particular, which are scheduled and have been ordered to supply more than 58 % of the total quantity desired by the brewery, and which delivered almost 60 % of the total quantity received. The calculated entropies and entropy ratios for this subset of suppliers using formulas (7) are depicted on the Table 2.

supplier	**D_1**	**D_3**	**D_5**	**D_{13}**	**D_{18}**
$H(\pi)$	4.48386	5.015	4.12193	4.31582	4.1066
$h(\pi)=H(\pi)/I_U$	0.965546	0.977719	0.953724	0.918175	0.934951

Table 2 Entropies $H(\pi)$ and entropy ratios $h(\pi)$ for major suppliers (D_1, D_3, D_5, D_{13}, D_{18}) in 2008Q1-2011Q1

The rather high values of entropy ratios $h(\pi)$ express large variability of deliveries, i.e. uncertainties in supplies of such important commodity of any brewery as malted barley. However, relatively small number of particular deliveries of each supplier could also cause such warning values by low size of samples for construction of empirical distributions. Sure, and last but not at least, there is a fact, that top quality malted barley as an important agricultural commodity depends upon seasonal weather, as well, which may support relatively high number of suppliers on another side. Hence, we summarized deliveries of all twenty suppliers D*i*, *i*=1,..,20, in order to analyze their particular delivery contributions.

Figure 7 show plots of total delivery volumes scheduled Q_s and their excesses Q_p-Q_s in [t] of all suppliers D*i*, *i*=1,..,20 in the period investigated, where Q_p denotes volumes received.

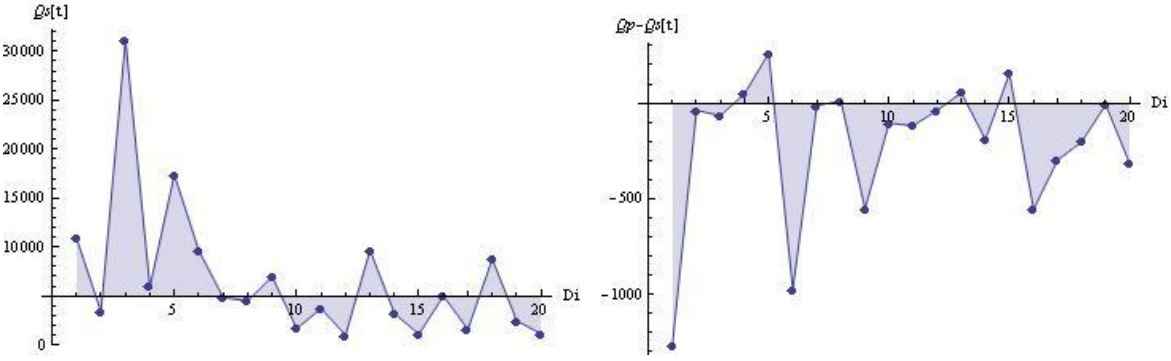

Figure 7 Suppliers D_1-D_{20}: – left: Q_s total volumes scheduled, – right: $\Delta Q_{p,s} = Q_p$-Q_s delivery variations

First, we can clearly confirm an identification of malted barley major suppliers (D_1, D_3, D_5, D_{13}, D_{18}), and further, we directly inspect large variability of $\Delta Q_{p,s}$ if being tackled as a global random variable describing uncertainty of delivery that commodity from the brewery management point of view.

Making total balances of malted barley both scheduled & ordered Q_s and received Q_p by brewery from all deliveries of all suppliers during 2008Q1-2011Q1, we can conclude that only 96.87 % of the requested volume was really delivered, which means that brewery faces a global demand excess 3.13 % of global scheduled & ordered quantity. The global results together with results of major suppliers are summarized on the Table 3 below.

Balance [t]	**Ω_s**	**%**	**Ω_p**	**%**	**$\Delta = \Omega_p - \Omega_s$**	**%**
Sum D*i*, *i*=1,..,20	134 200	100.00	129 998	100.00	–4 202	100.00
$D_1+D_3+D_5+D_{13}+D_{18}$	77 990	58.11	76 776	59.06	–1 214	28.89

Table 3 Malted barley: total volume scheduled Ω_s, received Ω_p, and excess $\Delta = \Omega_p - \Omega_s$ in 2008Q1-2011Q1

Finally, Figure 8 shows relative variations $(Q_p\text{-}Q_s)/Q_s$ given in percents for all suppliers together with 8 % tolerance bands, which are usually applied by the brewery management when making negotiation and settlement of particular delivery agreement with each supplier. We can directly inspect also five out-of-tolerance suppliers during the investigated period: D_1, D_6, D_{16}, D_{17}, and D_{20}.

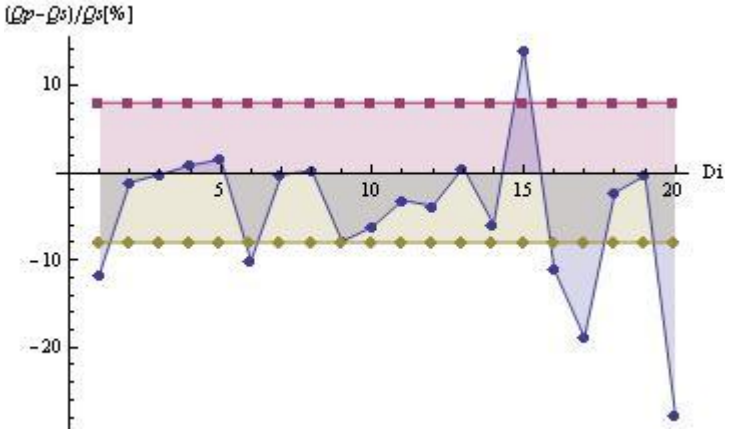

Figure 8 $(Q_p\text{-}Q_s)/Q_s$ [%] total volumes scheduled, suppliers D*i*, *i*=1,..,20

4 Conclusion

The measure of operational complexity based upon entropy provides versatile instrument for supplier-customer system analysis. We have presented not only well-known Shannon entropy, but also a more general approach based upon α-power entropy, which as we hope can open new fields of applications, e.g. case dependent quantitative measuring of various flow variations of as volume as term oriented quantities in supplier-customer systems. In the case study presented, we have analyzed uncertainty of malted barley deliveries into brewery and some aspects related therewith. We focused our research primarily to volume variations of deliveries, since they play a dominant role from the brewery management point of view. In the on-going research, we will try first, to couple volume and term variations together in order to get complex quantitative characteristics, and second, to implement the α-power entropy in our Mathematica notebooks in order to enrich our tools for managerial analysis and decision-making.

Acknowledgements

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Imprecise input data and option valuation problem

Michal Holčapek[1], Tomáš Tichý[2]

Abstract. During last decades the stochastic simulation approach, both via MC and QMC has been vastly applied and subsequently analyzed in almost all branches of science. Very nice applications can be found in areas that rely on modeling via stochastic processes, such as finance. However, as for any other approach, the most crucial step is feeding the assumed model with data and estimation of model parameters. It is a matter of fact that financial processes are instable in time and often switch their regimes. Several scholars therefor suggest to specify some parts of financial models by means of fuzzy set theory. In this contribution the recent knowledge of fuzzy numbers and their approximation is utilized in order to suggest fuzzy-MC simulation to option price modeling in terms of fuzzy-random variables. In particular, we suggest three distinct fuzzy-random processes as an alternative to a standard crisp model and show application possibilities of one of them on illustrative example.

Keywords: Fuzzy numbers, fuzzy random variable, option, simulation

JEL classification: C46, E37, G17, G24
AMS classification: 90C15

1 Introduction

Options, a specific nonlinear type of a financial derivative, play an important role in the economy. In particular, the usage of options allows one to reach a higher level of efficiency in terms of risk-return trade-off, whether through speculation or hedging strategy. The options holder can exercise his right, eg. buy or sell an underlying asset, when he finds it useful. Obviously, it is the case of positive cash flowing from the option exercising. Otherwise the option matures unexploited. By contrast, the seller of the option has to act according to the instructions of the holder. This asymmetry of buyer/seller rights implies the needs of advanced technique for option pricing and hedging. Denoting the underlying asset price at maturity time as S_T we can write the payoff function for European call and put options as $\Psi_T^{(vanilla\ call)} = (S_T - K)^+$ and $\Psi_T^{(vanilla\ put)} = (K - S_T)^+$, respectively.

For example, for the call option price f_t at time $t < T$ it generally holds that $f_t = e^{d_\tau} E[(S_T - K)^+]$, where a discount factor d_τ relates to the probability measure under which the expectation operator E is evaluated and $\tau = T - t$ is the remaining time to maturity. Commonly, E^P denotes the real world expectation (under physical probability measure), while E^Q is used within the risk-neutral world, i.e., where r is a riskless rate valid over time interval τ. Since financial asset prices are often restricted to positive values only, geometric processes are commonly preferred. If, for example, $Z(t)$ denotes a stochastic process for log-returns of financial asset S, e.g., a non-dividend paying stock, to model its price in time we have to evaluate the exponential function of $Z(t)$. It follows that under Q the key formula above can be rewritten into:

$$f_t = e^{-r\tau} E^Q[(S_T e^{r\tau + Z_\tau^Q} - K)^+],$$

where Z_τ^Q is a (potentially compensated) realization of a suitable stochastic process over ? such that it is ensured that S is a martingale.

Actually, the assumption about Z determines suitable approaches for option pricing. The standard ways to option pricing, as well as replication and hedging, dates back to 70s to the seminal papers of Black

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and Scholes [2] and Merton [11] or Boyle [3]. While Black and Scholes [2] and Merton [11](1973) derived their respective models within continuous time by solving partial differential equations (and thereafter called Black-Scholes-Merton partial differential equations) for risk free portfolio consisting of option itself and its underlying asset, Boyle [3] suggested that in order to obtain the (discounted) expectation of the option payoff function the Monte Carlo simulation technique can be useful, i.e., instead of riskless portfolio construction and utilization of no-arbitrage principle the risk-neutral world is assumed. It is a well known result of quantitative finance that these approaches are equivalent under the assumption of complete markets, or, at least, when an equivalent martingale measure exists.

Although the original approaches slightly differ in details the underlying process is derived from Gaussian distribution and all parameters are either deterministic or probabilistic, ie. particular probabilities are assigned to the set of real numbers. However, in the real world, it is often difficult to obtain reliable estimates to input parameters. The reason can be that sufficiently long time series of data is lacking or the data are too heterogenous. Several research papers collected by Ribeira et al. [13] suggested that the fuzzy set theory proposed by Zadeh [16] can be useful for financial engineering problems of such kind.

One of the first attempts to utilize the fuzzy set theory in option pricing dates back to Cherubini [4]. Further results were reviewed eg. by Holčapek and Tichý [8]. However, except recent, but brief contribution of Nowak and Romaniuk [12] there was no attempt to value an option with fuzzy parameters via Monte Carlo simulation approach.

In this paper, we try to fill the gap by suggesting three distinct types of potential underlying processes defined on the basis of fuzzy-random variables. More particularly, we assume (geometric) Brownian motion with fuzzy volatility and (geometric) Brownian motion with time t replaced either by fuzzy process or by fuzzified gamma subordinator, which allows us to redefine in finance well known and commonly used Lévy type variance gamma model [5] in terms of fuzzy-random subordinator. In the next section, we provide brief details about option pricing via plain Monte Carlo simulation within the risk neutral setting. Next, LU-fuzzy numbers and relevant operations with them are defined. After that, three potential candidates to option underlying asset price model are suggested. Finally, a European option price is evaluated assuming all three processes.

2 Cubic rational spline approximation of LU-fuzzy numbers

In the solving of practical problems it is useful to approximate fuzzy numbers by simpler functions. The main reason is that the definition of arithmetic with fuzzy numbers naturally based on Zadeh's extension principle (see [6]) is complex from the computation perspective. Among the basic approximations of fuzzy numbers there belong well known triangular or rectangular approximations (see, e.g., [1] and references therein). Note that the arithmetic with such kind of fuzzy numbers is computationally simple, but often does not correspond to the results obtained by Zadeh's extension principle. Therefore, in recent works on fuzzy numbers (more precisely, LU-fuzzy numbers that are expressed in terms of α-cuts systems), a very popular approach to the approximation of fuzzy numbers is based on rational splines (see [14, 15])) that helps to overcome these difficulties. The idea of approximation of fuzzy numbers using rational splines is based on an advanced system of parameters in contrast to three or four applied in the case of triangular or rectangular models, respectively. In the sequel we briefly recall the basic construction of parametrized LU- fuzzy numbers, for details, we refer to [7] (see also [14]).

Let $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_N = 1$ be real numbers for a finite decomposition of the unit interval. Considering the differentiable case, a parametrized LU-fuzzy number A is represented by the following system of vectors of parameters

$$A = (\alpha_i; u_i^-, \delta u_i^-, u_i^+, \delta u_i^+)_{i=0,\ldots,N}, \tag{1}$$

with the data $u_0^- \le u_1^- \le \cdots \le u_N^- \le u_N^+ \le u_{N-1}^+ \le \cdots \le u_0^+$ and the slopes $\delta u_i^- \ge 0$ and $\delta u_i^+ \le 0$, for all $i = 0, \ldots, N$. So, each parametrized fuzzy number is expressed as a finite system of intervals (α_i-cuts) with slopes defined for each endpoint. Note that the use of slopes is crucial here and enables us to complete the remaining α-cuts using the rational spline interpolation and to obtain a special case of LU-fuzzy number. It is easy to see that each LU-fuzzy numbers expressed as an infinite system of α-cuts can be successfully approximated by a finite system of α_i-cuts with suitable slopes for endpoints.

In this paper, we consider the rational cubic splines. Recall that *a piecewise rational cubic Hermite parametric function (spline) $P \in C^1[\alpha_0, \alpha_n]$ with parameters v_i, w_i, $i = 0, \ldots, n-1$* is defined for any

$\alpha \in [\alpha_i, \alpha_{i+1}]$, $i = 0, \ldots, n-1$, by

$$P(\alpha) = P_i(\alpha, v_i, w_i) = \frac{(1-\theta)^3 u_i + \theta(1-\theta)^2(v_i u_i + h_i \delta u_i) + \theta^2(1-\theta)(w_i u_{i+1} - h_i \delta u_{i+1}) + \theta^3 u_{i+1}}{(1-\theta)^3 + v_i \theta(1-\theta)^2 + w_i \theta^2(1-\theta) + \theta^3},$$

where the notations u_i and δu_i are, respectively, the data values and the first derivative values (slopes) at the nodes α_i, $i = 0, \ldots, n$ with $\alpha_0 < \cdots < \alpha_n$, $h_i = \alpha_{i+1} - \alpha_i$, $\theta = (\alpha - \alpha_i)/h_i$ and $v_i, w_i \geq 0$. Note that v_i and w_i are called tension parameters, and if $v_i = w_i$, we obtain the ordinary cubic spline. Here, we suppose

$$v_i = w_i = \begin{cases} \frac{\delta u_{i+1} + \delta u_i}{u_{i+1} - u_i}, & \text{for } u_{i+1} \neq u_i, \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

which guarantees the global monotonicity (see [14]). Further, we restrict ourselves to uniform partitions of the unit interval, i.e., $\alpha_{i+1} - \alpha_i$ is a constant for any $i = 0, \ldots, N-1$. Finally, we express the system (1) in terms of special matrices as follows

$$A = \begin{pmatrix} (u_0^-, \delta u_0^-) & \cdots & (u_N^-, \delta u_N^-) \\ (u_0^+, \delta u_0^+) & \cdots & (u_N^+, \delta u_N^+) \end{pmatrix}.$$

The matrix expression allows us to use some of the basic operations with matrices to introduce operations with fuzzy numbers. First, let us define the operations of addition and multiplication on the set $\mathbb{R}^2$ by

$$(u_1, u_2) \oplus (v_1, v_2) = (u_1 + v_1, u_2 + v_2) \quad \text{and} \quad (u_1, u_2) \otimes (v_1, v_2) = (u_1 v_1, u_1 v_2 + u_2 v_1),$$

where the standard operations of sum and multiplication of reals are used on the right side of the equality. Further, let us denote by $\leq$ the lexicographical ordering on $\mathbb{R}^2$. One could demonstrate that the algebraic structure $\mathbb{R}^2 = (\mathbb{R}^2, \oplus, \otimes, \leq)$ possesses nearly all properties of fields. The only difference is that $(\mathbb{R}^2 \setminus \{(0,0)\}, \oplus)$ does not form an abelian group, because there is no inversion for elements of type $(0, r)$.[1] Obviously, $\mathbb{R}$ can be embedded into $\mathbb{R}^2$ using $f(r) = (r, 0)$.

Thus, parametrized LU-fuzzy numbers can be naturally expressed using matrices over $\mathbb{R}^2$ in the same respect as over an ordered field. Now, we can define the addition and multiplication of parametrized LU-fuzzy numbers as follows (consider the i-th column):

$$(A \oplus B)_i = \begin{pmatrix} (u_i^-, \delta u_i^-) \oplus (v_i^-, \delta v_i^-) \\ (u_i^+, \delta u_i^+) \oplus (v_i^+, \delta v_i^+) \end{pmatrix}_i \quad \text{and} \quad (A \otimes B)_i = \begin{pmatrix} \min\limits_{p,q \in \{+,-\}} ((u_i^p, \delta u_i^p) \otimes (v_i^q, \delta v_i^q)) \\ \max\limits_{p,q \in \{+,-\}} (((u_i^p, \delta u_i^p) \otimes (v_i^q, \delta v_i^q)) \end{pmatrix}_i$$

where min and max are defined with respect to the lexicographic ordering $\leq$ on $\mathbb{P}$. The scalar multiplication (by reals) can be derived using the multiplication of matrices, where each real number r is interpreted as a matrix of the form

$$r = \begin{pmatrix} (r, 0), \ldots, (r, 0) \\ (r, 0), \ldots, (r, 0) \end{pmatrix}.$$

Hence, we simply obtain (consider the i-th column)

$$(r \otimes A)_i = \begin{pmatrix} (r u_i^{sgn(r)-}, r \delta u_i^{sgn(r)-}) \\ (r u_i^{sgn(r)+}, r \delta u_i^{sgn(r)+}) \end{pmatrix}_i$$

where $sgn(r) = +$, if $r \geq 0$, $sgn(r) = -$, otherwise, and the following rules are applied: $++ = -- = -$ and $+- = -+ = -$. It should be noted that all operation introduced above coincide with the operations defined in [14].

[1] Note that $(\mathbb{R}^2, \oplus, \otimes)$ is an MI-field, where MI-fields generalize the concept of field in the respect that a structure of so-called pseudoidentities is admitted. In the case of $\mathbb{R}^2$, the elements of type $(0, r)$ are pseudoindentities and $(\mathbb{R} \setminus E, \otimes)$, where $E = \{(0, r \mid r \in \mathbb{R})\}$, forms an abelian group. For details about MI-algebras, we refere to [9, 10].

3 Random parametrized LU-fuzzy numbers

To apply the Monte-Carlo techniques on problems with uncertainty modeled using parametrized fuzzy numbers, we need the concept of fuzzy random variable which values are expressed by parametrized LU-fuzzy numbers. For simplicity, such kind of fuzzy random variables will be called as random parametrized LU-fuzzy numbers. In [14], the authors proposed to defined random prametrized LU-fuzzy numbers in such a way that both the data and slopes are random variables satisfying all conditions stated for the parametrized LU-fuzzy numbers (see (1). Similarly, we can defined a random matrix (i.e., its values are random variables) with values expressed by parametrized LU-fuzzy numbers.

Definition 1. A random matrix

$$X = \begin{pmatrix} (X_0^-, \delta X_0^-) \cdots (X_N^-, \delta X_N^-) \\ (X_0^+, \delta X_0^+) \cdots (X_N^+, \delta X_N^+) \end{pmatrix} \tag{3}$$

where $X_0^- \leq X_1^- \leq \cdots \leq X_N^- \leq X_N^+ \leq X_{N-1}^+ \leq \cdots \leq X_0^+$ and $\delta X_i^- \geq 0$ and $\delta X_i^+ \leq 0$, $i = 0, \ldots, N$, is called a *random parametrized LU-fuzzy number*.

4 Potential candidates for price modelling

As we have already argued, it can be very difficult to obtain reliable estimates for the parameters (e.g., volatility or intensity of jumps) of the stochastic process $Z(t)$. It is the reason why many researchers suggest to define the underlying process in terms of fuzzy or fuzzy-random variables. In this section, three distinct fuzzy-random models are suggested as potential candidates to describe the option underlying asset price process; in particular, we assume (i) standard market model (Brownian motion) with fuzzy parameter, (ii) Brownian motion with fuzzy subordinator, and (iii) Brownian motion with fuzzified gamma subordinator.

Model 1 (standard market model with fuzzy parameter) Let σ_{LU} be an LU-fuzzy number defined around crisp estimation of σ. Then, we can model price returns by the following fuzzy-stochastic model:

$$Z(t) = \mu t + \sigma_{LU}\sqrt{t\varepsilon}.$$

Model 2 (Brownian motion with fuzzy subordinator) Let x_{LU} be a non-negative LU-fuzzy number centered around t so that it can be a subordinator. Then, we get the following alternative to the common assumption of Brownian motion:

$$Z(t) = \theta g(t) + \xi\sqrt{g(t)\varepsilon}.$$

Model 3 (Brownian motion with fuzzified gamma subordinator) Let g_{LU} be an LU-fuzzy number centered around a random gamma variable. Then, we can get another alternative model by using g_{LU} as a subordinator to the Brownian motion:

$$Z(t) = \theta x_{LU}(t) + \sigma\sqrt{x_{LU}(t)\varepsilon}.$$

5 Results

In order to evaluate the risk-neutral expectation via Monte Carlo simulation, we need to get models of the preceding section into the exponential and choose a proper ω_{LU} such that the complex process will be martingale when discounted by the riskless rate.

$$f_t = e^{-r\tau}E^Q\left[\left(S_T e^{r\tau+Z_\tau^Q} - K\right)^+\right] \approx \frac{e^{-r\tau}}{N}\sum_{i=1}^{N}\left(S_T e^{r\tau+Z_\tau^{Q^{(i)}}} - K\right)^+,$$

where the superscript (i) refers to the i-th scenario from a given probability space.

Comparative results of particular models for various input data are provided in Table 1. Let us assume put options written on stock price index in the form of a mutual fund price. For the illustrative example we derive the input data from the price observations of a Pioneer stock fund over 5 years.

Generally, we assume crisp values of initial price of the underlying asset ($S_0 = 100$), exercise price ($K = 100$), riskless rate ($r = 0$) and maturity ($T = 1$). Model 1 is similar to BS model, except that the volatility of underlying asset price returns is defined as a fuzzy random number over normal distribution $N(0.15; 0.1)$, with $s = 0.15$ being the most commonly observed value. For sensitivity reasons, we also consider $N(0.15; 0.05)$ and $N(0.20; 0.1)$. By contrast, Model 2 (first panel) provides us the results of Brownian motion with subordinator defined as a fuzzy random number over uniform distribution $U(0.5; 1.5)$. Within the model, a symmetry of log-returns can be assumed or it can be relaxed by setting suitable θ to obtain either the positive and the negative skew. Similarly, in the second panel, the fuzzy-option price assuming Model 3 is depicted for fuzzy random gamma process with variance parameter 0.85 allowing again both, symmetry and asymmetry of log returns by setting suitable θ.

Model 1 (BS model with fuzzy volatility σ_{LU}) $S_0 = 100$, $K = 100$, $r = 0$, $T = 1$		
	$(\sigma_m = 0.15)$, $(\sigma_m = 0.25)$	
$\sigma_s = 0.1$	$\begin{pmatrix} (1.22, 227) & (3.10, 209) & (6.76, 7023) \\ (22.56, -138) & (14.66, -161) & (6.76, -7023) \end{pmatrix}$, $\begin{pmatrix} (1.15, 235) & (3.78, 202) & (8.85, 8773) \\ (26.75, -128) & (17.22, -148) & (8.85, -8773) \end{pmatrix}$	
$\sigma_s = 0.05$	$\begin{pmatrix} (2.89, 221) & (5.98, 194) & (10.05, 9849) \\ (19.64, -148) & (14.69, -156) & (10.05, -9849) \end{pmatrix}$, $\begin{pmatrix} (1.04, 208) & (2.90, 207) & (5.73, 35241) \\ (14.33, -155) & (9.28, -180) & (5.73, -35241) \end{pmatrix}$	

Table 1 Output table of pricing algorithm for put options when various models are considered

More complex figure about the distribution of option values due to the particular models can be obtained by inspection of fuzzy-histograms (see Figure 1). Apparently, both subordinators lead to very different results. While fuzz-subordinator makes the distribution clearly decreasing with slight discontinuity for very low values, the fuzzyfication of gamma subordinator leads to some sort of normal distribution.

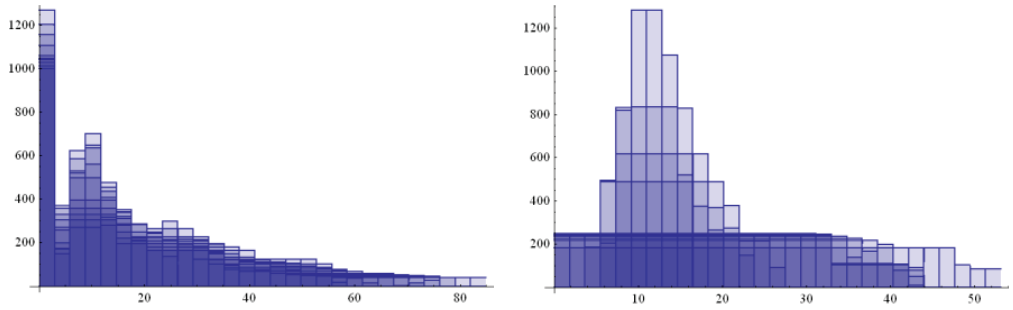

Figure 1 Comparison of fuzzy histograms of option price for both models fuzzy subordinator on the left and fuzzyfied gamma subordinator on the right

6 Conclusion

Many issues of financial modeling and decision making require some knowledge about the future states. However, sometimes it is very difficult to get reliable parametrization of stochastic models. In this contribution we suggested an alternative approach to option valuation problem via Monte Carlo simulation by specifying three distinct types of fuzzy-random processes. Suggested models of financial returns can have very interesting impact on option pricing and hedging. First we should note that the BS option price should always be around the midpoint. The results there-fore indicated that increasing the fuzzy-volatility, we get wider spread of fuzzy-option price. Similarly, we could observed the same effect for additional skewness. Obviously, positive skewness had higher impact. In subsequent research It can be interesting to study the effect of particular parameters on fuzzy-option price, compare it to real market data as well analyze the convergence of fuzzy-Monte Carlo simulation.

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The SGP - faulty by design or made faulty by politicians? An assessment based on simulation model.

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Abstract. By joining the European Monetary Union (the "EMU") member countries lost the ability to use the monetary policy as a tool of macroeconomic regulation. The attention was than focused on the regulation of the fiscal policy and the agreed instrument was the Stability and Growth Pact (the "SGP"). States of the EMU have agreed to satisfy the 3% of GDP for the maximum annual public budgets deficit. Based on the evolution of the public debt in member countries we can say that the SGP has failed as a tool of fiscal discipline. In this paper we answer the question whether the failure was given due to incorrect concept of the SGP or whether the development of the debt was affected more by arbitrary disrespect of agreed rules. The two reasons mentioned above are interdependent. To separate them, we construct a dynamic model of EU countries public debt. By using real data we simulate the potential values of public debt in a situation where the SGP rules had been respected in recent years. Comparing the results for the potential debt given by simulation of the model with the current real values we are able to quantify the impact of non-compliance for each country. The initial results indicate that there are both EU states where non-compliance has led to a negligible increase in public debt - up to 5% of GDP, as well as other states where this factor caused the growth of public debt by more than 25% of GDP.

Keywords: Fiscal policy, primary balance, public debt, European Monetary Union, Stability and Growth Pact, fiscal sustainability

JEL Classification: C63, E62, H62, H63
AMS Classification: 65C20, 68U20

Introduction

One of the motives of the Maastricht Treaty was the establishment of procedures for the functioning of the single currency and the European Monetary Union ("EMU"). By joining the EMU member countries dress surrendered part of their sovereignty and entrust monetary policy in the hands of the European Central Bank (ECB). Fiscal policy remained to them as a tool for macroeconomic stabilization. The need for coordination of fiscal policies is necessary to ensure the functioning of the common currency and the Stability and Growth Pact (SGP) was drawn as a tool of fiscal discipline. It was expected that the Maastricht criteria together with the SGP could ensure fiscal discipline in the EU. However, as we saw in last years, the expectations were not met. Von Hagen and Wolff [7] claim that the introduction of SGP together with Excessive Deficit Procedure (EDP) caused the use of "creative accountability" when member countries were falsifying the statistics or using nonstandard fiscal operations to meet the reference values.

The financial crisis, which has grown into a debt crisis in the euro area, clearly shows how important it is to pay more attention to the state of public debt. Despite Maastricht criteria and the SGP, many states failed to reduce the value of the debt to a sustainable level (Van Nieuwenhuyze [9]). Stoian and Alves [11] in their study also showed that the euro area does not perform such a fiscal policy that would be compatible with the requirements of the SGP.

One of the consequences of the debt crisis is that risk can be determined for each country separately, while bonds are denominated in the common currency. This is in a contradiction with the principles of the EMU, which were established in the 90-ties (Buiter [3]). The economy of the member countries is very heterogeneous and for more, we cannot talk about fiscal federalism in EU. The Euro seems overvalued for the southern European countries (including France) and underestimated for countries in Northern Europe, especially Germany (Duwicquet [5] or Coudert [4]). Financial markets may decide to refuse a loan for some countries, simply be-

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cause the operation would be too risky. Consequently heavily indebted countries may find themselves close to default.

The concept of fiscal sustainability can be defined variously. Blanchard et al. [2] for example, states that sustainable fiscal policy is when the ratio of debt / GDP after some excessive deflection returns to its original value. Afonso [1] simply claims that for sustainable fiscal policy, the government deficits have to be compensated by deficit surpluses in the future. ECB [6] notes that fiscal sustainability is *a country's ability to service all accumulated government debt at any point in time*.

Reference values in the SGP have been established specifically to ensure the sustainable fiscal discipline of the signatory countries. The value of 60% debt / GDP ratio was determined based on the average of members. 3% threshold for the deficit / GDP ratio is based on a simple dynamic model for the public debt. The respect of this value is necessary to avoid a constant growth of debt and the snowball effect.

In our paper, we use the simple public debt dynamic model to answer the question whether the failure was given due to incorrect concept of the SGP or whether the development of the debt was affected more by arbitrary disrespect of agreed rules. The two reasons mentioned above are interdependent. To separate them, we construct a dynamic model of EU countries public debt. By using real data we simulate the potential values of public debt in a situation where the SGP rules had been respected in recent years. Comparing the results for the potential debt given by simulation of the model with the current real values we are able to quantify the impact of non-compliance for each country.

Sotian and Alves [16] used similar model to calculate what level of government surplus would be necessary in order to stabilize public debt. Stoian [12] believes that the economic crisis may be launched due to improper fiscal policy. He uses the dynamic model of debt to determine whether the conduct of fiscal policy is vulnerable or not. Afonso [1] uses the model of a stationary test to count fiscal sustainability of all Eurozone member countries and like most authors he agrees that the implementation of fiscal policy is not sustainable. By using the model Izák [8] confirms that the government that runs budget deficit faces significantly higher costs of borrowing. Van den Noord [10] attempts to quantify the costs of financial crisis and the impact on the sustainability of public finances.

1 Methodology and data

In the following part of the paper we present a simulation model to assess the impact of deviating the terms agreed in SGP. When constructing the model, we assume the literature of fiscal sustainability, especially the already mentioned works by Blanchard et al. [2] or Izák [8]. This chapter contains also a brief description of the data used in the model.

We use two basic indices for the variables. The upper always marks a year for which the value is calculated or simulated. For the lower index we have two possibilities. In the first case, the lower index of variables is missing; than the variable is calculated from the available data (see next section). In the second case the lower index S is used to describe the simulated variable which is determined as described below.

The root of model (1) is based on the equation of time for the public debt, which stipulates that the level of public debt in current year (marked as B^i) is equal to the level of debt in the previous year and this year's change in debt.

Ideally, the change in this year's debt (equation 2) is equivalent to the level of public deficit in given year (D^i). Afonso [1] draws attention to the fact that the transfer of public deficit into debt is not necessarily time-consistent within one year. He also notes that the government policy can raise the debt by extra budgetary ways. Therefore we introduce the variable ADJ^i that captures all situations in which the growth of public debt in the reference year is different from the size of the public deficit.

The public deficit (in equation 3a) can be expressed as the sum of the primary balance (PD^i) and interest on the public debt (IB^i). At the same time we follow the standard practice of using capital letters to indicate nominal values of the monitored variables and lowercase letters denote their share in gross domestic product (see equation 3b).

$$B^i = B^{i-1} + \Delta B^i \tag{1}$$

$$\Delta B^i = D^i + ADJ^i \tag{2}$$

$$D^i = PD^i + IB^i \tag{3a}$$

$$d^i = \frac{D^i}{GDP^i} \tag{3b}$$

In the real economy the amount of interest on public debt depends on a combination of instruments used by the government to finance the debt and their cost. We use a simplified relationship in the model (equation 4) to show that the amount of interests in a given year depends on the amount of the debt at the end of the previous year and the average implicit interest rate (r^i) of the debt. In equation 5 is expressed in annual real growth of gross domestic product (y^i) a nominal growth adjusted for inflation (π^i).

$$IB^i = r^i * B^{i-1} \tag{4}$$

$$y^i = \frac{GDP^i}{GDP^{i-1}} - 1 - \pi^i \tag{5}$$

The rules of the SGP assume that the public deficit in the medium term has to converge to zero and must not exceed 3% of GDP in a given year. While we cannot clearly interpret what „the public deficit in the medium term has to converge to zero „ means, the 3% limit is easily controllable.

If the conduct of budgetary policy does not respect this limit, we cannot blame the approved rules of the SGP for the consequences. The equation 6 therefore provides a basic assumption on which the model stands, namely that the value of the public deficit in each year should be lower than 3% of GDP.

As such, we simulate this policy in the form of the ratio indicator (d_S^i) or nominal indicators (D_S^i) in the equation 7. The S indicates that these values are simulated to satisfy the rules of the SGP and at the same time to not reach worse values than achieved in actual policy.

$$d_S^i \leq 3\% \tag{6}$$

$$d_S^i = \frac{D_S^i}{GDP^i} \tag{7}$$

Assuming that the simulated policy of the state in each year complied with the rules of the SGP and it would not achieve the worse values than realpolitik, it is expected that the simulated level of public debt B_S^{i-1} should be reduced, or at least at the same level as the actual debt. In this situation, however, the level of simulated interests on the public debt IB_S^i was also on the lower or the same level as for the actual level of public debt. We assume (see equation 8) that the simulated level of interests on public debt is due to the simulated level of public debt and the amount of the implicit interest rate.

$$IB_S^i = r^i * B_S^{i-1} \tag{8}$$

Simulated level of public deficit in a given year is determined by simulated interest on the public debt and the amount of simulated primary deficit (equation 9a and 9b). If the government succeeds to maintain the public deficit below the reference value of 3% of GDP, there is no reason for the simulated primary balance to be different from the real state. However, if the budgetary policy of the government does not comply with the rules of the SGP we simulate the primary balance so that it reaches together with the simulated interest on public debt the worst permissible level - 3% of GDP. Another constraint for the simulated primary balance says that it cannot be worse than the actual primary balance (such term is described in equation 9a).

$$if\,(d^i \leq 3\%; PD_S^i = PD^i; PD_S^i = MIN(PD^i; 3\% * GDP^i - IB_S^i)) \tag{9a}$$

In this situation, the overall deficit is equal to the sum of the simulated primary balance and simulated interest on public debt. This deficit never exceeds 3% of GDP in the given year and is in accordance with the rules of the SGP. The rules, however, allow an exception for the 3% reference limit in case when the real GDP growth is negative or the annual growth is slowed by 2 percentage points. For such situations the model allows that simulated value of public budget deficit is equal to the actual value of the primary balance and simulated interest on the public debt (this exception is described in equation 9b).

$$if\ ((y^i < 0 \vee (y^i - y^{i-1}) < -2\%); D_S^i = PD^i + IB_S^i; D_S^i = PD_S^i + IB_S^i) \tag{9b}$$

Equations 1 and 2 are used to describe the decomposition of actual government debt, equations 10 and 11, then return to compose the simulated level of debt. Equation 10 therefore says that the simulated increase in public debt is equal to the simulated deficit and "time discrepancies" that we have excluded in equation 2 from other calculations. Finally, equation 11 describes the simulated amount of debt at the end of the budget period is equal to the simulated value of the debt at the beginning of the period and the simulated growth of public debt in the given period.

$$\Delta B_S^i = D_S^i + ADJ^i \tag{10}$$

$$B_S^i = B_S^{i-1} + \Delta B_S^i \tag{11}$$

In order to perform the above mentioned calculations, it is necessary to set the period from when the quality of budgetary policy of the government will be followed up. Due to the date of the introduction of the Stability and Growth Pact and the availability of the necessary data for each country, the first simulation for the public deficit is for the year 1996. At the same time, we set of the same level for simulated and actual indebtedness in 1995 (equation 12). The most recent data in our model used from the Eurostat server are for the year 2012. We evaluate the quality of fiscal policy of governments in the past 17 years, when we compare the amount of real and simulated level of public debt (equation 13). For reasons of comparability we use always a GDP ratio.

$$B_S^{1995} = B^{1995} \tag{12}$$

$$A1_j = \frac{B^{2012} - B_S^{2012}}{GDP^{2012}} \tag{13}$$

As already mentioned the model is based on Eurostat data for the period 1995 - 2012 and assesses the quality of budgetary policies of national governments in 1996 - 2012. Unfortunately, not all data are available in time series 1995 - 2012. In the case of Romania the first data are available in 1996, in Bulgaria in 1998 and in Denmark, Poland and Greece up from 2000. In these cases, the evaluation period of fiscal policy is shorter. In the model, the following data from the Eurostat server were used: gross domestic product, public debt, public deficit and public debt interest costs. All other values are calculated according to the equations mentioned above. The values for inflation are calculated from the development of the HICP annual average.

2 Results

The main results are shown in the following Figure 1 The dark column reveals the amount of debt in 1995, the hatched column shows the change in the amount of debt up to 2012 in the situation when states would implement responsible fiscal policy in the sense that they follow the rules of the SGP. The black indicator A1 (values on the secondary axis) indicates the effect of failure to comply with these rules to change of the level of deb. The states in the chart are sorted according to this indicator. Finally, the sum of all three values (green dot) gives the actual level of debt in 2012. All values are given in% of GDP.

Even in 1995, the debt of the most of the countries was below or near the level of the Maastricht convergence criteria (60% of GDP). The average value (weighted average with weights according to the economic performance of each state) for the above-mentioned 27 countries reached 66.2% of GDP. Even then the states such as Greece (94.0% GDP), Italy (120.9% of GDP) and Belgium (130.2% of GDP) protruded above that average.

If states apply responsible budgetary policy, it is surprising that the greater part of them would meet the increase of public debt. The average increase in the value of the above-mentioned 27 countries is 9.0% of GDP. In some countries the application of responsible budgetary policy brings excepted results i.e. the decrease in public debt. The indicator A1 shows the effect of non-compliance with the agreed rules on the increase in the debt level. A1 for the budgetary responsible countries reaches up to 10% of GDP (countries on the right of the chart), for less fiscally responsible countries the indicator reaches 30% and more. (the left part of the chart). The average value of this indicator monitored for 27 countries is 11.4% of GDP. This indicator thus explains the 56% of increase in the average debt of the monitored countries for the period 1995-2012. Let's assume that the rules of the SGP do not allow derogation from the responsible budgetary policy in case of economic decline. The average value of A1indicator would be 15.5% of GDP and explains 76% of increase in the average debt.

As we said the indicator A1 is able to quantify the extent to which the budgetary policy was responsible in the period 1995 – 2012. It may be surprising that in the case of disciplined countries (the right side of the chart) the impact on the reduction of the debt was greater than in undisciplined countries. The graph shows that these states would not succeed to reduce the public debt even if their budgetary policies were responsible. In contrary, the level of public debt would decrease in all of them except Hungary who is not a Eurozone member. It is hard to explain this paradoxical situation, our next paper will be focused on detailed analysis. Hagen and Wolff [7] offer a partial explanation by saying that: "SGP rules have induced governments to use stock-flow adjustments. This tendency is especially strong for situations when reducing the deficit is particularly large". Governments that had problems to deal with public deficits (and therefore with the rules of the SGP), probably solved this situation with using the stock-flow adjustment (creative accounting) to avoid the accusation of not fulfilling the agreed rules.

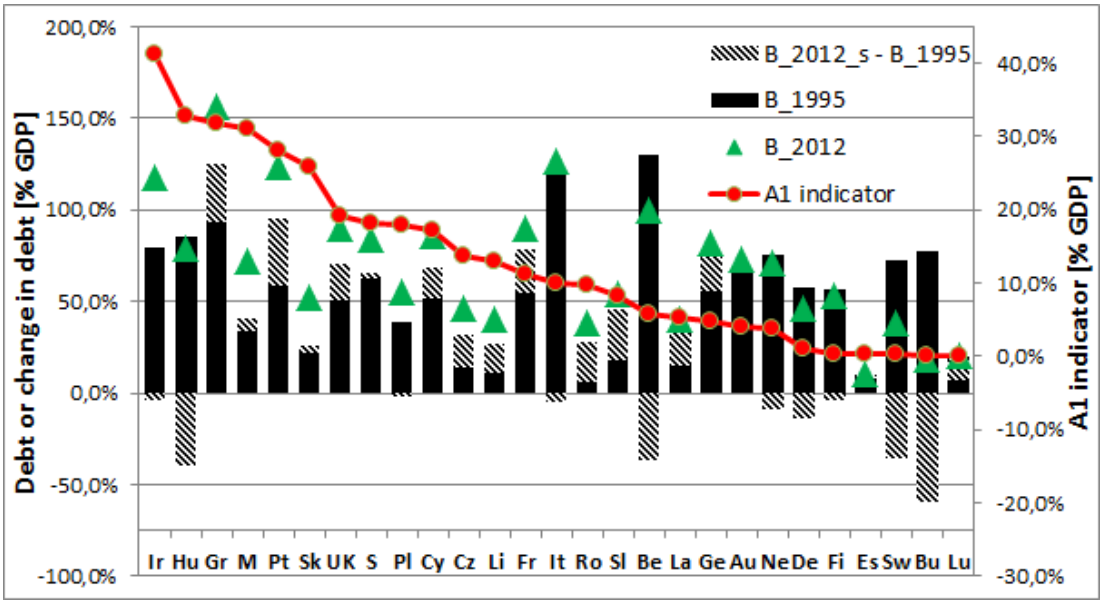

Figure 1 Values for debt, change of the debt and A1 indicator of EU member countries (% of GDP)

Another explanation that can be given is the existence of exceptions to the SGP. We take in to account only the influence of the economic downturn. The annual decline of more than 2% does not require maintaining the deficit below the 3% of GDP. The results of our model show that this is the case especially for the counties from the left side of the chart. Our model shows that the average implicit interest rate on public debt was still close to 8% in 1996 (see Figure 2).The greater frequency of situations when the countries can avoid the agreed rules causes that the effect of budgetary responsible policy will be outweighed by influence of the period when the states do not pursue the level of deficit.

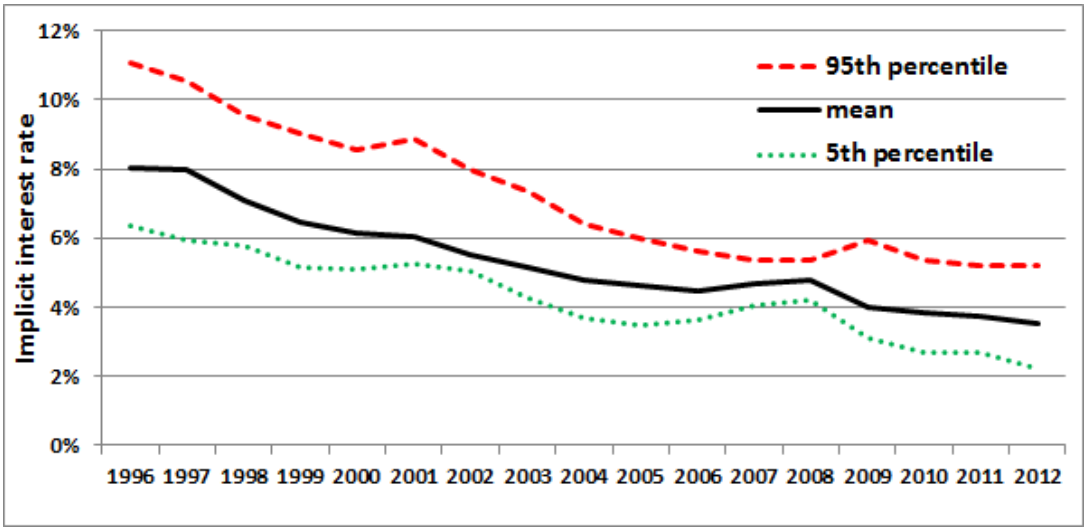

Figure 2 The average implicit interest rate and the development of values corresponding to the 5th and 95th percentile

Along with the average public debt around 60% of GDP, this meant that countries with higher levels of debt had to achieve primary surpluses to thrive comply with the Maastricht convergence criteria for entry into the Eurozone. Their settings - maximum 3% deficit and 60% debt of GDP - were adopted with regard to the implicit interest rate on public debt in the amount of 5%. One of the positive effects of the introduction of the common currency was the convergence of interest rates and their reduction to below 4%. As the figure shows, the greatest degree of convergence was achieved in 2008, from that year due to the financial and economic crisis, interest rates diverge more. The situation of lower interest cost and primary surpluses in the countries could be used to sharp reduction of fiscal imbalances. But it turned out that one component of the deficit (interest costs) was continuously substituted by another component of the deficit (primary deficit).

The results of our previous research indicated that good intentions about themselves are not sufficient to ensure good outcomes of the budgetary policy. The rules of the Stability and Growth Pact allowed the short-term deficit financing. In the medium term the public finances should be balanced, however it never happened in the most of the countries. Even the application of the maximum allowable limit for the deficit was not observed by some states. According to our calculations, even if the fiscally responsible policies took place between 1995 and 2012 we would observe an average increase in the level of public debt by 9.0% of GDP. The disrespect of the 3% rule caused the increase in the level of public debt of 11.4% of GDP in the reference period. We should also notice the influence of variable stock-flow adjustment, which in some countries was used to conceal the real state of public finances. The rules of SGP were set simply and uniformly so they led to the decline in interest costs on public debt. But the countries are not forced to profit from this favourable situation and did not reduce their public deficit. If we summarize the above, we must conclude that the actual setting of the SGP not guarantee adequate decrease or stabilization in the level of public debt. This effect was further enhanced by non-compliance with agreed rules - both by Member States and by the European Commission, respectively representatives of ECOFIN who did not enforce the respect of SGP.

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The spectral properties of the lag operator

Richard Horský[1]

Abstract. In the theory of stochastic processes some operators are employed. Among them the lag operator plays remarkable role. It enables us to get formally familiar descriptions of the stochastic processes. On the other hand we can deal with this operator as the object of the functional analysis. This approach provides more particular view on the problems connected with the convergence of stochastic processes. We will analyze the spectral properties of the lag operator in different spaces.

Keywords: stochastic process, stationarity, lag operator, spaces of sequences, spectral analysis, spectral radius.

1 The stochastic process, its convergence and stationarity

The space $L_2(\Omega,\pi)$ is the space of all functions defined (almost everywhere) on the measurable space Ω equipped by the probability measure π. It is a complete space with respect to the norm derived from the scalar product $(X,Y)=\int_\Omega XYd\pi$, i.e. the Hilbert space. The elements of this space are random variables with a finite mean and a finite variance. The space of all measurable functions defined (almost everywhere) on the measurable space Ω with the probability measure π is usually denoted as $L_0(\Omega,\pi)$. The convergence structure is given by the convergence in probability.

Stochastic process is a mapping $\mathbf{X}: T \to L_0(\Omega,\pi)$, domain of which is the set of all integers *T*. The values of this mapping are random variables $X_t, t=0,\pm1,\pm2,\ldots$ We will write $\mathbf{X}=(X_t)$. The domain *T* is usually interpreted as time set. On the other hand one can only observe results of a stochastic process. The *time series* is a sample realization of such a process from an infinite population of possible realizations. We often talk about a model of time series. The model denotes the same as the stochastic process. It is a form in which the time series is generated.

1.1 Stationary stochastic process

We usually take some assumptions on a stochastic process $\mathbf{X}=(X_t)$ to ensure its „good"properties:

1) All the random variables X_t have the same finite mean: $EX_t=\mu$.
2) All the random variables X_t have the same finite variance: $\sigma^2=DX_t=EX_t^2-\mu^2$.
3) The covariance is invariant with respect to any time shift, i.e. $\mathrm{cov}(X_t,X_s)=\mathrm{cov}(X_{t+k},X_{s+k})$ for all integers *t, s, k*.

Such a process is called *stationary stochastic process*. One often assumes the case of the *normal distribution* within a process. Then the stochastic process is called *normal* and stationarity is satisfied. It is convenient to define the *autocovariance function*

$$\gamma_k=\mathrm{cov}(X_t,X_{t+k}).$$

This function, the domain of which is the time set *T*, is even and so we can regard it only for nonnegative integers. Its value at zero is obviously the variance $\gamma_0=DX_t=\sigma^2$. We can notice that the requirement 1) means that $X_t\in L_1(\Omega,\pi)$ and the requirement (2) that $X_t\in L_2(\Omega,\pi)$. The autocovariance is the scalar product of so called *centered variables* $X_t-\mu$ (its mean is zero) in the Hilbert space $L_2(\Omega,\pi)$.

1.2 The white noise process and general linear process

If a stochastic process has zero mean, a finite positive variance σ_ε^2 and consists of noncorrelated random variables (shocks) ε_t (it means $\gamma_k=0$ for any positive *k*), then it is called the *white noise process*. This is the basic

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stochastic process, from which other models are derived. This process is obviously stationary. The white noise process is an orthogonal system in $L_2(\Omega,\pi)$.

The *general linear process* has the form

$$X_t = \mu + \sum_{n=0}^{\infty} \psi_n \varepsilon_{t-n}, \tag{1}$$

where μ is the common mean of all X_t and (ψ_n) is a sequence of real numbers that are called *weights of the process*, $\psi_0 = 1$. We can suppose w.l.o.g. that $\mu = 0$. The stochastic process with $\mu = 0$ is called the *centered process.* The partial sums of the series in (1) are random variables

$$S_t(n) = \sum_{j=0}^{n} \psi_j \varepsilon_{t-j}. \tag{2}$$

The convergence of the series (1) is regarded as the *convergence in the square mean.* It means

$$E(S_t(n) - X_t)^2 \to 0, \text{ for } n \to \infty. \tag{3}$$

The convergence (3) is derived from the norm of the Hilbert space $L_2(\Omega,\pi)$. We will show that this type of convergence is equivalent to the stationarity of the process (1). If the process is not stationary then the series on the right side in (1), or equivalently the sequence (2) respectively, are not convergent.

Let us take the function $S_t(n) - S_t(m)$, $n > m$. The partial sums (2) are elements of the space $L_2(\Omega,\pi)$ that is complete with respect to the norm $\|X\|_2^2 = E(X^2) = \int_{\Omega} X^2 d\pi$. Hence we obtain

$$E(S_t(n) - S_t(m))^2 = \sum_{i=m+1}^{n} \sum_{j=m+1}^{n} \psi_i \psi_j E(\varepsilon_{t-i}\varepsilon_{t-j}) = \sum_{i=m+1}^{n} \psi_i^2 E(\varepsilon_{t-i}^2) = \sigma_\varepsilon^2 \sum_{i=m+1}^{n} \psi_i^2. \tag{4}$$

Thus if

$$\sum_{n=0}^{\infty} \psi_n^2 < \infty \tag{5}$$

holds, the right side of (4) tends to zero, $\sum_{i=m+1}^{n} \psi_i^2 \to 0$ for $m, n \to \infty$. It follows the sequence (2) is Cauchy in $L_2(\Omega,\pi)$ and hence the series in (1) is convergent. If the condition (5) is satisfied, the linear process is stationary. We can express the autocovariance function of the process (1) in terms of its weights

$$\gamma_k = E(X_t X_{t+k}) = E\left(\sum_{n=0}^{\infty} \psi_n \varepsilon_{t-n}\right)\left(\sum_{n=0}^{\infty} \psi_n \varepsilon_{t+k-n}\right) = \sigma_\varepsilon^2 \sum_{n=0}^{\infty} \psi_n \psi_{n+k}. \tag{6}$$

The condition (5) is equivalent to the fact that the sequence of the weights lies in the Hilbert space l_2. The series on the right side in (6) is a scalar product of the sequences (ψ_n) and (ψ_{n+k}) in the space l_2. With respect to the well-known Schwartz inequality and the fact the l_2-norm of (ψ_n) is equal or greater than the l_2-norm of (ψ_{n+k}) we obtain

$$|\gamma_k| = \sigma_\varepsilon^2 \left|\sum_{n=0}^{\infty} \psi_n \psi_{n+k}\right| \le \sigma_\varepsilon^2 \sum_{n=0}^{\infty} \psi_n^2 = \gamma_0. \tag{7}$$

The inequality (7) proves that the centered process (1) is stationary the autocovariances being upper bounded by the finite variance $\gamma_0 = EX_t^2$ of the centered process (1).

In the end of this section we can notice that a sufficient condition for (5) is the absolute convergence of the series $\sum_{n=0}^{\infty}\psi_n$, in other words

$$\sum_{n=0}^{\infty}|\psi_n|<\infty. \tag{8}$$

The condition (8) is equivalent to the fact the sequence (ψ_n) lies in l_1-space. (8) is only sufficient but not necessary for (5). It is obvious from the trivial example $\psi_n=\frac{1}{n}$.

2 The lag operator

The introduction of the lag operator provides a simplification for formal writing of the stochastic processes at least in the case of models of the type (1). But as we said it is not only formal abbreviation. This operator bears the typical properties of any linear operator similarly to operators of derivative, difference, summation or integral and so on. That shows it is an object which may be studied by the means of the functional analysis.

2.1 The algebra of the lag operator

As to the formal introduction of the notion of the lag operator it is not necessary to solve the problem of its domain and range (see [2] or [3]). However, we cannot avoid to specify these sets since the operator is a mapping after all. The natural access with respect to the definition of the stochastic process or its special case the general linear process seems to define

$$B:\Pi\to\Pi,\quad B(X_t)=(X_{t-1}) \tag{9}$$

for any stochastic process $\mathbf{X}=(X_t)\in\Pi$, where Π denotes the set of all stochastic processes. The mapping B is the so called *lag operator*. We will use this operator in the form $BX_t=X_{t-1}$, which means its application on the particular random variable in the stochastic process. It allows to write some formulae in more comfortable way.

The set Π is in actually the linear space. The addition and scalar multiplication are defined in the analogical way as in the case of the spaces of sequences or functions, it means pointwise. It can be easily seen that the mapping (9) is a linear operator. Its powers, defined by the induction

$$B^0=I\ ,\ B^{n+1}=B\left[B^n\right] \tag{10}$$

for all nonnegative integers n, are also linear operators. The symbol I in (10) denotes the identity on Π. Thus by (10) we have $B^k(X_t)=(X_{t-k})$ or $BX_t=X_{t-k}$. The space of all linear operators on Π forms the algebra (see [1], p.9). The algebra of all (finite) linear combinations of the operators (10) is its subalgebra.

2.2 Algebraic and analytical properties of the lag operator

As to the infinite linear combinations, i.e. the power series in B, we can apply the results from the previous part 1.2. However, the infinite linear combinations are connected with the structure of convergence. We restrict to the stochastic processes which are bounded, i.e.

$$\|(X_t)\|=\sup\{|X_t|:t\in T\}<\infty. \tag{11}$$

The stochastic processes satisfying (11) forms a space that we denote B(T).The function (11) is the norm in this space and this space is complete (Banach) space with respect to this norm. B(T) is a subspace of Π. To avoid formal difficulties we denote the restriction of the operator (9) on B(T) by the same symbol, i.e. B. The lag operator is now continuous (or bounded) operator which norm is

$$\|B\|=\sup_{\|(X_t)\|=1}\|B(X_t)\|=1. \tag{12}$$

To see this it is sufficient to take a constant process $\mathbf{E}=(1)$ which norm (11) is obviously 1 and $B\mathbf{E}=\mathbf{E}$. The same argument may be used to show that any operator (10) has the norm equal to 1. Thus all of the operators

(10) lie in the space [B(T)] of all continuous operators on the space B(T). The algebra [B(T)] contains a subalgebra of all finite linear combinations of the operators (10), i.e. the polynomials in B.

Let us draw our attention to the case of the general linear process (1). Using the lag operator B we can formally write

$$X_t = \sum_{n=0}^{\infty} \psi_n B^n \varepsilon_t = \psi(B)\varepsilon_t$$

at any point $t \in T$. The abbreviation $\psi(B)$ stands for $\sum_{n=0}^{\infty} \psi_n B^n$. Now we realize whether this abbreviation is an operator from [B(T)]. Let us compute

$$\|\psi(B)\| \le \sum_{n=0}^{\infty} |\psi_n| \|B^n\| = \sum_{n=0}^{\infty} |\psi_n|. \tag{13}$$

Hence if the condition (8) is satisfied then

$$\psi(B) = \sum_{n=0}^{\infty} \psi_n B^n \in [\mathrm{B}(T)]. \tag{14}$$

The operator (14) is called *linear filter*. The condition (8) is substantial.

Example 1. If we set $\psi_n = \frac{1}{n}$ then $(\psi_n) \in l_2 - l_1$ and $\psi(B)$ is not continuous operator on B(T). It is obvious if we take $\mathbf{E} = (1)$ then $\psi(B)\mathbf{E} \notin \mathrm{B}(T)$.

Example 2. If we set $\psi_n = \lambda^n$ then $(\psi_n) \in l_1$ iff

$$|\lambda| < 1. \tag{15}$$

The condition (15) is equivalent to the fact $\psi(B) = \sum_{n=0}^{\infty} \lambda^n B^n \in [\mathrm{B}(T)]$. $\psi(B)$ is so called *geometric lag operator*. The name is derived from the fact the sequence of the weights is geometric. Let us notice that the choose $\lambda = 1$ leads to the divergent series (in [B(T)]) $\sum_{n=0}^{\infty} B^n$. This process is called *random walk process.*

Example 3. If we set $\psi_n = 0$ for any $n > q$, q *is* fixed nonnegative integer, then $(\psi_n) \in l_1$ and $\psi(B)$ is obviously continuous operator on B(T). Here $\psi(B) = \sum_{n=0}^{q} \psi_n B^n$ is so called *polynomial lag operator (of the degree q).*

Example 4. Suppose $\psi(B) = \frac{\Theta(B)}{\Phi(B)}$, where $\Phi(B)$ is a polynomial lag operator of positive degree p, $\Theta(B)$ is a polynomial lag operator of degree q and polynomials $\Phi(B)$ and $\Theta(B)$ are coprime. The fraction means the composition of the inverse to $\Phi(B)$ and the operator $\Theta(B)$. The problem of the existence of the inverse operator $\Phi^{-1}(B)$ is meaningful. It is shown e.g. in [4], p.27, that $\Phi^{-1}(B)$ exists iff the polynomial $\Phi(z)$ of the complex variable z has all its roots outside the unit circle. This linear filter is called *rational lag operator*.

In terminology of the classical Box Jenkins methodology the linear filter in the Example 3 defines the process denoted MA(q) (*moving average process of the order q*) or that in the Example 4 defines the process ARMA(p,q) (*autoregressive moving average process of the order (p, q)*) respectively.

Besides the convergence of (1) there is another important problem: problem of its invertibility. Using the terminology of the lag operator it is a question of the invertibility of the operator (14). The operator

$$\pi(B)=\sum_{n=0}^{\infty}\pi_n B^n \tag{16}$$

is the inverse to the linear filter (14) iff $\pi(B)\psi(B)=I$ or equivalently

$$\pi_0=1,\ \sum_{k=0}^{n}\psi_k\pi_{n-k}=0, \tag{17}$$

n positive integer. The problem (17) is a difference equation with the initial condition. It determinates the sequence (π_n) uniquely. The condition (8) formulated for (π_n) assures $\pi(B)\in[\mathrm{B}(T)]$. The important case is that in Example 4 when $\Theta(B)=I$. Then $\pi(B)=\Phi(B)$.The model of the form $\Phi(B)X_t=\varepsilon_t$ is well-known as the *autoregressive model of the order p*, abbreviation AR(p). If (14) is of the type from Example 3 and the polynomial $\Theta(z)$ has all roots outside the unit circle then $\pi(B)\in[\mathrm{B}(T)]$ is of the type of rational lag operator $\pi(B)=\Theta^{-1}(B)$.

3 Spectral properties of the lag operator

The lag operator (9) can be also interpreted in another way. The matrix representation of the lag operator requires another conception of the stochastic process. From now on we will regard the stochastic process as the sequence $\mathbf{X}=(X_t,X_{t-1},\ldots)=(x_1,x_2,\ldots)$. This access is justified by the knowledge of the process up to the time *t* (future values are unknown at present time *t* while the previous one are available). We will consider these process as the elements of the spaces $l_p(L_2(\Omega,\pi)),1\le p\le\infty$, i.e. the spaces as the classical l_p (see [7], p.94) but there is a difference: members of the sequences from $l_p(L_2(\Omega,\pi))$ are functions (random variables) from $L_2(\Omega,\pi)$. We will denote them, nevertheless briefly l_p. These spaces are complete and (except l_∞) separable.

In the final dimension the linear operators are represented by matrices. The situation in ifinite dimension is rather more complicated. However if the spaces are complete and have a countable basis the matrix representation is usual (see [7], p.208).

The lag operator *B* is in this context expressed by and coincided with the matrix

$$\mathbf{B}=\begin{bmatrix}0&1&0&\cdots\\0&0&1&\cdots\\0&0&0&\cdots\\\vdots&\vdots&\vdots&\ddots\end{bmatrix}. \tag{18}$$

We will make a classification of the spectrum of (18).

3.1 The elements of the spectral analysis

If we deal with a spectrum of a linear operator $A:V\to V$ we study the properties of the operator

$$\lambda I-A, \tag{19}$$

where λ is a scalar (real or complex). The set of all values λ for which the inverse of the operator (19) exists and its range is dense in *V* is called the *resolvent set of the operator A,* and is denoted $\rho(A)$. Its complement (in the field of scalars) is called the *spectrum of the operator A*, and is denoted $\sigma(A)$. The basic properties of these sets are those $\rho(A)$ is an open subset of the field of scalars, $\{\lambda:|\lambda|>\|A\|\}\subset\rho(A)$, and $\sigma(A)$ is a closed set for which $\{\lambda:|\lambda|\le\|A\|\}\subset\sigma(A)$. The number $r(A)=\sup_{\lambda\in\sigma(A)}|\lambda|$ is called the *spectral radius of the operator A.* It generally holds $r(A)\le\|A\|$. The classification of the spectrum of the operator *A* is as follows:

1. The *point spectrum* $\sigma_P(A)$ is created by the points λ for which $(\lambda I-A)^{-1}$ does not exist. Its elements are so called *eigenvalues of the operator A.*
2. The *continuous spectrum* $\sigma_C(A)$: $(\lambda I-A)^{-1}$ exists but it is not continuous and the range of (19) is dense in *V.*

3. The *residual spectrum* $\sigma_R(A)$: $(\lambda I - A)^{-1}$ exists and the range of (19) is not dense in *V*.

3.2 The spectrum of the lag operator

The operator *B* is coincided with the matrix (18). Suppose $V = l_p$, $1 \le p \le \infty$. The kernel of *B*, i.e. $\{x \in V : B\mathbf{X} = \mathbf{0}\}$, consists of the vectors $\mathbf{X} = x_1\mathbf{E}_1$, where $\mathbf{E}_1 = (1,0,0,\ldots)$. If $\mathbf{Y} = (y_n) \in V$ then $\mathbf{Y} = B\mathbf{X}$ for any $\mathbf{X} = (x_1, y_1, y_2, \ldots)$, x_1 any scalar, and thus the range of *B* is the whole space *V*. Spectral analysis starts by the claim that $\lambda \in \rho(B)$ for any $|\lambda| > 1$. Let us consider the geometric sequence

$$\Lambda = (1, \lambda, \lambda^2, \ldots). \tag{20}$$

The sequence (20) is the eigenvector of *B* whenever $|\lambda| < 1$. The sequence (20) belongs to the eigenvalue λ. It follows that $r(B) = \|B\| = 1$. If $|\lambda| = 1$ then $\Lambda \in l_p$ iff $p = \infty$. It means that $\sigma(B) = \sigma_P(B) = \{\lambda : |\lambda| \le 1\}$ iff $V = l_\infty$. In any case the spectrum of the lag operator *B* is the compact unit circle (in the field of complex numbers).

In the following consideration we will suppose $V = l_p, 1 \le p < \infty$ and $|\lambda| = 1$. We describe the range of $\lambda I - B$. If $\mathbf{Y} = (y_n) \in V$ is given then the equation $(\lambda I - B)\mathbf{X} = \mathbf{Y}$ has in the components form

$$\lambda x_n - x_{n+1} = y_n, \tag{21}$$

The general solution of the linear difference equation (21) is

$$x_n = .C\lambda^{n-1} - y_1\lambda^{n-2} - \ldots - y_1, \tag{22}$$

where *C* is a constant. The space *W* of sequences with a finite nonzero elements is a dense subspace in l_p, $p < \infty$. If $\mathbf{Y} = (y_n) \in W$, i.e. there is a positive integer N such that $y_N = y_{N+1} = \ldots = 0$, we can find an element $\mathbf{X} = (x_n) \in V$ like this: we choose the constant *C* so that $C\lambda^{N-1} - y_1\lambda^{N-2} - \ldots - y_{N-1} = 0$. This choice leads with respect to (22) and (21) to the sequence $\mathbf{X} = (x_n) \in W$ for $x_N = x_{N+1} = \ldots = 0$. This consideration shows that $\sigma_C(B) = \{\lambda : |\lambda| = 1\}$ in l_p, $1 \le p < \infty$.

3.3 Conclusion

The spectral analysis presented in previous section is in accord with the results given in such monographies as [2], [3] or [4]. There is an interesting result as to the existence (and continuity) of the inverse to the difference operator $\Delta = I - B$ which is employed in the theory of stochastic processes. As we have found $\Delta^{-1} = (I - B)^{-1}$ does not exist on the space l_∞. It shows that the AR(1) process with Δ as the autoregressive operator, i.e. $\Delta X_t = \varepsilon_t$, is not stationary linear process. If we restrict Δ on a l_p, $1 \le p < \infty$, Δ^{-1} exists but it is not continuous. That shows a small perturbation of ε_t has generally a great influence on X_t. However the considered restriction means to give up the concept of stationarity. The stochastic process in l_p, $1 \le p < \infty$, may have a constant variance up to some point in the past, further its variance has to tend to zero. However it is not a problem since an observed time series in distant past is not available.

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The Impact of the Technological Progress on the Development of the Slovak Economy – Neoclassical approach

Peter Horvát[1], Brian König[2], Filip Ostrihoň[3]

Abstract. The following paper introduces a simple RBC model, as a tool for evaluation of the technological progress in the conditions of Slovakia, and its consequent effect on the development of the Slovak economy.
The first section outlines the model itself and contains the derivation of the equilibrium conditions and the steady state values. The second section describes the estimation procedure and the data used for estimation. The Bayesian method was used to estimate the model according to the real values of Slovak GDP and Consumption in order to capture, to the highest possible extent, the complexity of the economic reality with the DSGE model. The final section describes the use of estimated DSGE model in order to assess the impact of technological progress and further evaluate the effect of technological shocks on the selected variables.
According to the estimated model the immediate response of the Slovak GDP to a technological shock, at the magnitude of 1% positive deviation is approximately 1.4% growth of GDP. The impact of the one period shock lasts at least next 12 years, and its aftermath on GDP can be visible for another 13 years.

Keywords: Technological progress, DSGE model, Neoclassical approach, the Bayesian estimation, Slovakia.

JEL Classification: C11, E13
AMS Classification: 91B16

1 Introduction

Based on the neoclassical assumptions the *technological progress* was the main driving force of the per capita GDP growth in developed countries through last several decades. Since Slovakia, as a transitive economy, is seemingly catching up with mentioned countries we can expect stronger influence of the *technological progress* on the economy of the country. The aim of this paper is not to examine the change of the impact of *technological progress* throughout the years, but merely to capture the current state and describe it for possible future analysis.

In the second section we will introduce the model itself, and derive the equilibrium conditions and the steady state values. In following two sections we will describe the estimation procedure and the data used for estimation. We have used the *Bayesian method* to estimate the model according to the actual values of Slovak gross domestic product (GDP) and final household consumption (FHC), in order to capture the complexity of the economic reality, with the mentioned *Dynamic Stochastic General Equilibrium* (DSGE) model to the highest possible extent. In the fifth section we will describe the utilization of the estimated DSGE model to assess the impact of *technological progress* and further evaluate the effect of *technological shocks* on the selected variables. The results of the described analysis are discussed in the final chapter.

2 Neoclassical RBC model

The presented model was inspired by the Kydland and Prescots *real business cycle* (RBC) framework, further methodically reprocessed and described by DeJong and Dave [1]. Therefore we have extensively drawn from DeJong and Dave, particularly relations (1) - (8) and (11) – (16) are contained in the source literature [1]. The model features the labor / leisure trade-off decision process, with the notion of economic growth.

The model assumes a large number of households, which may be aggregated as a representative household. Their aim is to maximize the utility *U*, which yields from the leisure and the consumption of goods. The con-

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sumer also assigns higher utility to the more recent consumption instead of the future consumption. In the model the total consumers' utility is modeled as a sum of discounted instantaneous utility from present to infinity.

$$U = E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{{c_t}^{\varphi} {l_t}^{1-\varphi}}{1-\phi} \right)^{1-\phi} \rightarrow \max_{c_t, l_t} \tag{1}$$

Where E_0 is the expectations operator, conditioned to the present information (time $t = 0$), β is the discount factor, used to asses the future utility, c_t is the particular consumption of goods and l_t is the amount of leisure at time t. The utility itself is distributed according to *Constant relative risk aversion function* (CRRA), where parameter φ determines the ratio of both the consumption and the leisure on the consumer's immediate utility. The parameter ϕ denotes the risk aversion and the intertemporal elasticity of substitution.

The representative household is also capable to produce goods, according to the production technology at its disposal. As an abstraction goods are modeled through single representative good. The production technology is described, according to the framework of neoclassical modeling, as a *Cobb-Douglas production function*.

$$y_t = z_t {k_t}^{\alpha} {n_t}^{1-\alpha} \tag{2}$$

Where the y_t is the level of output or the production of a single aggregated good, z_t is denoting the level of productivity or *technological progress*. The k_t and n_t are the level of capital stock and the level of labor input respectively. The coefficient α denotes the elasticity of capital and, due to constant returns of scale, also the elasticity of labor of the output. Furthermore we assume that the representative household has to its disposal, at each given moment, one unit of time, which the household can distribute into work or leisure.

$$1 = n_t + l_t \tag{3}$$

Where the n_t denotes the amount of time spent with labor and l_t represents the time left for the leisure. Similarly the output generated at time t can be consumed (c_t) or left for the future, to increase the stock of capital, as an investment (i_t).

$$y_t = c_t + i_t \tag{4}$$

The capital stock itself diminishes over time, through deprecation. That's why it has to be replenished through investment.

$$k_{t+1} = i_t + (1-\delta) k_t \tag{5}$$

Where δ denotes the depreciation rate of the capital (k_t). The evolution of *technological progress* is described as a first order autoregressive process (AR1), in order to capture persistence of the technological shock, to some degree.

$$\log z_t = (1-\rho) \log \bar{z} + \rho \log z_{t-1} + \varepsilon_t \tag{6}$$

Where ρ is the autoregressive coefficient and ε_t is the innovation or *technological shock* at the time t. We assume that ε_t can be described as independently and identically distributed IID (0,σ).

2.1 Optimality conditions

The optimality conditions may be derived through rearranging the constraints (2) – (5) to denote the capital stock in the next period (time $t + 1$) as a function of current capital stock, leisure and consumption. As a solution to consumers constrained optimization problem, we've found the set of Euler's equations or intertemporal optimality conditions.

$${c_t}^{(1-\phi)\varphi-1} {l_t}^{(1-\phi)(1-\varphi)} = \beta E_t \left\{ {c_{t+1}}^{(1-\phi)\varphi-1} {l_{t+1}}^{(1-\phi)(1-\varphi)} \left[1 - \delta + \alpha z_{t+1} \left(\frac{k_{t+1}}{1-l_{t+1}} \right)^{\alpha-1} \right] \right\} \tag{7}$$

$$\frac{(1-\varphi)}{\varphi} \frac{c_t}{l_t} = (1-\alpha) z_t \left(\frac{k_t}{1-l_t} \right)^{\alpha} \tag{8}$$

$$k_t = z_{t-1} {k_{t-1}}^{\alpha} (1-l_{t-1})^{1-\alpha} + (1-\delta) k_{t-1} - c_{t-1} \tag{9}$$

If we expand the third intertemporal optimality condition (9) back into constraints (2) – (5), we will obtain the first outline of the used model. For purpose of estimation we've introduced another stochastic variable in equation (7), denoting the expectation error among the observations. The equation (7) may be than denoted as follows.

$$c_t^{(1-\phi)\varphi-1} l_t^{(1-\phi)(1-\varphi)} = \beta \left\{ c_{t+1}^{(1-\phi)\varphi-1} l_{t+1}^{(1-\phi)(1-\varphi)} \left[1 - \delta + \alpha z_{t+1} \left(\frac{k_{t+1}}{1 - l_{t+1}} \right)^{\alpha-1} \right] \right\} \cdot e^{\epsilon_t} \quad (10)$$

2.2 The Steady state

Stable equilibrium may not be achieved for every model, but for this particular model (7), (8), (2) – (6) the steady state does exist and we may compute it analytically when we assume that the steady state value of the $\bar{z}$ is equal to 1. From the first intertemporal optimality condition we may find (under the assumption that in steady state the variables are no longer changing) the steady states value of the quotient of capital and labor and subsequently derive steady state values for the other variables.

$$\frac{\bar{k}}{\bar{n}} = \left(\frac{\alpha}{\frac{1}{\beta} - 1 + \delta} \right)^{\frac{1}{1-\alpha}} \quad (11)$$

$$\frac{\bar{y}}{\bar{n}} = \left(\frac{\bar{k}}{\bar{n}} \right)^{\alpha} = \left(\frac{\alpha}{\frac{1}{\beta} - 1 + \delta} \right)^{\frac{\alpha}{1-\alpha}} \quad (12)$$

$$\frac{\bar{\imath}}{\bar{n}} = \delta \frac{\bar{k}}{\bar{n}} = \delta \left(\frac{\alpha}{\frac{1}{\beta} - 1 + \delta} \right)^{\frac{1}{1-\alpha}} \quad (13)$$

$$\frac{\bar{c}}{\bar{n}} = \left(\frac{\alpha}{\frac{1}{\beta} - 1 + \delta} \right)^{\frac{\alpha}{1-\alpha}} - \delta \left(\frac{\alpha}{\frac{1}{\beta} - 1 + \delta} \right)^{\frac{1}{1-\alpha}} \quad (14)$$

$$\bar{n} = \frac{1}{1 + \frac{1}{1-\alpha} \cdot \frac{(1-\varphi)}{\varphi} \cdot \left[1 - \delta \left(\frac{\bar{k}}{\bar{n}} \right)^{\alpha} \right]} \quad (15)$$

$$\bar{l} = 1 - \bar{n} \quad (16)$$

2.3 Log – linearized model

As a final step of this section we've performed logarithmic approximation of the model, and modeled the denoted variables as the deviations from the steady state. Consequently, we've obtained the final form of the model as followed.

$$0 = \tilde{c}_t - \tilde{l}_t - \tilde{z}_t - \alpha \tilde{k}_t - \alpha \tilde{n}_t \quad (17)$$

$$\begin{aligned} 0 = {} & \big((1-\phi)\varphi - 1\big)\tilde{c}_t + (1-\phi)(1-\varphi)\tilde{l}_t - \big((1-\phi)\varphi - 1\big)\tilde{c}_{t+1} - (1-\phi)(1-\varphi)\tilde{l}_{t+1} \\ & - \log\left[\alpha e^{\tilde{z}_{t+1}} \frac{e^{(1-\alpha)(\tilde{n}_{t+1} + \log \bar{n})}}{e^{(1-\alpha)(\tilde{k}_{t+1} + \log \bar{k})}} + (1-\delta) \right] + \log\left[\alpha \frac{e^{(1-\alpha)\log \bar{n}}}{e^{(1-\alpha)\log \bar{k}}} + (1-\delta) \right] \end{aligned} \quad (18)$$

$$0 = \tilde{y}_t - \tilde{z}_t - \alpha \tilde{k}_t - (1-\alpha)\tilde{n}_t \quad (19)$$

$$0 = \log\left(\bar{n} e^{\tilde{n}_t} + \bar{l} e^{\tilde{l}_t} \right) \quad (20)$$

$$0 = \tilde{k}_t - \log\left(\bar{\imath} e^{\tilde{\imath}_t} + (1-\delta)\bar{k} e^{\tilde{k}_{t-1}} \right) + \log\left(\bar{\imath} + (1-\delta)\bar{k} \right) \quad (21)$$

$$0 = \tilde{y}_t - \log\left(\bar{\imath} e^{\tilde{\imath}_t} + \bar{c} e^{\tilde{c}_t} \right) - \log(\bar{\imath} + \bar{c}) \quad (22)$$

$$0 = \tilde{z}_t - \rho \tilde{z}_{t-1} \tag{23}$$

Where arbitrary variable $\tilde{x}_t$ denotes the deviation of that variable from its steady state $\tilde{x}_t = \log(x_t / \bar{x})$.

3 The Data and their transformation

We have used the data available at Slovak statistical office [4] for the estimation of the model, particularly the time series of real gross domestic product (GDP), real government consumption, real export, real import and real final household consumption (FHC). All mentioned series were available quarterly, from 1996Q1 to 2012Q4. Thus our sample consists out of 68 observations.

According to the model, stated in the previous section, we are abstracting from any governmental or foreign influence. In order to represent this abstraction in the data, we had to remove the governmental and the external balance component of the GDP. Thus we've added import and subtracted government spending and export from the GDP. This way we've obtained GDP component representing only the consumption of households and firms. As a second observed variable we have used FHC, which didn't need any additional adjustments.

Even after obtaining the amended GDP and the FHC values, we still weren't able to utilize them to estimate derived model. As we may see, the model (17) – (23) uses logarithms of the deviations from the steady state values of GDP and FHC, instead of the actual values. Therefore we took logarithms of mentioned variables, and applied seasonal filter (Census X12). Afterwards we've filtered the data using *Hodrick – Prescot* (HP) filter, set to the default settings for quarterly data ($\lambda = 1600$). Through mentioned step we intent to capture the dynamic evolution of the steady state values represented in our model as the trend component of the GDP and FHC series. Further the modeled logarithms of the deviations from the steady state values were represented as the cyclical component, obtained from the HP filter.

4 Model estimation

For the estimation procedure we've used the *Metropolis – Hastings Markov chain Monte Carlo* (MH-MCMC) algorithm, preprogrammed for the software environment of Octave in so called *Dynare* [2] computational tool.

MH-MCMC, which may be utilized for *Bayesian estimation* of the parameters of particular model, requests prior information about the distribution of parameters, hereinafter denoted as the "priors". In order to obtain such distributions we made an assumption partially based on the information contained in literature, such as Szomolányi, Lukáčik and Lukáčiková [5] and partially on our own insight. After a series of trials and errors we have identified following set of priors (Table 1) for which we have computed the corresponding steady states, which we've obtained from the relations (11) – (16).

Parameters			Steady state values	
Notion	**Mean**	**Distribution**	**Notion**	**Value**
A	0.56	Beta	$\bar{n}$	0.771729886
B	0.95	Beta	$\bar{l}$	0.228270114
Δ	0.005	Beta	$\bar{c}$	13.2644347
Φ	0.3	Beta	$\bar{i}$	0.677354508
Φ	1.27	Beta	$\bar{y}$	13.94178921
P	0.95	Beta	$\bar{k}$	135.4709016
σ_ε	0.011	Inv_gamma		
σ_ε	0.05	Inv_gamma		

Table 1 Priors and corresponding Steady state

In the final run of the estimation we have used the following settings of the MH-MCMC algorithm. We have set the algorithm to perform 50 000 replications of the Monte Carlo simulation, the number of Markov chains blocks were 4, the algorithm's drop rate was set to 0.45 and the jump scale to 0.25. We have used the Sims computational method. As indicated above, we have also included the observed series of deviation of GDP and FHC from theirs steady state values, which contained 68 observations (1996Q1 – 2012Q4). The results of the last run of the Bayesian (MH-MCMC) algorithm are presented in the Table 2.

Parameters	Prior mean	Post. mean	Conf. inter.		Prior	Post. Dev.
α	0.56	0.56	0.5584	0.5617	Beta	0.001
β	0.95	0.9498	0.9482	0.9515	Beta	0.001
φ	0.3	0.3138	0.2812	0.3465	Beta	0.02
ϕ	1.27	1.2825	1.2486	1.3167	Beta	0.02
δ	0.005	0.0051	0.0033	0.0068	Beta	0.001
ρ	0.95	0.9487	0.9327	0.965	Beta	0.01
σ_ε	0.011	0.0112	0.01	0.0123	Inv_gamma	0.001
σ_ε	0.05	0.0488	0.0473	0.0503	Inv_gamma	0.001

Table 2 Estimation results

Since all of the prior means are contained in the estimated 90% confidential intervals, we can conclude that set priors were chosen suitably and we will proceed with describing them instead of the posteriors. The parameter α with value of approximately 0.56 says that the production technology is more demanding on the stock of capital than labor force. The value of β indicates that the consumption of identical good consumed one year from now, would yield only 81.45% utility compared to the current consumption. The φ parameter with the value 0.3 indicates that household strongly prefers leisure compared to consumption. According to the estimated value of ϕ the overall intertemporal substitution rate is equal to -0.27. Estimated parameter δ, at the level of 0.005 indicates that the stock of capital is mainly composed of long-term durable items, which fully depreciate after more than 50 years. We are aware that estimated value of δ is merely a tenth of standardly used value for depreciation, but for reasonable interpretation of steady state values we weren't able to use greater values. The last parameter ρ at the level of 0.95 signalizes a strong persistence of the effect of possible technological shock.

When looking closer at estimated steady state values of various variables we see another picture. First it is important to point out that these aren't the actual values of the steady state. Due to the application of the HP filter in the section 3 we set the steady state values of the *technological progress* equal to the evolving trend component of the HP filter at each given moment. Therefore the steady state values obtained from the estimated model are deviations of the evolution of selected variable relative to the evolution of the *technological progress*. With this in mind we start to interpret particular parameters of the model. The steady state of labor indicates that approximately 77.17% of the time in given moment is spent with labor. That leaves 22.83% of the time for leisure. The steady state value of the consumption indicates that the actual consumption is oscillating around 95.14% of the overall production. Analogically the steady state of investment is relatively at 4.86% of the GDP. Such result corresponds with relatively low rate of depreciation, which implies high durability of capital goods and thus only very small need for additional investment. Therefore almost all of the production of the country may be consumed by the household. Finally the steady state value of the capital stock is oscillating around the 9.72 times the level of quarterly GDP of Slovakia. Thus capital stock of Slovakia may in steady state correspond to the 2.43 multiple of annual GDP, which is in line with work of Kubíček [3]

5 The impact of the technological progress

In the next step we have used the parameters, confirmed through the Bayesian estimation, to analyze the effect of *technological progress* on Slovak economy. In order to do so we have performed stochastic simulations with the estimated model. We have set the number of replications to 100 000. This way we have obtained the impulse response function (IRF), which depicts the adjustment of particular variable to the effect of a given shock. In our case we have set the number of observations, during which we will be tracking the IRF to 200 periods. Consequently, we have obtained the evolution of IRF for all of the variables present in the model, visible at Figure 1.

Conclusively, we can interpret the obtained results. The highest impact of the *technological shock* or an innovation to productivity can be seen on investment. Slovak households temporarily substitute consumption with investment, while they are motivated with higher future returns. Although the effect is only short-lived it is definitely of highest magnitude. Through the investment we can also explain the effect on the stock of capital, which is positive as well, but keeps longer persistence than investment. Mentioned phenomenon can be explained by the multiplicative effect of both investment and *technological progress*, which in turn are responsible for higher output, of which greater share was transformed into capital. Even after the retraction of the investment, the original investment keeps the growth of the capital positive. Similar is the interpretation of consumption, which rises moderately due to the growth in overall output.

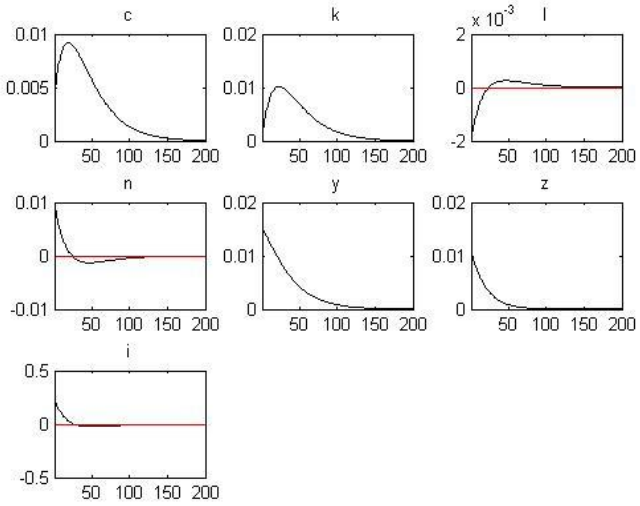

Figure 1 Impulse response function off Technological shock

Speaking of the overall output, rise of its immediate value is firstly fueled by the instant growth of productivity, which translates into rise of output both directly, through the higher output at the same level of inputs and indirectly, through the immediate growth in the rate of consumers substation, for whom is the labor (and with it the future consumption) far more interesting than leisure. When immediate substation effect wares off the overall output is still fueled with the increase of capital (as explained before). The leisure naturally evolves precisely in the opposite direction than labor does, but with different magnitude. And finally the persistence of the technical shock in the technical progress variable is determined solely by the value of autoregressive coefficient ρ.

6 Conclusion

In the presented paper we have described application of the RBC model in the conditions of Slovakia, as a tool to evaluate the impact of arbitrary *technological shock* on the selected variables of the country. The immediate impact of 1% positive shock on the total output is rise in approximately 1.382%. Similarly the 1% positive *technological shock* affects all other model variables. In case of FHC it results in approximately 0.4% increase. The greatest increase was recorded for the investment, where the magnitude is approximately 20.709%. In contrary the lowest increase was identified for the stock of capital, at the approximate magnitude of 0.1%. As mentioned in section 5 the impact on the labor market demonstrates the substitution effect among the time spent in work and leisure. The magnitude of 1% earlier mentioned shock results in 0.873% increase in labor input and 0.173% decrease in the leisure. The direct effect on the economy of the *technological shock* will diminish over time and it is significant for at least 47 periods (which corresponds with approximately 12 years), after which the value of deviation from the steady state of the technological progress is less than 0.001. The momentum of the shock lasts for another 49 periods (which corresponds with approximately 13 years), after which the deviation from the steady state of the GDP is insignificant (less than 0.001).

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Implementation of logistic regression into technical analysis

Juraj Hruška[1]

Abstract. Most of the investment strategies based on technical analysis are based on the principle that your strategy should be as easy as it is possible; in order to simplify your decision making. Goal of this paper is to use more sophisticated methods to combine the signals from indicators of technical analysis to create advanced form of investing strategy, which will be sustainable in long run. This can be ensured by self-correcting mechanisms build-in the strategy itself. Econometrical methods will be used to determine whether some kind of indicator has it relevance on the chosen type of asset. All input variables will be time series of dummy variables showing whether the indicator is suggesting taking a long position or not and of course their lags. Explained variable will be the successful trade (the price movement upwards is greater then spread and commissions). For this kind of purposes logistic regression seems to be essential, which is widely used in credit scoring. Basically the problem whether to invest or not is the same issue as whether to give a customer a loan or not. The only difference will be in the type of data. Credit scoring use mostly panel data, however it will be handled solely with time series in this paper.
Keywords: logistic regression, technical analysis, moving averages, automated trading.

JEL Classification: G11
AMS Classification: 62J02

1 Introduction

Technical analysis is one of the most popular and most efficient tools in trading business. It uses quantum of price-based information and indicators, which should bring at least trustworthy hints when to buy or sell certain asset. In this paper I will focus on one specific indicator of technical analysis; moving average. It is one of the easiest and most basic indicators. On the other hand its performance in the past has proved its significance. Many traders build their trading strategies on this indicator, or on the indicators derived from it, such as Moving average convergence divergence (MACD) or more complicated types of moving averages and their combinations. I have chosen the simple moving averages for my analysis. I used logistic regression to determine which combinations of moving averages give the most suitable signals for automated trading strategies.

Moving averages are basic indicator, but they are still used as a benchmark in many research analysis. Gradojevic and Gençay [4] analyze fuzzy technical indicators and moving averages implemented on the EUR/USD exchange rate. Pavlov [7] used moving averages in bootstrap simulations as a tool in portfolio selection process. Shintani, Yabu and Nagakura[9] have used indicators based on moving averages as predictors in time series modeling and testing hypothesis that indicators are falsely supported and regression is spurious.

2 Methodology and data

I based this paper on historically confirmed assumptions about indicators. I did my analysis on time series of 5 minute close spot exchange rate of EUR/USD from 15.10.2012 till 5.5.2013 (all data available from Bloomberg database with such high frequency). And I validated the results on the data since 15.4.2013 to 10.5.2013. Intervals are overlapping due to necessity of creating lagged variables and rather small original validating sample. Consequently, I tested all possible combinations of two simple moving averages.

Explained variable for purposes of this paper, so called successful trade i.e. system, was able to close the trade with take profit order with profit level 0,1% without being canceled out by stop-loss order. If any of the levels had been triggered, system waited for 100 minutes and then the trade was automatically considered as unsuccessful. Logistic regression was used to test, how combinations of these moving averages can explain the trade being successful.

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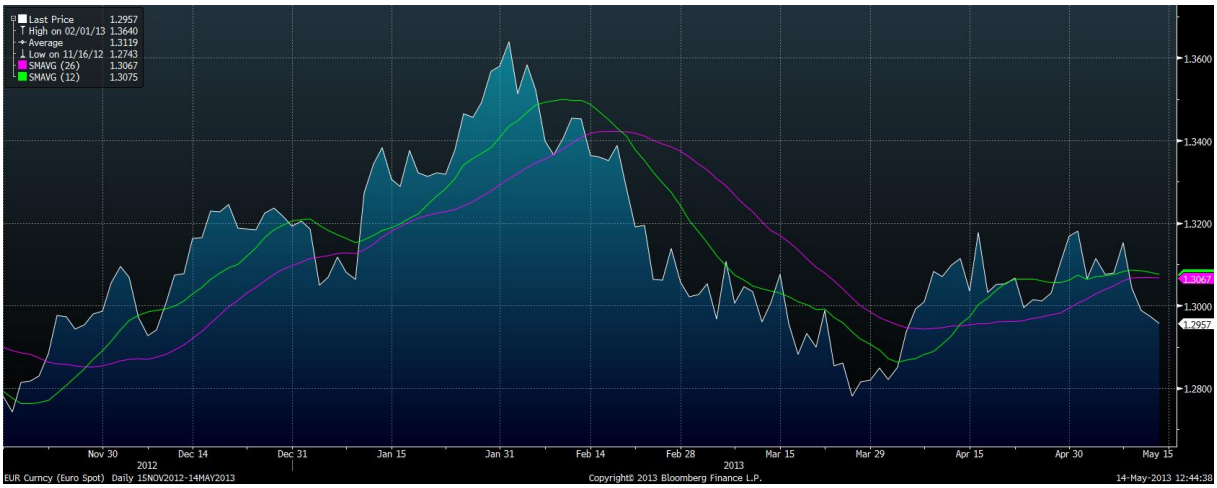

Figure 1 EUR/USD exchange rate development (15.10.2012-5.5.2013) from Bloomberg

Moving averages are used in combination with the line of spot prices. When the spot price crosses the line of moving average upwards, it is considered as a signal to take a long position. On the other hand when the line of the spot price crosses the moving average downwards, it is a signal to take a short position or to close a long position. The same logic holds in case we do not use spot price but another moving average (it should be shorter than the first one, or the positions should change).

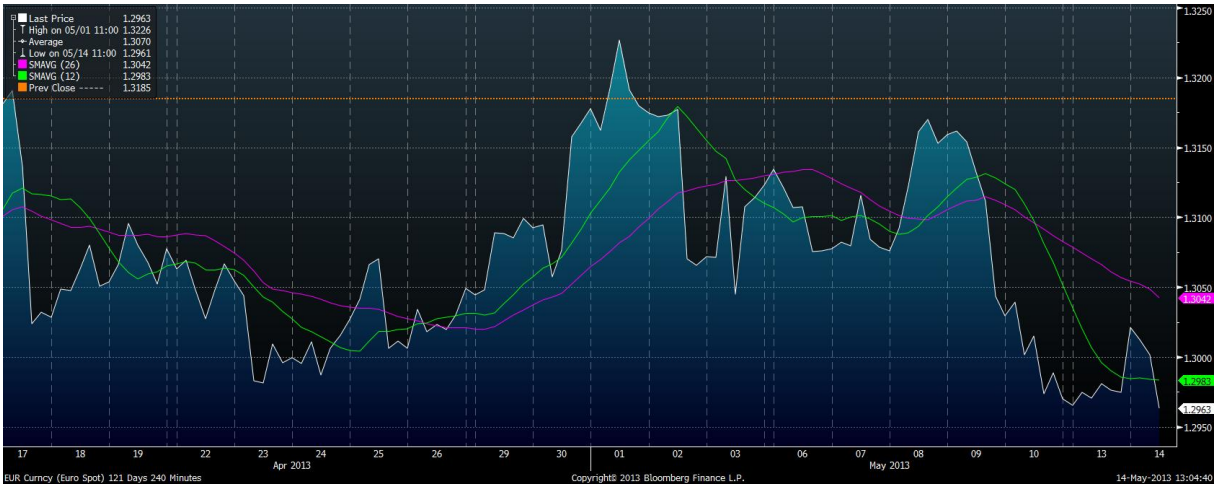

Figure 2 EUR/USD exchange rate development (1.4.2013-15.5.2013)-validating data from Bloomberg

Moving average is calculated as the average of current price of asset and several spot prices directly following each other:

$$MA\{n\} = \frac{1}{n}\sum_{i=0}^{n} S_{t-i} \tag{1}$$

Usually the Combination of 25 and 12 prices and 12 and 5 prices are considered as the most efficient ones according to Brada[1]. But it always depends on the frequency of evaluated data. For closing price of the day different combinations of moving averages could be used, than for the tick prices. Also the number of signals is an important factor when choosing a type of data we want to analyze. If we want to use them for automated trading, it would be efficient to use averages that produce more signals. This model tested all combinations of moving averages up to length of 40.

Logistic regression was used for analysis of the successful trades by the moving averages as explaining variables. It is based on maximum likelihood function and does not have such strong requirements for the variables as Ordinary least square estimation. The general formula of the model is:

$$y_i = \beta_1 + \beta_2 X + \epsilon_i \tag{2}$$

Where, y_i is the dependent variable and X is the matrix of explaining variables and $\beta_!$ is intercept and β_2 is a vector of coefficient of explaining variables and ϵ_i is residual.

Logistic function is derived from the formula of normal standard distribution and has form of a maximum likelihood function:

$$f(y_i|X;\beta,\sigma^2) = \frac{1}{\sqrt{(2\pi\sigma^2)}} exp\left\{-\frac{(y_i-\beta_1-\beta_2X)^2}{2\sigma^2}\right\} \tag{3}$$

The maximum likelihood function is transformed into a log-likelihood function, with unknown parameters β and σ^2, which we are trying to estimate:

$$logL(y_i|X;\beta,\sigma^2) = -\frac{N}{2}log(2\pi\sigma^2) - \sum_{i=1}^{n}\frac{(y_i-\beta_i-\beta_2X)^2}{2\sigma^2} \tag{4}$$

After differentiating the log-likelihood function with respect to all parameters and equaling them to 0, we get estimations of all unknown coefficients in form:

$$\frac{\partial logL(y_i|X;\beta,\sigma^2)}{\partial\beta} = 0 \tag{5}$$

Logistic regression is mostly used for binary choice models. Model can be enhanced to the form, where the explaining variable would express also signal to take short position. This would certainly require some multinomial form of the model. In practice we are modeling a probability that the explaining variable is 1. Mostly the cut-off probability is 0.5, but it can be modified due to requirements to the significance of the model. Logistic regression transforms probabilities to *logits*.

$$logit(p_i) = ln\left(\frac{p_i}{1-p_i}\right) \tag{6}$$

Where p_i is the probability that observation *i* will be evaluated as 1. So we are transforming the explained variable to the *logits*.

$$logit(p_i) = \beta_1 + \beta_2X \tag{7}$$

These parameters are estimated by the Maximum likelihood estimation. Probabilities would be obtained from the function:

$$p_i = \frac{1}{1+e^{-\beta_1-\beta_2X}} \tag{8}$$

More about logistic regression theory can be found in Heij [3].

3 Definition of the model

At first I created explaining variable a "successful trade" (Trade) which is the situation, when profit from acquisition in time *t* is higher than 0,1% in following 20 observations (100 minutes), without triggering the stop-loss set on the level of 99% of Ask price. Then I calculated moving averages up to length of 40. Then I used them to create explaining variables in a form of dummy variables that should signalize profit if we take a long position in the asset. Theory says that moving averages suggest submitting buying order if the faster moving average crosses the slower moving average upwards. After the comparison of all possible combinations 780 explaining variables were created. Then 5 lags of every variable were added, which sum up to over 5000 regressors altogether. That is rather extensive amount of variables for regression, but if I wanted to identify as much successful trades as possible, I simply could not omit any variable. The longer moving averages are not that interesting for the model, because since certain level they all behave approximately the same and give the same information. One of the solutions might be adding moving averages with higher sequence.
Model can be extended with different types of indicators of technical analysis such as, Moving average Convergence/Divergence, Relative Strength Index, CCI, or some trend lines or any kind of indicator based on the volume of trading.

The resulting model after, ignoring all insignificant variables consisted of around 250 predicators. The model can be accepted according to the global tests.

Testing Global Null Hypothesis: BETA=0			
Test	**Chi-Square**	**DF**	**Pr > ChiSq**
Likelihood Ratio	939.6593	276	<.0001
Score	916.9869	276	<.0001
Wald	891.9390	276	<.0001

Table 1 Significance statistics of model

According to the summary statistics of the model, it can be considered as rather weak model, because its Somer's D, calculated as ratio of concordant and disconcordant pairs, has small value and the value of the area under the ROC curve is small as well, compared to the area under the curve of a random model. The area under the curve is only 0.5973 which means only 19.46 % of cases are explained by chosen variables. Transformation of the explained might be helpful in a way, that it would contain more trend profits and less random profits.

ROC Association Statistics							
	Mann-Whitney						
ROC Model	**Area**	**Standard Error**	**95% Wald Confidence Limits**		**Somers' D (Gini)**	**Gamma**	**Tau-a**
Model	0.5973	0.00325	0.5909	0.6036	0.1946	0.1946	0.0691
ROC1	0.5000	0	0.5000	0.5000	0	.	0

Table 2 ROC statistics for the model with main data

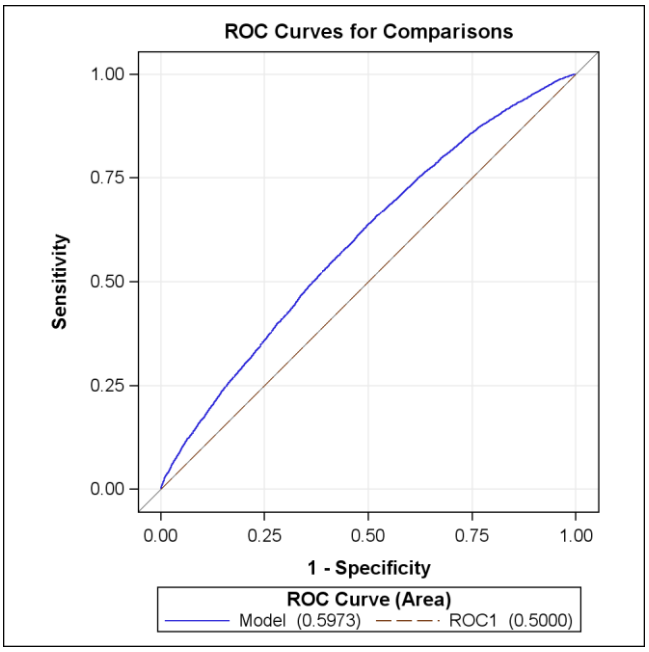

Figure 3 ROC curve for the model with main data

Nevertheless, after analyzing the frequency table of Trade observed values and fitted values, we can see that in spite of the fact that model found only a small fraction of successful trades; it was quite effective in preferring profitable trades to trades with loss. The rate of success was nearly 65%. If we kept investing according to this model we would gain 2.94% after selected period of time.

Table of Trade by Model			
	Model-fitted values		
Trade	0	1	Total
0	31876 76.85 99.95 76.94	16 0.04 0.05 35.56	31892 76.89
1	9556 23.04 99.70 23.06	29 0.07 0.30 64.44	9585 23.11
Total	41432 99.89	45 0.11	41477 100.00

Table 3 Frequency table of Trade and fitted values

If we decrease the cut-off ratio of the log-likelihood function from 0.5 to 0.45 the number of trades will increase to 170, which is nearly four times as much. However, the successfulness of trades will also fall to 53.22%. In any of these cases the unsuccessful trade was created by triggering stop-loss barrier. On the other hand the gain after this period was around 3.59 %, which is even more than in the previous model. That is caused by the fact, that many trades did not cross the profit level 0.1%, but they were still profitable.

After testing the model on validating data set of EUR/USD exchange rate, slightly better results were obtained.

ROC Association Statistics							
	Mann-Whitney						
ROC Model	Area	Standard Error	95% Wald Confidence Limits		Somers' D (Gini)	Gamma	Tau-a
Model	0.6240	0.00675	0.6108	0.6372	0.2480	0.2480	0.0883
ROC1	0.5000	0	0.5000	0.5000	0	.	0

Table 4 ROC statistics for the model with validating data

With area under ROC curve 0.624 and Gini coefficient 0.248, calculated model seems to have medium quality on validating data. That suggests that the model is not overlearnt even with such a number of explaining variables.

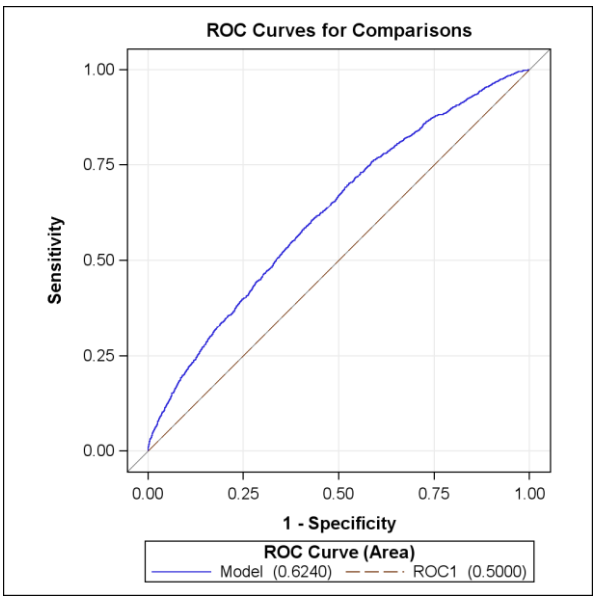

Figure 4 ROC curve for the model with validating data

The frequency table shows that even the effectiveness of predicating trades was somewhat better (65.63% of the trades were correctly selected). These results are for the model with cut-off with basic level of 0.5. An interesting fact is that more trades were picked from quarter amount of observations. The overall of profit after the validating period (month and half) was 4.287%. With the cut-off level 0.45, the model was effective on 53 %, so it can be concluded that it is more efficient to stick with stricter model. Profit was only 3.684% for the more benevolent model with cut-off 0.45.

Table of Trade by Model			
	Model-fitted values		
Trade	0	1	Total
0	7238 76.61 99.70 77.13	22 0.23 0.30 34.38	7260 76.84
1	2146 22.71 98.08 22.87	42 0.44 1.92 65.63	2188 23.16
Total	9384 99.32	64 0.68	9448 100.00

Table 5 Frequency table of Trade and fitted values - validation data

4 Conclusion

After these calculations I can proclaim that logistic regression can be successfully used as a decision-making tool in automated trading. Profits gained by this strategy are higher than using just ordinary moving averages strategies. My model produced only few signals, but successfully avoided many of false signals, that would be created by moving averages themselves. These models were weak, but still profitable. They can be improved by using more indicators as explaining variables and even moving averages can be exchanged for the exponential moving averages. In view of automated trading this model was creating only few signals, compared to what would be necessary for application in such a field; it would also require higher frequency of observations.

Acknowledgements

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Comparison of the AHP and ANP approach to Investment Decision Making

Eva Chalúpková[1], Jiří Franek[2]

Abstract. The decision making can be performed according to perspectives, which may also have a conflicting nature. Models that are helpful in dealing with such problems can be found in decomposition multiple attribute decision making methods (MADM). In this paper the authors are dealing with an investment decision making problem that can be divided into hierarchy of levels thus MADM methods the AHP (Analytic Hierarchy Process) and the ANP (Analytic Network Process) are applied. The aim of the paper is to compare both approaches their results and to analyse which of them has a better fit for actual investment decision making from entrepreneurs perspective. The sensitivity analysis is performed on results from AHP to assess sensitivity of alternatives ranking and weights values. The ANP method includes also the interrelationships among selected criteria. Results are compared.

Keywords: leasing, analytic hierarchy process, analytic network process, MADM

JEL Classification: M10, C44
AMS Classification: 90B50, 90C29, 91B06

1 Introduction

Decision making belongs to day to day activity of entrepreneurs. It represents a process in which an individual or a group selects the best or optimal alternative with regard to a set of attributes (criteria). Capital investment decisions are among the most important decision making activities in business. Most of these capital investments are focused towards renewal of fixtures or expansion of business tangible and intangible assets [3]. Investment decisions have a profound and frequently long term impact on the business. It is therefore necessary to consider more than few decision criteria. That is why this multiple attribute decision making methods can be used. This paper is focused on the application of multi level decomposition decision making methods of Analytic Hierarchy Process (AHP) and Analytic Network Process (ANP) developed by [7], [8]. These methods were largely studied and analysed in various business decision making problems [2], [6], [10]. The goal of this paper is to present both methodologies (AHP and ANP) in comparison manner and to assess their appropriate utilization for entrepreneurs. The AHP decision making process is extended by sensitivity analysis of its results.

2 Multi level decomposition decision making methods

The problem of multi-criteria evaluation of alternatives is foremost a task of finding the best (optimal, compromise) alternative, raking of the alternatives from the best to the worst conceivable, alignment of alternatives to hierarchical clusters, separation of alternatives into two groups of acceptable and not acceptable, determination of the effectives (non dominated, pareto) set of alternatives or exclusion of the ineffective alternative [12]. In short it is the optimization problem. The rank of alternatives and selection of the optimal one is based on weighted sum criteria (total weighted utility) of the alternative that can be calculated using to the eq. (1)

$$U(a_i) = \frac{\sum_{j=1}^{m} x_{i,j}.v_j}{\sum_{j=1}^{m} v_j}, \tag{1}$$

where v_j represents non-normalized weight of the j^{th} criterion, and stands for the sum of all (non-normalized) criteria weights, x_{ij} represents the evaluation of the i^{th} alternative according to the j^{th} criterion. Then for the weighted sum criteria of normalized weights following formula (2) can be applied:

$$U\left(a_i\right) = \sum_{j=1}^{m} w_j.x_{i,j}, \tag{2}$$

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where the w_j represents the normalized weight of the j^{th} criterion.

The advantages of multi level decomposition decision making methods can be found in the decision maker's ability to evaluate each alternative using a large number of criteria. Methods can be accordingly classified based on the information about the criteria or alternatives preferences into following categories: methods with cardinal information, methods with ordinal information about criteria preferences, methods with nominal information [5]. Multi level decomposition methods are among the most convenient when it comes to evaluation of finite number of alternatives. One of the basic and most widely used methods is the analytic hierarchy process (AHP) [9].

2.1 Analytic Hierarchy Process (AHP)

The AHP solves multi-criteria decision making problem based on hierarchy. Generally the hierarchy has three levels: the goal, criteria and alternatives. Criteria can be broken down to sub-criteria to make lower level. The theoretical procedure of the AHP method consists of four steps: hierarchy design, identification of priorities, combination and evaluation. The method itself is based on Saaty method of pair-wise comparison [7]. The actual weight estimation can be calculated according to several approaches. The most frequent and appropriate are the Row Geometric Mean Method (RGMM) and Eigenvalue method (EVM). In order to provide sufficient close results in most situations, [9] suggested the approach of the geometric mean of axis' row. The geometric consistency index suggested is to measure the individual consistency of judgment matrices. For an approximation method that provides sufficiently close results in most situations. In the RGMM the weights w_j can be obtained through an algorithm based on the geometric mean method (method of least logarithmic squares) under the same necessary condition then the solution is a normalized geometrical mean of the matrix as follows

$$w_j = \left[\prod_{j=1}^{i} s_{ij}\right]^{\frac{1}{i}} \Bigg/ \sum_{j=1}^{i} \left[\prod_{j=1}^{i} s_{ij}\right]^{\frac{1}{i}}. \tag{3}$$

The geometrical mean can be calculated using MS Excel function *GEOMEAN*. This function will be employed for calculations in the application part. On the other hand Saaty applied the main of eigenvalue S as a desirable vectors priority of ω, where the linear system (4) determines the vector as it is stated below,

$$S\omega = \lambda\omega, \;\; e^T\omega = 1, \tag{4}$$

where λ is the principal of eigenvalue towards A. If the decision maker is constant, the value of λ should be equal to the n. We can also say that the value of λ is greater to the value of n ($\lambda > n$). The estimation of eigenvalues for the main vector inconsistent matrix can be done by normalizing the number of lines of each element. Various researchers have proved that for small deviations around consistent ratio ω_i/ω_j of the EVM method gives a good approximation of the priority vector. The difference between above mentioned methods is evident when estimated utilities of alternatives are closer to each other and small change in values can influence the ranking, then it is recommended to calculate and compare both methods. Results from the EVM can be considered more precise. However, the evaluation requires a certain level of matrix consistency, i.e. that the elements are linear independent. That can be assessed employing consistency ratio C.R. as follows

$$C.R. = \frac{C.I.}{R.I.} = \frac{\frac{\lambda_{\max} - m}{m-1}}{R.I.}, \quad \text{where } \lambda_{\max} = \sum_{j=1}^{m} \frac{(\mathbf{S}\cdot\mathbf{v})_j}{m \cdot v_j} \tag{5}$$

where *C.I.* is the consistency index, λ_{max} is the highest eigenvalue of the matrix and m represents the number of independent rows of the matrix. The λ_{max} can be calculated using the matrix **S** that represents pair-wise comparison matrix and **v** means the matrix eigenvector. *R.I.* represents the random index that can be obtained from i.e. [1]. Solution of AHP can be found using supermatrices as follows

$$W = \begin{matrix} \\ goal \\ criteria \\ alternatives \end{matrix} \begin{bmatrix} 0 & 0 & 0 \\ W_{21} & 0 & 0 \\ 0 & W_{32} & I \end{bmatrix}, \tag{6}$$

where $\boldsymbol{W_{21}}$ is the matrix of criteria's pair-wise comparisons, $\boldsymbol{W_{32}}$ is the matrix of alternatives and criteria pair-wise comparisons and $\boldsymbol{I}$ is the identity matrix. We take these comparisons as independent. This leads us to the more complex ANP method where the decision maker has to consider criteria and alternatives interdependency.

2.2 Analytic Network Process (ANP)

The ANP structures the problem related to options in reverse logistics in a hierarchical form. With the ANP, the interdependencies among criteria, sub-criteria and determinants for the options can be considered. The original

analytical network process (ANP) was proposed in [8]. ANP is the extension of analytic hierarchy process (AHP) and is a more general form of AHP. Many decision problems cannot be structured hierarchically because they involve the interaction and dependence of higher level elements on low level elements. Saaty in [8] applied ANP to handle dependence among criteria and alternatives without assuming independent decision criteria. The ANP feedback approach replaces hierarchies with networks, and emphasizes interdependent relationships among various decision-making and also interdependencies among the decision criteria and allows more systematic analysis. This model can be defined by a supermatrix (7) see the differences in relation to (6)

$$W = \begin{matrix} goal \\ criteria \\ sub-criteria \\ alternatives \end{matrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ W_{21} & 0 & 0 & 0 \\ 0 & W_{32} & W_{33} & W_{34} \\ 0 & 0 & W_{43} & W_{44} \end{bmatrix}, \tag{7}$$

where $\boldsymbol{W_{ij}}$ represents all possible and logical pair-wise comparison weights. The actual method of ANP can be characterized in three basic steps: (i) composition of the initial supermatrix $\boldsymbol{W}$, where the normalized weights w_{ij} are put in columns; (ii) then the initial supermatrix is transformed into weighted supermatrix $\overline{\boldsymbol{W}}$ where

$$\sum_{i=1}^{j} w_{ij} = 1, \tag{8}$$

that derives from the need to find the convergent solution and the results would then represent the global criteria weights; (iii) the third step involves a calculation of the limited (final) supermatrix $\overline{\boldsymbol{W}}^{\infty}$ that can be processed in the case of the non cyclic weighted matrix according to [12] as follows

$$\overline{W}^{\infty} = \lim_{k \to \infty} \overline{W}^{k}, \tag{9}$$

$$\overline{W}^{N} = \frac{1}{N} \sum_{k}^{N} \overline{W^{k}}, \tag{10}$$

where $\overline{W}^{\infty}$ is the limited supermatrix, $\overline{W}^{k}$ is the supermatrix without a cycle powered *k*-times (9). In the case of the cyclic matrix the formula can described as in (10).

2.3 Sensitivity analysis of investment project alternatives

Based on acquired values of the weighted sum criterion $U(a_i)$ of particular alternatives, and having in mind the rule of the highest value, the best alternative can be found. According to results all alternatives can be organized by different measures (e.g. best to worst alternative, etc.). Then it is possible to assess created ranking with various exclusion methods and select a number of alternatives to be considered further. To assess the stability of the rank of alternatives the sensitivity analysis of weights changes of alternatives can be used. With the aforementioned information in mind (the rank of alternatives) that was calculated by employing weighted sum criterion $U(a_i)$ with formulas (1) and (2), the following sensitivity analysis will be dealing with evaluation of alternatives according to weights estimated for this criterion [11]. The weighted sum criterion $U(a_i)$ has to be calculated for particular alternatives. The aim is to find a limited value that would cause a change in the ranking of alternatives *m* and *n*. Actual procedure of the sensitivity analysis that was used in this study can be found in [12] pp 48-49.

3 Capital investment decision making using AHP and ANP and its comparison

Following chapter deals with the application of both aforementioned methods on a capital investment example. The hierarchical decision-making process is drafted in the following Figure 1. It has 4 levels (goal, criteria, sub-criteria, alternatives). This hierarchy has two loops in the third level. In the AHP those loops were not used.

The decision making process with AHP/ANP has been done through following procedure:

1. Selection of alternatives for capital investment financing (a vehicle - fixed asset);
2. Financial and non-financial criteria perspectives were taken into consideration (C_1 and C_2);
3. Definition of sub-criteria (f_1,...,f_6);
4. Pair-wise comparisons of criteria where in the purchase of fixed assets the costs of whole transaction were considered. In this case the financial criteria are preferred before the non-financial.
5. Pair-wise comparison of sub-criteria among themselves.

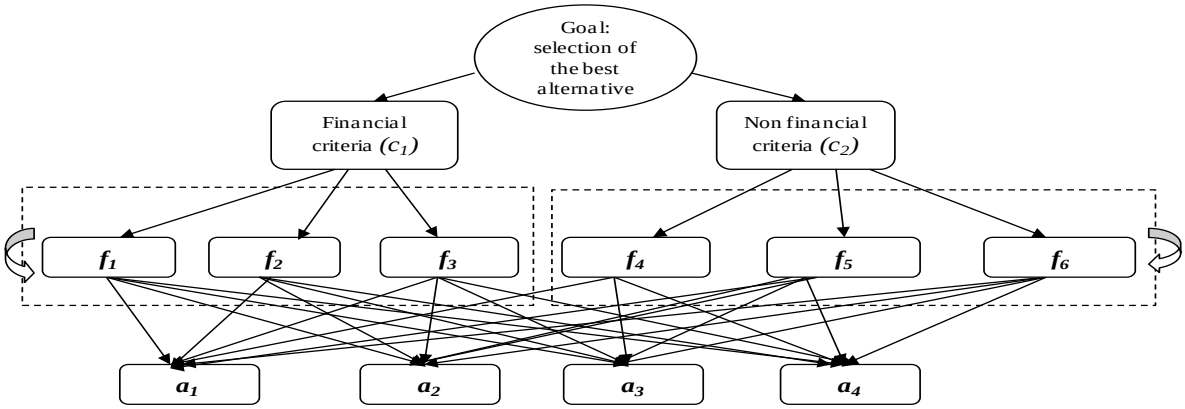

Figure 1 AHP/ANP decision making hierarchy

Data were gathered from several financial products that are available for assessment. These products serve as alternatives. The data are shown in the Table 1.

Criteria/ Alternatives	Financial criteria (c_1)			Non-financial criteria (c_2)		
	Max	Min	Min	Max	Min	Min
	Total cost [EUR]	Monthly payment [EUR]	Other fees [EUR]	Fixture period [months]	Administration	Office distance
Consumer loan 1 (a_1)	24 000	200	698	120	12	4
Consumer loan 2 (a_2)	18 000	288	9 010	36	13	9
Capital lease 1 (a_3)	21 600	285	4 600	60	5	8
Capital lease 2 (a_4)	18 000	250	6 800	48	5	6
Criteria symbol	f_1	f_2	f_3	f_4	f_5	f_6

Table 1 Capital investment decision data of alternatives

Table 2 shows the unweighted ANP supermatrix that consists of weights estimated by pair-wise comparisons and calculated using (3). All consistencies were within limits.

	goal	c_1	c_2	f_1	f_2	f_3	f_4	f_5	f_6	a_1	a_2	a_3	a_4
goal	0	0	0	0	0	0	0	0	0	0	0	0	0
c_1	**0,875**	**1**	0	0	0	0	0	0	0	0	0	0	0
c_2	**0,125**	0	**1**	0	0	0	0	0	0	0	0	0	0
f_1	0	**0,731**	0	0	**0,900**	**0,833**	0	0	0	**0,733**	**0,669**	**0,699**	**0,644**
f_2	0	**0,188**	0	**0,821**	0	**0,167**	0	0	0	**0,199**	**0,243**	**0,230**	**0,271**
f_3	0	**0,081**	0	**0,179**	**0,100**	0	0	0	0	**0,068**	**0,088**	**0,070**	**0,085**
f_4	0	0	**0,740**	0	0	0	0	**0,900**	**0,875**	**0,687**	**0,637**	**0,731**	**0,731**
f_5	0	0	**0,167**	0	0	0	**0,821**	0	**0,125**	**0,186**	**0,258**	**0,188**	**0,188**
f_6	0	0	**0,094**	0	0	0	**0,179**	**0,100**	0	**0,127**	**0,105**	**0,081**	**0,081**
a_1	0	0	0	**0,535**	**0,622**	**0,645**	**0,655**	**0,105**	**0,678**	0	**0,731**	**0,793**	**0,735**
a_2	0	0	0	**0,113**	**0,063**	**0,041**	**0,048**	**0,056**	**0,066**	**0,122**	0	**0,076**	**0,058**
a_3	0	0	0	**0,239**	**0,100**	**0,219**	**0,208**	**0,419**	**0,090**	**0,648**	**0,188**	0	**0,207**
a_4	0	0	0	**0,113**	**0,215**	**0,094**	**0,089**	**0,419**	**0,166**	**0,230**	**0,081**	**0,131**	0

Table 2 Unweighted ANP supermatrix

4 Results and discussion

In the case of pair-wise comparisons of financial criteria (c_1) the preference of the given sub-criteria the sub-criterion f_1 was selected (total cost) as the most important. It represents a value of money that has to be borrowed from the financial institution. It is a decisive factor because it has an influence on the decision maker's (entrepreneur's) sum of liabilities. The decision maker's (DM) goal is to purchase particular fixed asset with minimum use of liabilities. The amount of monthly payment (f_2) represents smaller financial demands from the decision maker than for a payment of the whole value of the asset. The DM's goal is to pay/have monthly low as possible. The actual purchase of the fixed asset comes together with miscellaneous costs (f_3) like other fees or insurance that changes the final price of the asset but not as much as the other sub-criteria.

The non-financial criteria (c_2) represent less preferred group of criteria. The sub-criterion f4 (fixture period) can be perceived as the most important, because this represents the payback time in total with fees that are bound to the period. Generally, longer period is preferred to shorter because the real value of money is decreasing.

Various financial institutions have different demands for client's credit score (in the case of loans or leasing) which has to be proclaimed by specific documents and makes it more difficult to administer. In this case these documents are represented in numbers (sub-criterion f5) so the less documents has to be delivered the more is this option preferred. The last sub-criterion that is considered is the distance to particular institution's office.

The evaluation of alternatives

In this step the DM has to evaluate alternatives according to given criteria. All estimated weights calculated using RGMM (3) method in excel can be put in the supermatrix (6) to calculate $U(a_i)$ of the AHP approach using equations (9) and (10). Results are shown in the table 4. In the next step the looping comparisons are made according to the supermatrix (7). Weights were again calculated using the RGMM approach (3). Application of ANP should provide an alternative perspective on dependencies among given criteria and alternatives. The ANP unweighted supermatrix is shown in Table 2. Then the limited ANP supermatrix in Table 3 is calculated using equations (9) and (10). The results were stable after 7th iteration. Actual results of the realized decision making using AHP and ANP approaches are presented in Table 4. Results show that the ranking of criteria has not changed except sub-criteria f_3 and f_4 that have swapped places. Priorities estimated by ANP are less variable and tend to be more equivalent. Small changes in priorities estimated by ANP had no impact on their ranking.

	Goal	c_1	c_2	f_1	f_2	f_3	f_4	f_5	f_6	a_1	a_2	a_3	a_4
Goal	0	0	0	0	0	0	0	0	0	0	0	0	0
c_1	0	0	0	0	0	0	0	0	0	0	0	0	0
c_2	0	0	0	0	0	0	0	0	0	0	0	0	0
f_1	**0.160**	0.160	0.160	0.160	0.160	0.160	0.160	0.160	0.160	0.160	0.160	0.160	0.160
f_2	**0.095**	0.095	0.095	0.095	0.095	0.095	0.095	0.095	0.095	0.095	0.095	0.095	0.095
f_3	**0.031**	0.031	0.031	0.031	0.031	0.031	0.031	0.031	0.031	0.031	0.031	0.031	0.031
f_4	**0.160**	0.160	0.160	0.160	0.160	0.160	0.160	0.160	0.160	0.160	0.160	0.160	0.160
f_5	**0.095**	0.095	0.095	0.095	0.095	0.095	0.095	0.095	0.095	0.095	0.095	0.095	0.095
f_6	**0.031**	0.031	0.031	0.031	0.031	0.031	0.031	0.031	0.031	0.031	0.031	0.031	0.031
a_1	**0.207**	0.207	0.207	0.207	0.207	0.207	0.207	0.207	0.207	0.207	0.207	0.207	0.207
a_2	**0.033**	0.033	0.033	0.033	0.033	0.033	0.033	0.033	0.033	0.033	0.033	0.033	0.033
a_3	**0.117**	0.117	0.117	0.117	0.117	0.117	0.117	0.117	0.117	0.117	0.117	0.117	0.117
a_4	**0.072**	0.072	0.072	0.072	0.072	0.072	0.072	0.072	0.072	0.072	0.072	0.072	0.072

Table 3 Final limited ANP supermatrix

Criteria		**Local priorities**		**Global priorities**	
		AHP	**ANP**	**AHP**	**ANP**
c_1	f_1	0,731	0,560	0.639	0.280
0,875	f_2	0,188	0,333	0.165	0.166
	f_3	0,081	0,107	0.071	0.054
c_2	f_4	0,740	0,561	0.092	0.281
0,125	f_5	0,167	0,331	0.021	0.166
	f_6	0,094	0,107	0.012	0.054
Alternatives	a_1	0,561	0,482		
	a_2	0,092	0,077		
	a_3	0,214	0,273		
	a_4	0,133	0,168		

Table 4 Comparison of AHP and ANP results

The Figure 2 shows significant changes in values of criteria priorities and values of utility function of given alternatives. Priorities estimated by ANP are more evenly distributed. This is a result of interdependencies.

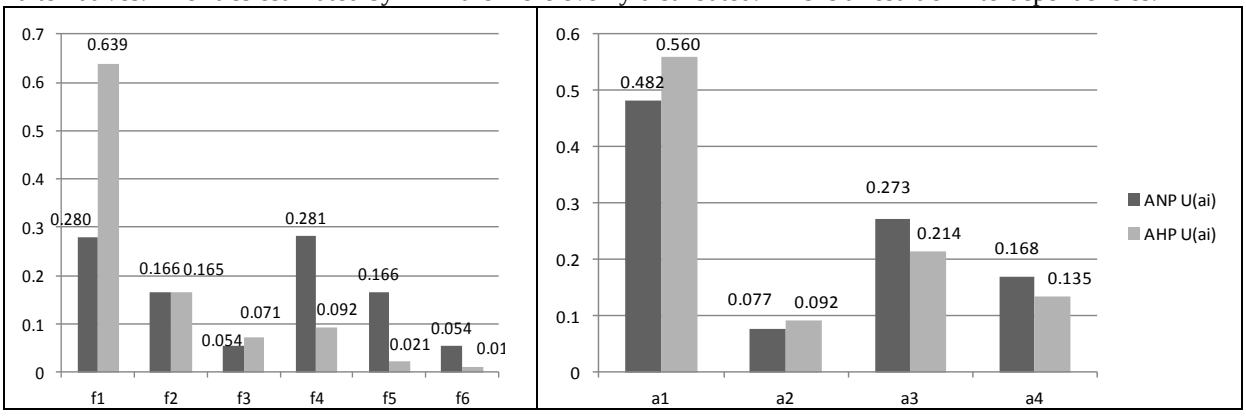

Figure 2 Comparison of criteria global weights and alternative's utility of AHP/ANP results

Results of the comparison led to a suggested conclusion that utilization of ANP on small scale and less complicated decision making problems is applicable, whereas AHP shows relative robust weight distribution that can be evaluated using sensitivity analysis as in [12] pp 48-49, thus it is not very efficient to spend time and make computations that are necessary for ANP method. In the case of decision making about alternatives the AHP is sufficient tool. When the estimation of criteria weights is considered, the ANP can be applied to evaluate possible interdependencies and their influences. AHP can be easily exercised using for example MS Excel add-on DAME [4].

5 Conclusion

The aim of the paper was to compare two MADM approaches to decision making, the analytic hierarchy process and analytic network process that were applied on a decision making problem of financial banking of a capital investment. The problem had four levels. Financial and nonfinancial decision-making criteria were considered and their weights estimated using RGMM method. Final results were calculated by limited supermatrices. Based on the results gathered from the application in a decision making problem example it can be suggested that AHP method is sufficient and more efficient that ANP when dealing with less complicated decision making with a goal to select the best alternative. It is also advisable to apply the sensitivity analysis to evaluate the robustness of results. In the case of criteria and alternatives weight distribution (importance) the ANP can be considered for further evaluation to cover the consideration of possible interdependencies among given criteria and alternatives. Employment of MS Excel also demonstrated its practical use for small firms that can not afford expensive decision support software. Results of this study were verified by Super Decision software and have not revealed any significant differences.

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Using Pearson distribution system to monitor a process, which distribution is unknown

Magdalena Chmielińska[1]

Abstract. A control chart is a tool of statistical quality control, which is commonly used in factories. Meeting its basic assumptions, guarantee faultless estimation of correctness of monitored production process. Infringement of classical control charts assumptions may cause false signals in case of regulated process, or lack of signal as well as signal delayed in time in case of inregulated process.
In this paper, based on true data, the trial of calculation the control limits for production process, whose controlled feature has a distribution, which is essentially different to a normal distribution, was made. A proposed method of calculation the control limits based on the Pearson distribution system, regards in first step to selection from the Pearson family distribution, the correct one, which can by used to estimation the data empirical distribution. Next the level of control limits are fitted on based on the quantile of this distribution. The features of the proposed method were testing by the simulations carried out in R.

Keywords: Control charts, the Pearson's System of Distributions

JEL Classification: C44
AMS Classification: 90C15

1 The control chart

Control chart is a tool, which enables statistical monitoring of production process. It gives information about run of process, as well as it contributes to the rise of a production quality. Its characteristic features are simplicity of construction and effectiveness.

The classical control charts are base on an assumption, that the diagnostic variable, which is observed during process monitoring has a normal distribution with an expected value μ and a standard dividation σ [6]. They assume also, that the measurements in successives time periods are independent. The control chart is a graph dedicated to register the results of current quality control of products. On its are plotted the central line – defining the expected value of the monitored characteristicts, the control limits (upper and lower) and the warning limits (upper and lower).

The control lines are set at such a level that the statistical probability of the values below the lower and above the upper control line, when the process is in – control, was appropriately small. Classically, assuming a normal distribution of the characteristic, these lines are three standard deviations of the monitored manufacturing process away from the central line. Warning lines, and are usually placed at the level of the center line plus and minus 2σ [4]. Due to the violation of the assumption of normal distribution of monitored characteristic, the position of the control lines is increasingly set on the basis of the values of the appropriate quantile of known theoretical distributions that can be used to model the actual data.

Points configuration on a control chart in relation to the level of control and warning limits is the basis for reasoning about the correctness of the course of the controlled process. In the literature [9] the many signs of a deregulation process are considered, while in this paper as a signal of process disturbance just a point above (below) the upper (lower) control line is considered.

The control charts are characterized by the ARL (Average Run Length). In case of considering only exceeded upper or lower control lines, it is the inverse of the probability of the signal (p_s) for a single observation (sample) [6]:

$$ARL = \frac{1}{p_s} \tag{1}$$

ARL for regulated process is marked as ARL_0 and for process beeing out – of – control as ARL_1.

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2 The Pearson's system of distributions

Elaborated toward the end of XIX century, by Carl Pearson a system of distributions of random variability, came into begin by solve a differential equation in the form [3]:

$$\frac{df(x)}{dx} = -\frac{a+x}{c_0 + c_1 x + c_2 x^2} f(x) \tag{2}$$

As results of this equation are density functions in the form:

$$f(x) = Ce^{-\int \frac{a+x}{c_0+c_1x+c_2x^2}dx}. \tag{3}$$

Miszczak [8] announces that, the atributtes of those density functions depend on squer trinomial in the form of:

$$M(x) = c_0 + c_1 x + c_2 x^2 \tag{4}$$

Defining the square of the skewness coefficient (β_1) and flattening coefficient (β_2) as:

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3}, \qquad \beta_2 = \frac{\mu_4}{\mu_2^2},$$

where μ_i – central moment of *i*-th order,

classification of seven of the 12 awarded by the Pearson type of greatest practical use, can be represented graphically (Fig. 1). It is considered that the main types are types I, IV and VI, the types of transition are, types III (straight line) and type V (almost in a straight line). There are also types of symmetry, which include:

- Type 0 (in the literature referred to as type N) at the point (0,3) to denote the normal distribution,
- Type II includes a vertical line on the Y axis for $\beta_2 < 3$ of which is a special case of type I,
- Type VII covering a vertical line on the Y axis for $\beta_2 > 3$ of which is a special case of type IV.

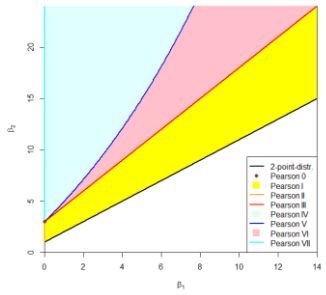

Figure 1 Pearson types of distributions of the system according to β_1, β_2
Source: software R (R i386 3.0.0)

The normal distribution, designated on the figure 1 as "Pearson 0" is a limiting distribution of all designed by Pearson types [5]. I type the Pearson's distribution is obtained when the two elements trinomial square (4) are real and have different signs. Denoting elements as x_1 and x_2, probability density function is:

$$f(x) = C(x - x_1)^{\frac{a+x_1}{c_2x_2 - c_2x_1}}(x - x_2)^{\frac{a+x_1}{c_2x_1 - c_2x_2}} \tag{5}$$

The solution (5) may be distributions U – shaped, when both exponent are negative, or distributions J – shaped, as one of the powers is negative and the other is positive. Distributions belonging to this family are distributions of beta class.

Type II, which is a particular case of distribution of type I, is characterized in that the module of value of square roots of the trinomial are equal. Distribution of this type is symmetrical, and its density function is:

$$f(x) = C(x^2 - x_1^2)^{-\frac{1}{2c_2}} \tag{6}$$

Another type of Pearson's distributions obtained when $c_2 = 0$, is the type III. Density function of this distribution is given by:

$$f(x) = Ce^{-\frac{x}{c_1}}(c_0 + c_1 x)^{\frac{c_0 - ac_1}{c_1^2}} \tag{7}$$

Since the reduction $c_2 = 0$ is equivalent to the union $\beta_2 = 1{,}5\beta_1 + 3$, the density distribution of type III expressed by the parameter β_1 can be written as:

$$f(x)=C\left(\frac{12+6x\sqrt{\beta_1}+3\beta_1+1{,}5x\beta_1\sqrt{\beta_1}}{12+3\beta_1}\right)^{-\frac{\beta_1-4}{\beta_1}} e^{-\frac{2x}{\sqrt{\beta_1}}} \tag{8}$$

Particular cases of this type are distributions of gamma distributions.

Type IV of Pearson's distribution system, which is a basic type, we get when the trinomial square (4) does not have real roots. Its density function is given by:

$$f(x)=Ce^{\frac{(c_1-2ac_2)arctg\left(\frac{c_1+2c_2x}{\sqrt{4c_0c_2-c_1^2}}\right)}{c_2\sqrt{4c_0c_2-c_1^2}}}\left(c_0+x(c_1+c_{2x})\right)^{-\frac{1}{2c_2}} \tag{9}$$

This type is in fact an asymmetric version of the Student's t distribution. His special case, according to Heinrich [11], the Cauchy distribution is.

Another type of Pearson's curve (Type V) is obtained when trinomial square (4) has only one root. The density distribution of this type is:

$$f(x)=Ce^{\frac{2ac_2-c_1}{c_2(c_1+2c_2x)}}\left(c_1+2c_2x\right)^{--\frac{1}{c_2}} \tag{10}$$

Type VI of distribution is obtained, however, the real roots of trinomial square (4) exist and have the same sign. Density of Pearson's distributions type VI is as follows:

$$f(x)=C(x-x_1)^{\frac{a+x_1}{c_2x_2-c_2x_1}}(x-x_2)^{\frac{a+x_2}{c_2x_1-c_2x_2}} \tag{11}$$

Symmetrical type VII of distribution is a special case of Pearson's type VI distribution. It is obtained when $c_1=a=0$. Its density function is expressed by the formula:

$$f(x)=C\left(c_0+c_2x^2\right)^{--\frac{1}{2c_2}} \tag{12}$$

A special case of this family of distributions is the Student – t distribution.

A full list of all 12 types of curves Pearson's Magiera [7].

3 Description of the data

Presented in this paper considerations was based on available data collected and processed by a plant representing the automotive industry leading its activities in the province of Silesia. The analyzed data derived from a middle – operating control of chosen characteristic of produced there element. Shared data cover a range of information about their controls selected characteristics of each batch produced the selected item in the period from 10.09.2012 to 15.09.2012. In total, this is 10969 records.

During this period, there was no disturbance of the production process, since all obtained during the audit, the values were situated in determined for the controlled characteristic standards. The analyzed data can be treated so as data from a process of correct course. Therefore, this data can help determine the control limits of the controlled characteristic.

These data were previously similar analysis. In [2] tested the compatibility of the empirical distribution of the analyzed data with normal distribution and gamma distribution. In both cases verified hypothesis was rejected.

4 Analysis of the correctness of the manufacturing process

The analysis of the production process should start by checking (usually based on historical data) assumptions of control chart. As the Chmielińska [2] analyzed the empirical distribution of the data in this study can not be approximated by known theoretical distributions, including the normal distribution.

Due to the lack of normal distribution of the controlled diagnostic variable classical control charts, requiring compliance of the empirical distribution of the data with a normal distribution, can lead to an incorrect assessment of the correctness of the production process. This situation occurs in the case of the analyzed data. The classical control chart (Figure 2) indicates that the production process is out – of – control. ARL_0 is equal $ARL_0=79$, while it is known that the process runs correctly, that the ARL_0 should be the order of 370.

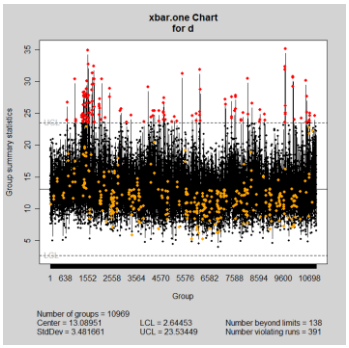

Figure 2 The classical control card removed for the analyzed data

4.1 Non-classical method for determining the control limits

In view of the risk of an erroneous assessment of the production process for the empirical data on the distribution significantly different from a normal distribution the control limits on the control chart should be set in a different way. The article [2] studied the consequences of monitoring the analyzed production process using control charts based on the quantile of the theoretical distribution, whose sharp is close to the shape of the distribution of empirical data and based on the quantile of estimated by a kernel estimation density function of empirical data. Also Chakraborti, van der Laan, van de Wiel [1] provide an effective solution to this problem. In this paper a method for determining the control lines based on quantiles suitably selected curve of the Pearson family of distributions is proposed.

The preparation of the proposed chard, should start by selecting the type of curve that approximates the empirical distribution of the data the best from among the family of Pearson's distributions. It is necessary to determine the parameters of all functions and their mutual comparison. In this case, the analysis carried out in the R. In the package PearsonDS, which is dedicated Pearson curves, is a function *pearsonMSC* that sets the parameters for all seven basic Pearson curves and makes their comparison using information criteria (ML, AIC, AICC, BIC, HQC). All benchmarks show that the type IV curve of Pearson (Pearson IV) models the empirical distribution of the data the best of all.

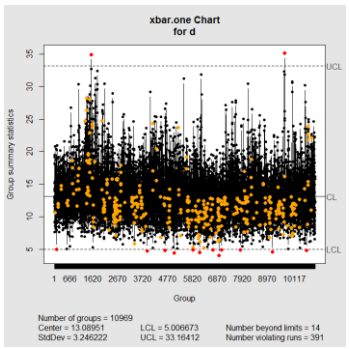

Figure 3 Control chard for the analyzed data plotted where control lines were determined on the basis of quantile of Pearson curve type IV

Figure 3 shows the control chart where the control limits are based on the values of the corresponding quantiles fitted to the data Pearson curve type IV. On this basis it can be concluded that the proposed method is effective for determining the control limits for the data of non-normal probability distribution. It is determined at a such level that the probability of obtaining a signal of deregulation for in – control process is of the order of $p_s = 0{,}001$. ARL_0 is equal $ARL_0 = 784$.

4.2 Verification of the proposed method for determining the control limits based on the quantile of corresponding curve form the Pearson family of distributions

In order to verify the accuracy of the proposed method for determining the control lines based on the quantile of suitable Pearson's curve the computer simulations was carried out. The simulation procedure, including two variants proceed in stages. The first option assumes property run of process. In the first step, from the available data samples consist of n – element were randomized. This samples form the basis of the determining of levels of control lines. Considered discussed two methods of determining the control limits, i.e. the classical method and a method based on the values of the quantile of corresponding Pearson distribution curve. In case of the

proposed method parameters for all types of distributions were determined, their fitting to empirical data assessed using the information criteria (ML, AIC, AICC, the BIC, HQC) to select the best type, then of the type indicated by the five considered information criteria chosen this dominant recognizing it as the type, which is the best suited to the empirical data. In the next step, control charts were plotted with the specified control limits and for the available data, the actual number of exceedances of the control line, and the signal probability and ARL_0 were counted. The procedure was repeated 1000 times, averaging the results. Control lines determined based on samples consist of 100, 500, 1000, 2000, 4000 and 5000 elements.

In the second option assuming the out – of – control process, as in the first variant *n* - elements samples were used for the determination of the control lines in accordance with an algorithms of the analyzed methods. Then the generated values from a normal distribution with parameters $\mu = 11$ and $\sigma = 2{,}9$. These values were applied to the prepared control charts, then number of exceedances of control lines were counted, the probability of a signal and ARL_1 were calculated. The procedure was repeated 1000 times, averaging the results.

Sample size *n*	The in – control process		The out – of – control process	
	The classical control chart	The chart based on the quantile of corresponding Pearson's curve	The classical control chart	The chart based on the quantile of corresponding Pearson's curve
100	69,54	104,67	318,77	23,43
500	76,86	334,73	454,58	39,83
1000	78,68	495,10	474,56	47,54
2000	79,31	569,94	487,34	50,54
4000	79,45	626,44	494,23	51,06
5000	79,47	614,44	498,00	52,46

Table 1 ARL_0 and ARL_1 for the analyzed data according to the method of determining the level of control lines and the number of initial sample

Table 1 shows the average run length in depend on the method for determining the control lines and the number of the initial sample of analyzed data for a process of correct run and data derived from of the normal distribution with parameters $\mu = 11$ and $\sigma = 2{,}9$ for the out – of – control process. On this basis it can be concluded that regardless of the method for determining the control lines and accuracy of the process, along with an increase in the initial sample size ARL increases. Assuming correct run of manufacturing process, this means that with the increase in the sample size precision of the assumed methods increase. For the in – control process ARL_0 value should be large enough (about 370) to false signals about the deregulation process did not occur very often, and when the process is out – of – control ARL_1 value should be small enough that the deregulation can be readily detected [10]. Comparing the ARL values obtained by the classical control chart and the proposed chart should recognize that the non – classical method, based on the corresponding curve form the Pearson's family of distributions guarantee the ARL_0 and ARL_1 at the expected level. Classic control chart for the analyzed data generates the exact opposite – low ARL_0 value to regulated process and high ARL_1 value to process deregulated. This means that in case of regulated process, numerous false signals of deregulation are occurred, while, in case of deregulated process this chart, based on the assumption that the monitored characteristic has a normal distribution with parameters similar to the values set at the sample, the value of which, in the present case are at a level of $\bar{x} = 13{,}09$ and $s = 3{,}48$, regardless of the initial sample size is insensitive to the disorder introduced in the process. Adverse results for classical control chart are the result of the lack of fulfillment of its basic assumption – the assumption of normal distribution of the controlled characteristic.

Analyzing the ARL values generated by the proposed control chart should be aware that they depend on the goodness of fit of Pearson's distribution to the empirical distribution of the data. This goodness depends on the choice of the appropriate type curve and the value of the estimated parameters. Choosing Pearson's curve used to estimate the accuracy of the analyzed process and at the same time the value of the parameters is determined by the initial sample drawn out by lot. Analyzing the information contained in Table 2, showing the distribution of a function of the Pearson's family of curves depending on the size of the initial sample, it should be noted that the correct (same as for the totality of the available data) type of curve with probability $p = 0{,}9$ is chosen from among the available options for sample numbering a minimum 1,000 items. For 500 – elemental initial sample, which is a true dominant type IV, but the probability of selecting is row $p = 0{,}64$. Too small initial sample size makes the appropriate type (for the analyzed data – type IV) is chosen very rare. For the 100 – elemental initial sample type IV is considered to be the most appropriate for modeling the analyzed data average fifteen times for each hundred attempts made.

Sample size *n*	The Pearson's cave type							
	0	**I**	**II**	**III**	**IV**	**V**	**VI**	**VII**
100	0,0715	0,0285	0,0045	0,256	0,152	0,471	0	0,0165
500	0	0	0	0,0165	0,641	0,3425	0	0
1000	0	0	0	0,0005	0,907	0,0925	0	0
2000	0	0	0	0	0,9895	0,0105	0	0
4000	0	0	0	0	0,995	0,005	0	0
5000	0	0	0	0	0,9955	0,0045	0	0

Table 2 Distribution of Pearson's curves type for the analyzed data, depending on the size of the initial sample

Analyzing the information contained in the tables above, it should be noted that the proposed method to with the assumed high probability (at least $p = 0{,}9$) chose the correct Pearson's distribution giving acceptable results (comparable to the results achieved by classical control chart with the fulfillment of assumptions) requires a relatively large initial batch of samples least 1,000 items.

5 Conclusions

Presented in this paper reflections on the monitoring of production processes characterized by a non – normal distribution of the controlled characteristics lead to the conclusion that the use of the classical approach based on the common infringed assumption about the normality of distribution of controlled characteristics, leading to an incorrect assessment of the correctness of the process being monitored. Proposed in this paper method for determining control limits for the monitored process based on modeling the empirical distribution of diagnostic variable using Pearson distributions and estimating about proper conduct of the process being monitored on the basis of the resulting distribution, belongs to distribution – free methods.

On the basis of carried out simulation analysis can be concluded that the proposed method has the desirable features of good control chart. The proposed chart responds quickly to increase of defective in manufacturing process and does not indicate too often the disturbance, when the process is in – control. The proposed method seems to be an effective method for determining the control lines for a process, which monitored characteristic has unknown distribution. The huge advantage of it is to not assuming any kind of distribution. The disadvantage of the proposed method is a need for a large initial sample required to estimate the levels of control lines.

Moved in this paper problem of process monitoring, which distribution is unknown, is important from the control of the production process point of view. Therefore, the presented analysis will be further developed. The use of stable distributions to approximate the unknown distribution of the monitored characteristic is considered.

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The Global Financial Crisis and Stock Returns: Evidence from the Czech Republic, Hungary and Poland

Michaela Chocholatá[1]

Abstract. The paper deals with the analysis of the stock market co-movements of the Central Eastern European (CEE) stock markets (Czech, Hungarian and Polish) vis-à-vis the Western European stock market based on BEKK-GARCH models using daily closing values of individual stock market indices during the period from January 8, 1996 to January 21, 2013, and studies also the impact of the current global financial crisis on the corresponding conditional correlations.
The existence of the co-movement of CEE stock markets vis-à-vis Western European markets was proved. The average conditional correlations varied in the whole analysed period as follows: Hungary 0,438 – 0,470; the Czech Republic 0,415 – 0,472 and Poland 0,496 – 0,525. Although the stock market integration strengthened during the whole analysed period, it surprisingly weakened in the crisis period (the only exception was the co-movement of the Polish WIG with German DAX).

Keywords: financial integration, BEKK-GARCH, global financial crisis, conditional correlation

JEL Classification: C58, G15, G01, C22
AMS Classification: 91G70, 91B84

1 Introduction

Nowadays the term "crisis" can be heard almost everywhere. Although during the recent history we can find different periods of various crises, e. g. Asian currency crisis of 1997, Russian crises of 1998, the Brazilian devaluation in 1999, sub-prime mortgages crises in US in 2007, the current global financial crisis is often characterized to be the worst one since the Great Depression. Since the question of financial integration between economies of different countries has recently become more popular (in order to diversify the risks effectively and to improve the capital allocation efficiency), therefore it seems to be very attractive for analysts to assess the impact of current financial crisis on various especially emerging markets. There are plenty of studies dealing with this issue using various ways and methods of analysis in order to capture how shocks from one market can be transmitted to another market(s). Schmukler [18] tries to present a balanced view of financial globalization, dealing with the pros and cons that globalization entails for developing countries. Asongu [1] deals with the linkages between financial integration (globalization) and crises and also provides the survey of various definitions of contagion. We will use the definition of Forbes and Rigobon [14] who distinguish the stock market co-movement during the periods of stability and during the periods after a shock or crisis. They use the term contagion to define „a significant increase in cross-market linkages after a shock to one country (or group of countries)". So, in case that the co-movement does not increase significantly after a shock or crisis, they speak about interdependence. In their paper [14] they also present different methodologies which can be used to analyse the stock market co-movement, e. g. cross-market correlation coefficients, Autoregressive Conditional Heteroscedasticity (ARCH) and Generalized ARCH (GARCH) models, cointegration techniques and direct estimation of specific transmission mechanisms. In recent analyses the dominant role play various variants of the multivariate GARCH (MGARCH) models which are superior to univariate GARCH versions, because they are able to capture also the development of covariances over time. Different types of multivariate GARCH models can be used, e. g. VECH model [5], CCC model [6], BEKK model [12], DCC model [13] and AG-DCC model [10].[2]

The main aim of this paper is to study the stock market co-movements of the Central Eastern European (CEE) stock markets (Czech, Hungarian and Polish) vis-à-vis the Western European stock market (as a benchmark the French CAC40, German DAX and STOXX Europe 600[3] were used) based on BEKK-GARCH models[4] and also to assess the impact of the current global financial crisis on the corresponding conditional correlations.

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[2] For an extensive survey of multivariate GARCH models see e. g. Bauwens et al. [3].
[3] The STOXX Europe 600 Index represents large, mid and small capitalisation companies across 18 countries of the European region: Austria, Belgium, Denmark, Finland, France, Germany, Greece, Iceland, Ireland, Italy, Luxembourg, the Netherlands, Norway, Portugal, Spain, Sweden, Switzerland and the United Kingdom.

The CEE stock markets can be characterized as emerging stock markets and it is commonly known that the emerging stock markets are usually more volatile than the developed stock markets. Concerning the impact of the crisis, majority of the emerging markets are easily affected. Although there exist some studies dealing with the co-movements of CEE stock markets and their integration with the Western European stock markets using various benchmark stock markets as well as various techniques and also giving mixed results, not so many studies dealt with the impact of the crisis on the degree of the stock market integration. Some studies based on multivariate GARCH models are as follows: Égert and Kočenda [11] studied co-movements between three developed stock markets (France, Germany, the UK) and the three CEE stock markets based on five-minute tick intraday stock price data. They detected very little systematic positive correlation between the Western European stock markets and the three CEE stock markets. Wang and Moore [19] investigated the extent of integration of three CEE stock markets with the aggregate eurozone market. They proved a higher level of the stock market correlation during and after the Asian and Russian crisis and also during the period after integration of the CEE countries into the EU. Baumöhl et al. [2] analysed the integration of the stock markets of V4 countries with the German market and confirmed that the correlations of the stock market indices of V4 countries (the only exception was the Slovak SAX) with German DAX became higher during the analyzed period. Horvath and Petrovski [15] proved a quite high level of stock market integration between the analysed CEE countries and Western Europe (corresponding conditional correlations were around 0,6). They did not confirm the impact of the crisis on the degree of the stock market integration between the analysed group of countries.

The paper is organised as follows: section 2 deals with the methodology of MGARCH models, section 3 describes the data used for analysis and estimation results, section 4 concludes.

2 Methodology

The methodology in this paper is based on MGARCH models in order to capture the market interdependencies in the conditional mean and/or conditional variance equations. As it was already mentioned in the introduction, nowadays there exist several various multivariate GARCH model specifications with certain advantages and disadvantages, and with different restrictions and specifications concerning the conditional variance. The basic MGARCH model was firstly introduced by Bollerslev et al. [5] who extended the univariate GARCH model [4] to include a vectorised conditional covariance matrix. To estimate the unrestricted model, known as the VECH-GARCH model, it is quite a hard issue because of the large number of parameters contained within it. Since that time several other modifications of MGARCH models have been developed (some of them were mentioned in the introduction) in order to reduce the number of parameters needed to be estimated. The most widely used models for modelling of the conditional covariances and correlations are the BEKK model [12] and the DCC model [13]. There exist also some studies in order to determine whether the BEKK or the DCC model should be applied in practical analyses, e. g. Caporin and McAleer [9] and Huang et al. [16] came to the conclusion that each of these models has its advantages and disadvantages, but it is not possible to judge in general about which one of these models to prefer. In this paper we will apply the bivariate BEKK-GARCH (1,1) model. To capture the dynamic relationship in returns we use the vector autoregressive (VAR) model with *k* lags:

$$\mathbf{r_t} = \boldsymbol{\omega} + \sum_{i=1}^{k} \boldsymbol{\Gamma_i} \mathbf{r_{t-i}} + \boldsymbol{\varepsilon_t} \tag{1}$$

where $\mathbf{r_t}$ is a 2×1 dimensional vector of daily stock returns, $\boldsymbol{\omega}$ is a 2×1 dimensional vector of constants, $\boldsymbol{\Gamma_i}$ ($i = 1, 2, \ldots k$) are a 2×2 dimensional matrices of parameters and $\boldsymbol{\varepsilon_t}$ is a vector of innovations (disturbances) conditional on information at time $t-1$. The conditional distribution of $\boldsymbol{\varepsilon_t}$ is assumed to be multivariate normal with the mean zero and the conditional variance-covariance matrix $\mathbf{H_t}$, i. e.

$$\boldsymbol{\varepsilon_t} | \Omega_{t-1} \sim N(0, \mathbf{H_t}) \tag{2}$$

where Ω_{t-1} represents the information set at time $t-1$.

It is possible to specify $\mathbf{H_t}$ in different ways. Taking into account that $\mathbf{H_t}$ is a conditional variance-covariance matrix, positive definiteness has to be ensured. We will concentrate on the BEKK-GARCH model in which the matrix $\mathbf{H_t}$ can be received through the generalization of the univariate GARCH model of Bollerslev [4]. The matrix $\mathbf{H_t}$ of the BEKK-GARCH (1,1) model is defined as follows:

[4] For simplicity we will consider only the GARCH (1,1) models without further emphasising of this fact.

$$\mathbf{H_t} = \mathbf{C}'\mathbf{C} + \mathbf{A}'\boldsymbol{\varepsilon}_{\mathbf{t-1}}\boldsymbol{\varepsilon}'_{\mathbf{t-1}}\mathbf{A} + \mathbf{B}'\mathbf{H_{t-1}}\mathbf{B} \tag{3}$$

where **C** denotes a 2×2 dimensional upper triangular matrix of parameters and **A** and **B** are 2×2 dimensional matrices of parameters.

Since the presented model is an extension of a univariate one, the parameters can be similarly estimated based on the maximum likelihood method by replacing the one-dimensional function of the sample by its multi-dimensional counterpart. Taking into account the fact, that the conditional distribution of $\boldsymbol{\varepsilon}_\mathbf{t}$ is assumed to be multivariate normal, the parameters can be estimated by maximising of the log-likelihood function which is in case of bivariate models as follows (see e. g. [7], [8]):

$$L(\boldsymbol{\theta}) = -T\log 2\pi - \frac{1}{2}\sum_{t=1}^{T}\left(\ln|\mathbf{H_t}| + \hat{\boldsymbol{\varepsilon}}_{\mathbf{t-1}}\mathbf{H_t^{-1}}\hat{\boldsymbol{\varepsilon}}'_{\mathbf{t-1}}\right) \tag{4}$$

where T represents the number of observations, $\boldsymbol{\theta}$ represents the vector of all the unknown parameters and all the other symbols are the same as above.

3 Data and Estimation Results

Daily stock price indices of Central Eastern European (CEE) countries - the Czech PX, Hungarian BUX and Polish WIG20 were used for analysis. As a benchmark for Western European stock markets the French CAC40, German DAX and STOXX Europe 600 Index were used. The analysis in this paper is based on daily closing values of individual stock market indices during the period from January 8, 1996 to January 21, 2013 (totally 4088 observations)[5] and was carried out in econometric software EViews. The source of data is as follows: Czech PX [23], Hungarian BUX and Polish WIG20 [21], French CAC40 and German DAX [22], STOXX Europe 600 Index [20].

At the beginning of the analysis it is necessary to be aware of the fact that in the stock market analysis we have to do with the financial time series, one of the basic features of which is the non-stationarity. The non-stationarity in variance can be quite easily solved e. g. by using the logarithmic transformation of the corresponding time series. Since the non-stationarity in mean also represents a serious problem in econometric analysis, it is necessary to deal with it. We also started by testing the existence of the unit root in individual logarithmic stock price indices based on the Augmented Dickey – Fuller (ADF) test. At 1 % significance level we failed to reject the null hypothesis about the existence of unit root, i. e. all the series were identified to be non-stationary. The first differences of all analysed logarithmic stock indices, i. e. logarithmic stock returns, were already stationary[6]. In further analysis we also concentrated on modelling of logarithmic returns.

Descriptive statistics for logarithmic returns[7] together with the Jarque-Bera test statistics testing the normality are summarized in Table 1. The mean values vary around zero, concerning the values of standard deviations in CEE countries, the Hungarian market is most volatile with standard deviation around 1,87 %, followed by the Polish market with standard deviation 1,83 % and the Czech market with volatility of around 1,52%. Taking into account the Western European logarithmic stock returns, the highest volatility was identified in case of German DAX (1,62%) and less volatile were the logarithmic stock returns of STOXX Europe 600 Index (1,31%). All the distributions were negatively skewed and leptokurtic, the normality hypothesis can be rejected (see the Jarque-Bera statistics and corresponding probability values).

	DLBUX	DLPX	DLWIG	DLCAC	DLDAX	DLSTOXX
Mean	0.058789	0.020921	0.026020	0.016508	0.029471	0.017154
Median	0.092686	0.052739	0.022334	0.056160	0.099445	0.079673
Maximum	22.01627	12.36405	10.89614	10.59459	10.79747	9.409957
Minimum	-15.52589	-19.90195	-11.68553	-9.471537	-8.396307	-7.929709
Std. Dev.	1.875661	1.521893	1.829898	1.557932	1.616628	1.310741
Skewness	-0.183046	-0.938140	-0.140966	-0.031006	-0.059671	-0.152245
Kurtosis	14.39155	19.64185	6.544326	7.811220	6.874363	8.458169
Jarque-Bera	22121.15	47761.98	2152.782	3942.539	2558.621	5089.052
Probability	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
Observations	4087	4087	4087	4087	4087	4087

Table 1 Descriptive statistics of logarithmic stock returns

[5] Although the data span available was longer, we started from January 8, 1996 in order to overcome the infrequent trading of the Czech PX index. The number of available observations was in individual cases different, but only those data were used which were defined for each of the analysed index series.

[6] The results are available from the author upon request.

[7] Logarithmic returns were calculated as the first difference of logarithmic index series and thereafter multiplied by 100.

First of all the stock market interdependencies were measured by unconditional correlation coefficients. Since we wanted also to assess the impact of the crisis onto the market co-movements, the calculation was done for the whole analysed period as well as for the crisis period from October 10, 2008 till the end[8]. The corresponding unconditional correlations are presented in Table 2. Since the unconditional correlations of CEE markets vis-à-vis Western European markets varied between 0,487 and 0,567 during the whole analysed period (the upper triangular' portion of a matrix in Table 2), the values became higher during the crisis period October 10, 2008 – January 21, 2013 (the lower triangular' portion of a matrix) and ranged between 0,628 and 0,699. Also the unconditional mutual correlations between the Western European markets became higher during the crisis period.

	DLBUX	**DLPX**	**DLWIG**	**DLCAC**	**DLDAX**	**DLSTOXX**
DLBUX	1.000000	0.545360	0.561432	0.510925	0.493172	0.545071
DLPX	***0.653513***	1.000000	0.559377	0.531238	0.486659	0.566797
DLWIG	***0.660533***	***0.717716***	1.000000	0.526481	0.510933	0.556419
DLCAC	***0.660280***	***0.642111***	***0.683735***	1.000000	0.875543	0.946946
DLDAX	***0.650396***	***0.627895***	***0.695947***	***0.951720***	1.000000	0.900765
DLSTOXX	***0.661247***	***0.672521***	***0.698551***	***0.969245***	***0.950260***	1.000000

Table 2 Unconditional correlation matrix – the upper triangular' portion of a matrix (period: January 8, 1996 – January 21, 2013), the lower triangular' portion of a matrix (period: October 10, 2008 – January 21, 2013)

In order to assess the development of conditional correlations against time, the VAR(*k*)-BEKK-GARCH(1,1) models were estimated by non-linear maximum likelihood method assuming the multivariate normal distribution for the innovations. The number of lags *k* used in individual VAR models is in Table 3 and was specified by the Schwarz information criterion, but in case that the serial correlation was detected, more lags were used in order to ensure the uncorrelatedness.

	DLCAC	**DLDAX**	**DLSTOXX**
DLBUX	9	9	6
DLPX	10	4	10
DLWIG	10	1	8

Table 3 Number of lags *k* used in individual VAR(*k*) models

Concerning the estimation results[9], based on the statistically significant parameters of the individual VAR(*k*) models we can conclude, that there exists a price spillover effect from the Western European stock market (using each of CAC, DAX and STOXX) to Hungary and the Czech Republic but not vice versa. For Poland the price spillover effect in neither direction was confirmed. Concerning the variance equation parameters, i. e. the parameters of the BEKK-GARCH(1,1) models, all of them were statistically significant in all analysed cases which confirms the adequate use of these models.[10] Figure 1 plots the conditional correlations from the estimation of BEKK-GARCH(1,1) models for the logarithmic stock returns of CEE countries vis-à-vis Western European markets. The average conditional correlations for individual CEE countries vis-à-vis Western European markets were in the whole analysed period as follows: Hungary 0,438 – 0,470; the Czech Republic 0,415 – 0,472 and Poland 0,496 – 0,525. Similarly as Horvath and Petrovski [15] we can conclude that although the values of the conditional correlations fluctuate, it is evident that there is the co-movement of CEE stock markets vis-à-vis Western European markets.

Furthermore we tested the character of market co-movement (i. e. the intensity of the stock market integration) in individual cases by estimating the trend models for the whole analysed period and for the crisis period. The estimated values of the corresponding trend parameters are summarized in Table 4. Taking into account the results, we can conclude, that although the stock market integration strengthened during the whole analysed period, it surprisingly weakened in the crisis period (the only exception was the co-movement of the Polish WIG with German DAX). It means that the contagion effect [14] was not confirmed.

[8] It is problematic to identify the start date of the current global financial crisis. We chose the date October 10, 2008, since the analysed stock market indices rapidly declined for the first time on this day.
[9] The estimation results are not reported from the space reasons, but can be provided by the author upon request.
[10] Standardized residuals of all models were successfully tested for autocorrelation using the Portmanteau statistics.

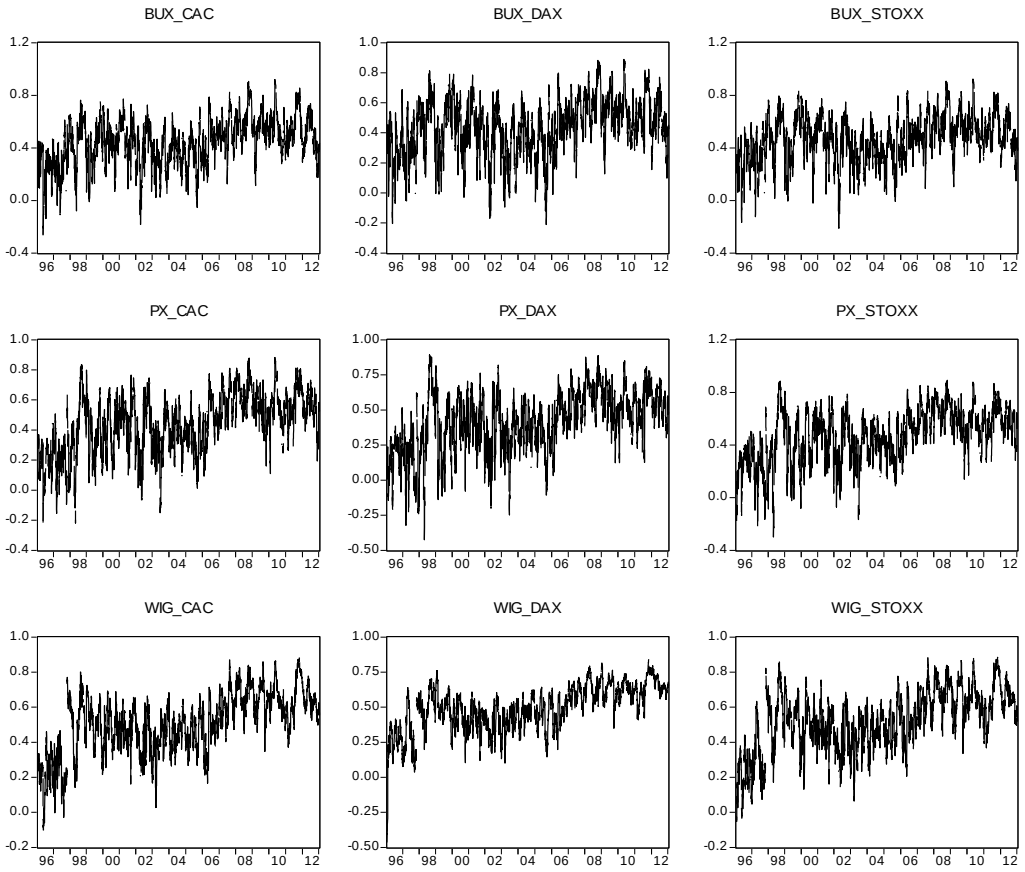

Figure 1 Conditional correlations from the BEKK-GARCH(1,1) model

	DLCAC		DLDAX		DLSTOXX	
DLBUX	$6{,}36.10^{-5}$***	-0,000123***	$5{,}08.10^{-5}$***	-0,000148***	$4{,}44.10^{-5}$***	-0,000124***
DLPX	$8{,}79.10^{-5}$***	$-3{,}09.10^{-5}$**	$9{,}26.10^{-5}$***	$-5{,}83.10^{-5}$***	$8{,}45.10^{-5}$***	$-4{,}22.10^{-5}$***
DLWIG	$8{,}97.10^{-5}$***	$-3{,}38.10^{-5}$***	$8{,}82.10^{-5}$***	$2{,}95.10^{-5}$***	$8{,}24.10^{-5}$***	$-1{,}10.10^{-5}$

Table 4 Trend in conditional correlations – the first columns (period: January 8, 1996 – January 21, 2013), the second columns (period: October 10, 2008 – January 21, 2013)[11]

4 Conclusion

In this paper we analysed the stock market integration of the Central Eastern European (CEE) stock markets (Czech, Hungarian and Polish) vis-à-vis the Western European stock market based on BEKK-GARCH models using daily closing values of individual stock market indices during the period from January 8, 1996 to January 21, 2013. We also investigated the impact of the current global financial crisis on the corresponding conditional correlations.

The whole analysis was based on logarithmic stock returns using the unconditional correlations as well as the conditional correlations from the BEKK-GARCH(1,1) models. Concerning the unconditional correlation coefficients of CEE markets vis-à-vis Western European markets these varied between 0,487 and 0,567 during the whole analysed period, but became higher during the crisis period (0,628 – 0,699). Forbes and Rigobon [14] showed that although this is the typical conclusion of the studies based on correlation coefficients, these coefficients are biased and inaccurate due to heteroscedasticity in market returns. In case of conditional correlations from the BEKK-GARCH(1,1) models the results were different. Since the values ranged during the whole analysed period between 0,415 and 0,525, no higher intensity of the stock market integration was proved during the crisis period. The results concerning the height of conditional correlation coefficients are more or less comparable to those of other studies (see e. g. [2], [15], [19]). In order to assess the impact of the crisis period, the con-

[11] Note: The symbols *** and ** denote the statistical significance of the parameter at the 0,01 and 0,05 significance level.

cluding remarks of Horvath and Petrovski [15] are similar as ours. Although they did not investigate separately the pre-crisis and crisis periods, just based on the graphical illustration they came to the conclusion that the conditional correlations in the Czech Republic and Hungary decreased somewhat during the crisis period and in general the crisis does not have impact on the strength of stock market integration between CEE countries and Western Europe. On the other side, Wang and Moore in [19] tested the impact of various crises and confirmed strengthened integration between CEE countries and Western European countries. They also provide the survey of several other studies which confirmed that international correlation increased in crisis periods.

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Production Function and Human Capital: The Case of Visegrad Countries

Silvie Chudárková[1], Tomáš Verner[2]

Abstract. The Classical economists already emphasized the effect of education, more precisely skills and knowledge (human capital), to increase individual productivity and wealth of nation. Comprehensive concepts of theory of human capital appeared in the early Sixties of 20th century. In the last two decades, the human capital influence on economic growth and productivity was confirmed by many economic studies. The aim of this paper is to find out whether there is significant human capital effect on economic output in Visegrad group countries during the period 1999-2011. Augmented standard aggregate production function linking output to productive inputs – labour, physical capital, human capital and total factor productivity is used according to the new theories of economic growth. Many researchers express human capital as investment in education, schooling enrollments or average length of education. From our point of view, these variables do not seem to be appropriate indicators of human capital. The value of it is expressed by tertiary graduates. Annual data collected from Eurostat were employed. Due to shortcoming of observations panel data model was performed. We found out negligible, but positive and statistically significant influence of human capital on economic growth.

Keywords: economic growth, human capital, panel data model, production function.

JEL Classification: J24, C30
AMS Classification: 62P20

1 Introduction

The production function determines the interdependence of input production factors and output. The production function specifies the maximum volume of output which could be produced for a certain time period and for a given amount of inputs and with the combination of inputs. According to the considered number of factors which indicate the volume of production, the production functions are divided into single-factor and two-factor, more precisely multifactor production function. The aggregate production function describes the relationship of the volume of used input production factors, technological progress and output. The aggregate production function became the core of models of economic growth. The issue of economic growth accompanies economic theory for its whole existence. In the 20th century we can distinguish two important milestones in the development of the theories of economic growth. Firstly, Forties and Fifties brought Keynesian models of growth followed by the neoclassical models of growth. Solow's model is the best known neoclassical growth model and became the subject of the further research. Assumption that labour is identical in all countries belongs to the contentious issues of Solow's model. This model assumes that education and qualifications are coincident in all countries. Equally qualified labour allows apply the best available technologies to all countries, which is not really true. The poorer countries often have worse educational systems and suffer from a lack of qualified workers, who hinder the use of advanced technologies. This shortcoming should be removed by including the human capital in the production function, as an additional explanatory variable. The second wave of interest in the issue of growth appeared at the end of Eighties and Nineties. In particular this was reflected by developing models of endogenous growth, which contain human capital, inter alia.

The main aim of this paper is to find out whether there is significant positive human capital effect on economic output in Visegrad group countries. Furthermore, we can consider the higher effect of human capital the greater improvement in the competitiveness at the national level. Education, especially higher education, and training represents the fifth of twelve pillars of competitiveness at macroeconomic level [17]. Besides, national competitiveness the increasing significance of regions in concept of European Union deserves more attention

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especially because of the economic efficiency of regions representing the basis of competitiveness of the country [15].

The paper is structured as follows. In Section 2, we define production function and the position and importance of human capital in this function. In Section 3, we introduce and describe the dataset and specify the used methods. In Section 4, we present and discuss the results obtained from testing. Section 5 concludes the paper.

2 Human capital as a production factor

Becker [7] defined human capital as skills and adequate motivation to apply these skills. The main premise, which the human capital theory is based on, postulates that education increases the productivity of the individual. Each individual tries to optimise the return on their investment in education and will continue in the study until the rate of return on her investment in education will exceed the rate of returns of alternative investments. The benefit to the society is then the increased labor productivity of better educated members of society and also technological progress. Generally, economists of the 1960's were trying to determine how various factors contribute to economic growth.

As already been mentioned, we distinguish single-factor and two-factor, more precisely multifactor production functions. The single-factor production function expresses the relationship between one variable input used in the production (usually labour) and a range of output. However, two-factor production function is examined and discussed in the economic theory more often. The two-factor production function can be represented graphically by using isoquant. Generally the two-factor production function can be expressed as follows:

$$Y = f(L, K) \tag{1}$$

where Y is the real product, L is the labour and K is the capital.

A Cobb-Douglas production function became a significant and widely used two-factor production function. Its authors belong to representatives of the neoclassical growth theory. Cobb-Douglas production function thus involves two input production factors – labour and capital. Labour and capital are included in the production function in various proportions and combinations, with the possibility of their mutual and unlimited substitution and with a free pricing of production factors depending on the dynamics of labour and capital. In basic expression is constructed the Cobb-Douglas production function so that the production effect of one production factor could be represented by the effect of another [12]:

$$Y = f(A, L^{\alpha}, K^{\beta}) \tag{2}$$

where Y is the real product, A is transformation parameter, L is the labour, K is the physical capital, α is labour elasticity coefficient and β is capital elasticity coefficient. If $\alpha + \beta < 1$, then there are decreasing returns to scale (if $\alpha + \beta = 1$, constant returns are present, and $\alpha + \beta > 1$, increasing returns are observed). The transformation parameter A is a coefficient of the aggregate productivity of production factors, which refers to both factors (also called total factor productivity). We discover size of the parameters α and β through the statistical methods.

Human capital as a production factor is included in the new growth theory models. Total product in the endogenous growth models is determined by both physical capital and labor and human capital which is accumulated in through education in every individual. The new growth theory applies the extended Solow's model (3) [12] and production function presented in the following form (4) [5]:

$$Y = A \cdot L^{\alpha} \cdot K^{\beta} \cdot e^{rt} \tag{3}$$

where Y is the real product, A is transformation parameter, L is the labour, K is the physical capital, α is labour elasticity coefficient and β is capital elasticity coefficient and e^{rt} is time factor, which reflects the influence of quantitative changes in production, including technological progress,

$$Y = A_t (L_t)^{1-\alpha-\beta} K_t^{\alpha} H_t^{\beta} \tag{4}$$

where Y is the product, A shows the level of technology, L is the labour, K is the physical capital, H is the human capital, α and β determine the proportions of individual factors on the overall product.

P. M. Romer, R. E. Lucas (see [13]) and also N. G. Mankiw, D. Romer and D. Weil [14] contributed significantly to the development of endogenous growth models. Overall, the research in this area confirms the existence of a relationship between the development of education and economic growth. One of the conclusions of endogenous growth models is that economic growth depends partly on the level of human capital. It assumes that human capital is the source of production of new ideas. It is true that the more developed economy, the stronger the relationship of education to the economic growth. While in less developed countries the primary task of starting economic growth nationwide is to ensure primary education, in the developed countries on the other hand is to drive further economic growth primarily on ensuring tertiary education. Romer [16] in his work addresses the issue of differences between the education and experience on the one hand, and technological progress on the other. The main source of economic growth is technological progress, in his opinion. Mankiw, Romer and Weil [14] in their work tried to eliminate shortcomings of the Solow's model by including the human capital expressed as an investment in education. Simplified representation of the value of human capital, respectively identification of human capital investment in education, with the achieved level of education or the number of students in various stages of study, is often a prerequisite in empirical studies examining the human capital at the macroeconomic level.

Barro [3] and others find a strong positive correlation between schooling enrollment and the subsequent growth rate gross domestic product (GDP) per capita. Barro [4] states that the growth of human capital expressed as an average length of education by one year corresponds to an increase of GDP growth by four percentage points a year. Bassanini and Scarpetta [6] states, that their results point to a positive and significant impact of human capital accumulation to output per capita growth. If the average length of study period is ten years, one additional year of study will increase production by six per cent. The existence of correlations between human capital, in this case the number of university graduates, and economic growth in their work was also confirmed by De la Fuente and Donénech [9]. Through that research the need for investment in human capital can be justified. However, there are views that refute or do not confirm the influence of human capital on economic growth. Bils and Klenow [8] in their study do not disprove any correlation between economic growth and human capital. However, they concluded that it is the level of gross domestic product, respectively its growth, leading to a higher level of human capital in the economy. Unlike previous studies on the causality of these variables this one is seen in the reverse order. Söderbom and Teal [18] came to the conclusion that human capital has a small, and not statistically significant effect, on the level of output. In addition, a more detailed analysis of the labour market during the period 1999 – 2010 in Visegrad group countries can be found in Tvrdon [19].

3 Data and econometric methodology

We employ annual data for Visegrad Group countries between 1999 and 2011. GDP at constant prices (2005) in EUR was used as a proxy variable of real product (Y), gross fixed capital formation at constant prices (2005) in EUR for physical capital (K), while for human capital (H) we used sum of graduates in upper secondary education, post-secondary non-tertiary education, first and second stages of tertiary education (i.e. ISCED'97 levels 3, 4, 5, 6). Worked hours were used for labour (L). All of the variables were collected from Eurostat database.

To examine the above mentioned relationship we perform panel data analysis. Panel data (or longitudinal data) cover both a time series and a cross-sectional dimension compared to pure time series or cross-sectional data [20]. Panel data models have become more and more popular among researchers because of their capacity for capturing the complexity of human behavior as contrasted to cross-sectional or time series data models [10]. Klevmarken [11], Hsiao [10] and Baltagi [2] list a number of panel data's benefits, e.g. (i) controlling for individual heterogeneity, (ii) give more informative data, more variability, less collinearity among the variables, more degrees of freedom and more efficiency, (iii) are better able to study dynamics of adjustment, (iv) are better capable to identify and measure effects that are simply not detectable in pure cross-section or pure time series data, (v) allow to construct and test more complicated behavioral models than purely cross-section or time series data and thus allow a researcher to analyze a number of important economic questions that con not be addressed using one dimensional data, and limitations, e.g. (i) design and data collection problem, (ii) distortions of measurement errors or (iii) selectivity problem.

A panel data set is formulated by a sample that contains N cross-sectional units (individuals, firms, households, countries etc.) that are observed at different time periods T [1], i.e. N = 4 and T = 13 in our case. Simple linear panel data model can be written as (5):

$$y_{it} = \alpha + \beta X'_{it} + u_{it} \tag{5}$$

where y represents the dependent variable, X vector of explanatory variables and subscript i denotes cross-section dimension (V4 countries) whereas t time series dimension (1999-2011), α, β are coefficients and u is a random disturbance term. In general, three different methods can be used to estimate linear panel data models by means of ordinary least squares: (i) common constant as in equation (5), (ii) fixed effects and (iii) random effects. The common constant method implies that there are no differences among variables of the cross-sectional dimension, so-called homogenous panel. Fixed or random effects allow us to capture the differences among units; hence the random disturbance term u is given by (6):

$$u_{it} = \mu_i + \nu_{it} \tag{6}$$

where μ_i denotes unobservable individual-specific effect which is time-invariant and is responsible for any individual-specific effect that is not contained in the regression. The term ν_{it} denotes remainder disturbance which varies over individuals and time [2], [10]. But the question, which model is more appropriate still remains. For common constant and fixed effect model we can apply standard F-test under the null hypothesis (H_0 thereinafter) that all the constants are the same [1]. In random effect model we assume zero correlation between explanatory variables and the unobserved effect. Hausman test can be employed to find out if this assumption is fulfilled under H_0: random effects are consistent and efficient.

Moreover, it should fulfill the assumptions for standard ordinary least squares error terms, i.e. the remained disturbance is homoskedastic, serially and spatial uncorrelated. In particular, the Cobb-Douglas production function is nonlinear in the parameters; hence we take the natural logarithm of (4) and obtain (7):

$$\ln Y = \ln A_t + (1-\alpha-\beta)\ln(L_t) + \alpha\ln(K_t) + \beta\ln(H_t) + \mu_i + \nu_{it} \tag{7}$$

4 Empirical results

In this section we present and discuss the results from Cobb-Douglas production function with human capital in the case of Visegrad countries. At first we estimate step by step all three models by means of least squares method. According to above mentioned tests the fixed effect model seems to be the most appropriate model to identify the effect of labour, physical and human capital on real product.

The coefficient of determination (R^2) = 0.98, the regression model and coefficients are statistically significant at 5 % level. The remained disturbance ν_{it} fulfills the assumptions for used methods.

When we estimate (7), anti-log, and we get (8) for the Czech Republic, (9) for the Hungary, (10) for the Poland and (11) for the Slovakia. The unobservable country-specific effect μ_i is represented by changes in the intercept.

$$Y_{CZ} = 16.62(L_t)^{0.09}(K_t)^{0.7}(H_t)^{0.28}\nu_{it} \tag{8}$$

$$Y_{HU} = 17.71(L_t)^{0.09}(K_t)^{0.7}(H_t)^{0.28}\nu_{it} \tag{9}$$

$$Y_{PL} = 14.12(L_t)^{0.09}(K_t)^{0.7}(H_t)^{0.28}\nu_{it} \tag{10}$$

$$Y_{SK} = 16.14(L_t)^{0.09}(K_t)^{0.7}(H_t)^{0.28}\nu_{it} \tag{11}$$

The estimated coefficients and signs are in accordance with expectations. The intercept represents total factor productivity. It differs across countries, it includes, e.g. technology, resources, climate and institutions. If we increase worked hours by 1 %, output (gross domestic product) goes up by about 0.09 %, holding the other inputs constant. Similarly, one more per cent of physical capital (gross fixed capital formation) causes output in-

creasing about 0.7 % (ceteris paribus). Increasing human capital (secondary and tertiary graduates) is associated with raising the output about 0.28 per cent.

5 Conclusion

The aim of this paper was to find out whether human capital positively affected economic output. The augmented Cobb-Douglas production function was used. It expresses relations among inputs and output. Annual data of Visegrad group countries during 1999 – 2011 were employed. Gross domestic product for output, gross fixed capital formation for physical capital, sum of graduates in upper secondary education, post-secondary non-tertiary education, first and second stages of tertiary education for human capital and worked hours for labour.

The panel data analysis was conducted to examine that relationship. Fixed effect method was chosen as the most appropriate, and total factor productivity differs across countries. The positive link among inputs and output was confirmed for Visegrad countries. If we increase physical capital input by one per cent, respectively human capital, respectively worked hours, output goes up by about 0.7 %, 0.28 % or 0.09 % ceteris paribus.

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Analysis of efficiency in multi-period systems: an application to performance evaluation in education

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Abstract. Data envelopment analysis (DEA) is a non-parametric method for efficiency and performance analysis of decision making units. The paper deals with production systems where decision making units are described by their inputs and outputs in several consecutive periods. The paper presents multi-period DEA models that allow evaluation the efficiency of DMUs within the whole production chain. Efficiency and super-efficiency DEA models for multi-period systems are formulated and illustrated on a case study. The study consists in analysis of research and teaching performance of Czech economic faculties in four years period from 2007 until 2010. The model considers two inputs (number of academic employees and labour costs), two outputs for teaching efficiency (number of students and number of graduated) and the number of publications in various important categories and the number of RIV points for research efficiency.

Keywords: data envelopment analysis, performance, efficiency, multi-period models

JEL Classification: C44
AMS Classification: 90C15

1 Introduction

Data envelopment analysis (DEA) is a non-parametric technique for evaluation of relative efficiency of decision making units described by multiple inputs and outputs. This approach that was firstly introduced in (Charnes et al., 1978) is based on solving a sequence of linear programming problems. Conventional DEA models analyze relative technical efficiency of the set of n decision making units (DMUs) that are characterized by m inputs and r outputs in one period. The efficiency score θ_q of the DMU_q is defined as the weighted sum of outputs divided by the weighted sum of inputs as follows:

$$\theta_q = \frac{\sum_{k=1}^{r} u_k y_{kq}}{\sum_{i=1}^{m} v_i x_{iq}}, \tag{1}$$

where u_k, $k = 1, 2, \ldots, r$ is the positive weight of the k-th output, v_i, $i = 1, 2, \ldots, m$ is the positive weight of the i-th input, and x_{ij}, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$ and y_{kj}, $k = 1, 2, \ldots, r$, $j = 1, 2, \ldots, n$ are non-negative values for the DMU_j of the i-th input and the k-th output respectively. Conventional DEA models maximize the efficiency score (1) under the assumption that the efficiency scores of all other DMUs do not exceed 1 (100%). This problem must be solved for each DMU separately, i.e. in order to evaluate the efficiency of all DMUs the set of n optimization problems must be solved. The presented problem is not linear in objective function but it can be modified using Charnes-Cooper transformation into a linear optimization problem and then solved easily. The transformation consists in maximization of the nominator or minimization of the denominator in expression (1). The constraints of this LP optimization problem express the upper bound for efficiency scores of all DMUs except the DMU_q and the unit sum of the denominator/nominator in (1). The model that maximizes the nominator in (1) is referenced as DEA input oriented model, the model that minimizes the denominator is DEA output oriented model. In both cases the DMUs with $\theta_q = 1$ are lying on the efficient frontier estimated by the model and denoted as efficient units. Otherwise the units are inefficient and the efficiency score can be explained as a rate for increasing inputs or reduction outputs for reaching maximum efficiency.

The above mentioned models are conventional DEA models with input or output orientation. They can be further modified according to returns to scales and other assumptions. Among them are super-efficiency models that aim with ranking of efficient units as they receive the same maximum efficiency score in conventional

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models. Another category of DEA models are network models that evaluate efficiency of production units with parallel or serial structure. Another stream in theory and practice of DEA models deals with evaluation of efficiency in multi-period production systems. In this case the models estimate the total efficiency in the context of time serial data. There were proposed various models how to deal with time serial data in DEA models in the past. Malmquist index (Fare and Grosskopf, 1996) and dynamic analysis (Sueyoshi and Sekitani, 2005) are only two of them that are relatively widely applied. Sam Park and Park (2009) formulated a two-stage DEA model that evaluates aggregative efficiency in multi-period case (further referenced as PP model).

The paper aims at multi-period DEA models, formulates a modification of the model presented in (Sam Park and Park, 2009) and applies the presented models to evaluation of teaching and research efficiency of Czech economic faculties. The paper is organized as follows. Section 2 formulates PP model and its modification. Section 3 presents the application of the models presented in previous parts of the paper on the data set of Czech economic faculties. The results of numerical experiments are analyzed and discussed. The final section of the paper summarizes presented results.

2 Multi-period DEA model

Conventional DEA models can be formulated either in their primal or dual form - see e.g. (Zhu, 2003). In this section we will use the dual form that has several advantages with respect to formulation of multi-period models. Let us suppose that the DMUs are described by the same set of inputs and outputs in T consecutive time periods $t = 1, 2, \ldots, T$, and assume that x^t_{ij}, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, and y^t_{kj}, $k = 1, 2, \ldots, r$, $j = 1, 2, \ldots, n$ are the values of the i-th input and the k-th output in the t-th period of the DMUj. The first phase of the output oriented PP model with constant (variable) returns to scale assumption that evaluates the aggregative efficiency of the DMU$_q$ is formulated as follows:

$$
\begin{aligned}
&\text{Maximize} && \theta_q && \\
&\text{subject to} && \sum_{j=1}^{n} x^t_{ij}\lambda^t_j \le x^t_{iq}, && i=1,2,\ldots,m, t=1,2,\ldots,T \\
& && \sum_{j=1}^{n} y^t_{ij}\lambda^t_j \ge \theta_q y^t_{iq}, && k=1,2,\ldots,r, t=1,2,\ldots,T \\
& && \left(\sum_{j=1}^{n} \lambda^t_j = 1, \right. && \left. t=1,2,\ldots,T\right) \\
& && \lambda^t_j \ge 0, && j=1,2,\ldots,n, t=1,2,\ldots,T,
\end{aligned}
\qquad (2)
$$

where λ^t_j $j = 1,2,\ldots,n$, $t = 1, 2,\ldots, T$ are variables of the model – coefficients of linear (convex) combination of the DMUs in time period t, and θ_q is the total aggregative efficiency score of the DMU$_q$. Let θ^*_q is the optimal objective function value of model (2). Then the second phase of the PP model is as follows:

$$
\begin{aligned}
&\text{Maximize} && \sum_{t=1}^{T}\sum_{i=1}^{m} s^{t-}_i / x^t_{iq} + \sum_{t=1}^{T}\sum_{k=1}^{r} s^{t+}_k / y^t_{iq} && \\
&\text{subject to} && \sum_{j=1}^{n} x^t_{ij}\lambda^t_j + s^{t-}_i = x^t_{iq}, && i=1,2,\ldots,m, t=1,2,\ldots,T \\
& && \sum_{j=1}^{n} y^t_{ij}\lambda^t_j - s^{t+}_k = \theta^*_q y^t_{iq}, && k=1,2,\ldots,r, t=1,2,\ldots,T \\
& && \left(\sum_{j=1}^{n} \lambda^t_j = 1, \right. && \left. t=1,2,\ldots,T\right) \\
& && s^{t-}_i \ge 0, && i=1,2,\ldots,m, t=1,2,\ldots,T, \\
& && s^{t+}_k \ge 0, && k=1,2,\ldots,r, t=1,2,\ldots,T, \\
& && \lambda^t_j \ge 0, && j=1,2,\ldots,n, t=1,2,\ldots,T,
\end{aligned}
\qquad (3)
$$

where s_i^{t-} and s_k^{t+} are slack/surplus variables belonging to constraints of model (2). According to the results of both stages the DMUs can be classified as follows:

- (Fully) efficient DMUs – the optimal objective function value of model (2) $\theta^*_q = 1$ and matrices of slack and surplus variables are zero matrices.
- Weakly efficient DMUs - the optimal objective function value of model (2) $\theta^*_q = 1$ and at least one element of matrices of slack/surplus variables is positive.
- Inefficient units - the optimal objective function value of model (2) $\theta^*_q > 1$ (for output oriented models).

The second phase of the PP model is slightly modified comparing to its original formulation. The original formulation contains in its objective function a simple sum of slack and surplus variables which seems to be meaningless due to possible high differences in input/output values. That is why a sum of relative slack and surplus variables is used in objective function of model (3).

In order to evaluate the DMUs, model (2) must be solved in the first phase and depending on the optimal objective function value the second phase follows. This phase must be applied when the optimal objective function value of the first phase is $\theta^*_q = 1$, i.e. the DMU under evaluation is recognized as fully or weakly efficient. In this case the second phase allows determination whether this unit is fully or weakly efficient. The inefficient or weakly efficient DMUs can be ranked according to the optimal objective function values from the first or second phase respectively. Nevertheless, the efficient units cannot be ranked at all using the PP model as proposed in (Sam Park and Park, 2009). In single period systems, this problem is solved by using of super-efficiency models or any other models that allow ranking of efficient DMUs. Information about the most important models of this class can be found e.g. in (Jablonsky, 2009).

First super-efficiency model was introduced by Andersen and Petersen (1993). Their model is based on the idea to remove the unit under evaluation from the set of units and then measure its distance from the new efficient frontier. The super-efficiency score of an originally efficient DMU is lower than one (taking into account the output oriented model) and can be explained as a rate for possible worsening of outputs in order to keep the efficiency status. A similar approach as in Andersen and Petersen model can be used for multi-period models too. For these purposes model (2) can be extended by additional constraints that ensure zero values for all weights of the DMU under evaluation, i.e.

$$\lambda_q^t = 0, \qquad t = 1,2,\ldots,T, \tag{4}$$

The advantages of this modified model consist in a possibility to discriminate better among fully efficient DMUs and in its higher computational efficiency because the second phase, i.e. the model (3), has to be solved for weakly efficient units only. It is clear that the modified model must be solved for each DMU ones in its first phase and usually only in a few cases (weakly efficiency) in its second phase in order to get the complete ranking of DMUs.

Let us denote φ^*_q optimal objective function value of model (2) with the constraint set extended by (4). The DMUs can be ranked according to φ^*_q values as follows:

- (Fully) efficient DMUs, i.e. the units with $\varphi^*_q < 1$ (for output oriented models), can be ranked according to this value – lower values indicate higher rank.
- Weakly efficient DMUs, i.e. the units with $\varphi^*_q = 1$, can be ranked according to the results of model (3) – of course $\theta^*_q = \varphi^*_q = 1$ in this second phase model.
- Inefficient DMUs, i.e. units with $\varphi^*_q > 1$, can be ranked as the fully efficient units – lower values of φ^*_q indicate higher rank.

3 Multi-period efficiency analysis of Czech economic faculties

Evaluation of efficiency of teaching process and research outputs of higher educational institutions belongs to important and widely discussed problems. The importance of this task consists in the necessity to evaluate the performance of the institution in both educational and research areas in order to have appropriate records for allocation of funds for future periods. Due to continuing discussion about future modifications of educational system in higher education in the Czech Republic modelling in this field is of a high importance. There are many open problems connected with the mentioned task. One of them is the selection of appropriate indicators that describe the performance of the evaluated institutions − they can be both of quantitative and qualitative nature and it is not always easy to get their numerical expression. Another problem consists in the selection of a modelling tool for analysis of the given data set. Various econometric and statistical models that are usually based on estimation of parameters of production functions that explain the performance by several defined indicators are widely used in this context. In this section of the paper we are trying to apply DEA models presented in previous

section for teaching and research efficiency evaluation of 19 economic faculties of Czech public universities. Data set available for the analysis contains information about various economic, teaching and research characteristics in a four years period starting in 2007. Any attempt to apply DEA models in any field including applications in education has to contain several steps:

1. Definition of inputs. Two input variables used further in analysis are the number of academic staffs and labour cost of the faculty.
2. Definition of outputs. There are considered two groups of output variables - one for teaching efficiency and one for research efficiency. Teaching efficiency is described by the total number of students and the number of graduated students. Research efficiency is influenced by the number of publications in various categories (books, papers in journals with a positive impact factor - Jimp, and papers in proceedings indexed in Web of Science - CPCI). The overall quality of publication is measured by so called RIV points. Finally we decided to work with two sets of outputs. The first set contains two teaching outputs and research is measured by RIV points only. The second set of outputs consists of two teaching outputs as in the first set and three research outputs – numbers of publications in three important categories as mentioned above.
3. Selection of an appropriate DEA model and its assumptions (returns to scale, weight restrictions, etc.). The multi-period DEA model with constant returns to scale assumption was further applied in numerical experiments concerning evaluation of overall efficiency of Czech economic faculties from 2007 until 2010.
4. Analysis of the results given by the model(s).

The data set (two inputs, two "teaching" outputs and four "research" outputs) for all 19 economic faculties for the last year of the given time period (2010) is presented in Table 1. The complete data set for the remaining years cannot be presented due to the space limits.

Faculty	Academic staff	Labour costs	# of students	# of graduates	Books	Jimp	CPCI	RIV points
FSV UK	137	57831	4105	821	37.39	5.58	24.35	3632
EkF JČU	69	26842	1764	522	2.74	7.00	1.50	578
FSE UJEP	67	26246	2200	559	2.08	2.13	0.00	122
ESF MU	93	49739	4453	738	12.66	11.67	8.00	1064
OPF SU	108	44908	4385	882	3.77	14.07	9.33	853
FE ZČU	61	20063	2312	519	3.04	3.67	3.17	367
HF TUL	83	32510	2081	600	12.83	6.00	5.67	988
FES UP	78	35977	2639	556	5.39	9.17	3.07	933
FP VUT	81	30280	2758	821	4.53	6.50	2.63	538
EkF VŠB	175	71448	6539	1701	24.13	15.67	7.51	1684
FME Zlín	84	28277	3419	970	5.99	29.03	3.50	889
FFU VŠE	82	42899	3176	805	2.84	1.50	10.50	946
FMV VŠE	172	71074	4713	1301	13.30	0.00	0.00	902
FPH VŠE	106	47113	3778	1022	4.61	3.67	3.33	422
FIS VŠE	100	43880	3332	686	4.35	48.22	11.54	1274
NH VŠE	65	28621	2572	462	5.41	2.83	15.12	1064
FM VŠE	39	16542	1437	321	0.16	8.84	2.00	277
PEF ČZU	186	121546	9462	2822	12.79	15.00	5.62	1648
PEF MZLU	114	49361	3658	958	4.48	4.00	6.50	1151

Table 1 Data set – 2010.

Computational experiments were realized with two set of outputs (I and II) as mentioned above with two four-period DEA models under the assumption of constant returns to scale – the original PP model (2) and (3) and its super-efficiency modification that is extended by constraints (4). All calculations were performed using our own codes written in LINGO modelling language. The results are presented in Table 2. This table contains the following information:

- Efficiency scores of all DMUs for both sets of outputs given by the original PP model – as its output oriented version is used the higher values of the scores mean lower rank.
- Rank of the units (faculties) according to the efficiency scores given by the PP model.

- Super-efficiency scores of all DMUs for both sets of outputs given by modification of the PP model (2) – due to output orientation of models the lowest score means the highest rank.
- Rank of the units (faculties) according to the super-efficiency scores given by modification of the PP model.

Faculty	Eff.score I PP model	Eff.score II PP model	Rank I	Rank II	Eff.score I Super PP	Eff.score II Super PP	Rank I	Rank II
FSV UK	1.0000	1.0000	1	1	0.4536	0.4474	1	2
EkF JČU	1.0798	1.1439	12	17	1.0798	1.1439	12	17
FSE UJEP	1.0413	1.0413	8	12	1.0413	1.0413	8	12
ESF MU	1.0000	1.0000	1	1	0.9611	0.9043	6	8
OPF SU	1.0731	1.0506	10	13	1.0731	1.0506	10	13
FE ZČU	1.0492	1.0239	9	11	1.0492	1.0239	9	11
HF TUL	1.3595	1.1518	18	18	1.3595	1.1518	18	18
FES UP	1.2344	1.1175	17	16	1.2344	1.1175	17	16
FP VUT	1.0834	1.0565	13	14	1.0834	1.0565	13	14
EkF VŠB	1.1257	1.0000	14	1	1.1257	0.8934	14	7
FME Zlín	1.0000	1.0000	1	1	0.7002	0.6251	2	3
FFU VŠE	1.0000	1.0000	1	1	0.8167	0.7035	5	4
FMV VŠE	1.4650	1.3068	19	19	1.4650	1.3068	19	19
FPH VŠE	1.0767	1.0000	11	1	1.0767	0.9596	11	9
FIS VŠE	1.2142	1.0000	16	1	1.2142	0.4014	16	1
NH VŠE	1.0000	1.0000	1	1	0.7993	0.7355	4	5
FM VŠE	1.1907	1.0000	15	1	1.1907	0.9737	15	10
PEF ČZU	1.0000	1.0000	1	1	0.7580	0.7391	3	6
PEF MZLU	1.0298	1.1072	7	15	1.0298	1.1072	7	15

Table 2 Efficiency scores and rank of faculties given by the PP model

The comparison of results given by the original PP model and its super-efficiency modification is quite clear. The original procedure cannot rank fully efficient DMUs and as it is shown, among the DMUs 6 of them are efficient (10 of them in case of the second set of outputs), i.e. 6 or even 10 units cannot be ranked by using the original model. The modified procedure allows ranking of all units easily. Another question is a discussion about the contribution of the presented PP model and its modification. As it is clear the efficiency (super-efficiency) score of this multi-period model is given as the best (minimum in case of output oriented model) efficiency score of particular periods. That is why the advantage of the PP model consists in a possibility to get the multi-period efficiency score by solving one optimization problem for each DMU instead of solving T optimization problems for each DMU.

The results of models with two different sets of outputs presented in Table 2 are quite contradictory. It shows that it is very important to pay attention to a proper selection of input and output variables of the model. The models "I" contain only one output – the number of RIV points. This number is given directly as the weighted sum of the numbers of publications in particular categories (each category has its weight reflecting its importance). The models "II" contain three outputs – books, Jimp and CPCI proceedings. The results in Table 2 show that all DMUs that are fully efficient in models "I" are fully efficient in models "II" as well. Nevertheless, the opposite relation does not hold. Some of inefficient units in models "I" are efficient in models with three outputs. An extreme situation occurs for the Faculty of Informatics and Statistics, University of Economics, Prague (FIS VSE). This faculty is inefficient in models "I" and its efficiency score is quite high and that is why its rank is quite low (16). In the contrary FIS VSE is efficient in models "II" and its super-efficiency score is the lowest, this means that it is on the highest rank among all faculties. This difference can be explained quite easily by analysis of the source data set presented in Table 1. FIS VSE has the highest number of publication in CPCI proceedings in 2010, it is much higher than the other faculties have. This fact causes that the faculty is fully efficient and leads to a quite low value of the super-efficiency score as well. This example demonstrates very well that it is necessary to build the model carefully and explain its results in the context of reality.

4 Conclusions

Analysis of efficiency of DMUs within multiple periods is an important task and many various models were proposed for these purposes in the past. The models presented in this paper and applied for evaluation of efficiency of Czech economic faculties is one of the last contributions in this area. The advantage of the proposed super-efficiency multi-period DEA model consists in its computational efficiency because it is necessary to solve only one optimization problem for each DMU in order to get its aggregative (super-) efficiency score. Disadvantage is the fact that the final results depend on the efficiencies of the DMUs in particular periods without any interconnections among periods. Another problem consists in a possible infeasibility of the optimization problem (2) with additional constraints (4) under the assumption of variable returns to scale. This problem is discussed e.g. in (Lee and Zhu, 2012). A future research can be focused on analysis of multi-period production systems with interconnections among the periods and using other DEA models than the conventional ones are, e.g. SBM models proposed in (Tone, 2002).

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Public service system design with disutility relevance estimation

Jaroslav Janáček[1], Marek Kvet[2]

Abstract. This paper deals with the problem of designing the optimal structure of most public service systems such that the users´ discomfort is minimized. The discomfort is usually proportional to the sum of demand-weighted network distances between the users and the nearest source of provided service. We assume that all facilities have equal setup cost and enough capacity to serve all users. Real instances of the weighted *p*-median problem are characterized by big number of possible service center locations, which can take the value of several thousands. In such cases the location-allocation approach usually fails due to enormous computational time or huge memory demands. This weakness has led to the approximate covering approach based on specific model reformulation. The disutility values are here estimated by some upper and lower bounds given by so called dividing points. Deployment of dividing points influences the solution accuracy. It is based on the idea that some disutility values can be considered relevant and are expected to obtain in the optimal solution. Hereby, we are studying various approaches to the relevance with their impact on the solution accuracy and saved computational time.

Keywords: weighted *p*-median problem, approximate covering model, disutility relevance estimation

JEL Classification: C61
AMS Classification: 90C06, 90C10, 90C27

1 Introduction

Design of almost any public service system [2], [4], [10] includes determination of center locations, from which the associated service is distributed to all users of the system. The service providing facilities must be usually concentrated to a limited number of centers due to economic and technological reasons [8], [11]. Regardless of the case whether the service is delivered to the users or the users travel for the service to the nearest center along the shortest path on the transportation network, which covers the serviced area. Thus the public service system structure is formed by the deployment of limited number of service centers and the associated objective in the standard formulation is to minimize the social costs, which are proportional to the distances between serviced objects and the nearest service centers. The social costs can be denoted as a disutility, to which a system user is exposed. In other words, the standard approach to a public service system design assumes that the user is serviced from the nearest located service center or from the center, which offers the smallest disutility to the user. Thus a mathematical model of the public service system design problem often takes the form of a large weighted *p*-median problem, where the numbers of serviced users and possible service center locations take the value of several thousands. The number of possible service center locations seriously impacts the computational time [9]. The necessity of solving large instances of the *p*-median problem as a predecessor to the weighted *p*-median problem has led to the approximate approach, which enables to solve real-sized problems in admissible time [1], [3], [5], [6]. The approximate approach is based on the upper bound minimization and performs as a heuristic, where the lower bound of the optimal value of the objective function is easy to obtain. The accuracy of the approach is given by a convenient determination of so-called dividing points, which are used for the upper approximation of the objective function. Based on the radial formulation of the classical *p*-median problem, we suggest the approach to the design of a public service system. The approach generalizes the radial approaches to the classical *p*-median problem to obtain a solving technique for the large weighted *p*-median problem, which is the core of the approach to the public service system design, where the system disutility is to be minimized.

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2 Public service system design model

The public service system design with minimal disutility is a task of location of at most p service centers so that the sum of individual disutilities of each user coming only from the nearest located service center is minimal. To describe the problem, we denote I a set of possible service center locations and J the set of possible users' locations. The symbol b_j denotes the number of users located at the location j. The disutility for a user at the location j following from the possible center location i is denoted as d_{ij}. Under assumption that the user's disutility originates only from the nearest service center, we can state the problem as follows:

$$\textit{Minimize} \qquad \left\{ \sum_{j \in J} b_j \min\left\{ d_{ij} : i \in I_1 \right\} : I_1 \subset I, |I_1| \le p \right\} \tag{1}$$

The symbol I_1 denotes a subset of the former set of all possible service center locations. The problem (1) is also known as the weighted p-median problem, which is broadly discussed in [1], [3], [5], [6] from the viewpoint of solving techniques suggested for fast solving of the huge instances. Within this paper, we focus on so called radial formulation used in the above papers and we explore the approaches based on the set of dividing points.

The strategic decision in the problem (1) concerns location of centers at possible center locations from the set I. To model this decision at particular location, we introduce a zero-one variable $y_i \in \{0, 1\}$, which takes the value of 1, if a center should be located at the location i, and it takes the value of 0 otherwise. To obtain an upper or a lower bound of the original objective function, the range $[d_0, d_m]$ of all possible disutilities $d_0 < d_1 < \ldots < d_m$ from the matrix $\{d_{ij}\}$ is partitioned into $r+1$ zones. The zones are separated by a finite ascending sequence of so called *dividing points* $D_1, D_2 \ldots D_r$ chosen from the sequence $d_0 < d_1 < \ldots < d_m$, where $0 = d_0 = D_0 < D_1$ and also $D_r < D_m = d_m$. The zone k corresponds with the interval $(D_k, D_{k+1}\rangle$. The length of the k-th interval is denoted by e_k for $k = 0 \ldots r$. In addition, auxiliary zero-one variables x_{jk} for $k = 0 \ldots r$ are introduced. The variable x_{jk} takes the value of 1, if the disutility of the user at $j \in J$ from the nearest located center is greater than D_k and it takes the value of 0 otherwise. Then the expression $e_0x_{j0} + e_1x_{j1} + e_2x_{j2} + \ldots + e_rx_{jr}$ constitutes an upper approximation of the disutility d_{j*} from user location j to the nearest located service center. If the disutility d_{j*} belongs to the interval $(D_k, D_{k+1}\rangle$, then the value of D_{k+1} is the upper estimation of d_{j*} with the maximal possible deviation e_k. Let us introduce a zero-one constant a_{ij}^k for each triple $[i, j, k] \in I \times J \times \{0 \ldots r\}$. The constant a_{ij}^k is equal to 1, if the disutility d_{ij} between the user location j and the possible center location i is less or equal to D_k, otherwise a_{ij}^k is equal to 0. Then the radial-type weighted covering model can be formulated according to [5], [6] as follows:

$$\textit{Minimize} \qquad \sum_{j \in J} b_j \sum_{k=0}^{r} e_k x_{jk} \tag{2}$$

$$\textit{Subject to:} \quad x_{jk} + \sum_{i \in I} a_{ij}^k y_i \ge 1 \quad \forall\, j \in J,\ \forall\, k = 0, 1, \ldots, r \tag{3}$$

$$\sum_{i \in I} y_i \le p \tag{4}$$

$$x_{jk} \ge 0 \quad \forall\, j \in J,\ \forall\, k = 0, 1, \ldots, r \tag{5}$$

$$y_i \in \{0, 1\} \quad \forall\, i \in I \tag{6}$$

The objective function (2) gives the upper bound of the sum of original disutility values. The constraints (3) ensure that the variables x_{jk} are allowed to take the value of 0, if there is at least one center located in radius D_k from the user location j. The constraint (4) puts a limit p on the number of located facilities.

To obtain a lower bound of the objective function value of the optimal solution of the original problem, several approaches can be employed. We present here the simplest one, which can be introduced as the expression $e_0x_{j1} + e_1x_{j2} + e_2x_{j3} + \ldots + e_{r-1}x_{jr}$.

3 Dividing points deployment and disutility relevance

Deployment of the dividing points influences the size of the covering model (2) – (6) and the accuracy of the result. The dividing points can be chosen only from the set of values $d_0 < d_1 < \ldots < d_m$ of the disutility matrix

$\{d_{ij}\}$, where $D_0 = d_0$ and $D_m = d_m$. Let the value d_h have a frequency N_h of its occurrence in the matrix $\{d_{ij}\}$. In the suggested approaches, we start from the hypothesis that the disutility d_h from the sequence $d_0 < d_1 <\ldots< d_m$ occurs in the resulting solution n_h times and that is why the deviation of this disutility value from its approximation encumbers the total deviation proportionally to n_h. The disutility d for a user from the nearest located service center can be only estimated taking into account that it belongs to the interval $(D_k, D_{k+1}\rangle$. The maximal deviation of the upper estimation D_{k+1} from the exact value d is $D_{k+1} - D_k^1$, where D_k^1 is the minimal element of the sequence $d_0 < d_1 <\ldots< d_m$, which belongs to the interval $(D_k, D_{k+1}\rangle$. If we were able to anticipate a frequency n_h of each d_h in the unknown optimal solution, we could minimize the total deviation of the upper approximation from the unknown optimal solution by convenient deployment of dividing points. The dividing points for the upper approximation follow from the optimal solution of the problem described by (7) – (11). If the disutility value d_h belongs to the interval ending by the dividing point d_t then the decision variable z_{ht} takes the value of 1.

$$\textit{Minimize} \quad \sum_{t=1}^{m}\sum_{h=1}^{t}(d_t - d_h)n_h z_{ht} \tag{7}$$

$$\textit{Subject to}: \quad z_{(h-1)t} \le z_{ht} \quad \forall\, t = 2, 3, \ldots, m, \ \forall\, h = 2, 3, \ldots, t \tag{8}$$

$$\sum_{t=h}^{m} z_{ht} = 1 \quad \forall\, h = 1, 2, \ldots, m \tag{9}$$

$$\sum_{t=1}^{m-1} z_{tt} = r \tag{10}$$

$$z_{ht} \in \{0, 1\} \quad \forall\, t = 1, 2, \ldots, m, \ \forall\, h = 1, 2, \ldots, t \tag{11}$$

The link-up constraints (8) ensure that the disutility value d_{h-1} belongs to the interval ending with d_t only if each other disutility between d_{h-1} and d_t belongs to this interval. Constraints (9) assure that each disutility value d_h belongs to some interval and the constraint (10) enables only r dividing points to be chosen. After the problem (7) – (11) is solved, the nonzero values of z_{tt} indicate the disutility values d_t corresponding with dividing points.

The associated solving technique consists of the estimation of the relevance n_h, solving the dividing points deployment problem (7) – (11) and subsequently solving the radial-type weighted covering model (2) – (6).

The above approach is obviously based on the "relevance" of a disutility d_h, which expresses the strength of our expectation that the disutility value d_h will be a part of the unknown optimal solution, which is searched for. We suggested and explored several ways of the relevance estimation. In the first one, we started with the frequency N_h of d_h occurrence in the disutility matrix $\{d_{ij}\}$ and set the initial relevance at the value n_h given by the expression (12), where T is a positive shaping parameter.

$$n_h = N_h e^{\frac{-d_h}{T}} \tag{12}$$

The approach (12) follows the hypothesis that the relevance decreases exponentially with increasing disutility value [6]. The other approach to the relevance described in [7] uses the column ranking evaluation $L^{ts}_j(d_{ij})$ of the disutility value d_{ij} and the relevance n_h is defined according to (13).

$$n_h = L^{ts}_h(d_h) = \sum_{j\in J}\sum_{\substack{i\in I \\ d_h = d_{ij}}} L^{ts}_j(d_{ij}) \tag{13}$$

The linear column ranking function $L^{ts}_j(d_{ij})$ is defined as follows: Let $P_j(d_{ij})$ be the position of d_{ij} in the ascending sequence of the j-th column items of the disutility matrix $\{d_{ij}\}$ and let a denote the cardinality of I. Then $L^{ts}_j(d_{ij}) = a + s * (1 - P_j(d_{ij}))$ for $P_j(d_{ij}) < a + 1 - t$ and $L^{ts}_j(d_{ij}) = 0$ otherwise. The parameters t and s represent a *threshold* and a *step* respectively. The *threshold* influences the number of $t - 1$ largest disutility values of the j-th column, which are not taken into account and the *step* gives the difference between the contributions of the k-th and $(k - 1)$-th item of the ascending sequence of the j-th column items. The parameter t can vary over the range $[p \ \ldots \ a\text{-}1]$ of integers and the step s can take the value from the interval $\langle 0, a/(a\text{-}t)\rangle$. This ranking expression of the relevance can be modified using the exponential reduction. Then, the associated relevance n_h is defined in accordance to (14).

$$n_h = L^{ts}_h(d_h) e^{\frac{-d_h}{T}} \tag{14}$$

The last suggested approach to the relevance estimation comes from the exponential approach, but it takes into account that the slope of the exponential function is too steep in the neighborhood of zero disutility and that the relevant disutility values can be sparsely distributed in this neighborhood. To avoid the groundless reduction of the relevant disutility values, we moved the exponential function to the range of biggest disutility values. The shifted exponential approach computes the relevance n_h according to (15).

$$n_h = N_h \, g(h) \tag{15}$$

The function $g(h)$ is equal to 1 for each $h \leq h_{crit}$ and it is defined by (16) for $h > h_{crit}$.

$$g(h) = e^{-\frac{h-h_{crit}}{T}} \tag{16}$$

The constants T and h_{crit} are parameters of the approach. T is the shaping parameter and h_{crit} is the critical value, which can be determined according to (17) for some given parameter q.

$$h_{crit} = min\left\{h \in Z^+ : \sum_{u=0}^{h} N_u \geq \frac{q}{p} \sum_{t=0}^{m} N_t\right\} \tag{17}$$

4 Computational study

So far, we have presented the approximate covering method for public service system design problem. This method is based on the disutility relevance estimation, which can be performed in different ways. Hereby, we are presenting the results of numerical experiments, which were aimed at the comparison of four suggested approaches from the viewpoint of computational time and the solution accuracy. All experiments were performed using the optimization software FICO Xpress 7.3 (64-bit, release 2012). The associated code was run on a PC equipped with the Intel® Core™ i7 2630QM processor with the parameters: 2.0 GHz and 8 GB RAM.

Particular approaches to the disutility relevance estimation were tested on the pool of benchmarks, whose size varied from 300 to 1500 possible service center locations. The number of user locations of the designed system was the same as the cardinality of the set I. For each size of the problem, six instances were solved. These instances differ in the value of parameter p, which limits the number of located service centers. The value of p was set in such a way, that the ratio of $|I|$ to p equals 2, 3, 4, 5, 10 and 20 respectively. The reported results are the average values grouped by the number of possible service center locations. As the system disutility of the users located at j the network distance between the location and the nearest located service center was taken. The weight b_j of each $j \in J$ corresponds with the number of users located at the location j. Furthermore, we assume that each possible location of a service center has enough capacity to serve all users.

The optimal solution of the problem was obtained by solving the well-known location-allocation model in the optimization environment Xpress. Since the covering model provides only the approximation of the former objective function, its real value must be computed according to the values of location variables y_i, $i \in I$ and the disutility matrix $\{d_{ij}\}$. Mathematical formulation can be expressed by (18).

$$\sum_{j \in J} \min\left\{d_{ij} : i \in I,\, y_i = 1\right\} \tag{18}$$

In the covering model (2) – (6), the number r of dividing points was set to the value of 20 in all solved instances and the parameter T was set to the value of 1. The threshold t used in the expressions (13) and (14) was set to the quarter of its possible range, which means $t = (3p + a - 1)$ *div* 4, where the value of a is equal to $|I|$. The step s was set to the value of $\alpha / (\alpha - t)$. The value of q in the expression (17) was 2.

The quality of the solution obtained by the covering approach with any of the disutility relevance estimation is measured by *gap*, which is defined as follows: Let *ES* denote the objective function value of the **e**xact **s**olution of the problem and let *CS* be the real objective function value of the **c**overing **s**olution computed according to (18). The gap represents the difference between these values expressed in percentage of the exact solution.

$$gap = \frac{|ES - CS|}{ES} * 100 \tag{19}$$

Time comparison of the suggested approaches is plotted in Table 1, which contains average computational time in seconds for each approach and each size of the solved problems. It is important to note, that the reported value of the covering approach includes two optimization processes – dividing points deployment by model (7) –

(11) and solving covering problem (2) – (6). As we can see in the table, the approximate covering method provides the solution much faster than the exact method, which can be considered very useful mainly in large-scaled weighted *p*-median instances.

$\|I\|$	Exact solution	Exponential relevance (12)	Ranking relevance (13)	Ranking-exponential relevance (14)	Shifted exponential relevance (15)
300	4,79	4,20	12,57	4,79	5,04
500	23,10	4,38	16,75	4,96	5,72
700	75,20	5,67	19,66	7,08	5,64
900	169,25	5,90	30,79	7,18	6,78
1100	326,80	7,36	37,72	8,70	6,68
1300	519,53	8,79	41,95	10,08	7,05
1500	855,29	6,83	49,13	8,84	7,55

Table 1 Time comparison of different approaches to the disutility relevance estimation. Computational time is given in seconds

Quality of the covering solution is evaluated by *gap*, which is computed according to the formula (19). Average results of numerical experiments are reported in Table 2, which has the same structure as Table 1. As concerns the solution accuracy, the experiments have indicated, that the most appropriate way of the disutility relevance estimation is the shifted exponential approach described by (15).

$\|I\|$	Exponential relevance (12)	Ranking relevance (13)	Ranking-exponential relevance (14)	Shifted exponential relevance (15)
300	7,55	6,53	10,89	1,78
500	6,29	9,81	6,62	0,66
700	6,28	11,36	6,28	0,32
900	5,43	15,52	5,57	0,47
1100	4,49	17,47	4,49	0,14
1300	3,84	19,24	3,84	0,17
1500	1,73	21,95	1,73	0,11

Table 2 Average gaps between the approximate covering solutions and the exact ones expressed in percent of the exact solution

5 Conclusions

The main goal of this paper was to present and compare different approaches to the disutility relevance estimation, which constitutes the core problem of the approximate approach to large public service system design problems. Comparing the reported results in Table 1 and Table 2, we can conclude that the studied approaches can be successfully applied for the public service system design even if they were originally developed to solve the classical p-median problem. It follows from the presented results that the computational time of the approximate approaches is hundred times smaller in comparison with the exact method used for bigger problem instances. The accuracy of the approximate approaches is very good, especially when the shifted exponential estimation of the disutility relevance is employed. Thus we can conclude that we present useful tool for the large public service system design, which can be implemented using common commercial optimization software.

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Allocation of Trains to Platforms at Railway Station

Ľudmila Jánošíková[1], Michal Krempl[2]

Abstract. This paper deals with the organization of train traffic in a passenger railway stations. One of the important problems which a dispatcher has to solve every day is track allocation for every approaching train. There is a tool helping him in his job called the Track Occupancy Plan, which is attached to the train traffic diagram. The plans specify the scheduled times (of arrivals and departures), as well as the assignment of platform tracks to arriving or departing trains.
In this paper the problem of allocation of trains to platforms is formulated as a bi-criterion mixed integer programming problem. The first objective is to minimise the deviations of the arrival and departure times proposed by the model from the times specified by the timetable. The second criterion maximises the desirability of the platform tracks to be assigned to the trains. For finding a solution there is used a lexicographic approach and the local branching algorithm.

Keywords*:* routing trains, scheduling, mixed integer programming, multiple-objective programming

JEL Classification: C44
AMS Classification: 90C15

1 Introduction

The problem of routing and scheduling trains at a station is a subproblem of the generation of a timetable for a railway company. The generation of a timetable is a hierarchical process. At the first stage, a preliminary timetable for the whole network is proposed. In this phase, a macroscopic viewpoint at the railway network is applied. Stations are considered as black boxes. Capacity limits of particular stations and the movement of trains inside the stations are not taken into account. Then, at the second stage, a microscopic viewpoint related to stations is applied. At every station, the network timetable is checked whether it is feasible with respect to capacity, safety and train operators' preferences. To prove the feasibility, detailed routes and schedules for the trains are generated. If desired arrival and departure times are not feasible at the microscopic level, the process returns to the first stage, where the timetable must be adjusted.

In the Czech and Slovak Republic, planning train movements through the station is done by hand, using planner's experience and a set of rules determined by a railway company. The main goal of this research is to design a more sophisticated approach which would serve as a planner's decision supporting tool and result in better routing and scheduling plans. Such an approach can play an important role especially at large, busy stations with multiple platforms and multiple in-lines and out-lines. Improvement in the plan quality results in

1. better management of train operation in the station, namely:
   a) shorter times of routes occupation by arriving and departing trains,
   b) uniform workload of the infrastructure elements, such as tracks, switches, and platforms,
   which leads to a more robust plan resistant to random disturbances;
2. higher service quality perceived by passengers, namely:
   a) shorter distances needed for changing trains,
   b) more appropriate platforms (platforms near to ticket sales points and to the station entrance, platforms equipped by station shops or catering etc.),
   c) less probability of changing the planned platform when the train delays,
   which leads to a higher share of railway in public transport;
3. meeting train operators' requirements on arrival and departure times and platforms assigned to trains.

Routing and scheduling trains at a station has been studied by researchers in countries, where large, busy stations with capacity constraints can be found. Billionet addressed only the routing problem [2]. The problem was modelled using a graph theory and the integer programming formulation of the resulting graph colouring problem was solved. However, the *k* colouring problem is not indeed an optimisation problem, it means any

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feasible solution is acceptable and the problem formulation does not reflect the solution quality, such as route lengths or platform preferences for individual trains. In [7, 8] the problem of train routing was described as a weighted node packing problem, using bivalent programming, while the solution algorithm applied the branch-and-cut method. A disadvantage of the above presented models is that the calculations connected with them are computationally too complex and time consuming. Another, practically oriented approach has given up on applying the integer programming methods, and replaced them by the heuristics, solving the scheduling and routing problems at a time [3]. The algorithm incorporates, or considers, the operational rules, costs, preferences and trade-offs, which are applied by experts creating plans manually. The shortcoming of this approach is obvious: since it is a heuristics, the optimality of the resulting plan is not guaranteed.

Other way of research, e.g. [1, 4], has been directed at operational train management. In real time it is necessary to reflect the requirements of the operation burdened with irregularities, i.e. to re-schedule the arrivals and departures times, and/or re-route trains.

In this paper we propose a mixed integer programming (MIP), multiple criteria model of the routing and scheduling problem. The problem can be solved by a lexicographic approach, where particular criteria are ranked according to their importance.

2 Problem formulation

The problem of routing and scheduling trains, as described in the previous section, consists of the following partial issues subject to decision-making process. For each train,

- a platform track must be specified at which the train should arrive; the platform track assignment determines the route, on which the train approaches from an in-line (or from a depot) to the platform, or departs from the platform to an out-line (or to a depot),
- arrival time at the platform and departure time from the platform need to be determined.

The solution should minimise deviations from the planned arrival and departure times and maximise the total preferences for platforms and routes.

The inputs to the mathematical programming model are as follow:

1. track layout of the station, which is necessary for determining feasible platform tracks for a train and conflicting routes,
2. list of trains, where the data required for each train include:
   a) planned time of its arrival at the platform,
   b) planned time of its departure from the platform,
   c) line on which the train arrives (in-line) and departs (out-line),
   d) list of feasible platform tracks with their desirability for the train,
   e) category of the train.

All time data are given in minutes.

Further on we present the formulation of the MIP model. First we need to explain the symbols used:

Subscripts which in the mathematical model represent objects

i, i', j train

k, k' platform track

Input parameters (constants)

t_i^{Pa} planned arrival time of train i at the platform

t_i^{Pd} planned departure time of train i

t^{Cn} standard amount of time passengers take to change trains (depends on particular railway station)

I_i arrival line track (in-line) for train i

O_i departure line track (out-line) for train i

c_i category of train i; $c_i = 1$ for regional stopping trains and increases with the speed and distance travelled by the train

t^{min} minimum dwell time of a train at the platform

t^{max} maximum time interval, in which two train movements are tested for a conflict

p_{ik} preference coefficient; it reflects the desirability of the assignment of platform track k to train i

s_{ik} number of switches on the route of train i from the arrival line track to platform track k and from platform track k to the departure line track

$s_i^{\min}$ number of switches on the shortest train route in the station

$s_i^{\max}$ number of switches on the longest train route in the station

$a(l,k,l',k')$ coefficient, which has value *true*, if the route connecting line l to platform track k conflicts with the route connecting line l' to platform track k'; if there exists any route connecting line l to track k and any route connecting line l' to track k' such that these two routes do not conflict, then $a(l,k,l',k') = false$. $a(l,k,l',k') = true$ for $k = k'$ or $l = l'$. The existence of route conflicts can be identified in advance from a detailed map of the track layout.

We adopted the concept of conflicting routes and conflict solving from the source [3]. If two trains are on conflicting routes we must ensure that there is at least a required minimum headway (time interval) between them, for safety and signalling reasons. The minimum headway depends on the order, types, and lengths of the trains, on whether the trains are arriving or departing from the station, and on the platform track and line used by each train. For example, let $h(i,k,i',k')^{da}$ is the minimum headway required between train i departing from track k and the next train i' arriving at track k'. The superscripts d and a denote departure and arrival, and the order of the superscripts indicates the order of the trains, i.e., train i is followed by i'. Similarly we have $h(i,k,i',k')^{aa}$, $h(i,k,i',k')^{ad}$ and $h(i,k,i',k')^{dd}$ for combinations arrival – arrival, arrival – departure, departure – departure. We need not introduce subscripts to denote the in-lines or out-lines used by trains since for an arriving train i the in-line is already specified by I_i, and for a departing train i the out-line is specified by O_i.

The preference coefficient p_{ik} may reflect:

- operator's preferences of platforms,
- the distance of the track k to the connecting trains,
- the length of the route used by train i arriving to or departing from platform track k. The smoother and shorter the route is, the less the possibility of a conflict with other trains is, hence the probability of delay propagation decreases.

In our model, coefficient p_{ik} is set according to the following formula:

$$p_{ik} = \begin{cases} 1 & \text{if track } k \text{ is the planned (or desired) track for train } i \\ 0.9 & \text{if track } k \text{ is located at the same platform as the planned track} \\ 0.8\left(\dfrac{s_i^{\max} - s_{ik}}{s_i^{\max} - s_i^{\min}}\right) & \text{otherwise} \end{cases}$$

Sets of objects

K set of all platform tracks

$K(i)$ set of feasible platform tracks for train i

U set of all arriving, departing, and transit trains

$W(j)$ set of all connecting trains, which has to wait for train j

$V^{aa} = \{(i,j) : i,j \in U, i < j, |t_i^{Pa} - t_j^{Pa}| \le t^{\max}\}$ set of ordered pairs of those trains that may arrive concurrently

$V^{ad} = \{(i,j) : i,j \in U, i < j, |t_i^{Pa} - t_j^{Pd}| \le t^{\max}\}$ set of ordered pairs of those trains that arriving train i and departing train j may travel concurrently

$V^{da} = \{(i,j) : i,j \in U, i < j, |t_i^{Pd} - t_j^{Pa}| \le t^{\max}\}$ set of ordered pairs of those trains that departing train i and arriving train j may travel concurrently

$V^{dd} = \{(i,j) : i,j \in U, i < j, |t_i^{Pd} - t_j^{Pd}| \le t^{\max}\}$ set of ordered pairs of those trains that may depart concurrently

Decision and auxiliary variables of the model

$$\text{for } i \in U, k \in K(i): x_{ik} = \begin{cases} 1 & \text{if track } k \text{ is assigned to train } i \\ 0 & \text{otherwise} \end{cases}$$

u_i difference between the planned and real arrival time of train i at a platform, $i \in U$

v_i difference between the planned and real departure time of train i from a platform, $i \in U$

The following auxiliary variables $\boldsymbol{y}$ are introduced for the couple of those trains i and j that may travel concurrently. They enable to express safety headways between conflicting trains.

$$\text{for } i,j \in U, i<j: y_{ij}^{aa} = \begin{cases} 1 & \text{if train } i \text{ arrives before train } j \text{ arrives} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{for } (i,j) \in V^{ad}: y_{ij}^{ad} = \begin{cases} 1 & \text{if train } i \text{ arrives before train } j \text{ departs} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{for } (i,j) \in V^{da}: y_{ij}^{da} = \begin{cases} 1 & \text{if train } i \text{ departs before train } j \text{ arrives} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{for } (i,j) \in V^{dd}: y_{ij}^{dd} = \begin{cases} 1 & \text{if train } i \text{ departs before train } j \text{ departs} \\ 0 & \text{otherwise} \end{cases}$$

Model formulation

minimise
$$c_i \sum_{i \in U} (u_i + v_i) \tag{1}$$

maximise
$$\sum_{i \in U} \sum_{k \in K(i)} p_{ik} x_{ik} \tag{2}$$

subject to

$$v_i + t_i^{Pd} \geq u_i + t_i^{Pa} + t^{\min} \qquad \forall i \in U \tag{3}$$

$$v_i + t_i^{Pd} \geq u_j + t_j^{Pa} + t^{Cn} \qquad \forall j \in U; i \in W(j) \tag{4}$$

$$u_{i'} + t_{i'}^{Pa} \geq u_i + t_i^{Pa} + h(i,k,i',k')^{aa} - M\left(1 - y_{ii'}^{aa}\right) - M(1 - x_{ik}) - M(1 - x_{i'k'})$$
$$\forall (i,i') \in V^{aa}, k \in K(i), k' \in K(i'): a(I_i, k, I_{i'}, k') \tag{5}$$

$$u_i + t_i^{Pa} \geq u_{i'} + t_{i'}^{Pa} + h(i',k',i,k)^{aa} - My_{ii'}^{aa} - M(1 - x_{ik}) - M(1 - x_{i'k'})$$
$$\forall (i,i') \in V^{aa}, k \in K(i), k' \in K(i'): a(I_i, k, I_{i'}, k') \tag{6}$$

$$v_{i'} + t_{i'}^{Pd} \geq u_i + t_i^{Pa} + h(i,k,i',k')^{ad} - M\left(1 - y_{ii'}^{ad}\right) - M(1 - x_{ik}) - M(1 - x_{i'k'})$$
$$\forall (i,i') \in V^{ad}, k \in K(i), k' \in K(i'): a(I_i, k, O_{i'}, k') \tag{7}$$

$$u_i + t_i^{Pa} \geq v_{i'} + t_{i'}^{Pd} + h(i',k',i,k)^{da} - My_{ii'}^{ad} - M(1 - x_{ik}) - M(1 - x_{i'k'})$$
$$\forall (i,i') \in V^{ad}, k \in K(i), k' \in K(i'): a(I_i, k, O_{i'}, k') \tag{8}$$

$$u_{i'} + t_{i'}^{Pa} \geq v_i + t_i^{Pd} + h(i,k,i',k')^{da} - M\left(1 - y_{ii'}^{da}\right) - M(1 - x_{ik}) - M(1 - x_{i'k'})$$
$$\forall (i,i') \in V^{da}, k \in K(i), k' \in K(i'): a(O_i, k, I_{i'}, k') \tag{9}$$

$$v_i + t_i^{Pd} \geq u_{i'} + t_{i'}^{Pa} + h(i',k',i,k)^{ad} - My_{ii'}^{da} - M(1 - x_{ik}) - M(1 - x_{i'k'})$$
$$\forall (i,i') \in V^{da}, k \in K(i), k' \in K(i'): a(O_i, k, I_{i'}, k') \tag{10}$$

$$v_{i'} + t_{i'}^{Pd} \geq v_i + t_i^{Pd} + h(i,k,i',k')^{dd} - M\left(1 - y_{ii'}^{dd}\right) - M(1 - x_{ik}) - M(1 - x_{i'k'})$$
$$\forall (i,i') \in V^{dd}, k \in K(i), k' \in K(i'): a(O_i, k, O_{i'}, k') \tag{11}$$

$$v_i + t_i^{Pd} \geq v_{i'} + t_{i'}^{Pd} + h(i',k',i,k)^{dd} - My_{ii'}^{dd} - M(1 - x_{ik}) - M(1 - x_{i'k'})$$
$$\forall (i,i') \in V^{dd}, k \in K(i), k' \in K(i'): a(O_i, k, O_{i'}, k') \tag{12}$$

$$u_i + t_i^{Pa} \geq v_{i'} + t_{i'}^{Pd} + h(i',k,i,k)^{da} - My_{ii'}^{aa} - M(1 - x_{ik}) - M(1 - x_{i'k})$$
$$\forall i, j \in U, i < j, k \in K(i) \cap K(i') \tag{13}$$

$$u_{i'} + t_{i'}^{Pa} \geq v_i + t_i^{Pd} + h(i,k,i',k)^{da} - M\left(1 - y_{ii'}^{aa}\right) - M(1 - x_{ik}) - M(1 - x_{i'k})$$
$$\forall i, j \in U, i < j, k \in K(i) \cap K(i') \tag{14}$$

$$y_{ij}^{aa} = 1 \quad \forall i, j \in U, i \neq j, I_i = I_j, t_i^{Pa} \leq t_j^{Pa} \tag{15}$$

$$\sum_{k \in K(i)} x_{ik} = 1 \qquad \forall i \in U \tag{16}$$

$$u_i, v_i \geq 0 \qquad \forall i \in U \tag{17}$$

$$x_{ik} \in \{0,1\} \qquad \forall i \in U \ \forall k \in K(i) \tag{18}$$

$$y_{ij}^{aa} \in \{0,1\} \qquad \forall i, j \in U, i < j \tag{19}$$

$$y_{ij}^{ad} \in \{0,1\} \qquad \forall (i, j) \in V^{ad} \tag{20}$$

$$y_{ij}^{da} \in \{0,1\} \qquad \forall (i, j) \in V^{da} \tag{21}$$

$$y_{ij}^{dd} \in \{0,1\} \qquad \forall (i, j) \in V^{dd} \tag{22}$$

Model description

Objective function (1) minimises the weighted deviations of the arrival and departure times proposed by the model from the times specified by the timetable. The weights cause that long-distance/high- speed trains will respect planned times and regional trains will be postponed if necessary. The second criterion maximises the desirability of the platform tracks to be assigned to the trains.

Constraint (3) ensures that a minimum dwell time needed for boarding and alighting must be kept.

Constraint (4) states that connecting train i with real departure $v_i + t_i^{Pd}$ has to wait in station to time at least $u_j + t_j^{Pa} + t^{Cn}$.

Constraints (5) – (12) ensure that a minimum headway will be kept between conflicting trains. More precisely, constraint (5) states that if trains i and i' have planned arrival times within t^{max} and train i arrives at platform track k before train i' arrives at track k', i.e.

$$x_{ik} = 1, x_{i'k'} = 1, y_{ii'}^{aa} = 1, \tag{23}$$

and trains are on conflicting routes (i.e. $a(I_i, k, I_{i'}, k')$ is *true*), then train i' is allowed to arrive at least $h(i,k,i',k')^{aa}$ minutes later than train i. If at least one of the conditions (23) is not met (e.g. train i is not assigned to track k), then constraint (5) becomes irrelevant as the right-hand side is negative (M is a suitably picked high positive number). If train i' is followed by train i ($y_{ii'}^{aa} = 0$), then i is allowed to arrive at least $h(i',k',i,k)^{aa}$ minutes later than i', which is ensured by constraint (6). Constraints (7) – (12) have a similar meaning for the other combinations of arrival – departure.

Constraints (13) – (14) ensure that a train will not be dispatched to an occupied track. If train i' is followed by train i ($y_{ii'}^{aa} = 0$) and both trains arrive at the same track k, then i is allowed to arrive at least $h(i',k,i,k)^{da}$ minutes after train i' leaves track k, which is expressed by constraint (13). Constraint (14) holds for the reverse order of trains i, i'.

Constraint (15) states that y_{ij}^{aa} is 1 if train i is followed by train j at the arrival and both trains travel on the same in-line.

Constraint (16) ensures that each train is always dispatched to exactly one platform track.

The remaining obligatory constraints (17) – (22) specify the definition domains of the variables.

This multiple-criteria optimisation problem was solved using the lexicographic approach, where the objective functions are ranked according to their importance. In the problem at hand, the first objective function (i.e. to meet the timetable) is more important that the second one (i.e. to respect track preferences). This ordering reflects how decisions are currently made in practice. The solution technique consists of two steps. In the first step the problem (1), (3) – (22) is solved giving the best value of the weighted sum of deviations f_1^{best}. Then the constraint

$$c_i \sum_{i \in U} (u_i + v_i) \leq f_1^{best} \tag{24}$$

is added and the model (2) – (22), (24) is solved. Because both MIP problems are hard and the optimal solutions cannot be found within a reasonable time limit, we decided to implement Local Branching heuristic [6] using the general optimisation software *Xpress* [5].

3 Case study

The model was verified by using the real data of Prague main station and the timetable valid for the years 2004/2005. Prague main station is a large station that at the given time had 7 platforms, 17 platform tracks and 5 arrival/departure line tracks. According to the timetable 2004/2005, the station dealt with 288 regular passenger trains per a weekday.

The results of computational experiments show that the timetable 2004/2005 was not correct with regard to safety requirements. There were some trains travelling on conflicting routes concurrently. That is why their desired arriving or departing times could not be kept. The best solution proposed by the model delays 3 trains at arrival by 5 minutes and 12 trains at departure by 17 minutes in total, and dispatches 32 (11 %) trains to platform tracks different from the planned ones.

A more detailed discussion of computational experiments will be presented at the conference.

4 Conclusions

In the paper, a mixed integer programming model for routing and scheduling trains at a passenger railway station is described. The model gives a solution with regard to particular criteria ranked according to their importance, i.e. the solution with minimal deviation of the arrival and departure times from the timetable that respects the desirability of the platform tracks to be assigned to the trains as much as possible.

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On the constraints in optimal managing of natural resources systems

Jitka Janová [1], Jana Musilová[2]

Abstract. Optimal managing of natural resources systems has been permanently in the focus of interest of worldwide policymakers. Today, more than ever, identifying optimal policies is of strategic importance to address the problems crucial to the economy and the environment. Since the beginning of the mathematical programming science the up-to-date mathematical methods were applied to create decision support for natural resources management. However, where the optimal policies have been determined, these often fail. It was shown that this is due to the shortcomings of the models' structure stemming mainly from neglecting of nonlinearities of real systems and miss identification of constraints among relevant variables.

This contribution serves as an introductory study of constrained systems arising in the natural resources modeling. We focus on two non-equivalent treatments of such constrained dynamical systems (so called vakonomic and nonholonomic approaches), we discuss the possibility to improve the optimal control models for natural resources systems using nonlinear techniques and, as a test case, we formulate a generalized anti-pollution policy problem.

Keywords: natural resources, bioeconomics, constraints, nonholonomic, vakonomic, dynamic optimization

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AMS classification: 49N99

1 Introduction

Since the beginning of mathematical programming science the up-to-date mathematical methods were applied to create quality decision support systems for the natural resources management. The most spread approach commonly applied mainly to agriculture and forestry production planning, transportation or scheduling problems is based on linear programming methods. Simultaneously with the development of mathematical methods the models started to use the advanced methods of operations research and mathematical programming. Particularly, this has been the case especially in environmental studies, where such problems as landscape optimization, ecosystem modeling, water resource allocation, forest structure optimization or herd management, etc. were solved using advanced deterministic methods of nonlinear programming. Currently, there is running quite a broad research on stochastic programming models for production and harvesting planning. Let us mention that the methods of stochastic programming (unlike the deterministic methods) are still extensively developed from the mathematical point of view to meet the demand of real optimization problems.

All the above mentioned examples are able to determine static optimal solution, but the natural resources systems are dynamical. It is possible to consider one-year-ahead solution for particular or local needs, but from a long run sustainable perspective these problems must be treated by the dynamical methods. Note, that compared to the number of models developed for static optimization of natural resources systems, the dynamic optimization methods are infrequently used.

Generally, in identifying the optimum long run decision strategies, discrete and continuous approaches are distinguished, the former being connected with sequential decision processes solved via dynamic

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programming, the latter with calculus of variations and optimal control theory.

The calculus of variations is inherently connected with the classical mechanics and its fundamental equations of motion - Lagrange equations. Based on the Hamilton approach, an alternative description of a mechanical system was formulated. The Hamilton apparatus was used in formulation of the control theory, where the dynamic optimization problem is viewed as consisting of three (rather than two) types of variables. Aside from the time variable and the state variable, consideration is given to a control variable. The control theory is extensively used and studied in engineering applications (robotics, automation) and also in economic applications covering finance, production and inventory problems, marketing, machine maintenance and replacement. Currently, there exists a considerable amount of literature studying optimal control problems in environmental studies, ecology and natural resources (see [2]). Let us mention only several recent interesting problems studied: e.g. optimal control of invasive species in [5], optimal extraction of natural resources under various assumptions concerning pollution, waste or prices and costs of entering variables in [18], optimal PEST control in agriculture in [4], greenhouse control systems [18] and optimal control of trade offs between water use and fish harvesting [19]. The intense of current research stem from the strong belief that optimal control theory is a powerful tool to identify the long run strategies and to evaluate sustainability implications of alternative policy making practices in ecology and natural resources economics. While it should be useful for managing these systems, the results are still unsatisfactory. Where the optimal control policies have been determined, these often fail. It was shown, that the paradoxical aspects of control theory in natural resources economics result from the shortcomings of the models constructed, stemming mainly from neglecting non-linearities and miss-identification of constraints among the variables (see [15]).

This paper is an introduction to the research of a constraint structure and a solution of the natural resources continuous dynamic systems with constraints (for some notes on the constraint structures in agricultural and forestry see [9] and [12]). First, we introduce the dynamic optimization techniques appropriate for a solution of the natural resources problems. Then, the concept of more general (nonholonomic) constraints in dynamic optimization will be briefly described and the method of incorporating the constraints into the dynamic model will be presented together with examples of constraints in natural resource dynamic problems. As an addition to the explanation of the mathematical background of the constrained system solution, we open the challenging problem of alternative approaches to the constrained dynamic optimization (so called nonholonomic and vakonomic formulation).

2 Methods of continuous dynamic optimization

2.1 The calculus of variations

The fundamental problem of calculus of variations is (see e.g. [3])

$$\max \ (\min) \quad V = \int_{t_0}^{t_1} G(t, q(t), \dot{q}(t))\mathrm{d}t, \tag{1}$$

$$q^\sigma(t_0) = q_0^\sigma, \tag{2}$$

$$q^\sigma(t_1) = q_1^\sigma, 1 \leq \sigma \leq m, \tag{3}$$

where $q(t) = (q^1(t), q^2(t), \ldots, q^m(t))$ and $\dot{q}(t) = (\dot{q}^1(t), \dot{q}^2(t), \ldots, \dot{q}^m(t))$ are state variables and its derivative respectively, and t is time variable [1]. The goal of the fundamental problem (1-3) is to find an optimal trajectory $q^\star(t)$, that yields an extremal (in general stationary) value of the functional $\int_{t_0}^{t_1} G(t, q(t), \dot{q}(t))\mathrm{d}t$. The basic necessary condition in the calculus of variations are the Euler equations:

$$\frac{\partial G(t, q(t), \dot{q}(t))}{\partial q^\sigma} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial G(t, q(t), \dot{q}(t))}{\partial \dot{q}^\sigma} = 0, \ 1 \leq \sigma \leq m. \tag{4}$$

Note, that in particular problems where function G is strictly concave or convex, which is often the case both in economic and physical problems, are the Euler equations the necessary and sufficient conditions for finding the optimal trajectory.

[1] We consider the common assumptions of the calculus of variations: we restrict the set of admissible paths to those continuous curves with continuous derivatives and we assume that the integrand function G is twice differentiable.

Nonholonomic constraints in calculus of variations

Consider a dynamical system described by (1-3) with $G = G(t, q, \dot{q})$ subject to general constraints $f^i(t, q, \dot{q}) = 0, 1 \leq i \leq k$. In physics– the field which the calculus of variations was invented for– we can find two different approaches for incorporating the constraints into the original system. In the first one *the vakonomic approach* we search for the solution of (unconstrained) variational problem associated to the function

$$\bar{G} = G + \sum_{i=1}^{k} \lambda_i f^i,$$

hence the variational problem takes for the constrained system the form

$$\max \ (\min) \quad V = \int_{t_0}^{t_1} \left[G(t, q, \dot{q}) + \sum_{i=1}^{k} \lambda_i f^i(t, q, \dot{q}) \right] \mathrm{d}t, \tag{5}$$

$$q^\sigma(t_0) = q_0^\sigma, \tag{6}$$

$$q^\sigma(t_1) = q_1^\sigma, 1 \leq \sigma \leq m, \tag{7}$$

where multipliers λ_i are considered as additional variables. Namely, the vakonomic solutions can be obtained by the Euler-Lagrange equations

$$\frac{\partial \bar{G}}{\partial q^\sigma} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \bar{G}}{\partial \dot{q}^\sigma} = 0, \tag{8}$$

$$\frac{\partial \bar{G}}{\partial \lambda_i} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \bar{G}}{\partial \dot{\lambda}_i} = 0, \tag{9}$$

which gives

$$\frac{\partial G}{\partial q^\sigma} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial G}{\partial \dot{q}^\sigma} + \sum_{i=1}^{k} \frac{\partial f^i}{\partial \dot{q}^\sigma} \cdot \dot{\lambda}_i - \sum_{i=1}^{k} \left(\frac{\partial f^i}{\partial q^\sigma} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial f^i}{\partial \dot{q}^\sigma} \right) \cdot \lambda_i = 0, \tag{10}$$

$$f^i(t, q, \dot{q}) = 0. \tag{11}$$

The second approach – *nonholonomic* –consists in incorporating the constraint forces into the Euler-Lagrange equations:

$$\frac{\partial G}{\partial q_\sigma} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial G}{\partial \dot{q}_\sigma} = \sum_{i=1}^{k} \frac{\partial f^i}{\partial \dot{q}_\sigma} \cdot \mu_i, \tag{12}$$

$$f^i(t, q, \dot{q}) = 0, \tag{13}$$

where μ are the Lagrange multipliers. The nonholonomic approach is typically used for solving the mechanical systems with nonholonomic constraints in physics (for instructive nonholonomic mechanical problems see e.g. [10], [11]).

Comparing the nonholonomic (12) and vakonomic equations (10) we arrive to the conclusion that the arising systems of differential equations are not equivalent unless

$$\sum_{i=1}^{k} \lambda_i \left(\frac{\partial f^i}{\partial q_\sigma} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial f^i}{\partial \dot{q}_\sigma} \right) = 0. \tag{14}$$

Verifying (14) is complicated task because the multipliers λ_i are a priori unknown. Nevertheless, for special types of constrained systems the equivalence conditions (14) can be simply verified without explicitly solving the constrained dynamics.

It is obvious that for the Lagrangian mechanical system subject to holonomic constraints $u^i(t, q) = 0$, $1 \leq i \leq k$, coincide the nonholonomic (12) and vakonomic equations (10). Now we shall consider a very frequent type of constraint, called *linear integrable*, or *semiholonomic* (see [14]). This constraint has the form $f^i = \dot{q}^{m-k+i} - g^i(t, q^\sigma, \dot{q}^l)$, $1 \leq i \leq k$, $1 \leq k \leq m-1$, $1 \leq l \leq m-k$, where functions g^i obey the

following relations:

$$\varepsilon_l'(g^i) = 0, \quad \varepsilon_l'(g^i) = \left(\frac{\partial g^i}{\partial q^l} + \frac{\partial g^j}{\partial \dot{q}^l}\frac{\partial g^i}{\partial q^{m-k+j}}\right) - \left(\frac{\partial^2 g^i}{\partial t \partial \dot{q}^l} + \dot{q}^s \frac{\partial^2 g^i}{\partial q^s \partial \dot{q}^l} + g^j \frac{\partial^2 g^i}{\partial q^{m-k+j}\partial \dot{q}^l}\right), \quad (15)$$

$$\frac{\partial^2 g^i}{\partial \dot{q}^s \partial \dot{q}^l} = 0. \quad (16)$$

Following proposition concern a special case of semiholonomic constraint, for which functions g^i are independent of variables q^{m-k+j}.

Proposition 1. *For a mechanical system subjected to semiholonomic constraint $f^i = \dot{q}^{m-k+i} - g^i(t, q^l, \dot{q}^l) = 0$, $1 \leq l \leq m-k$, the equivalence condition (14) is satisfied.*

Proof. Let us analyze the expression

$$\frac{\partial f^i}{\partial q^\sigma} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f^i}{\partial \dot{q}^\sigma}$$

included in equivalence conditions (14).

According to (15) and (16) we get for $1 \leq l \leq m-k$

$$\frac{\partial f^i}{\partial q^l} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f^i}{\partial \dot{q}^l} = 0 \quad (17)$$

and

$$\frac{\partial f^i}{\partial q^{m-k+j}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f^i}{\partial \dot{q}^{m-k+j}} = 0 - \frac{\mathrm{d}}{\mathrm{d}t}\delta^i_j = 0. \quad (18)$$

Hence the equivalence conditions (14) hold independently of values of multipliers λ_i. □

Note that for general semiholonomic constraint $f^i = \dot{q}^{m-k+i} - g^i(t, q^\sigma, \dot{q}^l)$ the conditions (18) do not hold because $\frac{\partial f^i}{\partial q^{m-k+j}} \neq 0$. Nevertheless, the conditions (14) can be satisfied via the appropriate form of Lagrange multipliers. Note that even in technical applications (that are the most elaborated in the calculus of variations) one can find contradictions in the results for semiholonomic constraints. In [8] the coupled rolling motion is considered that is described by Chaplygin system. Since the constraint in the problem obeys the conditions (15-16) it is considered to be semiholonomic. Nevertheless, according to our calculations the vakonomic and nonholonomic systems seems not to be equivalent. This is an issue to be settled by our further research.

2.2 Optimal control theory

The optimal control formulation of a dynamic optimization problem focuses upon control variables, $u(t) = (u^1(t), u^2(t), \ldots, u^p(t))$ that serve as the instrument of optimization. Let us denote the state variables $y(t) = (y^1(t), y^2(t), \ldots, y^r(t))$. Optimal control theory determines the optimal time path for a control variable $u^*(t)$ and we can also find the optimal state path $y^*(t)$. The relation $\dot{y} = f(t, y, u)$ between state variables and control variables must be identified to formulate the optimal control problem. This relation reflects the principle of studied optimization problem. The control variable u is a subject to our discretionary choice through which we influence the state variable y. The criterion of the dynamic optimization problem is reflected by the objective functional

$$V = \int_{t_0}^{t_1} G(t, y, u)\mathrm{d}t. \quad (19)$$

The simplest optimal control problem is

$$\max\ V = \int_{t_0}^{t_1} G(t, y, u)\mathrm{d}t, \quad (20)$$

$$\dot{y} = f(t, y, u), \quad (21)$$

$$y(t_0) = y_0,\ y(t_1)\ \text{free}, u(t) \in U, \quad (22)$$

where (21) refers to as the equation of motion or state equation. While the objective functional (19) reflects the optimization criterion well known to the policy maker and therefore not complicated to compile, the constraint (21), playing a key role in representation of the real world system principles, is in natural resources problems of complex unknown structure. Generally, function f is non-linear, but since the solution of nonlinear systems is very difficult mathematically, the linear relationships are mostly used when formulating the state equations in natural resources systems (for simple nonlinear optimal control problem in the fishery bioeconomics with one state and one control variable see [13]). Note, that linear control system is defined by the state equations

$$\dot{y} = A(t)y + B(t)u. \tag{23}$$

We suggest to adopt nonholonomic optimal control techniques for representing accurately the nonlinear structure of natural resources dynamic systems. Nonholonomic optimal control represents a nonliear approach that enables to consider more complex relations among first derivatives of the state variables. A typical form of nonholonomic control system is given by state equations

$$\dot{y} = F(y)u. \tag{24}$$

We formulate a generalized antipollution policy that is able to reflect nonlinear structure of the real system (see [3] for linear control system). We use the symbol E to represent the energy use, P denotes the stock of pollution with $\dot{P}$ as its flow. The use of energy generates a flow of pollution. Let A stand for the level of antipollution activities that can reduce the pollution stock in a known manner. We assume that the implementation of antipollution activities A in itself requires the use of energy. The criterion of the decision making is maximization of utility $U = U(C(E), P)$ with $C(E)$ being the consumption. The dynamic optimization problem takes the form

$$\begin{aligned}
\max\ V &= \int_0^T U(C(E), P)\mathrm{d}t, && (25)\\
\dot{S} &= -E - A, && (26)\\
\dot{P} &= f_E(S, P)E + f_A(S, P)A, && (27)\\
&P(0) = P_0,\ S(0) = S_0,\ P(T), S(T)\ \text{free}, E \geq 0, 0 \leq A \leq \bar{A}. && (28)
\end{aligned}$$

This is a nonholonomic optimal control problem with state variables S, P and control variables E, A. In our research we focus on adopting the nonholonomic solution techniques from mathematical physics for bioeconomic and natural resources models of type (25-28) to obtain the mathematically correct solutions and considerably improve the applicability of the optimal control results.

3 Conclusion

Incorporating the techniques of variational calculus and optimal control for solving the constrained dynamical system in the area of natural resources systems appears to be promising. Nevertheless, there exist problematic issues in the underlaying mathematical structures. In our contribution we focused on the systems with nonholonomic constraints for which two non equivalent approaches to solution exist. While in economics mainly the vakonomic approach has been adopted for solving the dynamic optimization problems with velocity constraints (see e.g. [6], [17] or [7]), in physics there are strong supporting examples and arguments (e.g. in [20]) that justify the nonholonomic approach as the physically correct one. But in the area of natural resources systems the concept of constrained systems are still to be validated. Particularly, for our further research, we state a hypothesis that in the area of calculus of variations the correctness of vakonomic approach is expected to be shown, while the optimal control applications in natural resources will fit better to the physical approach of nonholonomic constraints.

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Examining relations between Slovakia and foreign partners: a cointegrated vector autoregressive approach

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Abstract. Slovakia as a small open economy is highly dependent on foreign environment. The focus of this study is oriented on Slovakia's relations with its most important foreign trade partners.
Based on relationships derived from economic theory by means of AA-DD model a small open economy structural model is constructed. As most of macroeconomic variables are non-stationary, the suitability of cointegration approach is verified. The study models relations with foreign partners using a Cointegrated Vector Autoregressive approach. The following five long-run equilibrium relations are examined and adjusted for the case of Slovakia: relative purchasing power parity, money supply, output relations, uncovered interest rate parity, and Fisher inflation parity. The foreign partners are characterized by their aggregate domestic demand, consumer prices and interest rate. Also the importance of convergence of Slovakia's economic development to its foreign trade partners is investigated considering its inclusion into the model.
The first version of the estimated model is used to investigate responses of domestic macroeconomic variables to foreign shocks by impulse response functions.

Keywords: small open economy, AA-DD model, cointegrated VAR, impulse response function.

JEL Classification: C51, C52, E17
AMS Classification: 62P20

1 Introduction

A small open economy is an economy that participates in international trade, but is small enough compared to its trading partners and does likely not impact world economy to a great extent. Thus the policy of small open economy does not alter world prices or interest rates, the countries with small open economies are rather price-takers and they are highly dependent on development of foreign environment. Slovakia is a typical example of a small open economy. Moreover, its openness has been increasing over the last decade very rapidly. In 2012 its rate of openness[2] achieved 186.3%, while before 2000 it was still around 120%.

The goal of this paper is to describe and verify macroeconomic relationships between small open economy of Slovakia and its foreign environment. The foreign environment of Slovakia is represented by its most important foreign trading partners as the fluctuations of a small and open economy are mainly driven by its main trading partners [3]. Modelling of small open economy is allowing exogenous assumptions of the conditions in the foreign environment. In this context the paper investigates the long-run structural modelling approach for the Slovak economy. One of the recent attempts to construct a long-run macroeconometric model for Slovakia is e.g. in [2]. Their paper applies a practical approach to incorporate theoretic long-run relationships in a structural vector error correction model of the Slovak and Czech Republic. Their results confirmed the similarity of both economies.

The paper is organized as follows. In the second chapter the theoretical foundations of the model for small open economy are introduced. The third part contains estimated cointegrated VAR model and interpretation of results. The fourth chapter concludes.

2 Long-run equilibrium relations in small open economy (AA-DD model)

As a starting point of structural modelling of small open economy we used the small open economy model developed by Garratt et al. [1] and verified e.g. by Schneider et al. [6]. They derived from the economic theory

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[2] The openness of economy is defined as the share of total imports and exports of goods and services in gross domestic product at current prices.

based on RBC (real business cycle) models the following five linearized long-run relations: the uncovered interest rate parity, the purchasing power parity, production function, trade balance and real money balance. In this study, these equations were verified and modified by the model AA-DD. The model AA-DD describes equilibrium relations in small open economy. This model represents a synthesis of three market models: the foreign exchange (Forex) market, the money market, and the goods and services market (more detailed description e.g. in [7]. The AA-DD model is described with a diagram consisting of two curves: an AA curve representing asset market equilibriums derived from the money market and foreign exchange markets and a DD curve representing goods market (or demand) equilibriums. One of the basic mechanisms working in these markets is the arbitrage concept and the law of one price. The law of one price states that identical goods sell for the same price worldwide. The existence of arbitrage means that if the prices after exchange-rate adjustment were not equal worldwide, arbitrage of the goods ensures eventually that they will be. The first relationship (1) results from these international arbitrage activities. The relative purchasing power parity theory assumes a relationship between the domestic inflation and foreign inflation over a specified period and the movement in the nominal exchange rate between their two currencies over the same period. This theory states that the exchange rate of one currency against another will adjust to reflect changes in the price levels of the two countries. It means that the currency with the higher inflation rate is expected to depreciate relative to the currency with the lower rate of inflation. However, in practice not all goods and services can be bought in both countries and therefore there remains a certain difference and this leads to introducing a constant term into the equation - the relative purchasing power parity has the form (1). The second relation describes the money market equilibrium when money supply is equal to money demand (2). The real domestic money supply is determined by both domestic output and interest rate. There can be regarded also the trend which can capture the possible effect of the changing nature of financial intermediation. The important relation explaining transmission of business cycle from foreign environment into the domestic economy is output relation. The main channel of business cycle transmission is foreign output and according to Obstfeld and Krugman [4] the current account is a function of exchange rate and output. This implies the inclusion of exchange rate variable into the equation (3). The constant term represents the gap between the domestic and foreign output. The equation (4) describes the interest rate parity which is similar to purchasing power parity, however stands for relation between domestic and foreign interest rates which should determine also the nominal exchange rate. Also this parity relationship is a result of arbitrage activities and is derived from AA curve and based on the international Fisher effect which states that the exchange rate adjusts to the interest rate differential between two countries. It implies that the currency with lower interest rate is expected to appreciate relative to one with a higher rate. The last relation derived from arbitrage conditions is the Fisher inflation parity (5), which describes the relation between domestic interest rates and inflation. It is assumed that higher nominal interest rates reflect inflation.

$$\Delta p_t - \Delta ps_t - e_t = b_{10} + \varepsilon_{1,t+1} \tag{1}$$

$$m_t - \Delta p_t = b_{20} + \beta_{21} y_t + \beta_{22}\left(1 + \frac{r_t}{100}\right) + \varepsilon_{2,t+1} \tag{2}$$

$$y_t = b_{30} + \beta_{31} ys_t + \beta_{32} e_t + \varepsilon_{3,t+1} \tag{3}$$

$$\left(1 + \frac{r_t}{100_t}\right) - \left(1 + \frac{rs_t}{100_t}\right) - e_t = b_{40} + \varepsilon_{4,t+1} \tag{4}$$

$$\left(1 + \frac{r_t}{100}\right) = b_{50} + \beta_{51} \Delta p_t + \varepsilon_{5,t+1} \tag{5}$$

where p_t is domestic price level, ps_t is foreign price level, e_t is nominal exchange rate, m_t is money stock, y_t is domestic output, ys_t is foreign output, r_t is domestic interest rate, rs_t is foreign interest rate, b_{i0}, $i=1,2,\ldots,5$ stands for intercept and $\varepsilon_{i,t+1}$ are long-run stochastic terms.

In all the relations (1) – (5) we can consider also inclusion of trend term, because it can represent e.g. a convergence of Slovak Republic towards its trading partners in equations (1), (3) and (4). However, the inclusion of trend is questionable mainly in current crisis period when all economies are influenced by crisis and thus their economic development is getting closer.

3 Cointegrated VAR model

In this empirical study we apply the modelling strategy to Slovak data to see how the formulated economic theory accounts for the case of the Slovak Republic.

3.1 Input data

The variables for the model are y_t, ys_t, p_t, ps_t, r_t, rs_t, e_t and m_t. A detailed description of these variables is given below. They are all quarterly and seasonally adjusted data and they cover the period 2000Q1-2012Q4. In the model the data are used after logarithm transformation. To measure the domestic inflation we use the consumer price index unlike the paper by Garratt [1] where retail price index is used. To measure the output of foreign trading partners we use their domestic demand instead of GDP. This way we avoid the bias due to their foreign demand if they are also open economies. The most important trading partners of the Slovak Republic are identified on the basis of their shares in the foreign trade of Slovakia.

The list of variables, their description and data source

- y_t – gross domestic product of Slovakia at market prices, millions of national currency, chain-linked volumes with reference year 2005, source: Eurostat;
- ys_t – domestic demand at market prices for the most significant trading partners, millions of national currency, chain-linked volumes with reference year 2005, source: Eurostat;
- p_t – harmonized consumer price index of Slovakia (2005=100), source: Eurostat;
- ps_t – harmonized consumer price index of the most significant trading partners (2005=100), source: Eurostat;
- r_t – short-term interest rate of Slovakia - 3-month rate (%), source: OECD;
- rs_t – short-term interest rate of the most significant trading partners - 3-month rate (%), source: OECD;
- e_t – nominal effective exchange rate (NEER)[3] of Slovakia (27 trading partners), index 2005=100, source: Eurostat;
- m_t – money stock M1 of Slovakia, millions of EUR, source: National bank of Slovakia.

The aggregate variables for the most significant trading partners are composed using weights based on the shares of individual countries in the Slovak foreign trade which are varying in individual years. The most significant trading partners and their average shares in the foreign trade of Slovak Republic are given in the Table 1.

Trading partner	The share in foreign trade (%)
Germany	34.5
Czech Republic	21.1
Italy	9.0
Austria	8.1
Poland	8.0
Hungary	7.9
France	7.2
United Kingdom	4.2

Table 1 Average shares of trading partners in the foreign trade of Slovak Republic for the period 2000-2012

Investigation of the non-stationarity of data

The graphical depiction of variables under consideration (in Annex) suggests that these variables are non-stationary. The Augmented Dickey-Fuller (ADF) test was used to verify the stationarity of original time series and their first differences. The ADF test statistics for the levels and first differences of the original variables in logarithms are reported in Table 2 (the t-statistics are computed using ADF regressions with an intercept and linear time trend for levels and with an intercept for the first differences). The results of the test suggest that it is reasonable to treat all variables in consideration as I(1) variables. For these variables the unit root hypothesis is rejected when applied to their first differences at the significance level of 1%.

[3] The NEER (or, equivalently, the "Trade-weighted currency index") of a country aims to track changes in the value of that country's currency relative to the currencies of its principal trading partners. It is calculated as a weighted geometric average of the bilateral exchange rates against the currencies of competing countries.

Variable	For the levels t-statistic(sign.)	For the first differences t-stat/sign.
y_t	-0.95(0.94)	-7.09(0.00)
ys_t	-1.51(0.81)	-4.63(0.00)
Δp_t	-6.61(0.00)	-6.97(0.00)
Δps_t	-4.900(0.00)	-9.20(0.00)
r_t	-5.49(0.00)	-4.61(0.00)
rs_t	-2.88(0.18)	-4.26(0.00)
e_t	-1.70(0.74)	-5.39(0.00)
m_t	-1.63(0.77)	-6.02(0.00)

Table 2 Augmented Dickey-Fuller unit root test applied to Slovak variables

3.2 Model estimation and verification

In the first stage the order of the unrestricted VAR model is selected. VAR order of one appears to be appropriate when using the AIC and also SIC as the model selection criteria. The next step is to test the cointegration rank. The purpose of the cointegration test is to determine whether a group of non-stationary series are cointegrated or not. We applied the test with the linear trend in the data, as some time series show linear trend, and with an intercept but no trend in the cointegrating equation. The trace test and also the maximum eigenvalue test identify five cointegrating relationships among the eight variables at 5% level (Table 3).

Variable	Trace statistic (critical value)	Max eigen statistic (critical value)
$r \leq 0$	276.3664 (159.5297)	83.26436 (52.36261)
$r \leq 1$	193.1020 (125.6154)	51.65636 (46.23142)
$r \leq 2$	141.4457 (95.75366)	45.35564 (40.07757)
$r \leq 3$	96.09003 (69.81889)	40.85650 (33.87687)
$r \leq 4$	55.23352 (47.85613)	28.88881 (27.58434)
$r \leq 5$	26.34471 (29.79707)	13.06977 (21.13162)

Table 3 Johansen test of cointegration rank

Since five cointegration relations are in line with our theoretical expectations, we proceed estimating vector error correction (VEC) model with five cointegrating relations. The VEC model for the Slovak economy has the following form:

$$\Delta z_t = a_0 + \sum_{i=1}^{p-1} \Gamma_i \Delta z_{t-i} - \alpha \beta' z_{t-1} + u_t \tag{6}$$

where $z_t = (e_t\ r_t\ p_t\ y_t\ ps_t\ m_t\ rs_t\ ys_t)$', a_0 is a vector of intercepts, p is order of underlying VAR model, Γ_i are matrices of short-run coefficients, $\beta' z_{t-1}$ are the error correction terms, α is a matrix of error-correction coefficients and u_t is a vector of disturbances assumed to be white noise.

The matrix β'can be used to impose all the theoretical restrictions necessary for the structural long-run relationships:

$$\beta' = \begin{pmatrix} -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & \beta_{22} & -1 & \beta_{24} & 0 & 1 & 0 & 0 \\ \beta_{31} & 0 & 0 & 1 & 0 & 0 & 0 & \beta_{38} \\ -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & \beta_{53} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The estimation of a VEC model is carried out in two steps. In the first step, we estimate the cointegrating relations from the Johansen procedure as used in the cointegration test. We then construct the error correction terms from the estimated cointegrating relations and estimate a VAR in first differences including the error cor-

rection terms as regressors. The estimated long-run relationships for the Slovak economy, incorporating all the restrictions (35 restrictions) suggested by the theory are the following (with *t*-statistics in []):

$$\Delta p_t - \Delta ps_t - e_t = 4.678 + \varepsilon_{1,t+1} \tag{7}$$

$$m_t - \Delta p_t = 3.982 - 1.47 y_t + 7.409\left(1 + \frac{r_t}{100}\right) + \varepsilon_{2,t+1} \tag{8}$$

[-20.650] [16.159]

$$y_t = -20.483 + 1.479 ys_t - 2.658 e_t + \varepsilon_{3,t+1} \tag{9}$$

[3.410] [-14.819]

$$\left(1 + \frac{r_t}{100_t}\right) - \left(1 + \frac{rs_t}{100_t}\right) - e_t = 4.678 + \varepsilon_{4,t+1} \tag{10}$$

$$\left(1 + \frac{r_t}{100}\right) = 0.405 - 44.287 \Delta p_t + \varepsilon_{5,t+1} \tag{11}$$

[-9.914]

The first relation confirms the existence of cointegration relationship in purchasing power parity relation, thus empirical evidence on Slovak data supports this relationship. The value of intercept is also very close to that estimated for UK in [1]. The signs of long-run elasticities of the influence of the domestic output and interest rate on the real money supply are the opposite than expected according to the theory. The third long-run output relationship describes the influence of foreign output on the domestic output. The long-run elasticity of the influence of foreign output on domestic output is about 1.5%. The interest rate parity relation includes the intercept, which can be interpreted as the deterministic component of the risk premium, which is approx. 18.7% per annum. However, this value is rather high. The constant in Fisher inflation parity relation implies that the average long-run Slovak interest rate is about 4%. Also the importance of convergence of Slovakia to its foreign trade partners was investigated considering the inclusion of trend into the model. However, the results showed that the estimated parameters for trend terms are very low, though statistically significant and that we can consider the trend terms negligible in the model.

3.3 Impulse response analysis

The short-term dynamic reactions of domestic variables to foreign shock effects are examined by general impulse response functions. Generalized impulses as described by Pesaran and Shin [5] construct an orthogonal set of innovations that does not depend on the VAR ordering. We provide here the examples of the effects of exogenous shocks to a unit (one standard error) unexpected increase on the domestic endogenous variables. The Figure 1 includes the responses of Slovak variables to two shocks from its foreign trading partners: foreign demand shock and foreign monetary shock.

The responses of Slovak GDP to foreign shocks show very high persistence and the Slovak real output reacts positively to both foreign positive shocks in consideration – to the increase in foreign GDP and also in inflation. The temporary increase in GDP of the most important trading partners of Slovakia resulted in longer period of increasing Slovak GDP, short-term growth of Slovak prices a very short period of appreciation of the Slovak effective exchange rate followed by longer period of depreciation. A temporary increase in the foreign inflation causes a short-run rise in domestic inflation, a moderate growth of Slovak GDP and an appreciation of the currency, which is followed by longer period of depreciation. An increase in the foreign price level we can interpret in line with purchasing power parity relation in the following way. Increasing foreign price level causes the rise in domestic price level and the exchange rate depreciates in the long run. In that case it means that the Slovak inflation should be mostly higher than foreign inflation, which is evidenced also by the data used.

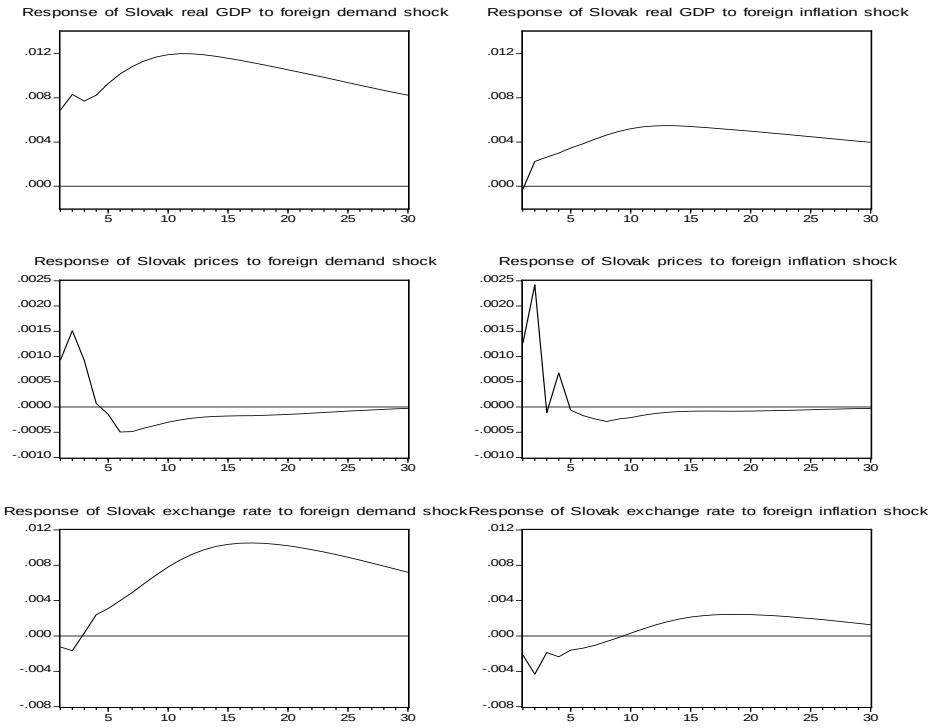

Figure 1 Impulse responses of unit foreign shocks

4 Conclusion

This paper examines long-run cointegration model on the Slovak data. The theoretical model is adjusted according to AA-DD model framework. The proposed relationships were confirmed to be cointegrated and can be used to examine the effects of a one-time rise in the foreign variables on the domestic variables. The results of impulse response analysis confirmed that the demand shock from the most important trading partners of Slovakia has high and persisting influence on the Slovak output. The positive foreign inflation shock is translated into immediate increase of Slovak prices and depreciation of Slovak currency in the long run. In future research the current cointegration model could be extended introducing a break variable.

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Interactions between sovereign credit default swaps and bonds: The case of EU countries

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Abstract. Credit default swap markets are considered as a leading indicator of the future development of creditworthiness, which can point out the potential situation in economy. The study examines the relationship between sovereign credit default swap and bond spreads of the EU countries in the period before, during and after financial crisis. The aim of the study is to find out if the price discovery process in the credit default swap and bond markets has been changed by financial crisis and continuing debt crisis in short-run. A vector autoregressive model in the context of Granger causality is employed to capture causal relationships between observed credit default swap and bond markets. Model is employed on daily and weekly data. Results can be beneficial for all participants in the financial markets, especially for regulators and investors as a possible indicator of credit risk. This research showed that the role of both markets has changed. We found out that the number of causal relations grew during the financial crisis period and has decreased during the debt crisis period. The credit default swap markets could be comprehended as a potential indicator of increasing credit risk during the period of financial crisis but they cannot be comprehended anymore in the presence.

Keywords: credit default swap market, bond market, Granger causality

JEL Classification: C32, G15
AMS Classification: 91G70

1 Introduction

Both government bonds and sovereign credit default swaps (CDS) offer investors exposure to the risk and return of sovereign debt. Increased attention to the relation between CDS and bonds started to be devoted after the outbreak of the financial crisis. Lead-lag analysis between CDS markets and bond markets is one of the centers of attention because of exploration the adjustment process between bond and CDS spreads, which can be beneficial for all participants in the financial markets.

First such analysis was published in empirical study by Blanco, Brennan and Marsh [2], which paid attention mainly to the relationship between investment grade bonds and CDS. It was the first study, in which a time series framework was used for credit derivatives. They used data for US and European firms. Besides other findings, they found out the CDS markets "lead" the bond markets. Except this contribution, there are several empirical papers which investigate lead-lag relationship between CDS markets and bond markets, see [5], [9], [7] or [11]. All of them confirm the leading role of CDS with respect to bond markets in majority of cases, which proves the CDS markets move ahead of the bond markets in price adjustment.

There are two main reasons why credit default swap markets are expected to lead the bond markets. 1) The information is reflected earlier in the credit default swap market than in the bond market. 2) Institutional features of the credit default swap markets facilitate a continuous flow of transactions, because short positions in the bond markets are more difficult to establish [9].

Another stream of research is focused on analysis of relationship between sovereign CDS and government bonds. Theoretically, credit default swap markets are considered as a leading indicator of the future development of creditworthiness, which can point out the potential situation in economy. It was confirmed in analysis employed by Palladini and Portes [10]. Contrary, this theoretical knowledge has not been proved in other empirical research, e.g. see [6], who found out that since 2008, in half of the sample countries, price discovery takes place in the bond market and half in the CDS market; or [1], who concluded that the price discovery process is state-dependent and that there are several significant factors in determining which market leads price discovery.

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In our study, we try to explore interaction between sovereign CDS and government bond spreads of EU countries. The aim of the study is to find out whether the price discovery process in the CDS and bond markets has been changed by financial crisis and continuing debt crisis.

2 Data

In our research, we pay attention to 20 members of the European Union for which data for 5 years sovereign CDS and bonds were available. All data were obtained from Bloomberg on daily and weekly basis.

The total sample period (March 2003 – May 2013) is divided into three sub-periods according to trends in development of the Markit iTraxx SovX Western Europe Index:

- Pre-crisis period (March 2003 – May 2007),
- Financial crisis period (June 2007 – October 2009),
- Debt crisis period (November 2009 – May 2013).

The start of observation differs for each country. It depends on availability of data for CDS.

The crisis period is understood as period of the biggest turmoil in the financial markets. Then the crisis has been transformed into a sovereign debt crisis that has not yet finished, therefore the period after the financial turmoil is denoted as the debt crisis period.

3 Methodology

Granger causality tests are employed to estimate linkages between prices of bond and CDS. It allows us to discover short-run causality. This causality test was developed by Granger [8]. General concept of Granger causality is related to prediction. In case that one time series has an influence on the second time series, the first one should help to improve predictions of the second one [4]. Granger causality means only correlation between present value of one variable and past values of other variables [3].

The standard Granger causality model for two variables can be represented as:

$$Y_t = \sum_{i=1}^{\rho} \alpha y_i \, Y_{t-1} + \sum_{i=1}^{\rho} \alpha x_i \, X_{t-1} + \varepsilon_t \quad (1)$$

$$X_t = \sum_{i=1}^{\rho} \beta x_i \, \Delta X_{t-1} + \sum_{i=1}^{\rho} \beta y_i \, Y_{t-1} + \varepsilon_t \quad (2)$$

where Y_t and X_t are stationary time series, ε_t is uncorrelated white noise, αx_i and βx_i are coefficients chosen to minimize σ^2 (variance), ρ is finite and shorter than the given time series (it can equal infinity but in practice, it is finite due to the length of the available data).

The null hypothesis "X does not Granger cause Y" (equation 1) or "Y does not Granger cause X" (equation 2) is rejected if the coefficients αx_i and βy_i are jointly significant.

In our research, Granger causality test tries to find if credit default swap prices do "Granger-cause" bond prices (past values of credit default swap prices improve the prediction of bond prices), and vice versa - if bond prices do "Granger-cause" credit default swap prices (past values of bond prices improve the prediction of credit default swap prices). We include five lags (in our case 5 days) within lag specification of Granger causality test.

4 Results

The Granger causality tests enabled us to examine two null hypotheses.

1) CDS "does not Granger cause" Bond.
2) Bond "does not Granger cause" CDS.

4.1 Daily data

Table 1 shows summary results of Granger causality tests in all periods employed on daily data.

Full Sample

For full sample data, results indicate that rates of return of credit default swap of six countries of European Union (from total 19 tested) "Granger causes" rates of return of bond. Detailed results are shown in Table 2. Probabilities are lower than 10 % significance level in these cases, therefore the null hypothesis "CDS does not Granger cause Bond" can be rejected in these six cases (Spain, Portugal, Sweden, Hungary, Poland, Czech Re-

public). The second null hypothesis "Bond does not Granger cause CDS" can be rejected for eight tested countries of European Union (Italy, Spain, Portugal, Sweden, Netherlands, Hungary, Poland, Czech Republic).

Period	No. of countries	No. of causal relationships		
		CDS -> bJe ond	Bond -> CDS	Mutual
Full sample	19	6	8	6
Pre-crisis period	12	0	0	0
Financial crisis	16	6	11	5
Debt crisis	19	5	2	1

Table 1 Summary of Granger causality test results on daily data (at 10% significance level)

A mutual relationship from CDS to Bond was proved in six cases (Spain, Portugal, Sweden, Hungary, Poland, Czech Republic). It means CDS is not strongly exogenous in any case. Bond is exogenous only in two cases. Both null hypotheses can be rejected in six cases simultaneously. It shows that there is a backward relation between variables. Full sample results do not indicate the relationships between variables were changing in the time, therefore our attention was paid to the particular periods.

Country	Null hypothesis	Start of the period	Probability			
			Full sample	Pre-crisis	Financial crisis	Debt crisis
United Kingdom	CDS -> Bond	08/11/2008	0,3858	-	0,9516	0,1738
	Bond -> CDS		0,1354	-	0,0397**	0,1340
France	CDS -> Bond	04/07/2003	0,8550	0,9734	0,5946	0,6535
	Bond -> CDS		0,1624	0,5941	0,5143	0,2229
Germany	CDS -> Bond	03/17/2003	0,9949	0,9177	0,7162	0,5547
	Bond -> CDS		0,6762	0,8125	0,8258	0,6722
Italy	CDS -> Bond	03/17/2003	0,1237	0,4727	0,2290	0,0416**
	Bond -> CDS		2*E-09***	0,6100	0,1538	0,4970
Spain	CDS -> Bond	04/12/2004	0,0952*	0,8200	0,1541	0,0092***
	Bond -> CDS		0,0005***	0,9072	0,0862*	0,2058
Portugal	CDS -> Bond	04/07/2003	1*E-06***	0,2564	0,3871	4*E-14***
	Bond -> CDS		2*E-07***	0,9568	0,0858*	0,1786
Sweden	CDS -> Bond	01/15/2007	0,0511*	0,4561	0,0009***	0,6387
	Bond -> CDS		0,0212**	0,4860	0,0024***	0,5381
Netherlands	CDS -> Bond	09/08/2008	0,2067	-	0,6356	0,4426
	Bond -> CDS		0,00933*	-	0,0870*	0,4833
Slovenia	CDS -> Bond	04/12/2010	0,1390	-	-	0,1390
	Bond -> CDS		0,1028	-	-	0,1028
Ireland	CDS -> Bond	03/17/2003	0,9846	-	0,5567	0,2530
	Bond -> CDS		0,5934	-	0,0002***	0,8558
Hungary	CDS -> Bond	03/05/2007	6*E-06***	0,5684	0,0011***	0,0009***
	Bond -> CDS		0,0002***	0,6177	0,0006***	0,2391
Poland	CDS -> Bond	03/05/2007	0,0818*	0,7480	0,0009***	0,6103
	Bond -> CDS		0,0003***	0,3642	0,0035***	0,0342**
Belgium	CDS -> Bond	03/17/2003	0,2628	0,5924	0,1304	0,1258
	Bond -> CDS		0,7940	0,9747	0,9474	0,6122
Romania	CDS -> Bond	04/28/2003	0,7577	-	-	0,7577
	Bond -> CDS		0,5132	-	-	0,5132
Czech Republic	CDS -> Bond	03/05/2007	0,0101**	0,5896	0,0019***	0,4783
	Bond -> CDS		0,0155**	0,8886	0,0005***	0,9162
Bulgaria	CDS -> Bond	06/28/2010	-	-	-	-
	Bond -> CDS		-	-	-	-
Denmark	CDS -> Bond	04/12/2004	0,9824	0,6691	0,0002***	0,4783
	Bond -> CDS		0,9931	0,1674	0,00005***	0,9162

Slovakia	CDS -> Bond	04/12/2010	0,2142	-	-	0,2142
	Bond -> CDS		0,9492	-	-	0,9492
Finland	CDS -> Bond	10/15/2007	0,8059	-	0,5067	0,5418
	Bond -> CDS		0,2063	-	0,0575*	0,3691
Greece	CDS -> Bond	03/05/2007	0,8516	0,4441	0,0023***	0,0471**
	Bond -> CDS		0,2855	0,8847	0,3447	0,0279**

Table 2 Detailed results of pairwise Granger causality tests (daily data)

*denotes significance at 10% level; ** denotes significance at 5% level; *** denotes significance at 1% level

Pre-crisis period

There was not any Granger causality detected in all 12 observed countries during the pre-crisis period. Both null hypotheses "CDS does not Granger cause" and "Bond does not Granger cause" cannot be rejected in any case. Financial markets were stable during this period without doubts about creditworthiness.

Financial crisis period

During the financial crisis period, we observed 16 countries of European Union. Results of the Granger causality test indicate that the null hypothesis "CDS does not Granger cause Bond" can be rejected in six cases (Sweden, Hungary, Poland, Czech Republic, Denmark, Greece). Only in one case (Greece) CDS is strongly exogenous variable.

The null hypothesis "Bond does not Granger cause CDS" can be rejected in 11 cases. Bond is strongly exogenous variable in five cases. Five relationships are mutual. Increased number of relationships can be caused as a consequence of turmoil in the financial markets.

Debt crisis period

The number of relationships has decreased compared to the financial crisis period. Only one relationship is mutual. In this period, we found five causal relationships (Italy, Spain, Portugal, Hungary, Greece) from 19 observed cases in total for the hypothesis "CDS does not Granger cause Bond". CDS is strongly exogenous in four cases. Two relationships were found for the hypothesis "Bond does not Granger cause CDS" (Poland, Greece). Bond is strongly exogenous only in one case.

4.2 Weekly data

Table 3 shows summary results of Granger causality tests in all periods employed on weekly data. Employing Granger causality tests on weekly data showed that they do not confirm results on daily data. The relationship between CDS and bonds is close and because of fast information absorption they do not have such importance in in short-run.

Full Sample

For full sample data, results indicate that rates of return of credit default swap of six countries of European Union (from total 20 tested) "Granger causes" rates of return of bond. Probabilities are lower than 10 % significance level in these cases, therefore the null hypothesis "CDS does not Granger cause Bond" can be rejected in six cases (Sweden, Hungary, Poland, Czech Republic, Slovakia, Greece). The second null hypothesis "Bond does not Granger cause CDS" can be rejected for two observed countries of European Union (Italy and Ireland).

A mutual relationship from CDS to Bond was not proved in any case. It means that CDS is strongly exogenous in six cases and Bond is exogenous in two cases.

Period	**No. of countries**	**No. of causal relationships**		
		CDS -> bond	**Bond -> CDS**	**Mutual**
Full sample	20	6	2	0
Pre-crisis period	12	1	2	0
Financial crisis	16	6	1	1
Debt crisis	20	2	0	0

Table 3 Summary of Granger causality test results on weekly data (at 10% significance level)

Pre-crisis period

Almost any Granger causality was not detected in all 12 observed countries during the pre-crisis period. The null hypothesis "CDS does not Granger cause Bond" can be rejected only in one case (Spain). The hypothesis "Bond does not Granger cause CDS" can be rejected in two cases (Italy and Greece). Results on weekly data are almost same as data on daily basis.

Financial crisis period

During the financial crisis period, we observed 16 countries of European Union. Results of the Granger causality test indicate that the null hypothesis "CDS does not Granger cause Bond" can be rejected in six cases (Spain, Portugal, Ireland, Czech Republic, Denmark, Greece). All cases except of Ireland are strongly exogenous variable. The null hypothesis "Bond does not Granger cause CDS" can be rejected in one case. Bond is not strongly exogenous variable in this case. Only one relationship is mutual.

Debt crisis period

The number of relationships has decreased compared to the financial crisis period. There is no mutual relationship. In this period we found two causal relationships (Czech Republic and Slovakia) from 20 observed cases in total for the hypothesis "CDS does not Granger cause Bond". CDS is strongly exogenous in these cases. No relationship was found for the hypothesis "Bond does not Granger cause CDS" (Poland, Greece). .

5 Conclusions

The aim of the study was to find out if the price discovery process in the CDS and bond markets has been changed by financial crisis and continuing debt crisis in short-run. Granger causality tests were employed to discover relations in the particular periods. Our research showed that the role of both markets has changed. We found out that the number of causal relations grew during the financial crisis period and has decreased during the debt crisis period. According to the obtained results the credit default swap markets could be comprehended as a potential indicator of increasing credit risk during the period of financial crisis but they cannot be comprehended anymore in the presence. The empirical studies showed important role of corporate CDS and corporate bond markets as a leading indicators of the future development of creditworthiness. Our research showed that this rule does not hold when it is examined on sovereign CDS and government bond markets. However, the role of the CDS and bond markets was changed during the financial crisis, the sovereign CDS and government bond markets could be considered as leading indicators, the role of them has changed during the period of debt crisis and the sovereign CDS and government bond markets cannot be considered as leading indicators of the future creditworthiness of the country anymore. These findings can be useful for all participants of the financial markets, policy makers and regulators. Attention in the future research should be paid to long-run causality and other determinants, which can affect prices of sovereign CDS and bonds.

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Economic and Financial Problems via Multiobjective Stochastic Optimization

Vlasta Kaňková[1]

Abstract. Multiobjective optimization problems depending on a probability measure correspond to many economic and financial activities. Evidently if the probability measure is completely known, then we can try to influence economic process employing methods of a multiobjective deterministic optimization theory. Since this assumption is fulfilled very seldom we have mostly to analyze the mathematical model and consequently also economic process on the data base. The aim of the talk will be to investigate a relationship between "characteristics" obtained on the base of complete knowledge of the probability measure and them obtained on the above mentioned data base. To this end, the results of the deterministic multiobjective optimization theory and the results obtained for stochastic one objective problems will be employed.

Keywords: Stochastic multiobjective optimization problems, efficient solution, Wasserstein metric, $\mathcal{L}_1$ norm, stability, empirical estimates, Lipschitz property.

JEL classification: C44
AMS classification: 90C15

1 Introduction

To introduce a "rather general" multiobjective stochastic programming problem, let $(\Omega, \mathcal{S}, P)$ be a probability space; $\xi := \xi(\omega) = (\xi_1(\omega), \ldots, \xi_s(\omega))$ s–dimensional random vector defined on $(\Omega, \mathcal{S}, P)$; $F(:= F(z),\, z \in R^s)$, P_F and Z_F denote the distribution function, the probability measure and the support corresponding to ξ. Let, moreover, $g_i := g_i(x,\, z)$, $i = 1, \ldots, l$, $l \geq 1$ be real–valued (say, continuous) functions defined on $R^n \times R^s$; $X_F \subset X \subset R^n$ be a nonempty set generally depending on F, and $X \subset R^n$ be a nonempty deterministic set. If the symbol E_F denotes the operator of mathematical expectation corresponding to F and if for every $x \in X$ there exist finite $\mathsf{E}_F g_i(x, \xi)$, $i = 1, \ldots, l$, then a rather general "multiobjective" one–stage stochastic programming problem can be introduced in the form:

$$\text{Find} \quad \min \mathsf{E}_F g_i(x, \xi),\ i = 1, \ldots, l \quad \text{subject to} \quad x \in X_F. \tag{1}$$

The multiobjective problem (1) corresponds evidently to economic situation in which a "result" of an economic process is simultaneously influenced by a random factor ξ and a decision parameter x, it is reasonable to evaluate this process by a few (say l, $l \geq 1$) objective functions. The decision vector has to be determined without knowledge of the random element realization and it seems to be reasonable to determine "the decision" with respect to the mathematical expectation of the objectives.

It is possible only very seldom to find out simultaneously the solution with respect to all criteria in (1) and moreover, these problems depend on a probability measure P_F that usually has to be estimated on the data base. Consequently, in applications very often the "underlying" probability measure P_F has to be replaced by empirical one. Evidently, then the "solution" and an analysis of the problem have to be done with respect to an empirical problem:

$$\text{Find} \quad \min \mathsf{E}_{F^N} g_i(x, \xi),\ i = 1, \ldots, l \quad \text{subject to} \quad x \in X_{F^N}, \tag{2}$$

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where F^N denotes an empirical distribution function determined by a random sample $\{\xi^i\}_{i=1}^N$ (not necessary independent) corresponding to the distribution function F.

To analyze the problem (1), first, the results of the multiobjective deterministic problems have to be recalled. Since, it follows from multiobjective theory that the results of uni–objective optimization theory can be useful (under rather general conditions) to investigate the relationship between the results obtained under complete knowledge of P_F and them obtained on the data base, we recall also the results obtained for uni–objective stochastic programming problems. Our aim will be to focus to "underlying" distributions with heavy tails, that correspond just to many economic and financial processes (for more details see e.g. [10] or [12]).

According to the above mentioned facts, the paper is organized as follows. First, we try to recall auxiliary assertions concerning deterministic multiobjective theory (subsection 2.1). Stability and empirical estimates obtained for uni–objective stochastic programming problems are recalled in section 2.2. Section 3 is devoted to the stability analysis of the problem (1). Essential results devoted to the multiobjective problems can be found in section 4.

2 Some Definition and Auxiliary Assertion

2.1 Deterministic Multiobjective Problems

To recall some results of the multiobjective deterministic optimization theory we consider a multiobjective deterministic optimization problem in the following form:

$$\text{Find} \quad \min f_i(x),\ i = 1, \ldots, l^{'} \quad \text{subject to} \quad x \in \mathcal{K}, \tag{3}$$

where $f_i(x)$, $i = 1, \ldots, l^{'}$ are real–valued functions defined on R^n, $\mathcal{K} \subset R^n$ is a nonempty set.

Definition 1. The vector x^* is an efficient solution of the problem (3) if and only if there exists no $x \in \mathcal{K}$ such that $f_i(x) \leq f_i(x^*)$ for $i = 1, \ldots, l^{'}$ and such that for at least one i_0 one has $f_{i_0}(x) < f_{i_0}(x^*)$.

Definition 2. The vector x^* is properly efficient solution of the multiobjective optimization problem (3) if and only if it is efficient and if there exists a scalar $M > 0$ such that for each i and each $x \in \mathcal{K}$ satisfying $f_i(x) < f_i(x^*)$ there exists at least one j such that $f_j(x^*) < f_j(x)$ and

$$\frac{f_i(x^*) - f_i(x)}{f_j(x) - f_j(x^*)} \leq M. \tag{4}$$

Proposition 1. ([4]) *Let $\mathcal{K} \subset R^n$ be a nonempty convex set and let $f_i(x)$, $i = 1, \ldots, r$ be convex functions on $\mathcal{K}$. Then x^0 is a properly efficient solution of the problem (3) if and only if x^0 is optimal in*

$$\min_{x \in \mathcal{K}} \sum_{i=1}^{r} \lambda_i f_i(x) \quad \text{for some} \quad \lambda_1, \ldots, \lambda_r > 0; \quad \sum_{i=1}^{r} \lambda_i = 1.$$

Definition 3. Let $\bar{h}(x)$ be a real–valued function defined on a nonempty convex se $\mathcal{K} \subset R^n$. $\bar{h}(x)$ is a strongly convex function with a parameter $\rho > 0$ if

$$\bar{h}(\lambda x^1 + (1-\lambda)x^2) \leq \lambda \bar{h}(x^1) + (1-\lambda)\bar{h}(x^2) - \lambda(1-\lambda)\rho\|x^1 - x^2\|^2 \quad \text{for every} \quad x^1, x^2 \in \mathcal{K},\ \lambda \in \langle 0, 1\rangle.$$

Proposition 2. ([6]) *Let $\mathcal{K} \subset R^n$ be a nonempty, compact, convex set. Let, moreover, $\bar{h}(x)$ be a strongly convex with a parameter $\rho > 0$, continuous, real–valued function defined on $\mathcal{K}$. If x^0 is defined by the relation $x^0 = \arg\min_{x \in \mathcal{K}} \bar{h}(x)$, then*

$$\|x - x^0\|^2 \leq \frac{2}{\rho}|\bar{h}(x) - \bar{h}(x^0|| \quad \text{for every} \quad x \in \mathcal{K}.$$

2.2 Uni–Objective Stochastic Programming Problems

To recall suitable for us assertions of one criteria stochastic optimization theory we start with the problem:

$$\text{Find} \quad \varphi(F, X_F) = \inf \mathsf{E}_F g_0(x, \xi) \quad \text{subject to} \quad x \in X_F, \tag{5}$$

where $g_0(x,\, z)$ is a real–valued function defined on $R^n \times R^s$.

First, if F and G are two s–dimensional distribution functions for which the Problem (5) is well defined, then we can obtain by the triangular inequality that

$$|\varphi(F,\, X_F) - \varphi(G,\, X_G)| \,\leq\, |\varphi(F,\, X_F) - \varphi(G,\, X_F)| \,+\, \varphi(G,\, X_F) - \varphi(G,\, X_G)|. \tag{6}$$

According to the relation (6) we can study separately stability of the problem (5) with respect to perturbation in the objective function and in constraints set. In this paper we restrict our consideration to the case $X_F = X$ independently of F. To this end we introduce the following assumptions:

A.1
- X is a convex set and there exists $\varepsilon > 0$ such that $g_0(x,\, z)$ is a convex bounded function on $X(\varepsilon)$ ($X(\varepsilon)$ denotes the ε–neighborhood of X),
- $g_0(x, z)$ is a Lipschitz function of $z \in R^s$ with the Lipschitz constant L (corresponding to the $\mathcal{L}_1$ norm) not depending on x.

To introduce the first assertion dealing with the stability of the problem (5) (with $X_F = X$) we denote by $F_i, i = 1, \ldots, s$ one–dimensional marginal distribution functions corresponding to F; $\mathcal{P}(R^s)$ the set of Borel measures on R^s, $\mathcal{M}_1(R^s) = \{P \in \mathcal{P}(R^s) : \int\limits_{R^s} \|z\|_s^1 P(dz) < \infty\}$, $\|\cdot\|_s^1$ denote $\mathcal{L}_1$ norm in R^s.

Proposition 3. ([7]) *Let $P_F, P_G \in \mathcal{M}_1(R^s)$, X be a nonempty set. If A.1 is fulfilled, then*

$$|\mathsf{E}_F g_0(x,\, \xi) - \mathsf{E}_G g_0(x,\, \xi)| \leq L \sum_{i=1}^{s} \int\limits_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| dz_i \quad \text{for every} \quad x \in X.$$

Proposition 3 reduces (from the mathematical point of view) s–dimensional case to one–dimensional. Of course, stochastic dependence between components of the random vector ξ is there neglected. Replacing G by an empirical estimate F^N of F we can employ Proposition 3 to investigate empirical estimates of Problem (5) (with $X_F = X$) and according to Proposition 1 also to analyze relationship between Problems (1) and (2). Evidently, according to Proposition 3 it is reasonable to investigate the behaviour of $\int\limits_{-\infty}^{\infty} |F_i(z_i) - F_i^N(z_i)| dz_i$, $i = 1, \ldots, s$. To this end, we recall the following assumptions:

A.2 $\{\xi^i\}_{i=1}^{\infty}$ is independent random sequence corresponding to F, F^N is an empirical distribution function determined by $\{\xi^i\}_{i=1}^{N}$,

A.3 $P_{F_i}, i = 1, \ldots, s$ are absolutely continuous w. r. t. the Lebesgue measure on R^1.

Proposition 4. ([15]) *Let $s = 1$ and $P_F \in \mathcal{M}_1(R^1)$. Let, moreover A.2 be fulfilled . Then*

$$P\{\omega : \int\limits_{-\infty}^{\infty} |F(z) - F^N(z)| dz \longrightarrow_{N \longrightarrow \infty} 0\} = 1.$$

Proposition 5. [8] *Let $s = 1$, $t > 0$ and Assumptions A.2, A.3 be fulfilled. If there exists $\beta > 0$, $R := R(N) > 0$ defined on $\mathcal{N}$ such that $R(N) \xrightarrow[N\to\infty]{} \infty$ and, moreover,*

$$\begin{aligned}
&N^\beta \int\limits_{-\infty}^{-R(N)} F(z) dz \xrightarrow[N\to\infty]{} 0, \qquad N^\beta \int\limits_{R(N)}^{\infty} [1 - F(z)] dz \xrightarrow[N\to\infty]{} 0,\\
&2NF(-R(N)) \xrightarrow[N\to\infty]{} 0, \qquad 2N[1 - F(R(N))] \xrightarrow[N\to\infty]{} 0,\\
&(\tfrac{12N^\beta R(N)}{t} + 1) \exp\{-2N(\tfrac{t}{12R(N)N^\beta})^2\} \xrightarrow[N\longrightarrow\infty]{} 0,
\end{aligned} \tag{7}$$

then

$$P\{\omega :\, N^\beta \int\limits_{-\infty}^{\infty} |F(z) - F^N(z)| dz > t\} \xrightarrow[N\to\infty]{} 0. \tag{8}$$

($\mathcal{N}$ denotes the set of natural numbers.)

Evidently, it follows from the relations (7), (8) and from the classical result of [2] that the validity of the relation (8) depends on the tails behaviour (for more details see e.g. [9]).

Proposition 6 ([5]). *Let $s = 1$, $t > 0$, $r > 0$, the assumptions A.2, A.3 be fulfilled. Let, moreover, ξ be a random variable such that $\mathsf{E}_F|\xi|^r < \infty$. If constants $\beta, \gamma > 0$ fulfil the inequalities $0 < \beta + \gamma < 1/2, \quad \gamma > 1/r, \quad \beta + (1-r)\gamma < 0$, then*

$$P\{\omega : N^\beta \int_{-\infty}^{\infty} |F(z) - F^N(z)|dz > t\} \longrightarrow_{N \longrightarrow \infty} 0.$$

Analyzing Proposition 6 we can obtain $\beta := \beta(r)$ fulfilling this assertion and simultaneously

$$\beta(r) \longrightarrow_{r \longrightarrow \infty} 1/2, \quad \beta(r) \longrightarrow_{r \longrightarrow 2^+} 0.$$

Proposition 6 covers also some cases of heavy tails distributions. Unfortunately, we cannot obtain by this Proposition any results for the case when there exist only $\mathsf{E}_F|\xi|^r$ for $r < 2$. But just this case corresponds to stable distributions with the tail (shape) parameter $\nu < 2$ (for more details see e.g. [11] or [13]). The shape parameter expresses how "heavy" tails of distribution are. The case $\nu = 2$ corresponds to normal distribution, when the second moment exists. To obtain at least weaker result for the case when the finite moment exists only for $r < 2\,(\nu < 2)$, we recall the results of [1].

Proposition 7 ([1]). *Let $s = 1, \quad \{\xi^i\}_{i=1}^N$, $N = 1, 2, \ldots$ be a sequence of independent random values corresponding to a heavy tailed distribution F with the shape parameter $\nu \in (1, 2)$ and let*

$$\sup_{t>0} t^\nu \, P\{\omega : |\xi| > t\} < \infty, \tag{9}$$

then

$$\lim_{\bar{M} \longrightarrow \infty} \sup_N P\{\omega : \frac{N}{N^{1/\nu}} \int_{-\infty}^{\infty} |F(z) - F^N(z)| > \bar{M}\} = 0. \tag{10}$$

3 Problem Analysis

To analyze the stability of the multiobjective stochastic problem (1) we define the sets $\mathcal{G}(F, X_F)$, $\bar{\mathcal{X}}(F, X_F)$, $\bar{\mathcal{G}}(F, X_F)$ and the function $\bar{g}(x, z, \lambda)$ by the relations

$$\begin{aligned}
\mathcal{G}(F, X_F) &= \{y \in R^l : y_j = \mathsf{E}_F g_j(x, \xi),\ j = 1, \ldots, l \quad \text{for some} \quad x \in X_F;\ y = (y_1, \ldots, y_l)\},\\
\bar{\mathcal{X}}(F, X_F) &= \{x \in X_F : x \quad \text{is a properly efficient point of the problem (1)}\},\\
\bar{\mathcal{G}}(F, X_F) &= \{y \in R^l : y_j = \mathsf{E}_F g_j(x, \xi),\ j = 1, \ldots, l \quad \text{for some} \quad x \in \bar{\mathcal{X}}(F, X_F)\}\\
\bar{g}(x, z, \lambda) &= \sum_{i=1}^{l} \lambda_i g_i(x, z), \quad x \in R^n,\ z \in R^s,\ \lambda = (\lambda_1, \ldots, \lambda_l),\ \lambda_i > 0,\ \sum_{i=1}^{l} \lambda_i = 1.
\end{aligned} \tag{11}$$

Evidently, if the following assumptions are fulfilled

B.1
- X is a convex set and, moreover, there exists $\varepsilon > 0$ such that $g_i(x, z)$, $i = 1, \ldots, s$ are for every $z \in R^s$ a convex functions on $X(\varepsilon)$,
- $g_i(x, z)$, $\imath = 1, \ldots, l$ are Lipschitz functions of $z \in R^s$ with the Lipschitz constant L (corresponding to $\mathcal{L}_1$ norm) not depending on x,

then $\bar{g}(x, z, \lambda)$ is a convex function on $X(\varepsilon)$ and, moreover, it is a Lipschitz function of z with the Lipschitz constant L not depending on x, λ. Consequently, according to to Proposition 3 we can obtain.

Proposition 8. *Let $P_F, P_G \in \mathcal{M}_1(R^s)$, X be a nonempty set. If B.1 is fulfilled, then*

$$|\mathsf{E}_F \bar{g}(x, \xi, \lambda) - \mathsf{E}_G \bar{g}(x, \xi, \lambda)| \le L \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)|dz_i, \quad x \in X,\ \lambda_i > 0,\ i = 1, \ldots, s,\ \sum_{i=1}^{s} \lambda_i = 1.$$

Proposition 9. *Let P_F, $P_G \in \mathcal{M}_1(R^s)$, X be a compact set. If B.1 is fulfilled, then*

$$\Delta_n[\mathcal{G}(F,\, X),\, \mathcal{G}(G,\, X)] \leq L \sum_{i=1}^{s} \int_{-\infty}^{+\infty} |F_i(z_i) - G_i(z_i)| dz_i,$$

where the symbol $\Delta n[\cdot,\, \cdot]$ is reserved for the Hausdorff distance of the subsets of R^n (for the definition of the Hausdorff distance see e.g. [14]).

Proposition 10. *Let X be a convex set. If $g_i(x,\, z)$, $i = 1,\, \ldots,\, l$ are strongly convex (with a parameter $\rho > 0$) function on X, then $\mathsf{E}_F \bar{g}(x,\, \xi, \lambda)$ is a strongly convex function on X with the parameter ρ.*

Proof. The assertion of Proposition 10 follows from Definition 3 and Relation (11). □

Employing the assertion of Proposition 1 we can investigate the relationship between the Problems (1), and (2).

4 Empirical Estimates

Theorem 11. *Let Assumptions B.1, A.2, and A.3 be fulfilled, $P_F \in \mathcal{M}_1(R^s)$, X be a compact set. Then*

$$P\{\omega : \Delta_n[\mathcal{G}(F,\, X),\, \mathcal{G}(F^N,\, X)] \longrightarrow_{N \longrightarrow \infty}\} = 1.$$

Proof. The assertion of Theorem 11 follows from Propositions 4, 9 and the relation (11). □

Theorem 12. *Let $t > 0$, $r > 0$, Assumptions B.1, A.2, A.3 be fulfilled. Let, moreover, ξ be a random vector with the components ξ_i, $i = 1,\, \ldots,\, s$ such that $\mathsf{E}_F |\xi_i|^r < \infty$. If constants $\beta,\, \gamma > 0$ fulfil the inequalities $0 < \beta + \gamma < 1/2$, $\gamma > 1/r$, $\beta + (1 - r)\gamma < 0$, then*

$$P\{\omega : N^{\beta}\, \Delta_n[\mathcal{G}(F,\, X),\, \mathcal{G}(F^N,\, X)] > t\} \longrightarrow_{N \longrightarrow \infty}\} = 0.$$

If, moreover, $g_i(x,\, z)$, $i = 1,\, \ldots,\, l$ are strongly convex with a parameter $\rho > 0$ function on X, then also

$$P\{\omega : N^{\beta} \Delta_n[\bar{\mathcal{X}}(F,\, X)),\, \bar{\mathcal{X}}(F^N,\, X)]^2 > t\} \longrightarrow_{N \longrightarrow \infty} 0.$$

Proof. First assertion of Theorem 12 follows from Propositions 6 and 9. The second assertion follows from the first one and from Proposition 2, 10 and the relation 10. □

Theorem 13. *Let Assumptions B.1, A.2 and A.3 be fulfilled, $P_F \in \mathcal{M}_1(R^s)$, $\bar{M} > 0$, X be a compact set. If one–dimensional components ξ_i, $i = 1,\, \ldots,\, s$ of the random vector ξ have distribution functions F_i with tails parameter $\nu_i \in (1,\, 2)$ fulfilling the relations*

$$\sup_{t>0} t^{\nu_i} P_F\{\omega : |\xi_i| > t\} < \infty, \quad i = 1,\, \ldots,\, s,$$

then

$$\lim_{\bar{M} \longrightarrow \infty} \sup_N P\{\omega : \frac{N}{N^{1/\nu}} \Delta_n[\mathcal{G}(F,\, X)),\, \mathcal{G}(F^N,\, X))], > \bar{M}\} = 0 \quad \text{with} \quad \nu = \min(\nu_1,\, \ldots,\, \nu_s\}.$$

If, moreover, $g_i(x,\, z)$, $i = 1,\, \ldots,\, l$ are strongly convex with a parameter $\rho > 0$ function on X, then also

$$\lim_{\bar{M} \longrightarrow \infty} \sup_N \{\omega : \frac{N}{N^{1/\nu}} \Delta_n[\bar{\mathcal{X}}(F,\, X)),\, \bar{\mathcal{X}}(F^N,\, X)]^2 > \bar{M}\} = 0 \quad \text{with} \quad \nu = \min(\nu_1,\, \ldots,\, \nu_s\}.$$

Proof. The assertion of Theorem 13 follows from the assertion of Propositions 2, 7 and 10. □

Remark 1. Let us assume that Assumptions of Theorem 13 are fulfilled and $\beta(\nu) := 1 - 1/\nu$. Then $\beta(\nu)$ is an increasing function of ν and holds up to

$$\lim_{\nu \longrightarrow 1^+} \beta(\nu) = 0, \quad \lim_{\nu \longrightarrow 2^-} \beta(\nu) = \frac{1}{2},$$

The assertions of Theorems 11, 12 and 13 are introduced (under the approach of properly efficient points and their functions mapping), however, since the set of properly efficient points is dense in the set of efficient points our results are not much restricted (for more details see e. g. [3]).

5 Conclusion

The paper deals with multiobjective stochastic programming problems, especially with a relationship between characteristics of these problems corresponding to complete knowledge of the probability measure and them determined on the data base. We have restricted ourselves to investigate the characteristics $\mathcal{G}(F, X)$, $\bar{\mathcal{X}}(F, X))$ and $X_F = X$, generally. Evidently the presented results can be generalized to the characteristic $\bar{\mathcal{G}}(F, X_F)$ and (employing Relation (6)) some type of constraints set depending on the probability measure (see the corresponding results achieved for one objective case [9]). However more detailed investigation in this direction is beyond the scope of this paper.

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Performance of simple heuristic algorithms for the clustering of countries with respect to food supply

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Abstract. Thanks to increasing availability of the computation power, some computationally intensive heuristic optimization techniques have become popular. These techniques include differential evolution algorithm or threshold accepting method. Results of application of simple general purpose heuristic methods are reported and the impact of values of control parameters on the quality of the solution is discussed. The problem addressed is cluster analysis of countries with respect to the food supply (various items per capita per day). The average silhouette width is used as the objective function for evaluation of the solution and the results are compared with the solution obtained with the commonly used partitioning algorithms. The results suggest that both the threshold accepting method and the differential evolution method are suitable for solving the optimal partitioning problem. These methods easily allow to supply the objective function preferred by the user and reach the solution rather quickly. The values of the control parameters did not exhibit any significant impact on the quality of the solution. The quality of the resulting classification was a little bit higher when using the differential evolution method than when using the threshold accepting method. And both heuristic methods slightly outperformed the standard implementations of the clustering methods (k-means, partitioning around medoids) available in the statistical software packages. The final classification has two groups, where the smaller cluster seems to include the countries usually treated as developing.

Keywords: clustering, silhouette width, threshold accepting, differential evolution.

JEL classification: C61
AMS classification: 90C26

1 Introduction

Many economic optimization problems are difficult to solve analytically (as there may be multiple local optima or there are other undesirable properties) and one has to resort to heuristic techniques, which may reach some suitable solution, even though not necessarily always the optimal one, within reasonable time. Heuristic optimization methods may include trajectory-based methods (there is just one candidate solution, which is modified in each iteration) or population-based methods (such methods work with a population of solutions in every iteration, which is often called a generation), see e.g. [1]. Evolutionary algorithms include the particle swarm optimization or the differential evolution method. Such computationally intensive methods have recently become popular thanks to the rise of the available computation power. Values of the control parameters of these general purpose algorithms often have to be tuned for the particular problem at hand (as there does not seem to be the rule for setting the optimal values of these parameters which would be suitable for all the objective functions). Regarding the differential evolution method, which seems to be a particularly efficient tool for solving difficult optimization tasks, the control parameters of the algorithm include the size of the population, the probability of crossover and the step size. The problem of finding optimal partitioning of multivariate data with respect to particular objective function with the use of heuristic algorithms is addressed in this paper. The average silhouette width is used as the objective function and the results of the application of a couple of simple general purpose heuristic methods are reported.

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The organization of the paper is as follows: after a brief introduction to the field of unsupervised classification, the principles of selected heuristic optimization methods are recalled and the description of the data set used in the analysis is provided in the Material and Methods section. Then the results of the analysis are reported.

2 Material and Methods

2.1 Data Description

The dataset used in the analysis is on food supply in particular countries[1]. The supply (in g per capita per day) of 10 food categories (i. e. there are ten variables in the data set) of 175 countries is analyzed with the aim to learn if there is some natural grouping of countries regarding the food supply. Clearly the variables have different scale and it does not make much sense using such variables, so the data preparation step included transforming the original values to ranks.

2.2 Clustering

The cluster analysis, or the unsupervised classification task, aims at the discovering of the best (whatever it means) groupings of the cases included in the analysis in such a way, that the cases in the same cluster are rather similar, whereas the cases from the different clusters are not.

There exist many algorithms for solving this problem, one commonly used clustering of such methods is given in [5]:

- hierarchical methods, which produce a set of clusterings
  - agglomerative hierarchical methods
    - the methods differ mostly in the way of calculating group-group differences from point-point dissimilarities
    - are computationally easy
  - divisive hierarchical methods
    - are computationally difficult
    - available methods are usually monothetic (i. e. the split is just on one variable at each stage)
- optimal partitioning methods
  - the number of clusters is fixed
  - needs some initial partitioning
  - there are various optimization criteria
  - include k-means, partitioning around medoids, CLARA, fuzzy cluster analysis or model-based methods

Some clustering methods have to operate on the original variables values, other methods can build the solution using just the distance matrix. The results of the clustering may be impacted by the choice of the distance measure, we use the usual 2 norm in this analysis.

2.3 The Objective Function

There exist several reasonable objective functions for evaluation of the quality of the clustering model. One of the possibilities is to choose the average silhouette width. We briefly recall how this objective function is defined, for more details see the original sources, such as [3].

[1] The data on food supply can be obtained at http://www.faostat.org.

The average silhouette width may be used for evaluation of the quality of the resulting clustering solution (and it may also help when deciding about the most suitable number of clusters to work with) when the partitioning clustering methods are used.

For every particular case i one defines the average dissimilarity of the case i and all the remaining cases which belong to the same cluster as i (let's denote this cluster as A and the number of cases assigned in this cluster as $|A|$):

$$a(i) = \frac{1}{|A| - 1} \sum_{j \in A, j \neq i} d(i, j).$$

Then for every cluster C other than the cluster A the average dissimilarity between the case i and the cluster C is calculated as

$$d(i, C) = \frac{1}{|C|} \sum_{j \in C} d(i, j).$$

Let's denote

$$b(i) = \min_{C \neq A} d(i, C).$$

The *silhouette width* of the case i is

$$s(i) = \frac{b(i) - a(i)}{\max\{a(i), b(i)\}}.$$

Obviously it is a number from interval $\langle -1, 1 \rangle$ and it can be interpreted as follows:

$s(i) \approx 1 \quad \rightarrow \quad$ the case is well assigned in the cluster considered,
$s(i) \approx 0 \quad \rightarrow \quad$ the case is somewhere between clusters,
$s(i) < 0 \quad \rightarrow \quad$ the case is probably assigned to wrong cluster.

The average silhouette width is then the silhouette width averaged over all the cases in the data set. The values near 0 suggest that the classification structure is weak and may be just artificial, values near 1 suggest the presence of a strong classification structure.

For the presentation of the quality assessment of the resulting clustering, we use the silhouette plot, which is available in the cluster package (see [3]) for the R computing environment (see [2]).

2.4 Threshold Accepting and Differential Evolution Methods

The threshold accepting algorithm is a trajectory-based optimization method, for its principles see [1]. The function for obtaining the neighbour (i. e. next candidate) solution has to be suitable for the particular task being solved and it has to be supplied by the user. The control parameters of the algorithm include the threshold value (usually some sequence with values decreasing with the iteration number).

The differential evolution algorithm, introduced in [4], is a population-based method and it includes several control parameters - the step size F, the probability of the crossover CR, the number of generations and the population size. For details see the original paper [4].

Both algorithms require some suitable choice of the number of iterations (or generations), which may be conveniently checked by observing the plot of the objective function value in response to the iteration number.

We use the implementation of the threshold accepting and the differential evolution methods available in NMOF package (see [1]) in the R computing environment (see [2]). Since the objective function is minimized in this implementation (as is the usual approach), the average silhouette width was always multiplied by -1.

3 Results

The optimal clustering with up to 4 clusters has been searched for. The solutions with just single cluster have been penalized through setting the objective function value to 0 for this case. The use of the heuristic methods allows automatic choice of the most suitable number of groups (2, 3 or 4).

Both the application of the threshold accepting method and the differential evolution method resulted in two clusters solutions. The values of the control parameters for the algorithms did not have any strong

impact on the results. The average silhouette width for the final solution obtained by the threshold accepting with 10000 iterations is 0.3033802 (see Figure 1 which shows how the objective function value changes with the iteration number) and for the differential evolution with F=0.5, CR=0.9, population size 40 and 200 generations it is 0.3033938 (see Figure 2). So the differential evolution did a slightly better job. The partitioning obtained with the differential evolution (let's call it the final clustering) gave one smaller, more clearly defined cluster and one larger cluster which does not seem to have a clear profile - see Figure 3 for the silhouette plot. When inspecting the assignments of particular countries to clusters, the smaller cluster seems to include the countries usually perceived as the developing ones. The prediction model (classification tree) was built to get some insight about the profiles of the clusters and a single predictive variable, which was the supply of Milk, was included in the resulting tree model. The countries with low rank in Milk supply got to the smaller cluster.

The final partitioning is superior to the solution obtained with the standard implementation of the k-means method (in the stats package in [2]), which gives the average silhouette width for two clusters solution 0.3001925, 0.20428 for 3 clusters and 0.1659714 for 4 clusters. Similarly for the partitioning around medoids method (in the cluster package in [2]), which gives 0.3010591, 0.1986875 and 0.2111096 respectively.

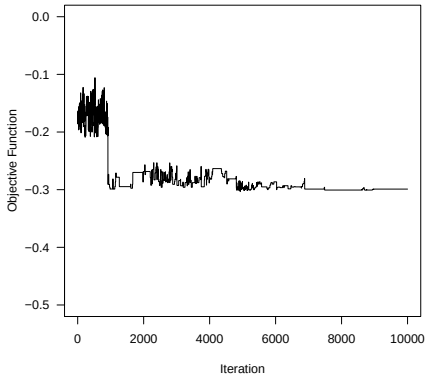

Figure 1 Threshold accepting

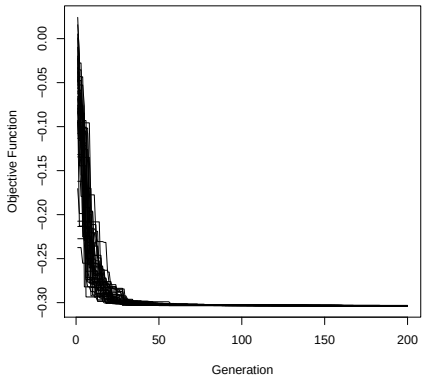

Figure 2 Differential evolution

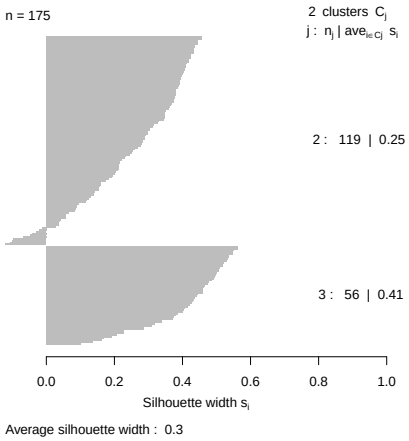

Figure 3 Silhouette plot for the resulting solution

4 Conclusions

It was shown that both the threshold accepting method and the differential evolution method are suitable for solving the optimal partitioning problem. These methods easily allow to supply the objective function preferred by the user and reach the solution rather quickly. The procedures can also automatically determine the best number of clusters, which is also an advantage. The values of the control parameters of the optimization algorithms did not exhibit any great impact on the quality of the solution. The quality of the resulting classification was slightly better when using the differential evolution method than when using the threshold accepting method. And in both cases the result obtained is slightly better than if the standard implementations of the clustering methods available in the software packages are used.

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Consistency versus transitivity in pair-wise comparison matrices

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Abstract: Pair-wise comparison matrix is an efficient instrument often used in decision making analysis. Usually, a decision-maker is capable of creating pair-wise comparisons of decision elements forming a pair-wise comparison matrix. Either in multiplicative or in additive approaches there may appear some "incompatibilities" in the pair-wise comparison matrix – inconsistency and/or intransitivity. These properties do not have to occur simultaneously. Inconsistency and/or intransitivity in multiplicative and additive pair-wise comparisons are detected and measured by using consistency index/ratio, transitivity index/ratio, respectively. In this contribution we propose the way how to transform additive (fuzzy) pair-wise comparison matrix into multiplicative one for which the consistency index/ratio and transitivity index/ratio are defined. An example illustrating the application of the above mentioned approaches is supplemented.

Keywords: AHP, decision analysis, fuzzy sets and systems.

JEL Classification: C44

AMS Classification: 90B50, 90C29, 91B06

1 Introduction

Pair-wise comparison matrix is an instrument often used in multcriteria decision making in effort to rank criteria/alternatives or select optimal one. The essence of pair-wise comparisons is to determine which of two objects is more preferred by the decision-maker. There are two approaches to pair-wise comparisons – multiplicative and additive. For both these approaches some properties of pair-wise comparison matrices are investigated in this contribution: reciprocity, consistency and transitivity. If these properties are not satisfied, i.e. the acceptable threshold is exceeded, the decision making process gives inappropriate conclusions. In that case it is necessary to reassess the preferences in pair-wise comparisons and change the pair-wise comparison matrix.

The aim of this contribution is to describe conditions whose satisfaction ensures consistency or transitivity of pair-wise comparison matrices and define how to measure the extent of potential inconsistency/intransitivity. It is necessary to mention that these two incompatibilities may appear independently of each other. This fact is demonstrated by illustrative examples.

2 Pair-wise comparison matrix

Pair-wise comparison matrix is a tool of multicriteria decision making (MCDM). It is an irreducible nonnegative $n \times n$ matrix which entries are results of pair-wise comparisons. To complete such a matrix it is enough to provide $\binom{n}{2} = \frac{n(n-1)}{2}$ pair-wise comparisons, the remaining entries are ensured by *multiplicative reciprocity* property:

$$a_{ij} \cdot a_{ji} = 1, \text{ for all } i, j = 1,2, \dots, n, \tag{1}$$

where a_{ij} are entries of pair-wise comparison matrix $A = \{a_{ij}\}$. A pair-wise comparison matrix $B = \{b_{ij}\}$ is called *additive reciprocal* (*a-reciprocal*), if

$$b_{ij} + b_{ji} = 1, \text{ for all } i, j = 1,2, \dots, n, \tag{2}$$

where b_{ij} are entries of pair-wise comparison matrix $B = \{b_{ij}\}$.

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There are two properties of pair-wise comparison matrices we are concerned in this contribution. These are consistency and transitivity properties.

3 Multiplicative pair-wise comparisons

In multiplicative approach of pair-wise comparisons there are elements $x_1, x_2, \dots, x_n$ to be compared in pairs to express preference intensity on the scale presented by the closed interval $[1/\sigma\,;\sigma]$, where $\sigma > 1$. In Saaty's approach [3], the scale is $[1/9\,;9]$. The i-th element is compared to the j-th element with respect to a particular property. If element i is more preferred than element j, the result a_{ij} satisfies the inequality $1 < a_{ij} \leq \sigma$. If element i is less preferred than element j, the entry a_{ij} satisfies the condition $1/\sigma \leq a_{ij} < 1$. If these two elements are equally preferred, then $a_{ij} = 1$. The equality $a_{ii} = 1$ for all i is obvious.

Usually, a decision-maker is able to make consistent pair-wise comparisons. *Multiplicative consistency* (*m-consistency*) expresses how much the pair-wise comparisons correspond to each other. A positive $n \times n$ matrix $A = \{a_{ij}\}$ is *m-consistent,* if

$$a_{ij} = a_{ik} \cdot a_{kj} \text{ for all } i, j, k. \tag{3}$$

Multiplicative transitivity (*m*-transitivity) is associated with *m*-consistency as follows. A positive $n \times n$ matrix $A = \{a_{ij}\}$ is *m-transitive,* if

$$\frac{a_{ij}}{a_{ji}} = \frac{a_{ik}}{a_{ki}} \cdot \frac{a_{kj}}{a_{jk}} \text{ for all } i, j, k, \tag{4}$$

where ratio $\frac{a_{ij}}{a_{ji}}$ denotes the relative efficiency of comparison of x_i over x_j.

Notice that if *A* is m-consistent then *A* is m-transitive. Moreover, if $A = \{a_{ij}\}$ is m-reciprocal, then *A* is m-transitive if and only if *A* is m-consistent. Here, we shall investigate m-transitive matrices with elements from the unit interval [0; 1] being not m-reciprocal, however, a-reciprocal.

4 Additive pair-wise comparisons

The multiplicative approach is not always the most acceptable way how to do pair-wise comparisons. Sometimes there is a more natural procedure how to do it – the additive approach. The principle is simple: 100% share of the property in question is divided into two parts, the first part, b_{ij}, is assigned to the i-th object and the second one, b_{ij}, to the j-th object. Moreover, b_{ij} belongs to the closed interval $[0; 1]$ for all i, j.

Let $B = \{b_{ij}\}$ be an $n \times n$ a-reciprocal matrix with $0 \leq b_{ij} \leq 1$ for all i, j, k.

B is called *additive transitive* (*a-transitive*) [4], if

$$b_{ij} - 0.5 = (b_{ik} - 0.5) + (b_{kj} - 0.5) \text{ for all } i, j, k. \tag{5}$$

If *B* is *a*-reciprocal and *m*-transitive, then *B* is called *additive consistent* (*a-consistent*).

There exist some relationships between m-reciprocal and a-reciprocal pair-wise comparison matrices. In order to show them, we define function $\phi(t)$ which transforms a-reciprocal matrix into an m-reciprocal one, see [2]:

$$\phi(t) = \frac{t}{1-t}, \text{ for } t \in (0,1). \tag{6}$$

Proposition 1. *Let $B = \{b_{ij}\}$ be an a-reciprocal $n \times n$ matrix with $0 < b_{ij} < 1$ for all i and j. If $B = \{b_{ij}\}$ is m-transitive then $A = \{a_{ij}\} = \{\phi(b_{ij})\}$ is m-consistent.*

According to Chiclana et al. [1] we shall use transformation function $\varphi_\sigma(t)$ for given $\sigma > 1$ as follows, see [2]:

$$\varphi_\sigma(t) = \sigma^{2t-1}, \text{ for } t \in [0,1]. \tag{7}$$

Proposition 2. *Let $B = \{b_{ij}\}$ be an $n \times n$ matrix with $0 \leq b_{ij} \leq 1$ for all i and j. If $B = \{b_{ij}\}$ is a-transitive then $A = \{a_{ij}\} = \{\varphi_\sigma(b_{ij})\}$ is m-consistent.*

Perfectly consistent or transitive pair-wise comparison matrices occur rarely, therefore, we need some tools for measuring of inconsistency/intransitivity. In practical situations we also need a threshold value such that if the extent of inconsistency/intransitivity of a matrix is smaller than this threshold then the matrix is considered consistent/transitive.

5 Measuring (in)consistency and (in)transitivity

If for some positive $n \times n$ matrix $A = \{a_{ij}\}$ and for some $i, j, k = 1,2, \dots, n$, multiplicative consistency condition (3) does not hold, then A is said to be *multiplicative-inconsistent* (*m-inconsistent*). If for some $n \times n$ matrix $B = \{b_{ij}\}$ with $0 \leq b_{ij} \leq 1$ for all i and j, and for some $i, j, k = 1,2, \dots, n$, (5) does not hold, then B is said to be *additive-intransitive* (*a-intransitive*) [2].

Existence of a positive *spectral radius* $\rho(A)$ and the corresponding positive vector w satisfying the characteristic equation $Aw = \rho(A)w$ follows from Perron-Frobenius theory for any irreducible nonnegative matrix A [2]. Vector $w = (w_1, w_2, \dots, w_n)$, $w_i > 0$ for all $i = 1,2, \dots, n$, is called the *principal eigenvector of* A. When normalized, i.e. $\sum_{i=1}^{n} w_i = 1$, element w_i of vector w represents the relative importance of alternative $x_i, i = 1,2, \dots n$. Then it is possible to rank all alternatives x_i and/or choose the optimal one(s) in this way. Then the spectral radius $\rho(A)$ is called the *principal eigenvalue of matrix A* and the principal eigenvector of A is called the *priority vector of A*.

The extent of m-consistency/m-inconsistency of m-reciprocal $n \times n$ matrix A is measured by the *m-consistency index* $I_{mc}(A)$ [3]:

$$I_{mc}(A) = \frac{\rho(A) - n}{n - 1}. \tag{8}$$

An *m*-reciprocal pair-wise comparison matrix A is *m*-consistent if and only if $I_{mc}(A) = 0$, i.e. $\rho(A) = n$.

By *m-consistency ratio* $CR_{mc}(A)$ we measure the inconsistency independently of the dimension n of A:

$$CR_{mc}(A) = \frac{I_{mc}(A)}{R_{mc}(n)}, \tag{9}$$

where $R_{mc}(n)$ is the *random consistency index* – the mean value of $I_{mc}(A)$ where the elements a_{ij} of A – *m*-reciprocal matrices of dimension n – are uniformly distributed random variables in the interval $[1/9; 9]$. An *m*-reciprocal pair-wise comparison matrix A is considered to be sufficiently *m*-consistent if the *m*-consistency ratio does not exceed the threshold value 0.1. In other words, A is considered to be sufficiently *m*-consistent if the *m*-consistency index $I_{mc}(A)$ is less than one tenth of the random consistency index $R_{mc}(n)$. A more detailed explanation and interpretation of this heuristic threshold can be found e.g. in [3].

By Propositions 1 and 2 we can measure the extent of inconsistency and intransitivity of a-reciprocal matrices.

Let $B = \{b_{ij}\}$ be an a-reciprocal $n \times n$ matrix with $0 < b_{ij} < 1$ for all i and j. The *a-consistency index* $I_{ac}(B)$ of B is defined as:

$$I_{ac}(B) = I_{mc}(A), \text{ where } A = \{\phi(b_{ij})\}. \tag{10}$$

Such matrix B is considered to be *a-consistent* if and only if $I_{ac}(B) = 0$.

Moreover, we define the *a-consistency ratio* $CR_{ac}(B)$ for measuring a-consistency independently on dimension n of the matrix as:

$$CR_{ac}(B) = \frac{I_{ac}(B)}{R_{mc}(n)}. \tag{11}$$

A-reciprocal pair-wise comparison matrix B is regarded to be sufficiently a-consistent if the a-consistency ratio does not exceed the value 0.1.

The corresponding priority vector $w^{ac} = (w_1^{ac}, w_2^{ac}, \dots, w_n^{ac})$ determining the ranking of alternatives $x_i, i = 1,2, \dots n$, is given by the characteristic equation $\phi(B)w^{ac} = \rho(\phi(B))w^{ac}$.

Let $B = \{b_{ij}\}$ be an a-reciprocal $n \times n$ matrix with $0 < b_{ij} < 1$ for all i and j. The *a-transitivity index* $I_{at}^{\sigma}(B)$ of B is defined as:

$$I_{at}^{\sigma}(B) = I_{mc}(A_{\sigma}), \text{ where } A_{\sigma} = \{\varphi_{\sigma}(b_{ij})\}, \tag{12}$$

where $\sigma > 1$ is given value.

Matrix B is *a-transitive* if and only if $I_{at}^{\sigma}(B) = 0$.

Similarly to m-consistency and a-consistency it is possible to measure a-transitivity independently of dimension n of the matrix. The *a-transitivity ratio* $CR_{at}^{\sigma}(B)$ is defined as:

$$CR_{at}^{\sigma}(B) = \frac{I_{at}^{\sigma}(B)}{R_{mc}(n)}. \tag{13}$$

If the a-transitivity ratio does not exceed 0.1, then the matrix B is considered to be sufficiently a-transitive, otherwise, a-intransitive is unacceptable.

The ranking of alternatives $x_i, i = 1,2, \dots n$ is given by the priority vector $w^{at} = (w_1^{at}, w_2^{at}, \dots, w_n^{at})$ which is calculated from the characteristic equation $\varphi_{\sigma}(B)w^{at} = \rho(\varphi_{\sigma}(B))w^{at}$.

6 Example

Let $X = \{x_1, x_2, x_3, x_4\}$ be a set of four alternatives. The alternatives should be ordered from the best to the worst. The preferences on X are given by matrix $B = \{b_{ij}\}$,

$$B = \begin{bmatrix} 0.5 & 0.1 & 0.25 & 0.15 \\ 0.9 & 0.5 & 0.9 & 0.85 \\ 0.75 & 0.1 & 0.5 & 0.15 \\ 0.85 & 0.15 & 0.85 & 0.5 \end{bmatrix}.$$

According to (2) and (3) matrix B is a-reciprocal but m-inconsistent as $b_{13} \cdot b_{32} \cdot b_{21} = 0.02 \neq 0.07 = b_{12} \cdot b_{23} \cdot b_{31}$. Applying (5) we find out that B is a-intransitive: $b_{13} - 0.5 = -0.25 \neq 0 = (b_{12} - 0.5) + (b_{23} - 0.5)$. We have to determine if the extent of inconsistency and intransitivity is unacceptable. Considering $\sigma = 9$ and $R_{mc}(4) = 0{,}89$ [3] we calculate matrices C and D according to (6) and (7):

$$C = \{\phi(b_{ij})\} = \begin{bmatrix} 1 & 0.11 & 0.33 & 0.18 \\ 9 & 1 & 9 & 5.67 \\ 3 & 0.11 & 1 & 0.18 \\ 5.67 & 0.18 & 5.67 & 1 \end{bmatrix},$$

$$D = \{\varphi_9(b_{ij})\} = \begin{bmatrix} 1 & 0.17 & 0.33 & 0.21 \\ 5.80 & 1 & 5.80 & 4.66 \\ 3 & 0.17 & 1 & 0.21 \\ 4.66 & 0.21 & 4.66 & 1 \end{bmatrix}.$$

We compute $\rho(C) = 4.36$, $\rho(D) = 4.38$. According to (8), (11) and (13) we get $CR_{ac}(B) = 0.14 > 0.1$ with the priority vector $w^{ac} = (0.04; 0.67; 0.07; 0.22)$ ranking alternatives $x_2 > x_4 > x_3 > x_1$. Similarly, $CR_{at}^{9}(B) = 0.14 > 0.1$ with the priority vector $w^{at} = (0.06; 0.60; 0.10; 0.25)$, which gives the same ranking of alternatives $x_2 > x_4 > x_3 > x_1$.

Obviously, matrix B is neither sufficiently *a-transitive* nor sufficiently *a-consistent* as the a-transitivity ratio and a-consistency ratio exceeds the value of 0.1.

The decision-maker reconsiders some of his/her preferences in pair-wise comparison matrix B and obtains new pair-wise comparison matrix $E = \{e_{ij}\}$:

$$E = \begin{bmatrix} 0.5 & 0.05 & 0.25 & 0.15 \\ 0.95 & 0.5 & 0.95 & 0.85 \\ 0.75 & 0.05 & 0.5 & 0.15 \\ 0.85 & 0.15 & 0.85 & 0.5 \end{bmatrix}.$$

By (2) and (3) we find out that E is a-reciprocal and m-inconsistent: $e_{14} \cdot e_{42} \cdot e_{21} = 0.02 \neq 0.04 = e_{12} \cdot e_{24} \cdot e_{41}$. From (5) it is obvious that E is also a-intransitive: $e_{14} - 0.5 = -0.35 \neq -0.10 = (e_{12} - 0.5) + (e_{24} - 0.5)$. We have to examine if the extent of inconsistency and intransitivity is unsatisfactory. Considering $\sigma = 9$ and $R_{mc}(4) = 0{,}89$ [3] we compute matrices F and G:

$$F = \{\phi(e_{ij})\} = \begin{bmatrix} 1 & 0.05 & 0.33 & 0.18 \\ 19 & 1 & 19 & 5.67 \\ 3 & 0.05 & 1 & 0.18 \\ 5.67 & 0.18 & 5.67 & 1 \end{bmatrix},$$

$$G = \{\varphi_9(e_{ij})\} = \begin{bmatrix} 1 & 0.14 & 0.33 & 0.21 \\ 7.22 & 1 & 7.22 & 4.66 \\ 3 & 0.14 & 1 & 0.21 \\ 4.66 & 0.21 & 4.66 & 1 \end{bmatrix}.$$

We calculate $\rho(F) = 4.19$ and $\rho(G) = 4.31$. Applying (8), (11) and (13) we obtain $CR_{ac}(E) = 0.07 < 0.1$ with the priority vector $w^{ac} = (0.03;\ 0.75;\ 0.05;\ 0.17)$ giving the ranking of alternatives $x_2 > x_4 > x_3 > x_1$. By a similar way, $CR_{at}^{9}(E) = 0.12 > 0.1$ with the priority vector $w^{at} = (0.05;\ 0.63;\ 0.09;\ 0.23)$ and final ranking of alternatives $x_2 > x_4 > x_3 > x_1$.

Matrix *E* is sufficiently *a-consistent* but unsatisfactory *a-transitive*.

The decision-maker considers more changes in his/her preferences and creates new pair-wise comparison matrix $H = \{h_{ij}\}$:

$$H = \begin{bmatrix} 0.5 & 0.05 & 0.25 & 0.15 \\ 0.95 & 0.5 & 0.6 & 0.7 \\ 0.75 & 0.4 & 0.5 & 0.2 \\ 0.85 & 0.3 & 0.8 & 0.5 \end{bmatrix}.$$

Applying (2) it becomes obvious that *H* is a-reciprocal. By (3) we find out that *H* is m-inconsistent: $h_{13} \cdot h_{32} \cdot h_{21} = 0.10 \neq 0.02 = h_{12} \cdot h_{23} \cdot h_{31}$. According to (5) matrix *H* is a-intransitive: $h_{13} - 0.5 = -0.25 \neq -0.05 = (h_{14} - 0.5) + (h_{43} - 0.5)$. Using (6) and (7) and considering $\sigma = 9$ we are able to find out whether the m-inconsistency and a-intransitivity is sufficient:

$$I = \{\phi(h_{ij})\} = \begin{bmatrix} 1 & 0.05 & 0.33 & 0.18 \\ 19 & 1 & 1.50 & 2.33 \\ 3 & 0.67 & 1 & 0.25 \\ 5.67 & 0.43 & 4 & 1 \end{bmatrix},$$

$$J = \{\varphi_9(h_{ij})\} = \begin{bmatrix} 1 & 0.14 & 0.33 & 0.21 \\ 7.22 & 1 & 1.55 & 2.41 \\ 3 & 0.64 & 1 & 0.27 \\ 4.66 & 0.42 & 3.74 & 1 \end{bmatrix}.$$

We compute $\rho(I) = 4.40$ and $\rho(J) = 4.28$. Using (8), (11) and (13) are got $CR_{ac}(H) = 0.15 > 0.1$ with the priority vector $w^{ac} = (0.04;\ 0.50;\ 0.15;\ 0.31)$ which ranks alternatives $x_2 > x_4 > x_3 > x_1$. Hence, $CR_{at}^{9}(H) = 0.10 \leq 0.1$, with the priority vector $w^{at} = (0.06;\ 0.45;\ 0.17;\ 0.33)$ and final ranking $x_2 > x_4 > x_3 > x_1$.

Matrix *H* is sufficiently *a-transitive* but unsatisfactory *a-consistent*.

The decision-maker once again reconsiders his/her pair-wise comparisons and forms the following pair-wise comparison matrix $K = \{k_{ij}\}$:

$$K = \begin{bmatrix} 0.5 & 0.1 & 0.25 & 0.15 \\ 0.9 & 0.5 & 0.65 & 0.7 \\ 0.75 & 0.35 & 0.5 & 0.2 \\ 0.85 & 0.3 & 0.8 & 0.5 \end{bmatrix}.$$

According to (2) matrix *K* is a-reciprocal. Using (3) it becomes evident matrix *K* is a-inconsistent: $k_{13} \cdot k_{34} \cdot k_{41} = 0.04 \neq 0.09 = k_{14} \cdot k_{43} \cdot k_{31}$. By (5) matrix *K* is a-intransitive: $k_{14} - 0.5 = -0.35 \neq -0.55 = (k_{13} - 0.5) + (k_{34} - 0.5)$. According to (6) and (7) we calculate matrices *L* and *M* considering $\sigma = 9$ and $R_{mc}(4) = 0{,}89$ [3] as

$$L = \{\phi(k_{ij})\} = \begin{bmatrix} 1 & 0.11 & 0.33 & 0.18 \\ 9 & 1 & 1.86 & 2.33 \\ 3 & 0.54 & 1 & 0.25 \\ 5.67 & 0.43 & 4 & 1 \end{bmatrix},$$

$$M = \{\varphi_9(k_{ij})\} = \begin{bmatrix} 1 & 0.17 & 0.33 & 0.21 \\ 5.80 & 1 & 1.93 & 2.41 \\ 3 & 0.52 & 1 & 0.27 \\ 4.66 & 0.42 & 3.74 & 1 \end{bmatrix}.$$

We calculate $\rho(L) = 4.23$ and $\rho(M) = 4.23$. By (8), (11) and (13) we obtain $CR_{ac}(K) = 0.09 < 0.1$ with the priority vector $w^{ac} = (0.05;\ 0.47;\ 0.15;\ 0.33)$, hence the final ranking of alternatives is $x_2 > x_4 > x_3 > x_1$. By the same way, $CR^9_{at}(K) = 0.09 < 0.1$ with the priority vector $w^{at} = (0.06;\ 0.45;\ 0.16;\ 0.33)$ giving again the ranking of alternatives $x_2 > x_4 > x_3 > x_1$.

Consequently, matrix K is sufficiently *a-transitive* and also sufficiently *a-consistent*.

Notice that accidentally the ranking of alternatives given by all four pair-wise comparison matrices *B*, *E*, *H* and *K* is the same.

7 Conclusion

In this contribution we investigated two types of incompatibility of pair-wise comparison matrices called inconsistency and intransitivity. These incompatibilities were measured by consistency index/ratio and transitivity index/ratio and the acceptable extent of inconsistency/intransitivity was defined. By the numerical example it was demonstrated that inconsistency and intransitivity may appear independently of each other, hence measuring inconsistency as well as intransitivity may be important in the real decision making.

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Some of the Results of the Mathematical Group in the Institute of Automation and Computer Science

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Abstract. Research interest within the Mathematics group in the Institute of Automation and Computer Science of the Brno UT, Faculty of Mechanical Engineering is focused on the creation and application of mathematical methods of operational research for project management and production control. Some results achieved by members of the mathematics group and its PhD students are published in Handbook of Optimization, Springer-Verlag, Berlin Heidelberg, 2013.

A heuristic method for the multicriteria selection of the group of project is being developed in the institute. This method enables the objectification of weights of the partial non-linear criteria functions, involves synergistic effects of second- and third- orders in benefit and cost criterion function, and in resource requirements respecting resource sharing and hierarchical contingency relationships among candidate projects, and dialogue of the solution.

In the project scheduling, the two-criteria time-cost minimization problem, solved by fuzzy linear programming, is a basis of the program system created here, as well as a technique for deriving a time-periodical production schedule for a group of products in a group of workplaces having limited capacity, performing cost optimization by lot size alteration. The manager could also observe, monitor or visualise effectively the differences between the scheduled and the actual courses of projects through the SSD Graph.

Keywords: project selection, scheduling, visualising, decision support, multicriterial optimisation, periodic production, lot sizes.

JEL classification: C44
AMS classification: 90B50

1 Decision Support System for Projects Selection

Computer support and mathematical modelling of project management consists (from the mathematical point of view) of the following basic stages (Klapka [6]): prognostic-planning stage, scheduling stage and the stage of analysis and chronicling.

One of the most important problems of prognostic-planning phase is the selection of projects by respecting appropriate criteria and by the limitations of the resources. In industrial enterprises and in the national economies there is often a need for a program system which would make it possible to carry out effectively multicriterial selection of hundreds of projects simultaneously, with tens of criterion functions including nonlinear ones, and tens of resources limitations with respect to the synergistic effects and the hierarchical interdependences between the projects. Since the tasks of this type usually belong to the "ill-defined" ones, it is advisable to use here the interactive dialogue approaches after the projects portfolio has been preliminarily optimized. Most often, the projects selected are the research and development (R&D) projects and the information systems projects. Some conditions for the formulation of the problems of this type are defined e.g. in Eilat, Golany and Shtub [2]. In Santhanam and Kyparisis [14] the problem is solved by means of the nonlinear goal programming. An excellent tool for the solution of the selection problems of great extent of input data is the idea of Stewart [15] who has created the scalarizing function

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by the modified reference point method, and for its optimization he has used a heuristic method of the effective gradient.

In [9] and [10] we have extended this approach to the possibility of utilizing criterion functions and constraints respecting synergistic effects of the second- and third-orders and hierarchical interdependencies between the projects. This work is based on the optimization by the modified method of effective gradient with the possibility of a dialogue improvement of the solution. The basic idea of the method is presented here.

The application of the presented decision support system proceeds under the cooperation with the Czech industrial enterprises and with other firms. The system is also applied for the teaching process in the Technical University of Brno. Problems of the similar type are still considered by a number of authors, of whom e.g. Litvinchev, López, Alvarez and Fernandez [12] introduced a fuzzyfication of some of the parameters. The lastly mentioned authors successfully managed effective optimization of portfolio of 25 000 projects, while solving the projects selection as two-criteria selection problem, where one criterion is the quality of the portfolio, and the second criterion is a number of projects in the portfolio. In our method, however, we solve a problem with tens of criterion functions with bivalent variables including polynomial ones and quotients of linear functions, tens of resources limitations and hundreds of projects.

1.1 Problem Formulation

The following problem is solved: to choose some of the s projects into the portfolio. Let i be a number of the project ($i = 1, 2, \ldots, s$). The projects belong to different categories (e.g. from the project type and the client type point of view). The categories need not be mutually exclusive. Let $S(k)$ be a set of the projects falling into category k ($k = 1, 2, \ldots, q$). The goal of the solution is to find for all i the values of bivalent variables δ_i for which $\delta_i = 1$ if the project i is selected for the portfolio, and $\delta_i = 0$ in the opposite case. The selection should be made so that all requirements of the solution can be fulfilled, which includes the following:

1. To satisfy the resource constraints

$$\sum_{i=1}^{s} a_{ij}\delta_i - \sum_{i=1}^{s-1}\sum_{k=i+1}^{s} a_{ijk}\delta_i\delta_k + \sum_{i=1}^{s-2}\sum_{k=i+1}^{s-1}\sum_{l=k+1}^{s} a_{ijkl}\delta_i\delta_k\delta_l \leq b_j \tag{1}$$

   where $b_j > 0$ is the total availability of resource j ($j = 1, 2, \ldots, m$), $a_{ij} \geq 0$ is the amount of resource j required by project i, $a_{ijk} \geq 0$ is the amount of resource j shared by projects i and k, a_{ijkl} is the amount of resource j shared by projects i, k, and l. In general it holds that, $a_{ij} \geq a_{ijk}$, $a_{kj} \geq a_{ijk}$, $a_{ijk} \geq a_{ijkl}$, $a_{ijl} \geq a_{ijkl}$, $a_{kjl} \geq a_{ijkl}$ for all i, k, l. In case of the absence of synergistic effect in resource sharing it holds $a_{ijk} = 0$, $a_{ijkl} = 0$.

2. To satisfy contingency constraints

$$\sum_{m \in A_i} \delta_m \geq |A_i|\delta_i \qquad \text{for all } i \in H\ , \tag{2}$$

   where H ($H \subset \{1, 2, \ldots, s\}$ is a set of all projects which are contingent upon the implementation of other projects, A_i ($A_i \subset \{1, 2, \ldots, s\}$ is a set of all projects upon the implementation of which the project i is contingent. $|A_i|$ is the number of elements in the set A_i.

3. To satisfy the directive constraints

$$\delta_i = \begin{cases} 1 & \text{for } i \in B \quad (B \subset \{1, 2, \ldots, s\})\ , \\ 0 & \text{for } i \in D \quad (D \subset \{1, 2, \ldots, s\})\ , \end{cases} \tag{3}$$

   where sets B, D are mandated due to internal and external restrictions.

4. To satisfy the restrictions for mutually exclusive projects: for some i, j ($i, j \in \{1, 2, \ldots, s\}$), can be required: if $\delta_i = 1$, then $\delta_j = 0$, and if $\delta_j = 1$, then $\delta_i = 0$ (e.g. in case when two projects represent alternative levels of activity on the same essential problem).

5. To obtain the highest possible values of criterion functions of gain (benefit)

$$z_j = \sum_{i=1}^{s} c_{ij}\delta_i + \sum_{i=1}^{s-1}\sum_{k=i+1}^{s} c_{ijk}\delta_i\delta_k + \sum_{i=1}^{s-2}\sum_{k=i+1}^{s-1}\sum_{l=k+1}^{s} c_{ijkl}\delta_i\delta_k\delta_l \quad (j = 1, 2, \ldots, p)\,, \tag{4}$$

where $c_{ij} \geq 0$ is the jth benefit derived from implementing project i alone, $c_{ijk} \geq 0$ is the additional jth benefit derived from implementing projects i and k together, and $c_{ijkl} \geq 0$ is the additional jth benefit derived from implementing projects i, k and l together.

Comments:

(a) In a similar way it is possible to formulate a cost-related objective, a negative value of which is maximized by approaching to zero.

(b) The special case when $c_{ijk} = 0$, $c_{ijkl} = 0$ corresponds to the absence of synergistic effect of benefit. Under the simplifying assumptions concerning the additivity of risk, the risk of the set of projects selected may be expressed by the first term in (4) where c_{ij} is now the risk of implementing project i. In this case, we minimize the total risk of portfolio of the selected projects by maximizing

$$\left\{-\sum_{i=1}^{s} c_{ij}\delta_i\right\}.$$

6. To obtain the smallest possible deviation of Stewart function

$$\Phi_k = \frac{\sum_{i\in S(k)} \mu_i\delta_i}{\sum_{i=1}^{s} \mu_i\delta_i} \qquad (k = 1, 2, \ldots, q)\,, \tag{5}$$

from its ideal value π_k (see the above definition of project categories) where μ_i is the cost related to the project i (or e.g. total manpower used by project i). It appears that $\Phi_k \in [0; 1]$, $\pi_k \in [0; 1]$. Let us assume that for at least one i it holds that $\delta_i = 1$.

1.2 Arrangement of the Formulation

Let us choose "asymmetric distance" of Φ_k from π_k, denoted by $\|\Phi_k - \pi_k\|$, in such a way that its value belongs to interval $[0; 1]$ and that for $\Phi_k = 1 \wedge \pi_k \neq 1 \vee \Phi_k = 0 \wedge \pi_k \neq 0$ it holds that $\|\Phi_k - \pi_k\| = 1$ and that for $\Phi_k = \pi_k$ it holds that $\|\Phi_k - \pi_k\| = 0$. For this purpose we define

$$\|\Phi_k - \pi_k\| = \begin{cases} 0 & (\Phi_k = \pi_k)\,, \\ \frac{\Phi_k - \pi_k}{l(\Phi_k - \pi_k) - \pi_k} & (\text{otherwise})\,, \end{cases}$$

where unit-step function $l(x) = 0$ for $x \leq 0$, $l(x) = 1$ for $x > 0$. This means that maximal possible deviations of Φ_k on either side of π_k are equally important.

Then, it is possible to reformulate the problem in the following way:

$$\text{"max"}\, z_j \qquad (j = 1, 2, \ldots, p + q)\,.$$

We solve this problem under constraints (1)–(3), where the criterion functions z_j $(j = 1, 2, \ldots, p)$ are given in (4) with signs of individual terms possibly changed with respect to Comments 5a and 5b. The criterion functions z_{p+k} $(k = 1, 2, \ldots, q)$ are now defined by

$$z_{p+k} = -\|\Phi_k - \pi_k\| = \begin{cases} 0 & (\Phi_k = \pi_k)\,, \\ \frac{\pi_k - \Phi_k}{l(\Phi_k - \pi_k) - \pi_k} & (\text{otherwise})\,, \end{cases} \qquad k = 1, 2, \ldots, q\,. \tag{6}$$

1.3 Optimization and dialogue

In Santhanam and Kyparisis [14] a project selection problem is solved with criterion functions of type (4), resource constraints (1) and contingency constraints (2) for 14 projects ($s = 14$) with accounting for interdependences (type of benefit, cost, resources sharing, and contingency) of up to the third-order

by means of the method based on the goal programming. Decision support system presented here by us enlarges the capabilities of Santhanam's system by the possibilities of solving the problems of more greater extent of input data, utilizing also the balance ratio functions of (5) type, then by the dialogue that makes it possible to solve also the ill-defined problems and by the utilization of the preliminary realistically assessed desired levels of individual criterion functions. With respect to this dialogue, it is not necessary to use accurate methods for optimization, but it is possible to use a heuristic method which makes it possible to enlarge the extent of input data of the solved problem.

For each criterion function z_j we determine the upper bound I_j (in the way of solving an appropriate monocriterial maximization problem with the criterion function z_j), and analogically the lower bound N_j of its optimum value through monocriterial minimization problem. It is very easy to find the bounds for each criterion function. At the same time, the user determines a realistically assessed desired level (reference level) R_j of each z_j. We require

$$N_j \leq R_j \leq I_j \ . \tag{7}$$

In case when the user is not able to order R_j, we can set the initial reference level as follows:

$$R_j = \frac{I_j + N_j}{2} \ .$$

The problem is now transformed to minimizing the scalarizing function

$$\sigma = \sum_{j=1}^{p+q} \left(\frac{I_j - z_j}{I_j - R_j} \right)^h$$

for some $h > 0$ under conditions (1)–(4), (6). In the system presented here, a method of effective gradient (Stewart, [15]), generalized by us for the case of the synergistic effects and hierarchical interdependences of projects, is used for the solution of this minimization problem. The selection $h = 4$ as a compromise between the sensitivity of the method and time and rounding off numerical difficulties proved to be equally right for us and the above-mentioned author. By means of this solution optimal values δ_i for all $i = 1, 2, \ldots, s$ are determined, which define a portfolio of projects selected.

The main idea of our generalized effective gradient method is as follows:

At the beginning of the process of solution, we select $\delta_i = 1$ for all $i = 1, 2, \ldots, s$ (except for a case when some projects must not take place simultaneously, in which case an arbitrary selection can be made between these). Then we calculate

$$u_j = \sum_{i=1}^{s} a_{ij}\delta_i - \sum_{i=1}^{s-1} \sum_{k=i+1}^{s} a_{ijk}\delta_i\delta_k + \sum_{i=1}^{s-2} \sum_{k=i+1}^{s-1} \sum_{l=k+1}^{s} a_{ijkl}\delta_i\delta_k\delta_l$$

for all $j = 1, 2, \ldots, m$, which is the amount of the jth resource requested by those projects that were incorporated in the portfolio. In case that a resource constraint is not satisfied, then for each i for which $\delta_i = 1$, we define $\Delta_i\sigma$ as an increase of function σ caused by dropping project i from the portfolio. For all such i we calculate

$$P_i = \Delta_i\sigma \frac{\sqrt{\sum_j (u_j - b_j)^2}}{\sum_j A_{ij}(u_j - b_j)} \ ,$$

where

$$A_{ij} = a_{ij} - \sum_{k=1,2,\ldots,i-1,i+1,i+2,\ldots,s} a_{ijk}\delta_k + \sum_{k=1,2,\ldots,i-1,i+1,i+2,\ldots,s-1} \ \sum_{l=k+1,k+2,\ldots,i-1,i+1\ldots,s} a_{ijkl}\delta_k\delta_l \tag{8}$$

is the amount of the jth resource consumed due to the implementation of the ith project. P_i is an effective gradient of the scalarizing function σ. The sums in the relation for P_i are realized for all $u_j \geq b_j$ only. The project giving the expression P_i a minimum value, will be dropped from the portfolio.

This step repeats until all resource limitations are fulfilled. Analogically, in a backward course, we introduce into portfolio a maximal number of projects that do not violate any resource constraint.

The dialogue between the user of the system and the person solving the problem, accomplished after the introductory optimization, influences also the value of reference levels R_j in an adaptive way, and

thus also the weights of components of the scalarizing function for the purposes of potential future reoptimization of portfolio.

This dialogue is described in [10]. The dependence of the calculation time for optimization on the number of projects is given in Figure 1.

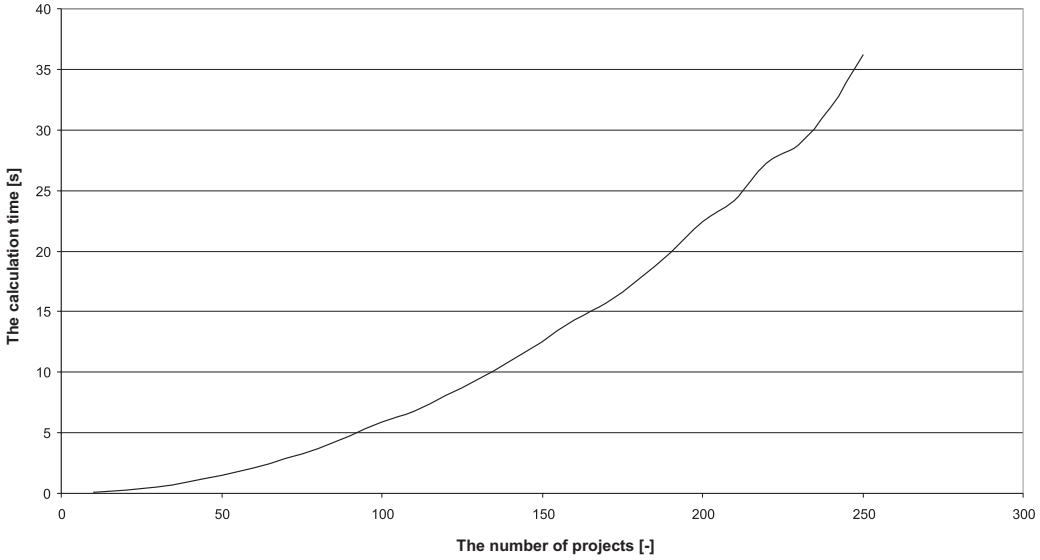

Figure 1 Graph of the dependence of the calculation time on the number of projects ($m = 50$, $p = 20$, $q = 20$).

2 Scheduling of the selected projects

Each project is given by its network graph. For every activity of the project resulting from node i and entering node j the following quantities are given: $y_{ij}^{(0)}$ = normal duration, m_{ij} = maximum crashing, k_{ij} = crashing cost. We search values of the following variables: x_i = time at event i, h_{ij} = crash time for activity (i, j). For each i, j the following conditions holds:

$$\begin{aligned} h_{ij} &\le m_{ij} \,, \\ h_{ij} &\ge 0 \,, \\ x_j - x_i + h_{ij} &\ge y_{ij}^{(0)} \,, \\ x_1 &= T_b && \text{(project start time)} \\ x_i &\ge 0 && (i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n) \end{aligned}$$

Then, partial criterion functions are $Z_1 = \sum k_{ij} h_{ij}$ (crashing cost), $Z_2 = x_n - x_1$ (duration of project). Let us denote $\min Z_1 = U_1$, $\min Z_2 = U_2$. For $Z_1 = U_1$ let us denote $Z_2 = L_2$. For $Z_2 = U_2$ let us denote $Z_1 = L_1$. Then, the two-criterial time-cost minimisation problem is to choose all x_i, h_{ij} which provide $\min Z$ where $Z = \lambda$ under the conditions

$$\lambda \ge \frac{Z_1 - U_1}{L_1 - U_1} \,, \qquad \lambda \ge \frac{Z_2 - U_2}{L_2 - U_2}$$

and the above-mentioned linear condition of the problem. The solution is made by *fuzzy linear programming*. As it is shown in [13] and in the diploma theses carried out at our institute, this method leads to substantially lower total affectable costs of a project than the heuristic "method of cost gradient" known until now.

3 Contemplating the Differences between Scheduled and Actual Course of Projects

For this purpose for each activity a_i of the project and time t we define a scheduled state $\overline{m}_{it}$ such as

$$\overline{m}_{it} = \begin{cases} -1 & (\text{if } a_i \text{ is not begun}) \\ 0 & (\text{if } a_i \text{ is in process}) \\ 1 & (\text{if } a_i \text{ is finished}) \end{cases}$$

and analogically an actual state m_{it}. Deviation d_{it} is defined as the difference between the actual and scheduled states on time t by $d_{it} = m_{it} - \overline{m}_{it}$.

$$d_{it} = \begin{cases} -2 & (\text{if } a_i \text{ is delayed by the second order}) \\ -1 & (\text{if } a_i \text{ is delayed by the first order}) \\ 0 & (\text{if } a_i \text{ is not delayed or in advance}) \\ 1 & (\text{if } a_i \text{ is in advance by the first order}) \\ 2 & (\text{if } a_i \text{ is in advance by the second order}) \end{cases}$$

A set of possible values of deviations d_{it} is $\{-2, -1, 0, 1, 2\}$. It is apparent that such concepts of state and deviations of project activities are very simple and efficient instruments for monitoring and visualising the course of project and the deviation of the actual course from the scheduled one. This non-traditional approach utilising according to [11] the synthesis of Petri nets with Gantt diagram may be combined both with the schedule described in § 2 and also with the commonly available system Microsoft Project. The program systems are prepared for use in Czech enterprises.

4 Time-Periodical Production Scheduling

Periodicity is often an important requirement supposed on the serial production e.g. in mechanical engineering. Scheduling plays an important role in manufacturing and production systems. In the last few decades, many algorithms to solve the production schedule of the group of products in the group of workplaces problem have been developed. Those techniques that are based on exact methods (e.g. [1]) are in most cases suitable for simultaneous production scheduling of a relatively small number of products manufactured on a small number of machines (production stages). For large scale problems, algorithms based on the modern heuristic methods were developed (e.g. [16]); however these approaches provide only an approximate solution and often without error estimation. Exact methods could be advantageously used for such an auxiliary computation as a computation of the lower and upper bound of a demanded criterion function [5, 7] and thus contribute to the efficiency of the heuristic method. The most used criteria are time (total processing time), controllable cost, and idle time. We try to find the optimal solution with respect to all of these measures. The resulting solution consists of the time schedule for all the manufacturing operations and their lot sizes.

In [8] we consider the problem of serial and time-periodical production in the engineering industry where the input parameters are deterministic. We propose a method of improving an existing schedule of the group of products (e.g. in a machinery and equipment manufacturing) with the goal of a cost optimisation through the lot sizes alteration using a multiplication by an appropriately selected constant. The practical benefit of this approach is also evident from the fact that its use can improve the schedule (after a condition of the periodicity of the production is introduced) created by virtually any suitable modern [3, 4, 16] or problem oriented heuristic method, while the mutual shares of lot sizes are preserved.

5 Conclusion

We tried to perfect current situation in using project management in production and manufacturing by making a system of decision support provided with a dialogue regime and by using non-traditional element such as fuzzy linear programming and the approach based upon the synthesis of Petri nets with Gantt diagrams. The newly constructed system makes the integrated application of these elements and functions possible.

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Multinomial Ordinal Response Models

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Abstract. For managing of credit risk in banks, it is fundamental to decide to which customer a loan or service will be offered and to which one it will not. The banks use many methods modeling and forecasting risk – application scorecards, probability of default etc. Inside these methods mainly binary models and logistic regression as technique are applied. The article intends to shortly illustrate and describe extension of these models on situations where response (dependent) variable has more than two categories. The binary models (or dichotomous, i.e. models in which dependent variable has just two categories) are not the only ones, it is possible to use models in which dependent variable has more than two categories – multinomial models.
There are two main groups of such models: with ordered (ordinal) categories and non-ordered (nominal) categories of the dependent variable. The article describes representatives from both groups, but focus on ordinal models, which are likely to be used in the financial institutions than nominal ones. For both groups practical examples are calculated and their results are discussed.

Keywords: Credit scoring, generalized linear regression model, logistic regression, logit, probit, binary, multinomial/multicategorical ordinal and nominal models.

JEL Classification: C35
AMS Classification: 62J12**,** 91G40

1 Introduction

One of the key tool of financial institutions for risk and also cost management is credit scoring and following risk segmentation of customers by the score.

Basic aim of the segmentation is to find good and bad clients, i.e. those who will probably pay, and those who will not. Most of used methods (e.g. binary logistic regression as mostly used) compute probability or number (score) that customer will be good or bad (will pay or not), hence transition from good to bad client is fluent (continuous). However, the way, how to obtain the probability or score, works only with purely good and bad customers, many cases without such easy classification (called indeterminate, intermediate) are excluded and unused. Or it works with all records/customers (each investigated case is decided as good or as bad) , but in such cases some expressive differences in characteristics could be diminished or even disappear. For that reason it is suitable to use multinomial (ordinal) models, in which the intermediate cases create the new (middle) category.

Needs of management or business units could be wider. Main target of firms is to increase sell and profit. But more reasonable is to attract good customers in both terms: of business and of risk. For the former aim as much they will react on offer. For such situations, nominal multinomial models can be used.

This article shortly introduces models and methods which use more than 2 attributes of response variable with focus on ordinal ones and shortly compare them with more used binary methods.

2 Theoretical part

2.1 Binary models

As representative of binary response models, binary logistic regression as most often used is introduced. It supposes that probability of event y_i (labeled as 1) under condition that random explanatory variables have concrete values x_i is derived by logistic distribution, or mathematically – see [1], [2], [5], [6]:

$$P(y_i = 1|x_i) = \pi_i = \frac{1}{1+\exp(-x_i'\beta)} = \frac{\exp(x_i'\beta)}{1+\exp(x_i'\beta)},$$

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where *i* means order of record and β is a vector of unknown parameters. Probability of complementary event is:

$$P(y_i = 0|x_i) = 1 - \pi_i = \frac{1}{1+\exp(x_i'\beta)}.$$

Based on previous relation a *logit* is defined as a logarithm of division of these two probabilities:

$$logit\ \pi_i \equiv \ln\left(\frac{\pi_i}{1-\pi_i}\right) = \ln\left(\frac{\exp(x_i'\beta)}{1+\exp(x_i'\beta)} \Big/ \frac{1}{1+\exp(x_i'\beta)}\right) = \ln(\exp(x_i'\beta)) = x_i'\beta .$$

Fraction inside the natural logarithm expresses relative chance that the alternative 1 will happen against that the alternative 0 will happen. The ratio is usually called *odds* or *odds ratio*.

One important comment: it is evident that extreme values for *x* are not problematic. Probability values according to definition must be from interval (0, 1).

Very similar, but not often used, is a *probit* model. For that model normal distribution and probability are used instead of the logistic ones. It could be defined by the following way:

$$P(y_i = 1|x_i) = \pi_i = \Phi(x_i'\beta),$$

where Φ is a standard normal distribution function N(0,1). It is rarely used due to very demanding computation.

While using simple probability linear model, we obtain:
$P(y_i = 1|x_i) = \pi_i = x_i\beta$ and $P(y_i = 0|x_i) = 1 - \pi_i = 1 - x_i\beta$. Here is evident, that it is hardly to ensure basic feature of probability – to be in an interval <0, 1>. Such models are very sensitive on extreme values.

2.2 Multinomial models

Multinomial models are extension of binary models. Models with multinomial dependent variable (i.e. with more than two categories) are divided to 2 basic groups:

- with nominal response variable – there is no ordering among alternatives
- with ordinal response variable – there is ordering among alternatives

Models with nominal response variable

Models with nominal (= non-ordered) response variable could be used when it is necessary to choose from a list of option, for which there is no or very small preference (deciding about mobile phone/operator, brand of car, university, place of holiday etc.).

Typical representative of nominal models is a multinomial logistic model. It can be written for *q* alternatives (categories) this way according to [1], [6]:

$P(y_i = r|x_i) = \pi_{i,r} = \frac{\exp(x_i'\beta_r)}{1+\sum_{j=1}^{q-1}\exp(x_i'\beta_j)}$ for each category $r = 1,\ldots, q\text{-}1$ and

$P(y_i = q|x_i) = \pi_{i,q} = \frac{1}{1+\sum_{j=1}^{q-1}\exp(x_i'\beta_j)}$ for a reference category *q*.

The logit is defined as division of probabilities of the category *r* and of reference category *q:*

$$logit\ \pi_{i,r} \equiv \ln\left(\frac{P(y_i = r|x_i)}{P(y_i = q|x_i)}\right) = x_i'\beta_r .$$

Models with ordinal response variable

There are several possible classifications of multinomial ordinal models and plenty of models. Fahrmeier and Tutz in [1, pp. 81-104] introduce cumulative and sequential models. While Hosmer and Lemeshow, focusing only on logistic regression in [3, pp. 288-291], define three groups of models: the adjacent category, the continuation ratio and proportional odds models. Representatives from all groups will be shortly introduced, but only cumulative models will be followed by practical examples.

Typical example of the cumulative models with ordinal response is the *cumulative logistic model*. It could be described in following way, see e.g. [1], [6]:

$P(y_i \le r|x_i) = \frac{\exp(\theta_r + x_i'\beta)}{1+\exp(\theta_r + x_i'\beta)}$, where $-\infty = \theta_0 < \theta_1 < \ldots < \theta_q = \infty$ and $r = 1, 2, \ldots, q$.

The model is defined through sum of probabilities, when dependent variable will be equal to 1, 2, …, *r* (left-hand side of the equality). Hence, why the term "cumulative" is used.

With respect to $P(y_i > r|x_i) = 1 - P(y_i \le r|x_i) = \frac{1}{1+\exp(\theta_r + x_i'\beta)}$, we can also write and derive:

$$\frac{P(y_i \le r|x_i)}{P(y_i > r|x_i)} = \exp(\theta_r + x_i'\beta) \text{ or } \ln\left(\frac{P(y_i \le r|x_i)}{P(y_i > r|x_i)}\right) = \theta_r + x_i'\beta .$$

For derivation of the model see e.g. Hušek [4, p.195]. By comparing probabilities of the event (one from several possibilities) and its complementary, inserting their ratio into natural logarithm, we again obtain the *logit*.

Other cumulative models are the *probit* model with normal distribution

$$P(y_i \le r|x_i) = \Phi(\theta_r + x_i'\beta), \text{ where } r = 1, 2, \ldots, q.$$

or *grouped Cox model* (or proportional hazards model) with extreme minimal value distribution, see also [1]:

$$P(y_i \le r|x_i) = 1 - \exp\{-\exp(\theta_r + x_i'\beta)\}, \text{ where } r = 1, 2, \ldots, q.$$

Or with *complementary log-log* link: $\log[-\log P(y_i > r|x_i)] = (\theta_r + x_i'\beta)$.

At the end of this part, classification and models derivation from Hosmer and Lemeshow [3] will be presented. The *adjacent-category* logistic model compares each response to the next larger response. Assuming that odds (and logit as well) does not depend on the response and the logit is linear in the coefficients the form of model could be following: $\ln\left(\frac{P(y_i = r|x_i)}{P(y_i = r-1|x_i)}\right) = \theta_r + x_i'\beta$

The second model compares each response to all lower responses. He is called *continuation-ratio* logistic model and could be described such a way: $\ln\left(\frac{P(y_i = r|x_i)}{P(y_i < r-1|x_i)}\right) = \theta_r + x_i'\beta.$

The third and last one compares probability of equal or lower response to probability of larger response and they call it as *proportional odds* model: $\ln\left(\frac{P(y_i \le r|x_i)}{P(y_i > r|x_i)}\right) = \theta_r + x_i'\beta$

This is exactly the same as cumulative logistic model. So, cumulative logistic model and proportional odds model are synonyms (e.g. used in [1]). For more details and important features of each model see [1], [2] and [3].

3 Practical part

Real data from a financial company were used for practical examples of using different methods including several transformations (grouping variables or transferring continuous variable to class one). The data were taken from closed or running contracts, for which one from 3 statuses (*good* - G, *bad* - B, or *intermediate* /or *indeterminate*/ - I) were be defined based on their payment histories. These free statutes have been transformed to statuses 0, 2, 1 (to keep appropriate ordering) for applying of multinomial models and related methods. For all models statistical software SAS were used, concretely procedures Logistic and GENMOD (including consultation on web pages in [7]).

Firstly logistic regression in statistical software SAS (PROC Logistic) was used, firstly for the binary model and status good and bad, then twice for logistic regression for all 3 statuses – concretely general (nominal) logistic regression and cumulative logistic regression. The first two model results (but with another data) were published in [5], all three in [6]. Then procedure GENMOD were used to calculate other cumulative model, namely cumulative probit and again cumulative logit for comparison (both introduced in theoretical section).

In this article, only part of standard SAS outputs is inserted due to limited space - only list of variables selected to the model with relevant statistics, part of parameters estimates, final statistics about model (e.g. several measures of association between estimated model and original data), where it has sense.

As for measures of association – for each model it is different. In the binary one it measures correct assignment only for good and bad, in model with ordinal response measures all three categories (good – indeterminate – bad), in model with nominal (categorical) response these statistics even are missing – in this model it is not stated which status is better and which worse, model only compares one status to reference one.

3.1 Binary model

For easy comparison and not so long outputs parameters alpha, which decide about insertion of concrete variable into a model (maximal probability that the variable is insignificant although I inserted it as significant into the model), was set for all models to 0.01 (1%).

For the binary model (5766 records: 5581 good, 185 bad), logistic regression gives only 5 variables and Somers' D (or GINI) coefficient 0.541 – see the SAS output below.

Parameter	*DF*	*Estimate*	*Standard error*	*Wald Chi-Square*	*Pr > ChiSq*
Intercept	1	0.8192	0.5892	1.9329	0.1644
bapaymth B	1	1.0946	0.2457	19.8493	<.0001
bahomtel Y	1	0.6532	0.2160	9.1491	0.0025
timjob 3	1	0.1116	0.2148	0.2702	0.6032
timjob 5	1	0.3279	0.2226	2.1700	0.1407
timjob 10	1	0.4646	0.2679	3.0072	0.0829
timjob 20	1	1.3118	0.3828	11.7460	0.0006
baaccomc CT	1	0.4551	0.5853	0.6047	0.4368
aaccomc HO	1	1.1697	0.5535	4.4653	0.0346
baaccomc LP	1	0.9226	0.6303	2.1423	0.1433
baaccomc O	1	0.3956	0.5706	0.4808	0.4880
baaccomc TF	1	0.4297	0.5678	0.5728	0.4492
timrel 1	1	3.7522	1.0079	13.8603	0.0002
timrel 5	1	0.8199	0.3427	5.7239	0.0167
timrel 10	1	0.7907	0.2026	15.2278	<.0001
timrel 15	1	0.6162	0.1904	10.4730	0.0012

Association of Predicted Probabilities and Observed Responses: Somers' D 0.541

3.2 Model with three categories – nominal response variable

Multinomial nominal model with 3 statuses (good, i.e. 0 – 5581 records, medium as 1 – 621 records, bad as 2 – 185 records) chose 9 variables – see the results below. But each has double degrees of freedom, because each response (except the reference one), i.e. responses 0 and 1, is compared and estimated against the reference category (response 2).

Effect	*DF*	*Chi-Square*	*Pr > ChiSq*
bac003	2	10.8364	0.0044
bapaymth	2	70.4523	<.0001
baemprol	18	36.6721	0.0058
bahomtel	2	20.5513	<.0001
timjob	8	39.7773	<.0001
baaccomc	10	94.4228	<.0001
appamt	8	40.3286	<.0001
grterm	8	72.8954	<.0001
timrel	8	230.6500	<.0001

Parameter	*target3*	*DF*	*Estimate*	*Standard Error*	*Wald Chi-Square*	*Pr > ChiSq*
Intercept	0	1	-0.5983	1.3571	0.1943	0.6593
Intercept	1	1	2.6207	1.4359	3.3314	0.0680
bac003 Y	0	1	0.3206	0.2156	2.2112	0.1370
bac003 Y	1	1	-0.0469	0.2412	0.0379	0.8457
bapaymth B	0	1	0.9638	0.2585	13.9046	0.0002
bapaymth B	1	1	-0.0510	0.2853	0.0320	0.8581

baemprol	C	0	1	1.6190	1.1940	1.8385	0.1751
baemprol	C	1	1	0.5434	1.2325	0.1944	0.6593
baemprol	MG	0	1	1.0241	1.1643	0.7737	0.3791 ….

The increased number of estimated parameters has impact on SC (Schwartz criterion) measuring model in terms of overfitting (too many variables and estimated parameters). The presented model has SC 5543.066, while the lowest value of SC (5361.163) is obtainable after insertion only 3 (!) variables.

3.3 Model with three categories – ordinal response variable (cumulative model)

Cumulative logistic regression using PROC Logistic

Step	*Effect*	*DF*	*Chi-Square*	*Pr > ChiSq*
1	baaccomc	5	228.1603	<.0001
2	bapaymth	1	95.4431	<.0001
3	timrel	4	80.1009	<.0001
4	grterm	4	58.2071	<.0001
5	appamt	4	46.4525	<.0001
6	timjob	4	34.1833	<.0001
7	bahomtel	1	16.0543	<.0001
8	bac003	1	7.7590	0.0053
9	baemprol	9	25.1272	0.0028

Parameter		*DF*	*Estimate*	*Standard Error*	*Chi-Square*	*Pr > ChiSq*
Intercept	0	1	-2.2369	0.7357	9.2456	0.0024
Intercept	1	1	-0.5547	0.7364	0.5673	0.4513
bac003	Y	1	0.3377	0.1078	9.8179	0.0017
bapaymth	B	1	0.9759	0.1187	67.5670	<.0001
timrel	1	1	-0.4587	0.1218	14.1794	0.0002
timrel	5	1	-0.6670	0.1380	23.3674	<.0001
timrel	10	1	0.4043	0.1216	11.0480	0.0009
timrel	15	1	0.4054	0.1184	11.7125	0.0006

…

Association of Predicted Probabilities and Observed Responses: Somers' D 0.249

The model selected again 9 variables (the same as in the nominal model), more than 5 as in the binary model. But you can see deep fall for Somers' D coefficient (0.541 in the binary model) – it is one of disadvantage of ordered models that the differences between categories are diminishing.

PROC GENMOD and cumulative logit

Procedure GENMOD estimates parameters for all included variables regardless of their significance. Hence, if one need direct comparison with previous models, it is necessary to manually exclude insignificant variables (e.g. exclude those where each parameter has "Pr > ChiSq" > 0.01).

This choice gives very similar result as PROC Logistic. Also it selected 9 variables, the differences are caused by different definition of reference categories of independent variables (PROC Genmod is not so "friendly" for human choice) – see part of the SAS output below.

Parameter		*DF*	*Estimate*	*Error*	*Confidence Interval*		*Chi-Square*	*Pr > ChiSq*
Intercept1		1	1.6893	0.7333	-0.1996	3.5782	5.31	0.0212
Intercept2		1	3.3716	0.7354	1.4772	5.2659	21.02	<.0001
bac003	Y	1	0.3377	0.1078	0.0601	0.6154	9.82	0.0017
bapaymth	A	1	-0.9759	0.1181	-1.2802	-0.6716	68.23	<.0001
timrel	1	1	-0.4588	0.1203	-0.7685	-0.1490	14.55	0.0001
timrel	5	1	-0.6670	0.1362	-1.0179	-0.3161	23.97	<.0001
timrel	10	1	0.4043	0.1228	0.0879	0.7206	10.83	0.0010
timrel	15	1	0.4053	0.1199	0.0965	0.7141	11.43	0.0007

…

Log likelihood -2569.9999

Looking at variable *timrel* we can see almost the same estimates. The estimates for two intercepts are different, but important is their distance measuring distance between "thresholds" for responses 0, 1 and 2 – in former model it is 1.6822, at latter one 1.6823. Almost the same difference.

PROC GENMOD and cumulative probit.

Parameter		*DF*	*Estimate*	*Standard Error*	*Confidence. interval*		*Chi-Square*	*Pr > ChiSq*
Intercept1		1	0.9364	0.4175	-0.1390	2.0118	5.03	0.0249
Intercept2		1	1.7627	0.4181	0.6857	2.8397	17.77	<.0001
bac003	Y	1	0.1711	0.0553	0.0286	0.3136	9.56	0.0020
bapaymth	A	1	-0.5050	0.0587	-0.6562	-0.3537	73.98	<.0001
timrel	1	1	-0.1311	0.0650	-0.2985	0.0364	4.06	0.0439
timrel	5	1	-0.2890	0.0753	-0.4828	-0.0951	14.75	0.0001
timrel	10	1	0.2450	0.0628	0.0834	0.4067	15.25	<.0001
timrel	15	1	0.2387	0.0615	0.0803	0.3972	15.07	0.0001
…								
Log likelihood			-2583.5944					

The result are not same as for cumulative logit, but they are similar as for relations (most of estimates are close to half of logit estimates – except parameter for value "1" of *timrel*, the parameter is insignificant on level 0.01)

4 Comments and conclusion

The presented models lead to very similar results (most significant variables are included in each model), there are some differences in the number of included variables and few included variables differ from model to model.

What is immediately visible, it is difference in structure of result: the nominal multinomial model has two sets of the same parameters (2 categories are compared to reference ones, also variables in the nominal model have double number of degrees of freedom). While the ordinal model has only one set of parameters, but it has 2 constants (thresholds), by which the categories are separated. One constant more is just difference against the basic binary model (except 2 different variables chosen in each model). Inside group of ordinal models, the results are similar, the estimates have different values (levels), but keep relation and basic behavior (significance, importance/weights of concrete estimates). Because the models differ only in used distribution (or link) functions.

The multinomial models give interesting results, they could be viewed as useful extension of the binary models. On possible usage of multinomial models is in identification of most predictive variables (mainly on a border of significance). One advantage is that they use all observations. They are also necessary for cases where the target is more complex (e.g. not only to find best customers in terms of risk, but also to find most active customers). For that purpose the nominal multinomial models are suitable (despite basic requirement of much higher number of estimated parameters). More difficult interpretation and sometimes technical issues for calculation can be chosen as disadvantages for those models.

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A note on quantiles for multidimensional data

Jana Klicnarová[1]

Abstract. In many economic applications, we need to compute multidimensional quantiles or to estimate the center of multidimensional distribution. Our approach to multidimensional quantiles is based on a modified version of interdirections which are given by halving hyperplanes. The methods working with halving hyperplanes do not require any knowledge about the location of the distribution, what is their main advantage in comparison with classical approaches.

Our aim is to study asymptotics for multidimensional quantiles. Under our approach to multidimensional quantiles, the very important tool for studying of these asymptotics are U-statistics. In fact, we need to study asymptotics for U-statistics (especially, weighted U-statistics and incomplete U-statistics).

Keywords: U-statistics, multidimensional quantile, interdirections

JEL classification: C13
AMS classification: 62G05

1 Introduction

In this paper, we focus on the problem of multidimensional quantiles. The theory of multidimensional quantiles is now very developing field. Recently, there were published a lot of interesting papers, see for example [7], [10].

In this paper, we study multidimensional quantiles using of so-called interdirections. Interdirections were presented by Randles in his paper [12]. In short, for two fixed points, the Randles's interdirections are such hyperplanes which go through $k-1$ points of observations and through an origin and which separate these fixed two points. Randles show that the number of such interdirections associated with two points y_1 and y_2 tends to angular distance between these two fixed points.

Through this paper, we suppose observations coming from symmetric distribution (without lost of generality with the center at an origin). Randles ([12]) worked with hyperplanes, which went through $k-1$ data points and the origin. In his paper, he introduced an empirical angular distance between two vectors and proposed a multivariate sign test based on this concept.

Similar concept close to Randles's one was developed also by Oja and Paindaveine in their paper [6]. Oja and Paindaveine obtained similar results to Randles, but they did not use hyperplanes going through $n-1$ observations and origin. Their results are based on all hyperplanes which go through all possible k distinct observation.

Our aim is to show similar results to the above mentioned. Our concept of interdirections is based on hyperplanes, which go through k points of observation and which divide the data set half-to-half. Surely, it is not easy to see, which hyperplanes divide the data set half-to-half, we need to define what it means "to be above" and "to be below" and in case of odd $n-p$ (where n is the number of observations and p is a dimension of our space), which hyperplane is the right one to divide out set half-to-half. The main advantage of our approach is that it leads to methods which do not require any knowledge about the location of the distribution.

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2 Notation

Let $X_1, X_2, \ldots, X_n$ be a observation of i.i.d. (independent and identically distributed) random variables in $\mathbf{R}^p$. By I_n, we will denote the subset of p indexes chosen from the set $\{1, 2, \ldots, n\}$, more precisely $I_n = \{i_1, i_2, \ldots, i_p\} \subset \{1, 2, \ldots, n\}$. We will write H^{I_n} for $H_{X_{i_1},\ldots,X_{i_p}} \subset \mathbf{R}^p$, where $H_{X_{i_1},\ldots,X_{i_p}}$ is the hyperplane going through p points – $X_{i_1}, \ldots, X_{i_p}$ – of the observation.

Then we can define the halving hyperplane.

Definition 1. We say, that the hyperplane H^{I_n} is a **halving hyperplane**, if

$$\begin{aligned}\sum_{i=1}^{n} \operatorname{sign}(\det((X_i, X^{I_n}, \mathbf{1}))) &= 0 \quad \text{for } n-p \text{ even},\\ &= 1 \quad \text{for } n-p \text{ odd},\end{aligned}$$

where $(X_i, X^{I_n}, \mathbf{1}) = (X_i, X_{i_1}, \ldots, X_{i_p}, \mathbf{1})$ is a matrix, where the first line is the vector $(X_i, 1)$, the second vector $(X_{i_1}, 1)$ and so on.

Let us denote by $\mathbf{d}(I_n)$ the vector whose components are the cofactors of the first line in the matrix $(x, X^{I_n}, \mathbf{1})$, where x is any p-vector (so, it is normal vector to the hyperplane H^{I_n}). We can see, that x belongs into the hyperplane H^{I_n} if $\mathbf{d}(I_n).(x, 1)$ is equal to zero.

Now, we can compute the distance between any point $x \in \mathbf{R}^p$ and hyperplane H^{I_n} as

$$d(H^{I_n}, x) = \frac{\mathbf{d}(I_n).(x, 1)}{||\mathbf{d}(I_n)||}.$$

(By $||\cdot||$, we denote the L_2-norm.)

Let us define the center given by halving hyperplanes, what is the point which is the nearest one to all halving hyperplanes. More precisely we can define.

Definition 2. The point $\hat{\theta}$ is called the center given by halving hyperplanes, iff

$$\hat{\theta} = \arg\min_{\theta \in \mathbf{R}^p} \left(\sum_{I_n \subset \{1,2,\ldots,n\}} d((\theta, H^{I_n}))\mathbf{I}_{\sum_{j=1}^{n} \det((X_j, X^{I_n}, \mathbf{1}))=0 \vee 1} \right). \tag{1}$$

So, we can see, that in fact, $\hat{\theta}$ is a function of observations. More precisely,

$$\hat{\theta} = \sum_{I_n=(i_1,i_2,\ldots,i_p)\subset\{1,2,\ldots,n\}} f(X^{I_n})\mathbf{I}_{\{H^{I_n} \text{ is halving hyperplane}\}}.$$

So, it is easy to see, that $\hat{\theta}$ is in fact incomplete U-statistic (see for example [8]).

Let us suppose that X_is come from any continuous symmetric distribution with any center $\mathbf{C}$. In the following, we can suppose – without lost of generality – the $\mathbf{C}$ to be equal $\mathbf{0}$.

At the end of this section, let us recall the idea of interdirections. For every two points y_1 and y_2 from $\mathbf{R}^p$, let us denote by $C_{1,2}^{(n)}$ a count of interdirections associated with y_1 and y_2 in the sample $X_1, X_2, \ldots, X_n$. The **interdirection associated with** y_1 **and** y_2 in the sample $X_1, X_2, \ldots, X_n$ is every halving hyperplane H^{I_n} going through p points of the collection $X_1, X_2, \ldots, X_n$ which separates the points y_1 and y_2.

3 Results

At the beginning of this section, let us show, that in case of symmetric distribution, halving hyperplanes tends to the origin (in the sense, that the distance between the origin and the halving hyperplane tends to zero).

Proposition 1. *In case of symmetric distribution with the center in origin and none zero density in a neighbourhood of the origin, we have for any halving hyperplanes H^{I_n}:*

$$d(H^{I_n}, \mathbf{0}) \to 0 \text{ as } n \to +\infty.$$

Proof. Let us suppose that for every n_0 and for every $\varepsilon > 0$ there exists $n > n_0$ such that

$$d(H^{I_n}, \mathbf{0}) > \varepsilon.$$

In such a case, put R^{I_n} for the hyperplane which is parallel to H^{I_n} and goes through the origin. And denote by $R^{I_n}_+$ the space, such that for all $x \in R^{I_n}_+$ there is satisfied: $\det(x, Y^{I_n}, \mathbf{1}) > 0$, where Y^{I_n} is any fixed basis of the hyperplane R^{I_n}. Hence, due to the condition on symmetrization of the distribution, we know, according to Strong Law of Large Numbers (SLLN), that for every choice of I_n

$$\frac{\#(X_i \in R^{I_n}_+)}{n} \to \frac{1}{2},$$

where $\#(\cdot)$ denotes the cardinality of set $(\cdot)$.

Hence, for every $\varepsilon_1 > 0$ we can find n_1 such that for all $n \geq n_1$:

$$\left| \frac{\#(X_i \in R^{I_n}_+)}{n} - \frac{1}{2} \right| \leq \varepsilon_1.$$

So, we obtained, that for every $\varepsilon_1 > 0$ the number of observation between the hyperplanes R^I_n and H^I_n is for n large enough less or equal to $\varepsilon_1 + 1/n$. It is in contradiction with SLLN, requirement of non zero density in the neighbourhood of the origin and condition that the distance between hyperplanes R^{I_n} and H^{I_n} is bigger or equal to fixed ε. □

Proposition 2. *Let $X_1, X_2, \ldots, X_n$ be a sequence of i.i.d. observations with symmetric distribution (with the center at the origin) in $\mathbf{R}^p$, where $p \geq 2$.*

For any fixed $\mathbf{y}_1$ and $\mathbf{y}_2$ in $\mathbf{R}^p$, we denote by

$$\alpha(\mathbf{y}_1, \mathbf{y}_2) := \arccos\left(\frac{\mathbf{y}_1' \mathbf{y}_2}{||\mathbf{y}_1|| \cdot ||\mathbf{y}_2||} \right)$$

the angle between $\mathbf{y}_1$ and $\mathbf{y}_2$, and by $C^{(n)}_{\mathbf{y}_1,\mathbf{y}_2}$ the number of the interdirections associated with $\mathbf{y}_1$ and $\mathbf{y}_2$ in the sample $X_1, X_2, \ldots, X_n$.

Then,

$$\frac{(n-p+1)C^{(n)}_{\mathbf{y}_1,\mathbf{y}_2}}{\binom{n}{p}}$$

converges in quadratic mean to $\frac{1}{\pi}\alpha(\mathbf{y}_1, \mathbf{y}_2)$ as $n \to \infty$.

The proof is similar to a proof of Lemma 1 given by Hallin and Paindaveine in [5].

Proof. For $p = 2$. For every fixed point x_1 of the observation $(X_1, X_2, \ldots, X_n)$, we can find just one other point x_2 from the selection such that the hyperplane going through these two points (x_1, x_2) is a halving hyperplane (in the sense of our definition 1). So, if we have two fixed points y_1 and y_2 which are not contained in the selection, then we can estimate the number of interdirections associated with these two points in this sample as the whole part of half of number of points from the sample between the points y_1 and y_2. By the number of points from the sample between the points y_1 and y_2, we mean the number of observations between two lines – one of them goes through the origin and the point y_1 and second one goes through the origin and the y_2. (This estimation is possible due to the proposition 1).

So, (for n tends to infinity) the halving hyperplane H^{I_n} separates the points y_1 and y_2 in case that the corresponding $\mathbf{d}(I_n) \in (\mathbf{d}(y_1, \mathbf{0}), \mathbf{d}(y_2, \mathbf{0}))$, where $\mathbf{d}(y_1, \mathbf{0})$, resp. $v(y_2, \mathbf{0})$ is a vector whose components are cofactors of the first line in the matrix $(x, y_1, \mathbf{0}, \mathbf{1})$, resp. $(x, y_2, \mathbf{0}, \mathbf{1})$ (normal vectors to hyperplane given by y_1 and the origin, resp. by y_2 and the origin). Because we know, that the distance between halving

hyperplane and the origin tends to zero, it follows from Law of Large Numbers and due to symmetrization of the distribution, that the number of halving hyperplanes for which $\mathbf{d}(I_n) \in (\mathbf{d}(y_1, \mathbf{0}), \mathbf{d}(y_2, \mathbf{0}))$ (in the sense, that $\mathbf{d}(I_n)$ is a convex combination of$\mathbf{d}(y_1, \mathbf{0})$ and $\mathbf{d}(y_2, \mathbf{0})$, more precisely there exists $k \in [0, 1]$ such that $\mathbf{d}(I_n) = k\mathbf{d}(y_1, \mathbf{0}) + (1 - k)\mathbf{d}(y_2, \mathbf{0}))$ tends to $\frac{1}{\pi}\alpha(\mathbf{y}_1, \mathbf{y}_2)$.

For $p \geq 3$. Let us denote:

$$\mathbf{Q} := \{\mathbf{q} = (i_1, i_2, \ldots, i_p) : 1 \leq i_1 < i_2 < \cdots < i_p \leq n\},$$

let H_q be the hyperplane $H_{X_{i_1}, X_{i_2}, \ldots, X_{i_p}}$ such that $q = (i_1, i_2, \ldots, i_p)$ and let $\det(X, H_q, \mathbf{1})$ be the determinant from the matrix with rows $(X, 1), (X_{i_1}, 1), \ldots, (X_{i_p}, 1)$, where $i_j \in q$ for all $j = 1, 2, \ldots p$.

With this notation we get (in case of $n - p$ even):

$$\left(\frac{\binom{n}{p}}{n-p+1}\right)^{-1} C^{(n)}_{\mathbf{y}_1, \mathbf{y}_2} =$$

$$= \left(\frac{\binom{n}{p}}{n-p+1}\right)^{-1} \sum_{\mathbf{q} \in \mathbf{Q}} \frac{1}{2} \mathbf{I}_{\{\sum_{k=1}^{n} \operatorname{sign}(\det(X_k, H_q, \mathbf{1}) = 0)\}} \left(1 - \operatorname{sign}(\det(\mathbf{y}_1, H_q, \mathbf{1})) \cdot \operatorname{sign}(\det(\mathbf{y}_2, H_q, \mathbf{1}))\right)$$

and in case of $n - p$ odd

$$\left(\frac{\binom{n}{p}}{n-p+1}\right)^{-1} C^{(n)}_{\mathbf{y}_1, \mathbf{y}_2} =$$

$$= \left(\frac{\binom{n}{p}}{n-p+1}\right)^{-1} \sum_{\mathbf{q} \in \mathbf{Q}} \frac{1}{2} \mathbf{I}_{\{\sum_{k=1}^{n} \operatorname{sign}(\det(X_k, H_q, \mathbf{1}) = 1)\}} \left(1 - \operatorname{sign}(\det(\mathbf{y}_1, H_q, \mathbf{1})) \cdot \operatorname{sign}(\det(\mathbf{y}_2, H_q, \mathbf{1}))\right).$$

Now, we can see that we obtained an incomplete U-statistic (see [8]) with a symmetric kernel

$$\begin{aligned} g(X_1, X_2, \ldots, X_n) &= \\ &= \frac{1}{2} \left(1 - \operatorname{sign}(\det(\mathbf{y}_1, H_q, \mathbf{1})) \cdot \operatorname{sign}(\det(\mathbf{y}_2, H_q, \mathbf{1}))\right). \end{aligned}$$

And with a finite second moment. It implies that $\left(\frac{\binom{n}{p}}{n-p+1}\right)^{-1} C^{(n)}_{\mathbf{y}_1, \mathbf{y}_2}$ converges in quadratic mean to $\mathrm{E}g(X_1, X_2, \ldots, X_n)$ as n tends to infinity.

To complete the proof we need to show that

$$\mathrm{E}g(X_1, X_2, \ldots, X_n) = \frac{1}{\pi}\alpha(\mathbf{y}_1, \mathbf{y}_2).$$

Let us consider a canonical basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_p\}$ in $\mathbf{R}^p$ such that $\mathbf{y}_1 = \mathbf{e}_1$ and $\mathbf{y}_2 := \lambda\mathbf{e}_1 + \mu\mathbf{e}_2$ for some $\lambda, \mu \in \mathbf{R}$. In such a case, we can see that

$$\begin{aligned} \mathrm{E}g(X_1, X_2, \ldots, X_n) &= \\ &= \mathrm{P}\left(\operatorname{sign}(\det(\mathbf{y}_1, H_q, \mathbf{1})) = -\operatorname{sign}(\det(\mathbf{y}_2, H_q, \mathbf{1}))\right). \end{aligned}$$

By A, we denote the subset in the space generated by $\mathbf{e}_1$ and $\mathbf{e}_2$, which is bounded by lines going through origin and projections of y_1 and y_2. Hence, according to lemma 1, we know that for every ε we can find n large enough such that with probability at least $1 - \varepsilon$ the distance of halving hyperplane from the origin is less or equal to ε. It is easily to seen that

$$\left|\mathrm{P}\left(\operatorname{sign}(\det(\mathbf{y}_1, H_q, \mathbf{1})) = -\operatorname{sign}(\det(\mathbf{y}_2, H_q, \mathbf{1}))\right) - \mathrm{P}(((\mathbf{e}_q)_1, (\mathbf{e}_q)_2) \in A\right| \leq$$

$$\leq \varepsilon + \mathrm{P}(((\mathbf{e}_q)_1, (\mathbf{e}_q)_2) \in A^{\varepsilon} \backslash A),$$

where A^ε is a subset of the space generated by $\mathbf{e}_1$ and $\mathbf{e}_2$ such that $x \in A^\varepsilon$ iff $d(x, A) \leq \varepsilon$. So, the right hand side of the inequality could be small enough.

Then, we need to compute

$$\mathrm{P}\left(((\mathbf{e}_q)_1, (\mathbf{e}_q)_2) \in A\right).$$

Using the same arguments as Hallin and Paindaveine we get that

$$\mathrm{P}(((\mathbf{e}_q)_1, (\mathbf{e}_q)_2) \in A) \to \frac{1}{\pi}\alpha(\mathbf{y}_1, \mathbf{y}_2)$$

for $n \to +\infty$.

□

From the proofs of previous propositions and the definition of the center given by halving hyperplanes, we can see, that the key step in proofs of asymptotics theorems for multidimensional quantiles is using of convergence theorem for incomplete U-statistics. The limit theorems for incomplete U-statistics based on i.i.d. data were studied in the seventies of the last century by many researchers. A lot of these results it is possible to find for example in [8].

However, there is a question, if it is possible to prove similar results on multidimensional quantiles in the case of dependent data, more precisely weakly dependent data. The reason for this research is, that in practice usage, we rarely have independent data, more often we handle with nonindependent data set. So, we need the theory of multidimensional quantiles for weakly dependent data, hence we need the asymptotics for U-statistics based on weakly dependent data set.

There are three main ways how to handle with weakly dependent data. The first possibility is to use some mixing conditions. Mixing conditions were introduced by Rosenblatt (see [13]) and are based on relations among σ-fields generated by random sequences. The problem of results based on mixing conditions is that it is quite difficult to verify their conditions, because they are very technical. On the other hand, there is a lot of results on asymptotics for U-statistics derived from mixing sequences, see for example [4], [14].

The second approach uses so-called associated random variables. The theory of associated random variables was introduced in sixties of the last century and it is still being developed by many researchers. Recently, there are published papers on asymptotics for U-statistics based on associated random variables, see for example [1].

The third approach studies the asymptotics of U-statistics which are based on stationary processes, without any requirement of satisfying any technical mixing condition. Such approach we can find in for example [3] and [9]. It seems to be very good approach – the proving techniques are more complicated then in the previous cases – but the conditions on dependent data, which must be verified, are quite easier than in above mentioned approaches.

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Representative firm's decisions and leading indicators

Miroslav Kľúčik[1]

Abstract. Leading indicators are potential carriers of signals of future changes in aggregate economic activity. They are mostly composed by means of empirical evidence found in development of aggregate time series. There is also a possibility to find their theoretical foundations by examining decisions of a representative firm in short run. The firm aims to minimize the producing costs for a given demand. The decisions involve change of employment, hours worked and stock of inventories according to the firm's expectations. The short run equilibrium is found by means of inter-temporal optimization. The evidence of lead is found by means of shock analysis.
The reduced form equations are utilized for empirical tests using available macroeconomic time series for EU-27 countries. The results show mixed evidence. While for almost every country there is one leading indicator found, the indicators proved the lead in average less than in half cases. The conclusions imply the theory contains evidence of leading indicators, which can be found also in real aggregate macroeconomic time series.

Keywords: leading indicators, economic theory, inter-temporal optimization.

JEL Classification: C62, D21, E32
AMS Classification: 35F20

1 Introduction

The concept of leading indicators arose in the first half of 20th century by empirical research at the National Bureau of Economic Research in the US [1]. It was based on co-movement of macroeconomic aggregate indicators, by definition, the so-called classical business cycles. Mainly, the analysis of business cycles was aimed at short-term economic forecasting, utilizing the lead of some variables against other found in actual data. The economic reasons (rationales) of the lead was systematically classified later by the OECD [2] or De Leeuw [3, 4] roughly into the early stage indicators, rapid response indicators, expectation sensitive indicators and prime movers.

The endeavour for investigation of economic theory behind the leading indicators was not pronounced for decades, probably because of their original empirical nature or, on the other side, their rather poor prognostic abilities [8] resulting into disinterest of most economists. The effort was mostly focused on the interpretation of observed business cycles; the basics of economic theory behind leading indicators are still to be found only in De Leeuw [3, 4]. The principal of profound business cycle research dealt with equilibrium economics based on microeconomic theory, such as real business cycle theory [6] and later New Keynesian economics [7]. De Leeuw used microeconomic equilibrium theory to provide some theoretical evidence of lead/lag behaviour of variables connected to short-term decisions of a representative firm for the three mentioned types of leading indicators - early stage indicators, rapid responsive indicators and expectation sensitive indicators [4]. To make a integration step of classical microeconomic analysis based on agents within general equilibrium economy (households, firms, government, monetary authority) and empirical indicators classified according the OECD a few links can be shed light upon.

Early stage indicators provide technical lead of some production processes against others, such as orders and actual production. This kind of indicators clearly relates only to firms, as shall be the only agent looked upon in this research paper. Rapid responsive indicators evidence the relatively rapid response of some macroeconomic variables to the development of aggregate economy, such as working hours fluctuations or delivery times in case of demand shocks. Also in this case the phenomenon is connected to the decisions of firms, which can alter working hours connected with lower costs in comparison with other decision variables such as employment (given the frictions on the labour market). The third type of leading indicators (expectation sensitive) is connected also with other classical agents, such as households and banks. These can behave differently based on their expectations that lead other major decisions of other agents on the market. Last type of leading indicator is

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connected with more profound reasons of business fluctuations such as government and central banks decisions about policies applied on the market (in case of no-anticipation of these steps) or the terms of trade between several countries.

Subsequent analysis will try to build on De Leeuw's approach, clarify the computation steps and enrich it both in theoretical and empirical way. Two types of leading indicators are examined - rapid responsive indicators (hours worked, changes in stock of inventories) and expectation sensitive indicators (expected shipments). Firstly, by deriving of original formulas the short-term behaviour of firm in equilibrium is dubbed, by example in an alternate definition of behaviour. Secondly, an exhaustive way of sensitivity analysis of particular parameters of the reduced form model is provided. Finally, an empirical analysis follows, unlike the De Leeuw's US practical example; it considers EU countries' macroeconomic data.

The following text is organized into the methodology chapter (2nd Chapter) with description of differentiation of short run equilibrium equations for the firm and utilisation of the reduced form equations for simulation of shocks, which shall provide some evidence of lead/lag behaviour of variables. The 3rd Chapter uses the structure of theoretical equations for real-world application of leading indicators for the EU countries. In the conclusions some future possibilities of dealing with this topic is elaborated upon.

2 Theoretical analysis

The original approach of De Leeuw is based on a short-term dynamic firm behaviour according to the work of Holt, Modigliani, Muth and Simon [5]. The aim was to find some theoretical evidence of lead and lag in decisions of a producing firm. The decisions are therefore looked upon in a theoretical state of short-term equilibrium, i.e. the firm decides about input and output in an optimal way.

The basic assumption is that the demand is given, i.e. the buyers can not alter their behaviour in short run, and also the prices are given. The firm decides about the current level of employment, working hours, stock of inventories to minimize the costs of production in order to meet the given demand. De Leeuw gives two alternatives; firstly, the firm produces solely for stock, secondly, solely for orders. For the forthcoming analysis only the stock alternative is taken, basically because no unfilled orders data are available for most EU countries utilized later in empirical analysis. The cost minimization follows:

$$\min_{C_t,r} \sum_{i}^{\infty} \frac{C_{t+i}}{(1+r)^i} \tag{1}$$

where $$C_{t+i} = a_0 + a_1 E_{t+i}\bar{H} + a_2[E_{t+i}(H_{t+i} - \bar{H})]^2 + a_3[\bar{H}(E_{t+i} - E_{t+i-1})]^2 + a_4[(J_{t+i} - c_1 S_{t+i})]^2 \tag{2}$$

$$J_{t+i} = J_{t-1+i} + S_{t+i} - Q_{t+i} \tag{3}$$

and $$Q_{t+i} = c_2 E_{t+i} H_{t+i} \tag{4}$$

C_t are the costs of the producing firm in an infinite time $t+i$, the variable E_t represents the employment costs (wages), H_t the actual working hours, $\bar{H}$ the average hours worked. Parameters indicated $a_0 ... a_4$ and c_1, c_2 are representing the elasticity of changing the costs. The last element in the equation (2) represents the variable stock of inventories (finished goods J_t minus realized supplies S_t). In this case the a_0 parameter can represent some fixed costs, the parameter a_1 indicates elasticity of changing the overall labour input costs, a_2 the costs related to changing of working hours, a_3 costs due to employment changes and a_4 costs associated with decisions about shipments and production time. Last three elements of the equation indicate higher associated costs (squared elements) in the production process. The actual production is restricted by the classical production function, missing capital as input (4). Capital is assumed to be fixed in short time, therefore the firm does not decide about the capital accumulation.

Another variation, still occurring in short run (leading indicators shall provide evidence only short-term lead), might be to maximize the so-called gross operating surplus, i.e. the gap between the actual turnover and producing costs given the same constraint of production function. The decision parameters would be disaggregated into decisions about raw materials, different staff costs etc., but this elements are in the case above (equation 1) included in the stock of inventories variable. This approach would produce the same results.

The minimisation problem has been described above in equations (1) to (4). Basic algebra transformation offers two approaches to solve the problem. First is the use of Lagrangian technique, second, the direct substitution approach. The second approach can be applied in this case.

Looking at the equation (1)-(4) we have the following unknown variables: Q_t, E_t, H_t, S_t, J_t and r. The discount rate is taken as given with no changes in time. The equation (3) can be substituted into equation (2), which leaves out one unknown variable J_t. Also the variable H_t can be easily eliminated by solving equation (4) for H_t and substituted into equation (2). Then there are three variables that affect the level of costs in the short run - expenditures related to employment - number of employees, level of working hours and expenditures related to stock of production, expressed as the shipments S_t, which affect the stocks - J_t.

With some variables having different lag at time *t*, the optimization problem becomes an inter-temporal one. While decision variables can be accounted for infinite time, simplification assumption is made as in [4] that no expectations are created more than 2 periods ahead and the working hours fall back to the average level in two periods (however, these constraint does not shorten the response time of variables in the simulations). Each variable in equation (2) must be dealt as unknown in each period of time *t*, *t+1* etc. Therefore each time shift (with the simplifying assumption now only *t+2*) must be taken into account when computing the equilibrium. The following equations arise taking into account the substitutions made for J_t and H_t:

$$C_t = a_0 + a_1 E_t \bar{H} + a_2 \left[\frac{Q_t}{c_2} - E_t \bar{H}\right]^2 + a_3[\bar{H}E_t - \bar{H}E_{t-1}]^2 + a_4[(J_{t-1} + Q_t - S_t - c_1 S_t)]^2 \tag{5}$$

$$\begin{aligned} C_{t+1} = a_0 + a_1 E_{t+1} \bar{H} + a_2 \left[\frac{Q_{t+1}}{c_2} - E_{t+1} \bar{H}\right]^2 + a_3[\bar{H}E_{t+1} - \bar{H}E_t]^2 \\ + a_4[(J_{t-1} + Q_t - S_t + Q_{t+1} - S_{t+1} - c_1 S_{t+1})]^2 \end{aligned} \tag{6}$$

$$\begin{aligned} C_{t+2} = a_0 + a_1 \frac{c_1 SE_{t+2} - J_{t-1} - Q_t + S_t - Q_{t+1} + SE_{t+1} + SE_{t+2}}{c_2} \\ + a_3 \left[\frac{c_1 SE_{t+2} - J_{t-1} - Q_t + S_t - Q_{t+1} + SE_{t+1} + SE_{t+2}}{c_2} - \bar{H}E_{t+1}\right]^2 \end{aligned} \tag{7}$$

The equations (5) and (6) contain the substituted expression Q_t/c_2 for the expression E_t*H_t. Also the expression after a_4 contains the substitution for J_t according to (3). From the equation (7) follows that the elements after parameters a_2 and a_4 fell out because of mentioned simplification assumptions. Initially, also in the third equation it is needed to get rid of unknowns Q_{t+2}. By using substitution by solving for Q_t in (3), four unknowns remain according to (5)-(7): E_t, E_{t+1}, Q_t, Q_{t+1} and the variables S_{t+1} and S_{t+2}. These are to be understood as expected shipments, i.e. they shall be known in time *t* (this assumption follows [4]).

The minimum costs are then found by differentiation of equation (5) - (7) with respect to E_t, E_{t+1}, Q_t and Q_{t+1}, and by setting the derivatives equal to zero. The solutions are equivalent to De Leeuw [4]. The solved equations for the variable Q_t and E_t represent the actual decision of a firm about production and employment, using the equations (3) and (4) also the current hours worked and level of stocks can be calculated. The solutions for E_{t+1} and Q_{t+1} are representing the planned level of employment and production. The available four equations with for unknowns can be solved either in the matrix form or by substitution analytically. In case of 0 value of discount rate and value of 1 for parameters a_0 to a_4, c_2 and 2 for parameter c_1 (De Leeuw's parameters), the following two equations with two unknowns will emerge[2]:

$$Q_t = 0.077 + 0.115E_{t-1} - 0.654J_{t-1} + 1.577S_t + 0.500SE_{t+1} + 0.115SE_{t+2} \tag{8}$$

$$E_t = -0.231 + 0.404E_{t-1} - 0.288J_{t-1} + 0.519S_t + 0.404SE_{t+1} + 0.250SE_{t+2} \tag{9}$$

The optimal conditions for short-term decisions about costs are now known, the next step is to examine the decisions by using assumptions about parameters and initial levels of employment, working hours, stocks, production and shipments.

2.1 Shock simulations

De Leeuw uses the value 1 for parameters a_0 to a_4 and c_2, and 2 for c_1. The baseline values for simulations are set to 100 for initial value of employment, shipments, production, 200 for initial level of stock of inventories and 1 for average and initial working hours. In the analysis the *r* discount rate is set to 0 for all periods. Then shocks are presented for step changes in demand (shipments - S_t), changes in expectations (expected shipments S_{t+1} and S_{t+2}) and changes of all parameters (including productivity). However, an important fact must be emphasized. De

[2] Both results are similar to De Leuuw's results, which were stated as the baseline parameters.

Leeuw uses the assumption that the shipments in time *t* are not the actual shipments, but expected. In this analysis a more standard case is examined, i.e. the shipments in time *t* are known, which, as it will be seen later, has crucial impact on the results in comparison to De Leeuw's work.

The expectations are modelled similarly as in De Leeuw's paper. The first period expectations are taken as the expectations from previous period plus half of the error of the previous period forecast. The second period expectation is calculated as the forecast for 2 periods ahead from the first period plus half of the difference against the previous period expectations.

Before taking look at the simulations, the expected results regarding the leading indicators approach can be stated. It is expected that the hours worked and also stock of inventories, both as rapid responsive indicators, are going to lead the production and employment in case of their growth and decline turning points (observed high and low point of the time series). The case of expected shipments as leading indicator is not elaborated as it is solely dependent on assumptions.

The level of simulation shocks is assumed in same manner as in De Leeuw's paper, i.e. simulation shock raises the shipments (and expected shipments) in period 6 from 100 to 120 and reduces the shipments back in period 16 to 100 etc.

The Table 1 contains the results of all two shocks on formation and behaviour of each variable regarding the time shift of variables (lagging, coincident, leading) in the turning points (after changes in period 6 and 16).

Type of shock	Turning point 1		Turning point 2	
	H_t	J_t	H_t	J_t
Demand shock	C, L	C, L	C, L	C, L
Expectations shock	-	C, L	C, L	C, L

Table 1 Lead/lag relationship for shocks[3]

The results show that neither working hours nor change in inventories have proven to be leading the production, but on the other side both are leading indicators for the level of employment in all cases. This is a contrary result to the De Leeuw work, which shows that hours worked and stock of inventories are leading indicators in some cases. In the expectations shock the hours worked had shown no conformity with the production and employment, therefore it is omitted.

Sensitivity analysis is performed regarding the values of parameters and shock levels (only demand case shock). The relative stability proved in De Leeuw analysis has been confirmed. The lead/lag behaviour provided in Table 1 is found to be strong and without change in case of parameter changes a_1 to a_3 (tested range 0.5-1.5), and also in the case of c_1 parameter (range 1.0-3.0). Changes in c_1 lead to very different initial state of the system, but in time of the first and second turning point the relationships between the variables is without changes. However, sensitivity of parameter a_4 revealed changes in lead behaviour - at the bottom of tested range (below 0.55) the hours worked variable became leading also to the production (this is the only case where this leading indicator proved his abilities theoretically). The same results showed the sensitivity of c_2 parameter. Below the value of 0.72 the hours worked became leading also the production variable. This implies stronger role of changes in work intensity when sensitivity to changes in stocks is low. Finally, the changes in level of shocks (tested range -50 to 50) and also the changes in discount rate (tested range 0.01 to 0.99) had no impact of lead/lag relationships stated in Table 1. The Figure 1 shows the leading behaviour of hours worked at the case of 0.7 value of c_2 parameter from sensitivity shocks.

3 Empirical Analysis

The forthcoming analysis takes as its basis the trial to prove the leading ability of working hours, stock of inventories and expected shipments in real world data. Empirical analysis shall encompass one solely firm as the representation of an average producing firm in national economy (an agent). Some general assumptions about the market in which the firm exists need to be stated. The labour and the produced good in the model are regarded homogenous. Other assumptions are the same as in the case of theoretical analysis, i.e. in the short term the prices and demand are given, and the firm cannot acquire capital. As no data about shipments are available, the firm is taken to be an exporter, i.e. the export of goods can be regarded as sufficient evidence of a shipment.

[3] C - coincident relationship, L - leading relationship.

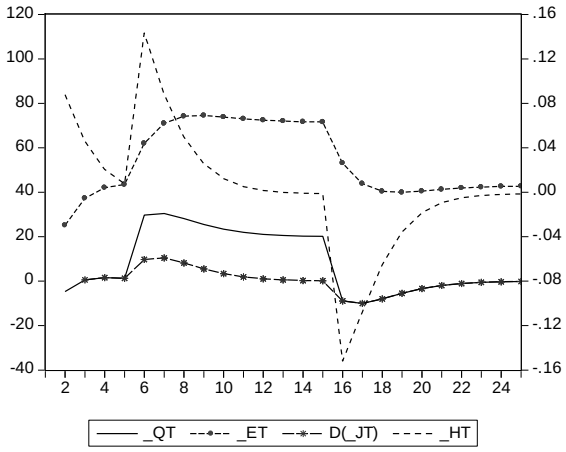

Figure 1 Shock observation and lead of hours worked at observation No. 6 and 16

The production is represented by industrial production index, average employment and working hours are also available for industry. The shipments are introduced through real exports of goods, while expectations can be approximated by expected export orders from business tendency surveys in industry branch. As stock of inventories is available for countries only at quarterly frequency this time series is imitated by estimated stock of inventories in industry, also available from business tendency surveys. All time series are available in monthly frequency. Several steps of transformation are needed for later use in standard regression. Firstly, the time series must be seasonal adjusted (EViews) then their stationarity is ensured through year-over-year transformation. The time series from business tendency surveys are available as balances (qualitative aggregates), and are taken in original form as deviations from trend (the so-called deviation cycle), therefore they do not need to be transformed into year over year growth rates. All time series are smoothed by Henderson Moving Average (X12-ARIMA method) available in EViews, while the smoothing parameter is by default chosen automatically according to the particular time series' variance.

To enable comparison between theoretical and empirical results the original regression is taken as the initial point (see equation 8 and 9), adding hours worked to the equation. Crucial is the observation of sign of each parameter, the lag is taken the same as in the theoretical analysis (1 month, 2 months for expectations). Time series mentioned in previous paragraph are available for 19 European countries from about 2001 until December 2012 (from data sources of Eurostat and National Statistical Institutes).

Country	Qt						Et					
	b_1	b_2	b_3	b_4	b_5	b_6	b_1	b_2	b_3	b_4	b_5	b_6
Bulgaria	-	-	-0.53	0.21	-	1.35	-0.06	0.98	0.06	0.02	-	-
Cyprus	0.56	0.59	-0.34	0.09	0.06	-	0.02	0.97	-0.01	0.01	0.01	
Czech R.	0.71	-0.69	-	0.52	0.22	0.23	0.06	0.85	-	0.03	0.02	0.05
Estonia	-0.13	0.20	-	0.57	-	0.36	0.04	0.95	-0.02	0.03	-	-
Finland	0.38	0.14	-0.09	0.54	0.04	-	0.05	0.92	-0.02	-	-	0.05
France	-	0.37	-0.10	0.72	-	-	0.03	0.93	-0.01	0.02	-	0.03
Germany	0.25	-0.21	-	0.65	-	0.31	0.08	0.91	-0.03	-	-	0.03
Greece	1.20	0.33	-0.62	0.04	-	-	0.08	0.97	-0.05	-	-	-
Hungary	-	-0.23	-	0.73	-	0.49	0.06	0.63	-	0.04	-	0.26
Latvia	0.14	0.25	-	0.59	-	-	-	0.83	-	0.10	-	0.06
Lithuania	0.24	0.25	-	0.23	-	0.30	0.05	0.96	-0.02	0.02	-	-
Netherlands	2.60	-1.62	-0.73	0.06	0.18	0.52	0.03	0.97	-0.01	0.01	-	-
Poland	-0.15	0.73	-	0.43	-	-	-	0.97	-	0.02	-	-
Romania	0.79	-1.72	-	-	0.14	1.81	0.10	1.01	-0.15	0.01	0.03	-
Slovenia	-0.63	-	-	0.43	0.09	1.10	-0.07	0.93	-	0.03	-	0.09
Slovakia	-	0.55	-	0.48	-	-	0.02	0.81	-	0.03	0.01	0.12
Spain	0.95	0.29	-0.54	0.29	-	-	0.11	0.93	-0.07	0.02	-	-
Sweden	0.12	0.13	-	0.74	-	-	0.08	0.89	-0.02	0.05	-	-
UK	1.36	-	-0.45	0.12	-	-	-	1.01	-0.01	-	-	-

Table 2 Regression results for empirical analysis

In the regression runs only variables with significant parameters are retained. For both production equation (8) and employment equation (9) the results are summarized in Table 2. The parameters are marked with b_1 (constant), b_2 (employment), b_3 (stock of inventories), b_4 (exports - shipments), b_5 (expected shipments - exports) and b_6 (hours worked).

No leading indicator is found for production for Latvia, Poland, Slovakia and Sweden. For employment this is the case for Poland. In the theoretical analysis the stock of inventories and hours worked showed to be leading for both production and employment in all cases. The empirical analysis agrees with this fact in 13 cases for stock of inventories and 8 cases for hours worked. The theory in previous chapter implied hours worked as leading indicator of production only in certain range of parameter values, while the same mixed results are evidenced in the empirical analysis. Table 2 shows that stock of inventories, expected shipments and hours worked are leading in less than half cases. All three indicators are significant only for industrial production of Netherlands.

The conclusions stated in Table 2 serve only for comparison between empirical and theoretical expected values. Given the differences between sectorial specialization of each country and different laws, one cannot truly compare the results of countries themselves. However, the market frictions in each country can manifest in different lead.

4 Conclusions

According to the theoretical search for an equilibrium in short run a firm does have an optimum connected to decisions about employment, hours worked and stock of inventories according to expectations, present at the point of minimal costs. The shocks applied to the reduced form equation imply the existence of leading behaviour in case of stock of inventories and hours worked in all cases for employment and in some levels of parameters for hours worked in case of production. The empirical analysis showed that except in four cases out of 40, each dependent variable has found its leading indicator among stock of inventories, expected shipments and hours worked. This proves the theoretical foundation provided, while simultaneously the empirical analysis shows mixed evidence for leading indicators varying among different countries.

The weak spot of the theoretical assumptions is the creation of expectations, which must be stated directly, but despite this fact the leading behaviour emerges as was proved in this paper. However, some new concepts shall be developed regarding the expectation creation, i.e. based on microeconomic theory. Also some cases of widening the period analysed shall be examined.

Acknowledgements

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Demographic ageing and its spatial consequences

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Abstract. This paper deals with an evaluation of demographic ageing in the Czech Republic and its spatial consequences. A regional analysis of the basic indicators of the economic demography (the age dependency ratio, the economic dependency ratio, the green dependency ratio, and the grey dependency ratio) comprises the main part of this article. The presented analysis can serve as a basic precondition for regional policy recommendations.

Keywords: demographic ageing, factor analysis, spatial autocorrelation, hot spots analysis

JEL classification: C44
AMS classification: 90C15

1 Introduction

Demographic ageing has affected more or less all countries in the world since the second half of the 20th century and has become one of several global problems. According to long-term population projections, the proportion of the elderly is going to increase, and this fact is often mentioned when talking about various social problems, such as pension systems and the increase of social welfare and health care costs.

Ageing, in the demographic sense, relates to the whole population. Only an increase in the proportion of young age groups can make a population that is getting older become younger. Demographic ageing is caused by changes in the character of demographic reproduction – changes in the share of the youthful and post-reproduction parts of the population. These changes could be caused by two factors. The first one is a relative slowdown of the increase in younger age groups, which is mostly a result of a decrease of birth and fertility rates. This type of ageing is called "relative ageing". The second one is an acceleration in the growth of older age groups, which is an outcome of a greater decrease of mortality rates in the higher age ranges. It results in an increase of life expectancy. This type of ageing is known as "absolute ageing". Both types usually occur at the same time.

The Czech Republic is one of the countries experiencing the demographic ageing intensively. According to the up-to-date prognosis [4], the population development of the Czech Republic to 2070 will have the following features:

- The interim period of the population growth by natural increase ends in 10–12 years. This period has been caused by the concurrence of low mortality and the realization of the transformed and postponed fertility. The future will depend on an amount of migration rate.
- The Czech population will further get older. The important factors will also be an expected decrease of fertility and a spin out of the human life length. The Czech population will firstly get older by absolute ageing. The relative ageing will also affect our ageing because changes of lower and higher fertility are expected.
- The expected migration will weaken the influence of anomalies in age structure.
- Expected migration gains will lead to a preservation of economic potential (labour force) in a medium-term perspective. Then we can expect a drop of this potential in a long-term perspective.

Thus, the appraisal of regional differences in the demographic ageing and its consequences is important. The found difference can be used as a recommendation for further regional policies.

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2 Material and Methods

Three economic generations comprise the age structure of the population:

- The pre-productive age: 0–19 years (Economic generation I),
- The productive age: 20–64 years (Economic generation II),
- The post-productive age: 65 years and older (Economic generation III).

Usually, the age of 15 years is used as the boundary for children in the literature, but many young people are not productive beyond this age, as they continue with their education. Thus, we used the age of 19 years as the lower boundary of the productive age which reflects better the current situation. We tried to include accessible characteristics which influence the demographic development, especially changes in the age structure. With respect to the availability of statistics sources, the basic characteristics of the economic population structure and some other chosen indicators were used for the evaluation of the Czech demographic ageing: the Economic dependency ratio (EDR) – a relation all of economic generations to Economic generation II (it shows how many people have to support one person in the productive age), the Green dependency ratio (GDR) – a relation of Economic generation I to the productive generation, the Grey dependency ratio (GYDR) – a relation of Economic generation II to the productive generation, the Ageing index (AI) – a relation of the post-productive age to children aged 0–15 years, the Mean age (MA), the Infant mortality rate (IMR) – death of a child less than one year of age per 1 000 live births, the Crude nuptiality rate (CNR) – the number of marriages per 1 000 inhabitants, the Crude divorce rate (CDR) – the number of divorces per 1 000 inhabitants, the Crude birth rate (CBR) – the number of births per 1 000 inhabitants, the Crude mortality rate (CMR) – the number of deaths per 1 000 inhabitants, the Incapacity for work (IW) – the notified cases of incapacity for work (cases per 100 sickness insur.), the Crude rate of the net migration (CMS) – migration saldo per 1 000 inhabitants, the cause-specific mortality – the mortality rate from a specified cause for a population per 100 000 inhabitants: cancer (C), cardiovascular disease (CV), breathing system (BS), external causes (EC); the density of population (DP) – the number of inhabitants per km^2, proportion (%) of inhabitants living in towns (IT), the proportion of recipients (OAP) of old-age pensions (in all the pensions), the proportion of foreigners (WF) with working permit (in all foreigners registered by employment bureau), the density of road system (DRS), the completed dwellings (CD) per 1 000 inhabitants, the unemployment rate (RU), the number of registered economic subjects (RES)per 1 000 inhabitants.

The data was obtained from the Czech Statistical Office and the Institute of Health Information and Statistics of the Czech Republic.

The number of characteristics led us to use the factor analysis. It is a method used to describe variability between observed, correlated variables in terms of a potentially lower number of unobserved variables called factors, eg. [5],[3], [10]. There are various rotational strategies that have been proposed. The goal of all of these strategies is to obtain a clear pattern of loadings, that is, the factors that are somehow clearly marked by high loadings for some of the variables and low loadings for the others. This general pattern is also sometimes referred to as the simple structure. We used the software Statistica for this analysis. The analysis was conducted for two years, 2006 and 2011, and then the results were compared in order to describe changes and dynamics of the demographic ageing in the Czech Republic. The main factors were extracted using the method of the principal components and then rotated.

The spatial distribution of demographic development was evaluated by Moran index of spatial autocorrelation I which we applied on the factors strongly correlated with the Economic Dependency Ratio:

$$I = \frac{n}{S_0} \cdot \frac{\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}(x_i - \mu)(x_j - \mu)}{\sum_{i=1}^{n} w_{ij}(x_i - \mu)^2}, \tag{1}$$

where w_{ij} is a spatial weight between feature i and j, n is equal to total number of features and S_0 is aggregate of all the spatial weights:

$$S_0 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}. \tag{2}$$

The spatial autocorrelation is characterized by a correlation in a sign among nearby locations in space. According to Fotheringham et. al [9], the spatial autocorrelation is more complex than the one-dimensional autocorrelation because the spatial correlation is multi-dimensional (i.e. 2 or 3 dimensions of space) and multi-directional. The computation of Moran's I is performed by dividing the spatial covariation by the total variation. The resulting values are in the range approximately from -1 to 1. The positive sign represents a positive spatial autocorrelation, while the converse is true for the negative sign. Zero result represents no spatial autocorrelation [2],[13].

In case of the global spatial autocorrelation, the null hypothesis states that "there is no spatial clustering of the values associated with the geographic features in the study area". When the p-value is small and the absolute value of Z score is large enough that it falls outside of the desired confidence level, the null hypothesis can be rejected. If the index value is greater than 0, the set of features exhibits a clustered pattern. If the value is less than 0, the set of features exhibits a dispersed pattern [12],[13].

The global Moran statistic indicates a tendency to clustering but cannot identify types of the clusters (high or low values). It is suitable to use a local statistic to identify clusters of high values (hot spots) or low values (cold spots)[1]. The Hot Spot Analysis calculates the Getis-Ord G_i^* statistic for each feature in a dataset. The resultant Z score indicates where the features with either high or low values cluster spatially. This tool works by looking at each the feature within the context of neighbouring features. A feature with a high value is interesting, but may not be a statistically significant hot spot. To be a statistically significant hot spot, a feature will have a high value and be surrounded by other features with high values as well [13]. The local sum for a feature and its neighbours is compared proportionally to the sum of all features; when the local sum is much different than the expected local sum, and that difference is too large to be the result of random chance, a statistically significant Z score results.

$$G_i^* = \frac{\sum_j w_{ij} d(x_j)}{\sum_j x_j}. \tag{3}$$

Various approaches to conceptualize spatial relationships (neighbourhood of the features used in the calculations). Three of them were tested in our analysis for the hot spots identification of the main factors: the fixed distance, the contiguity and K nearest neighbours. These concepts are described, for example, in [12] or [7].

3 Results and Discussion

The main factors were extracted using the method of the factor analysis. Six factors were chosen with eigenvalues greater than 1 according to the scree plots in both analysed years. The chosen factors explain 75.97% (2006), respectively 71.42% (2011) of the total variance. Tables 1 and 2 show factor loadings. Thus, they represent the most important information on which the interpretation of the factors is based.

We can try to interpret the factors influencing demographic development, with a certain amount of caution, in the following way: the main component "age structure" is the same for both years. The significance of the third factor in 2006, which can be labelled as "urbanization and business [1]", has become stronger in 2011 (as it can be seen in Table 3), whereas the second factor in 2006, "migration", lost its importance in 2011. Mortality played a significant role in demographic development in 2006. Its importance is still decreasing, which corresponds with the main trends discussed in the literature. Many authors use the term "mortality transition" [6],[8]. Immigrants working in the Czech Republic (factor 5 in 2006 and factor 6 in 2011) are also important for the economy in relation to demographic ageing. It also corresponds with various demographic prognoses which consider immigration as one of the future sources of the workforce [15].

The demographic development and its economic consequences are also influenced by particularities of "urban environment" (the population density, a higher crude divorce rate, some causes of mortality connected with the urban environment) which have emerged at the end of the analysed period. The "external causes of mortality" represents another factor of the demographic development playing an important role at the beginning of the period. It corresponds to the above mentioned features of the population development in the Czech Republic till 2070 [4].

[1] EV ... eigenvalue, CPTV ... cumulative percentage of total variation

variable	F1	F2	F3	F4	F5	F6
RU06	-0.49	**0.62**	0.10	0.35	0.03	-0.02
GDR06	**-0.72**	0.13	-0.50	-0.20	-0.02	0.23
GYDR06	**0.92**	0.06	-0.14	0.09	-0.08	0.20
AI06	**0.96**	0.07	0.14	0.12	-0.06	0.03
MA06	**0.94**	-0.12	0.16	0.22	0.01	0.04
IW06	-0.07	0.13	0.24	0.06	**0.72**	-0.01
CNR06	-0.19	-0.43	**0.69**	0.05	0.10	0.09
CDR06	-0.46	-0.19	**0.59**	0.29	0.32	-0.06
CBR06	-0.36	**-0.70**	0.34	-0.06	0.03	0.16
CMR06	0.49	0.06	-0.04	**0.79**	0.13	-0.15
IMR06	-0.29	0.31	0.07	**0.52**	0.02	0.14
CMS06	-0.07	**-0.89**	0.00	0.09	-0.03	-0.03
C06	-0.48	-0.11	0.39	**0.50**	0.09	0.11
CV06	0.31	0.11	-0.30	0.70	-0.25	0.04
BS06	0.49	0.25	0.04	-0.20	0.35	0.47
EC06	-0.09	0.26	-0.07	0.15	0.33	**-0.63**
WF06	0.00	0.00	-0.04	-0.05	**0.86**	-0.07
CD06	0.10	**-0.83**	0.05	-0.17	-0.26	0.10
DRS06	0.10	**-0.53**	-0.09	0.34	0.22	0.51
IT06	-0.02	0.27	**0.82**	0.08	0.24	0.03
PD06	0.34	0.09	**0.78**	-0.02	-0.07	-0.06
RES06	0.36	-0.44	**0.54**	-0.23	0.03	-0.30
OAP06	0.22	-0.16	**0.56**	-0.24	-0.14	0.28

Table 1: Factor loadings 2006

The next step of our analysis was spatial analysis of the main factors. We tested the hypotheses about their (global) spatial autocorrelation and tried to identify hot spots. Table 4 shows correlations of the Economic dependency ratio with the main factors. The values bolded represent significant correlations.

When calculating Moran statistics for highly correlated factors from the Table 4, we found out that all of them show a significant positive spatial autocorrelation, e.g. a tendency for clustering in space. We tried to identify clusters of high/low values (hot spots) in the next step. This part of the analysis was realized with the software GeoDA.

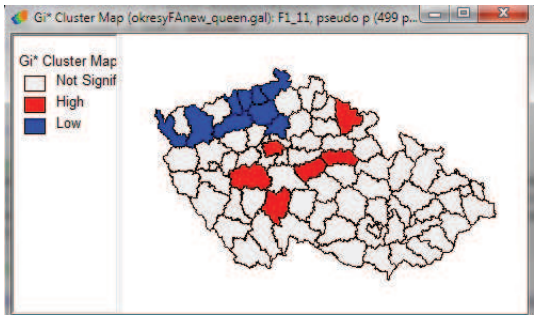

Figure 1: Hot spots - factor 1 - age structure - 2011

Figure 1 shows hot and cold spot of the age structure[2] in 2011 which corresponds with recent differences between districts: the districts in North Bohemia have more favourable age structure, but again there are some districts in the inner parts of the country, mainly along the borders of the administrative regions which have unfavourable age structure. In the Czech Republic, the inner peripheries are usually the

[2]Hot spots found out in 2006 are almost identical.

variable	F1	F2	F3	F4	F5	F6
RU11	-0.13	-0.23	0.31	**-0.57**	-0.52	-0.24
GDR11	**-0.66**	-0.42	-0.21	-0.15	0.37	-0.03
GYDR11	**0.91**	0.12	-0.22	-0.02	0.15	-0.07
AI11	**0.94**	0.23	-0.08	0.02	-0.11	-0.06
MA11	**0.94**	0.17	-0.01	-0.01	-0.18	-0.06
IW11	0.02	-0.17	0.22	**0.76**	0.08	0.08
CNR11	-0.24	0.18	0.01	**0.67**	0.06	0.03
CDR11	-0.31	0.12	**0.67**	0.24	0.07	0.10
CBR11	-0.52	0.30	-0.12	0.16	**0.62**	0.09
CMR11	**0.73**	-0.27	0.45	-0.17	-0.06	-0.08
IMR11	-0.12	0.06	0.49	-0.29	0.12	**-0.50**
CMS11	-0.53	0.12	-0.16	0.04	**0.64**	0.23
C11	0.39	-0.05	**0.69**	0.07	-0.07	0.16
CV11	**0.63**	-0.33	0.08	-0.40	0.02	0.14
BS11	0.32	-0.09	0.10	0.11	0.01	**-0.53**
EC11	0.12	-0.05	-0.07	-0.09	-0.27	-0.45
WF11	0.09	0.17	0.15	0.09	0.05	**0.71**
CD11	-0.39	0.12	-0.32	0.17	**0.71**	0.24
DRS11	0.10	0.00	0.29	0.00	**0.80**	-0.09
IT11	0.15	0.57	**0.61**	0.02	-0.20	-0.12
PD11	0.19	**0.79**	0.10	0.05	0.00	0.03
RES11	0.01	**0.81**	-0.02	0.22	0.16	0.18
OAP11	-0.06	0.66	-0.03	-0.26	0.23	0.18

Table 2: Factor loadings 2011

	2006			2011		
factor	EV	CPTV	interpretation	EV	CPTV	interpretation
F1	5.22	21.76	age structure	6.19	26.89	age structure
F2	4.55	40.73	migration	3.50	42.12	urbanization, business
F3	3.48	55.25	urbanization, business	2.25	51.94	urban environment
F4	2.14	64.17	internal causes of mortality	1.70	59.32	incapacity for work
F5	1.57	70.73	working foreigners	1.55	66.06	migration
F6	1.26	75.97	external causes of mortality	1.18	71.19	working foreigners

Table 3: Main factors of demographic development - comparison 2006 and 2011

year	F1	F2	F3	F4	F5	F6
2006	**0.30**	**-0.67**	0.16	-0.06	-0.12	**0.54**
2011	**0.46**	-0.09	**-0.38**	-0.04	-0.04	**0.45**

Table 4: Correlations of Economic dependency ratio and factors – comparison 2006 and 2011

peripheral zones of metropolitan areas and regional centre areas. In the mid 1990s, the populations stopped declining in some peripheries as a result of the sub-urbanisation processes, but in the other peripheries, the depopulation processes continued. This last category of the inner peripheries can be described as the hard core of Czech peripheral regions and according to Musil and Muller [14] they warrant the development of specific regional policy measures, stressing the creation of new jobs, the improvement of public transport, greater accessibility of service centres, and co-operation among communities.

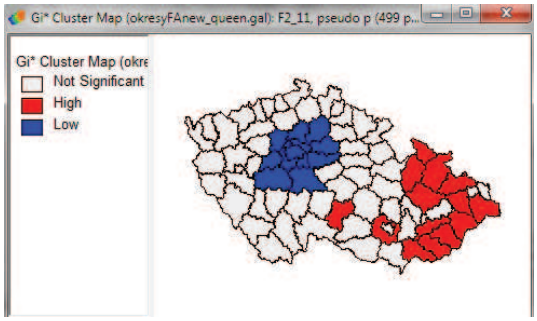

Figure 2: Hot spots - factor 2 - migration - 2006

Figure 2 shows hot and cold spots of migration in the Czech Republic in 2006. The sub-urbanization processes clearly crystallized and after the year 2000, it has become a decisive factor influencing migratory relations within the whole country. Vobecká (2010) stipulates that the sub-urbanization process has been, since 1995, clearly pronounced around primary centres and since 2001, also around smaller, secondary centres. These are so called residential sub-urbanization when people move out of the core cities but their still work there. The improving situation on the Czech housing market speeded up the whole sub-urbanization process. This is the period, in which de-concentration processes (sub-urbanization and partly de-urbanization) started being more important. According to Vobecká (2010), between the years 1995–2006, the most migratory attractive municipalities can be found mainly in close surroundings of big cities (namely Prague, Brno, and Pilsen) and in areas, which are closely tied to advantages offered by the city (within a reasonably short commuting distance) and, at the same time, which have better quality of the environment. On the other hand, there are regions with ecological and economical problems caused by emigration (the north parts of Moravia and areas at the Slovak border).

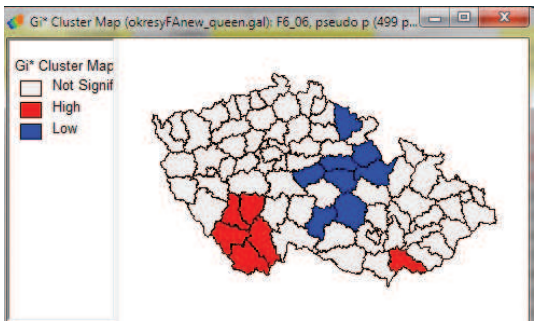

Figure 3: Hot spots - factor 6 - working foreigners - 2011

Figure 3 contains hot spots identified mainly on the basis of foreigners legally working in the Czech Republic (foreigners registered with the Job Center). They provide a picture of the spatial distribution of their activities.

Analysing the differences in the main factors among the regions, we can propose some measures of regional policy aimed at a reduction of the effects of demographic ageing. A detailed spatial analysis of the main factors could serve, for example, as a basis for suggestions on how to improve the Economic Dependency Ratio according to regional conditions.

4 Conclusion

The demographic profile of a region is usually seen as a slowly changing background phenomenon in the analysis of regional competitiveness and regional growth. However, regional demographic change can have a significant impact on regional competitiveness and such a change is often more rapid and profound than those at the national level. In turn, regional population size, growth, composition and distribution are endogenous to regional economic development.

Using spatial statistics, we proceeded to spatial demography. The aim of the paper was to show the usefulness of spatial analysis for regional policy makers. However, the results of any statistical analysis are always limited by the availability of suitable statistical data.

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Eliminating the ambiguity of composite indices in continuum economics

Jan Kodera [1], Tran Van Quang [1]

Abstract. The contemporary mainstream macroeconomics works on the assumption of a continuum of producers in which each of them produces just one differentiated good. Under this assumption, producers or goods are numbered by real numbers from interval [0,1] and it seems to be a good approximation of monopolistic competition economic system. But design of composite indices of production or consumption makes problems of ambiguity. As the usual way of aggregation is using Lebesgue integral, it raises questions like whether the integral exists and whether the conditions we impose have any meaningful economic interpretation. It turns out that these problems are difficult to solve and the answers, if found, are unsatisfactory. Even if the integral exists and a composite index for aggregating the size of economic quantities is available, the question of its ambiguity remains unanswered.

Keywords: DSGE models, continuum, aggregation, composite index, ambiguity, Lebesgue's integral

1 Introduction

The notion continuum in Mathematics has penetrated into Economics for more than a half of a century. Its usage is two-fold. First, to put it simply, it is used to indicate the arbitrary divisibility of an economic quantity. For example, to say capital is a continuum means that it can be arbitrarily any real number. In this sense, there is no ambiguity and its correctness is never disputed. But there is another usage of this notion in Economics which is related to the number of varieties of commodities or to the number of agents in an economy. Aumann [1] probably was the first author who uses the notion continuum in this context. In his work, he considers a continuum of agents in market. It is apparent that the number of agents in a market is finite and the idea of an uncountable set of agents does not coincide with reality and therefore it is refuted. But on the other hand, as in natural sciences, the notion a continuum of whatever is a useful modeling tool for analysis.

Dornbusch, Fischer and Samuelson [4] have further developed the original Heckscher-Ohlin $2 \times 2 \times 2$ model [7] in which they use a continuum of types of goods. They have drawn some interesting results which arise from the presence of a continuum of goods. But on the other hand, they also point to possible ambiguity resulting from the continuum assumption. They maintain that the results obtained from a model with a continuum of goods may differ from the ones drawn from a model with a finite number of types of goods. Later, New-Keynesian dynamic stochastic general equilibrium theory, while assuming a finite number of households, also frequently uses continuum of agents (producers of goods and services) and goods (see Gali [5], McCandless [8], Walsh [9]). For New-Keynesian theory the existence of a continuum of agents-producers and a continuum of sorts of goods is essential as important macroeconomic equations are drawn from the micro-economic foundation, i.e. from individual behavior of agents in economy. In order to do so, New Keynesian theorists need to aggregate the behavior of individual agents at the micro-economic level to the macro-economic level by integrating across those continua.

While the authors of DSGE models take the validity of such integration for granted, in our opinion, the validity is not so straightforward. Rather, it can be a stumbling block, if we do not handle properly, it can disprove the validity of the whole model. So far, to our knowledge, no-one has argued on the possible pitfall the aggregation may be exposed to in the literature. The only exception is Jablecki [6]. He rejects the introduction of a continuum of whatever in economics for its unrealistic nature and according to Jablecki, the conclusions on economic theory based on the continuum assumptions are

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incorrect and unapplicable. Unlike Jablecki, whose critical opinion on the continuum assumption may have deep philosophical roots, we accept the fact that when modeling an economy, one has to make some simplifying assumptions. Therefore we do not refuse to accept the usage of continuum approach. Our analysis is hence focused on how to make it applicable for modelling economy, especially in the DSGE model case. As a result, in our paper, we analyze the necessary conditions for the use of continuum approach and point to the possible ambiguity when using some mathematical construction based on the continuum notion. Finally, we would like to show how one can alleviate such ambiguity by using a decimal classification system.

2 Infinity and continuum

As we have already stated in the introduction, the subject of our interest in this paper is the notion continuum in Economics which is used to refer to the number of types of produced commodities, or the number of entities, either consumers or producers, in an economy. This is a very interesting problem whose theoretical foundation is not appropriately discussed in the current leading economic journals. Dynamic stochastic general equilibrium theory which dominates contemporary macroeconomic theory considers a continuum of producers or a continuum of sorts of goods (Gali [5], McCandless [8] and Walsh [9]). A producer and a sort of goods are closely related in the dynamic stochastic general equilibrium theory because the theory assumes that each sort of goods is produced by just one producer. It also assumes that each product has just one price, so the quantities of producers, sorts of goods and prices are the same and therefore, they can be indexed by the same numbers. As a result, it would be enough to deal only with a population of producers as the conclusions drawn from the model for them would be the same as for the whole sorts of goods and prices.

When dealing with a finite population of producers, we use the natural numbers to index them if the number of producers is finite. For example, if we have n agents in an economy, and each of them produces one type of product, we can sufficiently number them by natural numbers from one to n as well as the goods they produce by these numbers. The fact that we have n manufacturers and each of them is assigned to a number from interval of one to n, mathematically we can say that there is a bijection (each agent has just one number and each number corresponds just to one agent) of a set of producers to a set of numbers $\{1, 2, \ldots, n\}$. To put it formally: the cardinality of the set is n. Theoretically, this set can be "extended" to infinity (provided that the assumption that one commodity is produced by one manufacturer is retained). If we index all producers by a whole sequence of natural numbers $\{1, 2, 3, \ldots\}$, we talk about the fact that the set of producers is infinite, but countable (i.e. it is infinite, but it is ordered into a sequence).

If n denotes a natural number, it is possible to use the finite version of Dixit-Stiglitz aggregator [3] to express an aggregate economic quantity. For example, the composite index which aggregates individual production Y_i, $i = 1, \ldots, n$ is expressed by

$$Y = \left[\sum_{1}^{n} Y_i^{1-\frac{1}{\epsilon}} \right]^{\frac{\epsilon}{\epsilon-1}}, \ \epsilon > 1.$$

If we express the aggregate production this way, it is absolutely right and does require any other additional conditions. We can also release the assumption of finite n. Then we have an infinitely countable set of agents, one can used the infinite version of Dixit-Stiglitz aggregator:

$$Y = \left[\sum_{1}^{\infty} Y_i^{1-\frac{1}{\epsilon}} \right]^{\frac{\epsilon}{\epsilon-1}}, \ \epsilon > 1,$$

where Y_i, $i = 1, 2, \ldots$ is a infinite sequence of production indexed by natural numbers. The above composite index of production has an interpretation if the series of production converges. But the assumption of convergence is a little artificial in economic environment i.e. it is not implied by economic requirements.

Now, instead of using a sequence of natural numbers to index the whole population of producers, we use a continuum to describe them. Imagine an economy, in which we have a number of producers that are indexed by all real numbers from interval [0, 1]. In this case, we talk about a continuum of agents in an economy. It means that we assign each agent or each sort of commodity to a real number from

interval [0, 1] and each number from this interval corresponds to any agent. One then may ask what is the cardinality of a continuum in this case. The answer is the cardinality of a continuum is also infinity, but somehow "denser". Mathematically, we say that we map the continuum of producers (or equivalently the set of producers) on interval [0, 1].

But such mapping between continuum of producers and digits in interval $[0, 1]$ may bring up many problems. Let's assume that the production of an individual producer is $Y(i)$, $i \in [0, 1]$. The composite index of production Y is given by Dixit-Stiglitz aggregator for continuum economics, i.e. by integral:

$$Y = \left[\int_0^1 Y^{1-\frac{1}{\epsilon}}(i)\mathrm{d}i\right]^{\frac{\epsilon}{\epsilon-1}}, \quad \epsilon > 1.$$

Function $Y(i)$ on domain $[0, 1]$ can be arbitrary so it may not meet the integrability conditions in the sense of Lebesgue theory of integral. It is a very serious problem. For the aggregation purpose, New Keynesian theory uses mathematical structures (integrals), whose existence is not warranted. To solve this problem, it is necessary to find sufficient conditions for the integrability of function $Y(i)$ by giving them some economic interpretation.

For this purpose, it is necessary to begin with the discussion about the domain of function $Y(i)$. As the integral is used to obtain aggregate economic quantity Y, we have to study the measure on their domain $[0, 1]$. The most common approach is to use Borel measure defined on σ-algebra of Borel sets, which is generated by a system of all open subsets (subintervals (α, β), $\alpha, \beta \in [0, 1]$). In Economics, economic interpretation of each open subintervals (α, β) is very important because we solve the question if the agents or products indexed by close numbers, have also similar features. This question remains unanswered by New Keynesian Economics theorists.

3 The notion of continuum in economics

In the previous section, we have shown that we can index a producer in a population of them either by a natural number from a sequence of natural numbers or by a real number from interval [0,1]. One then may ask what makes economists-theorists to use the continuum of commodities, consumers, producers etc to cope with the whole population of agents in an economy. The introduction of a continuum of producers into economic modelling is clearly inspired by its use in Physics. In Physics, this approach is supported by human intuition.

Consider a meter-long metal stick with a varying density. The density differs from layer to layer, but it changes continuously which results from the physical properties of the rod. This variation in density causes no problem for calculating the weight of the rod as it is composed of a continuum of infinitely thin cross section slices putting side by side consecutively to make the final density. The total weight of the rod is the integral on the interval [0, 1] of the mass density function. In this case, each cross-section layer has its own size which corresponds to a real number from the interval [0, 1]. The density of infinitely thin slice is a function defined on interval [0, 1]. The rod weight is the integral of the density function from 0 to 1, which is the sum of infinite number of layers. As density function is continuous due to its physical nature, the weight must be the integral from zero to one of this density function.

Unlike in Physics, there is a danger that using a continuum of products or manufacturers to describe a population of agents in an economy when modeling may lead to other conclusions than those when the countable or infinitely countable sets are considered as Aumann [1] has already pointed to. The reason might be as follows. Let us recall how the concept of continuum is made. In an economy, each commodity is assigned to an index, which is a real number from interval $[0, 1]$ and vice versa. In this case, we talk about a continuum of types of products. Indexing commodity in this way can raise a serious problems when we introduce a function on domain $[0, 1]$ which for example captures the size of production of individual goods. It can be very obscure because without any mathematical restricting assumptions, and thus may not be integrable. If we want to explain economic meaning of function $Y(i)$, $i \in [0, 1]$, we must also explain economic characteristics of open subinterval (α, β). These subintervals should contain indices of individual product of similar feature. The structure of New Keynesian model and equilibrium conditions imply the fact that products with similar characteristics are produced at a similar amount. This implicitly guarantees the continuity of $Y(i)$ in a sufficiently small neighbourhood which means that integrability is secured.

The fact that theoreticians of New Keynesian Economics have no economic interpretation of open sets of $[0,1]$ matters more than we would expect. Individual real numbers thus indicate only indices of commodities without any ties to the surrounding numbers, which are, however, other commodities indices. In this case, when there is no other structure on the interval, it is possible to arbitrarily renumber types of products. Renumbering the types of products may cause the ambiguity of aggregate quantities displayed by composite indices as we will show in the following section.

4 Ambiguity of aggregation in continuum economics

In this section, we assume that the production of individual agents is given by a Lebesgue integrable function. Even after accepting this assumption which is not explained by New Keynesian Economics, there still remains the danger of ambiguity of aggregation. This indeterminacy results from the possibility to renumber the agents in the environment of a continuum of agents. We will show that one cannot obtain uniquely determined results for an aggregate quantity in the environment of a continuum-of-agents economics because macroeconomic aggregates depends on how the agents are indexed.

Let's denote the production of individual agents function as $Y(t)$. The domain of this function is interval $[0,1]$ and co-domain $R^+ \equiv x \in \Re : x \geq 0$. Let us assume (it is not too realistic assumption in this field) that $Y(i)$ is a Lebesgue integrable function. To express the aggregate of production Y with Dixit-Stiglitz aggregator

$$Y = \left[\int_0^1 Y^{1-\frac{1}{\epsilon}}(i)\mathrm{d}i\right]^{\frac{\epsilon}{\epsilon-1}} \tag{1}$$

where elasticity $\epsilon > 1$ is used. Now, we will argue by an example that the aggregator does not express the production of arbitrary economy unambiguously. Let the production of different goods numbered by real numbers $[0, 0.25)$ has the same size 4. Goods are indexed by numbers in interval $[0.25, 1]$ and they are produced in the same quantity 9. The graphical illustration is on Fig 1. Let us assume that elasticity of substitution for the economy $\epsilon = 2$. Integral (1) gets a form as follows:

$$Y = \left[\int_0^1 Y^{\frac{1}{2}}(i)\mathrm{d}i\right]^2. \tag{2}$$

Function Y in our example is a piece-wise constant function, so after replacing from our example we get:

$$Y = \left[\int_0^{0.25} Y^{\frac{1}{2}}(i)\mathrm{d}i + \int_{0.25}^1 Y^{\frac{1}{2}}(i)\mathrm{d}i\right]^2 = \left[0.25 \times 4^{\frac{1}{2}} + 0.75 \times 9^{\frac{1}{2}}\right]^2 = \left[\frac{1}{2} + \frac{9}{4}\right]^2 = \frac{121}{16}.$$

If we had a finite number of agents and goods and renumbered them, nothing would change as we know that addition is commutative. Renumbering in the case of continuum agents and goods corresponds to bijection from the interval $[0,1]$ on $[0,1]$. Let us consider function $j = \sqrt{i}$ which is a bijection from $[0,1]$ on $[0,1]$. Graphical illustration of function $Y(j)$ after renumbering is displayed on Fig 2. When apply aggregation with help of Dixit-Stiglitz aggregator and using data from our example we obtain:

$$Y = \left[\int_0^{0.5} Y^{\frac{1}{2}}(j)\mathrm{d}j + \int_{0.25}^1 Y^{\frac{1}{2}}(j)\mathrm{d}j\right]^2 = \left[0.5 \times 4^{\frac{1}{2}} + 0.5 \times 9^{\frac{1}{2}}\right]^2 = \left[1 + \frac{3}{2}\right]^2 = \frac{25}{4}.$$

The obtained result shows that renumbering goods has impact on the value of production index, which is a serious problem if we want to aggregate all individual productions by this approach. The total production of economy depends on a purely administrative act which is the mapping of a continuum of goods on interval [0, 1].

5 Decimal classification and following problems

As a solution to the problem we have described above we suggest a decimal classification of products. We label all products in consideration by real numbers from interval $[0,1]$. It means all numbers begin by digit 0 except the last one 1 which can be left out without losing any generality. The continuum of goods would be divided into ten categories $0.0, 0.1, ..., 0.9$ according to their qualitative features. Each category has its own ten subcategories and so on. By doing so, we obtain a topology on the set of all types of

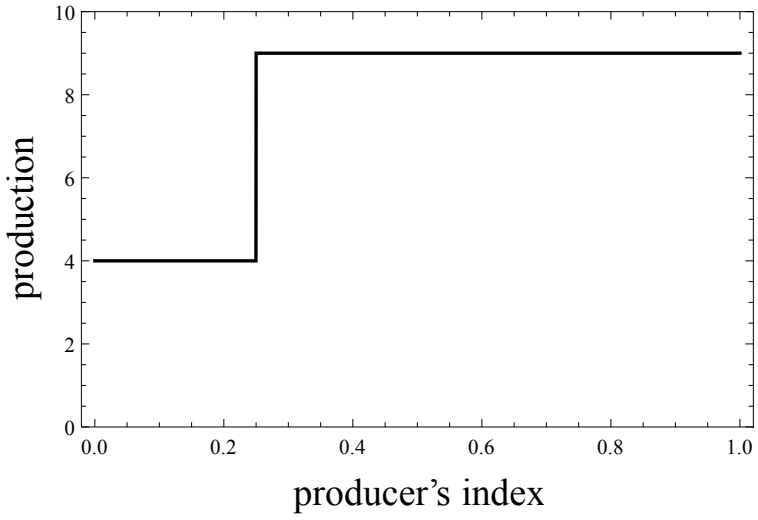

Figure 1 The aggregation of production with its original indices

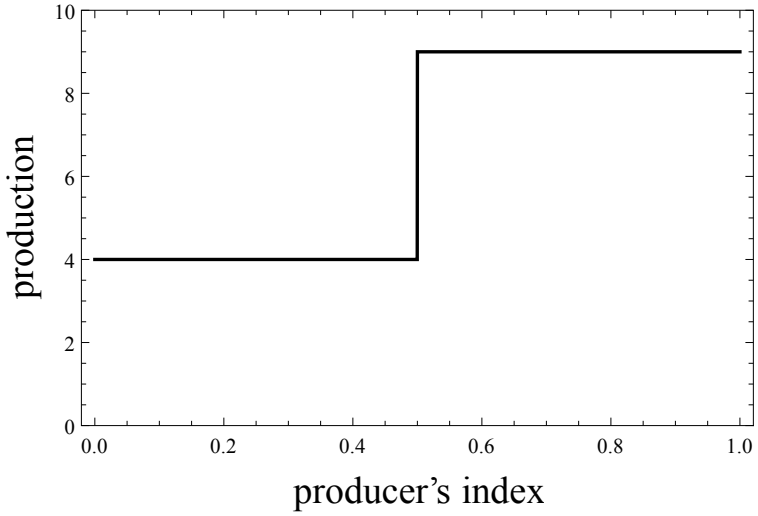

Figure 2 The aggregation of production after renumbering its indices

products, commodities close to each other according to their qualitative characteristics. The closeness of their corresponding indices assigned by a real number from interval (0, 1) in terms of their distance in this interval will depend only on their qualitative characteristics.

Assuming that the re-optimisation of prices (Gali [5]) is applied for goods of close characteristics, we can expect close goods will have a similar size of demand and in equilibrium, they would have an approximately equal size of production. If the size of production of somehow closely related products are not very different, then function of $Y(i)$ could be considered as a continuous one. Then the aggregation across all goods by integration would be feasible. In order to make the case more general, we may allow the existence of a finite number of points of discontinuity, or even an infinite but countable number of points of discontinuity. Such sets have a zero measure, i.e. the integral remains the same.

This idea of decimal classification we have suggested above should not be understood as an implementable proposal. It should be taken as a theoretical concept for how to solve the problem of ambiguity of aggregation across a continuum of agents or goods. As one may have noticed that decimal classification of product cannot entirely eliminate the problem of ambiguity, but it could bring some order into the indexation of different products by real numbers. It is obvious that the values of aggregates will depend on how decimal classification will be applied, i.e how categories subcategories and so on will be defined. But decimal classification introduces a systemic way of ordering of products of different sorts on the interval $[0, 1]$.

This process of sorting cannot continue to infinity, and it must be stopped after a finite number of

steps. The result of the process is finite system of disjunct semi-closed intervals which cover the interval $[0, 1]$. Elements of this intervals belong to agents or types of products with similar characteristics. This method, in our opinion, is suitable for indexing agents and goods and it also allows for aggregating across a whole population to get macro-economics aggregates for the need in New Keynesian Economics.

The other way to solve this problem is to abandon the assumption of a continuum of goods and agents. Instead of it, agents or goods are indexed by natural number and their numbers are finite. In this case, the act of indexing each good is a very simple operation and renumbering is possible without any danger as we have already shown when aggregating an economic quantity, one can use the finite Dixit-Stiglitz aggregator. However, it is necessary to find new methods and limit their use to finite versions to obtain the needed aggregates.

6 Conclusion

Economists have borrowed the notion continuum from Mathematics to capture a whole population of agents in an economy. Each agent in the continuum is arbitrarily indexed by a real number taken from a closed interval $[0, 1]$. To obtain an aggregate quantity, one just needs to integrate a corresponding quantity of an agent across the whole population. Though we do not a priori reject such kind of simplifying assumption for its unrealistic nature as others may, in our work we try to point to some problems this kind of simplification may get into.

In our opinion, the use of notion continuum in economics should be supported by a clear economic interpretation. It is important because it can give us a clue on the integrability of the quantity to be aggregated, no matter whether it is the output, the prices, the consumption and the individual labour force of different skills in a continuum. We have to say that modern New-Keynesian theorists does not bother to tackle this question.

Further, even if the existence of the integral is secured, the ambiguity of the aggregation across a whole continuum would still remain as it has been shown in the article. To solve this problem, we propose a solution which lies in the creation of relatively fixed structure. In this structure each individual quantity is numbered by a decimal classification as it has been shown. Instead of a continuum, there is an idea of finite number of agents, goods or whatever. To our knowledge, this approach not only eliminates the ambiguity connected with the aggregation across a continuum, but it also softens the unrealistic nature of the notion continuum in Economics.

Acknowledgements

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Impact of changes in fares and demand to the tariff zones design

Michal Kohání[1]

Abstract. When designing a tariff system, there are several approaches how to design it. One of them is to divide the region into tariff zones where the price of traveling depends on the number of traveled zones. There are many important factors with major influence on the solution of the problem such as the number of created zones or the price of traveling - fares. Changing of fares also affect the demand for traveling. Higher fare often occurs a drop in demand and number of carried passengers and it also affect the total income from passengers. We will introduce a mathematical model of the tariff zones design problem based on counting zones that contains parameters modeling the impact of fare changes on the demand for traveling. We will solve this model using a universal optimization tool Xpress. We will make a computational study on test data from selected region and compare the impact of obtained solutions to the tariff zones design and its parameters.

Keywords: tariff planning, tariff zones design, IP solver, demand, fares

JEL Classification: C44, C61
AMS Classification: 90C08

1 Introduction

One of important tasks in the designing of integrated transport system is the design of the tariff subsystem. There are several approaches of designing the tariff and one of them is dividing the region into the tariff zones. The price of traveling in such system is determined by the number of traveled tariff zones by the passenger [1][8].

There are many important factors with major influence on the solution of the problem such as the number of created zones or the price of traveling - fares. Changing of fares can change the demand for traveling. Higher fare often occur a drop in demand and number of carried passengers and it also affect the total income from passengers. However, if the price will be lower, it can motivate travelers to use public transport and may lead to the increasing of the number of transported passengers. Commonly used models don't take into account these factors.

In this article we analyze the problem of the tariff system design. In the third chapter we introduce a mathematical model of the tariff zones design problem based on counting zones. In the next chapter we introduce the adapted mathematical model that contains parameters modeling the impact of price changes on the demand for traveling. We make a computational study on test data from selected region and compare the impact of obtained solutions to the tariff zones design and its parameters. We perform numerical experiments using a universal optimization tool Xpress.

2 Tariff zones design problem

In the field of public transport there are several ways of designing the fare and prices for travelling. As was mentioned in [2], [3] and [8], frequently used way is a *distance tariff* system. In this system the price for a trip is calculated according to the length of the trip. We need to have the distance between origin and destination station to calculate the price for the trip. This type of tariff is often used in regional transportation and intercity lines. The *unit tariff* is the opposite of the distance tariff. In this system the prices for all trips are equal and are independent on the distance. The unit tariff is frequently used in city public transport, but it is not very suitable for regional public transportation, especially in large regions, as was mentioned in [8].

Third type of tariff system is a *zone tariff* system. This system combines advantages of both unit and distance tariff. In this type of tariff system the whole region is divided into smaller sub-regions (*the tariff zones*). The price for a trip depends on the starting and the ending zone of the trip. Inside one zone the unit tariff applies. For a trip that involves passing several zones, there are two ways of determining the price, as in [2] and [8].

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If the price is given arbitrarily for each pair of zones, we call it *a zone tariff with arbitrary prices*. An example for this tariff system is the Zilina Regional Integrated Transport System in Slovak Republic, see the tariff prices matrix in the Table 1 and Figure 1. Prices here depend on the pair of origin and destination zones and the number of travelled zones is not important, because prices are given for all pairs of zones separately.

Prices of basic fare single tickets in ŽRIDS (In Euro)

	To zone 1	**To zone 2**	**To zone 3**	**To zone 4**	**To zone 5**	**To zone 6**	**To zone 7**
From zone 1	x	x	x	1.00	1.00	1.30	1.50
From zone 2	x	x	x	0.90	0.90	1.20	1.40
From zone 3	x	x	x	0.45	0.45	0.75	0.95
From zone 4	1.00	0.90	0.45	0.45	0.45	0.70	0.90
From zone 5	1.00	0.90	0.45	0.45	0.45	0.50	0.70
From zone 6	1.30	1.20	0.75	0.70	0.50	0.45	0.45
From zone 7	1.50	1.40	0.95	0.90	0.70	0.45	0.45

Table 1 Price matrix in Zilina Regional Integrated Transport System *(www.dpmz.sk)*

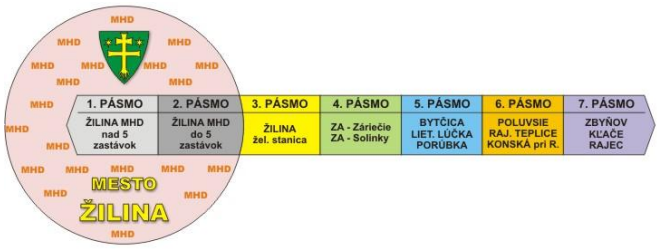

Figure 1 Zone tariff system in Zilina region *(www.dpmz.sk)*

The counting zone tariff system is the second variant of a zone tariff system. The price of trip in this system is calculated according to the number of crossed zones. The prices depend on the origin and the destination zone of the trip, but trips passing the same number of zones must have the same price. The example of a counting zone tariff system in Bratislava region in Slovakia is in the Table 2 and Figure 2.

Number of zones	**Price**
1 regional zones	14,90 €
2 regional zones	25,90 €
3 regional zones	31,90 €
4 regional zones	42,90 €
5 regional zones	52,90 €
6 regional zones	63,90 €
7 regional zones	73,90 €
8 regional zones	84,90 €
9 regional zones	95,90 €
10 regional zones	105,90 €
All zones	114,90 €

Table 2 Prices for travelling for 30 days period – Integrated system of Bratislava region *(www.bid.sk)*

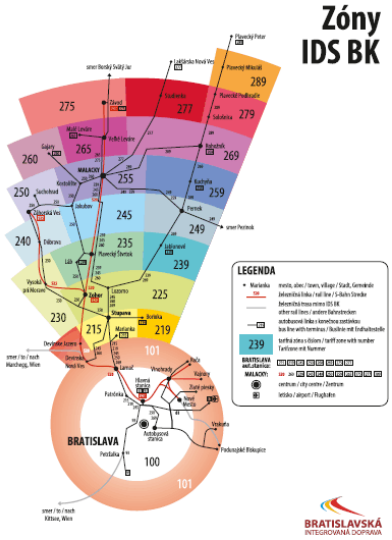

Figure 2 Zone tariff system in Bratislava region *(www.bid.sk)*

According to [2] and [8], zone tariff system is gaining popularity in integrated transport system across Europe due to its simplicity. When a public transportation company or the regional transportation office wants to change its tariff system to a zone tariff, it has to design the zones and to fix the new fares. The goal is to design in the way that resulting system must be accepted by the customers and does not decrease the income of the company. The goal then is to design the zones in that the new and the old price for most of the trips are as close as possible. This means that neither the public transportation company nor the customers will have major disadvantages when changing the current tariff system to a zone tariff. Another goal can be to design fair zones. This approach was described in [6].

3 Mathematical model of the tariff zones design problem

Let all stations in the network of public transport constitute the set I. The station i and j from set I are connected by the edge $(i,j) \in V$, if there is direct connection by public transport line between these two stations. Symbol V denotes the set of edges. The distance between stations i and j is denoted as d_{ij}. For each pair of stations i and j is c_{ij} the current price of travelling between these two stations. The number of passengers between stations i and j is b_{ij} (OD matrix).

If we want to calculate new price of the trip between nodes i and j in the counting zones tariff system, we need to calculate the number of zones crossed on this trip. The calculation of the number of crossed zones can be easily replaced by the calculation of crossed zone borders as was used in [2]. We assume that the node can be assigned only to one zone and then the border between zones is on the node. We will introduce the binary variable w_{rs} for each existing edge $(r, s) \in V$, which is equal to 1 if stations r and s are in different zones and is equal to 0 otherwise. For calculation of the number of crossed borders we need to determine the used path for traveling between stations i and j. We introduce a_{ij}^{rs}, where the used paths will be observed. a_{ij}^{rs} is equal to 1 if the edge (r,s) will be used for travelling between i and j and 0 otherwise.

As was mentioned in [3] and [4], there are two possibilities, how to set a new price for travelling. In the first case *(Price_uni)*, the unit price f for travelling in one zone will be established. This means that for the calculation of the new price it will be necessary to count the number of travelled zones and multiplied it by the unit price per one zone. In the second case *(Price_dif)* two different unit prices will be determined – price f_1 per travelling in the first zone and unit price f_2 for travelling in each additional zone. The final price will be calculated as a sum of the basic price for the first zone and number of other travelled zones multiplied by the unit price for additional zones. According to the numerical experiments in [4], we will use the *Price_dif* setting of prices for travelling. The current or fair price between stations i and j is denoted by c_{ij}. New price determined by the number of crossed zones will be calculated according to *Price_dif* setting as follows (1):

$$n_{ij} = f_1 + \sum_{(r,s)\in V} f_2 a_{ij}^{rs} w_{rs} \tag{1}$$

We introduce binary variables y_i, which represent the "fictional" centre of the zone. Variable y_i is equal to 1 if there is a centre of the zone in node *i* and 0 otherwise. For each pair of stations *i* and *j* we introduce variables z_{ij}. Variable z_{ij} is equal to 1 if the station *j* is assigned to the zone with centre in the node *i* and 0 otherwise. We expect to create at most *p* tariff zones.

According to [3] and [8] there are two different objective functions. First one will be the maximal deviation between the current or fair price and new price determined by the number of crossed zones for all passengers between *i* and *j*. Second one will be the average deviation between current and new price for all passengers. According to the previous research in [3] and advices of experts in [8], in this article we will use the average deviation between current and new price as a criterion in objective function.

The mathematical model *(Mod_simple)* can be written in the form:

$$Minimize\ dev_{avg} = \frac{\sum_{i\in I}\sum_{j\in J}\left|c_{ij}-n_{ij}\right|b_{ij}}{\sum_{i\in I}\sum_{j\in J}b_{ij}} \tag{2}$$

$$subject\ to \sum_{i\in I} z_{ij} = 1,\ for\ j \in I \tag{3}$$

$$z_{ij} \le y_i,\ for\ i, j \in I \tag{4}$$

$$z_{ij} - z_{ik} \le w_{jk},\ for\ i \in I, (j,k) \in V \tag{5}$$

$$\sum_{i\in I} y_i \le p \tag{6}$$

$$z_{ij} \in \{0,1\},\ for\ i,\ j \in I \tag{7}$$

$$y_i \in \{0,1\},\ for\ i \in I \tag{8}$$

$$w_{ij} \in \{0,1\},\ for\ (i,\ j) \in V \tag{9}$$

Conditions (3) ensure that each station will be assigned exactly to one zone. Conditions (4) ensure that the station *j* will be assigned only to the existing centre of the zone. Conditions (5) are coupling between variables for allocation of the station to the zone and the variables for determining the zone border on the edge *(j,k).* Condition (6) ensures that we will create the most *p* tariff zones.

This model will be solved using IP solver with exact methods, so we will obtain exact solution of the problem. Because the objective function (2) in this model is not a linear function, we need to modify this objective function to linear form. This adjustment was shown in [4].

4 Changes in fares and demand

According to [5] and [7], important factors which has major influence on demand in public transport are price, speed of travel, information about transport, safety, comfort and availability. In our model we can influence only the price for travelling. As was mentioned earlier, the goal is to design the zones in that the new and the old price for most of the trips are as close as possible. This means that neither the public transportation company nor the customers will have major disadvantages when changing the current tariff system to a zone tariff. The increase in prices can cause reducing the number of transported passengers. On the contrary, a price reduction may affect the attractiveness of transport and hence the slight increase in the demand and the number of transported passengers. Nevertheless, objective function (2) does not include this possibility so we will have to reformulate it.

We introduce new variables u_{ij}, $v_{ij.}$ Variables u_{ij} represent the calculated prices for travelling in case that new price is lower than current and variables v_{ij} represent the calculated prices for travelling in opposite case. We can also introduce the coefficients *d* and *e*. Coefficient *d* represents the percentage increase in number of passengers in the case of lover new prices, coefficient *e* represents the percentage decrease in number of passengers in the case of higher new prices.

Then we can reformulate mathematical model *(Mod_demand)* to the form:

$$Minimize\ dev_{\max} = \frac{\sum_{i\in I}\sum_{j\in J}(1+d).u_{ij}b_{ij} + \sum_{i\in I}\sum_{j\in J}(1-e).v_{ij}b_{ij}}{\sum_{i\in I}\sum_{j\in J}b_{ij}} \tag{10}$$

$$subject\ to\ (3)-(9)$$

$$c_{ij}-n_{ij} = u_{ij} - v_{ij}, for\ i,j \in I \tag{11}$$

$$u_{ij} \geq 0,\ for\ i,\ j \in I \tag{12}$$

$$v_{ij} \geq 0,\ for\ i,\ j \in I \tag{13}$$

5 Numerical experiments

The goal of numerical experiments was to compare the assignment of stations to the created zones and the average deviation in prices between both models. Numerical experiments were performed on the data of the Zvolen County in Slovak Republic. The stations in the network are represented by the 51 municipalities. Current prices were calculated according to real prices depending on the distance for travelling by regional buses. The OD matrix was estimated using the gravity model as in [4], where the number of passengers between nodes *i* and *j* is calculated as follows:

$$\frac{b_i b_j}{d_{ij}}$$

Parameter b_i represents the number of inhabitants in the node *i*. We assume that the decrease in number of passengers in the case of higher new prices will be higher than increase of passengers with lower new prices. Parameter *d* was set to 0.02 and the parameter *e* had value 0.05. Parameter f_1 was set to 0.7 and parameter f_2 to the value 0.4 and according to the best values obtained in [3]. We perform experiments for 8 different values of parameter *p*, denoted as p_{max} in Tables. To perform the computation we used the general optimization software tool FICO XPRESS 7.3 [9]. The experiments were performed on a personal computer equipped with Intel Core 2 Duo E6850 with parameters 3 GHz and 3.5 GB RAM. In the Table 3 and Table 4 there are results for both models *(Mod_simple)* and *(Mod_demand)* used in this study. In the Table 3 there are compared differences in assignments of stations (column *Difference*) and the number of created zones (column *Zones*). Total deviations between current prices and new prices for all passengers are calculated in the Table 4.

p_{max}	*Zones*		*Difference*
	Mod_simple	*Mod_demand*	
4	4	4	0
6	6	6	0
8	8	8	0
10	10	10	1
13	12	13	3
16	16	16	4
20	20	20	6
25	24	24	4

Table 3 Number of created zones and assignments

p_{max}	*Mod_simple*	*Mod_demand*
4	9780.9	9924.67
6	9655.7	9756.47
8	9628.7	9690.89
10	9230.2	9128.16
13	8455.2	8267.24
16	8266.1	8244.51
20	7859.5	7922.73
25	7691	7746.72

Table 4 Total deviations between current prices and new prices in the system

6 Conclusion

The results of numerical experiments show that the differences between two models (*Mod_simple, Mod_demand*) are very small. From the Table 3 we can see that the number of created zones is equal in almost all cases. There are some differences in assignment of stations to tariff zones but these differences in assignment are in the stations with smaller population, as was found by comparing the results. In the Table 4 we can see that the results obtained by *Mod_simple* model are better than in the model which includes the influence of demand on the number of transported passengers. To proof the above results, we want perform more experiments with real data of selected region in the future.

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Portfolio efficiency with respect to higher order stochastic dominance criteria

Miloš Kopa[1]

Abstract. This paper deals with portfolio efficiency testing with respect to higher order stochastic dominance criteria. Firstly, we test the second-order stochastic dominance (SSD) portfolio efficiency, that allows for risk averse decision makers, for portfolios from a regular grid. The grid is created from one-month US Treasury bill (a riskless asset) and 10 representative value-weighted active benchmark stock portfolios. They are formed, and annually rebalanced, based on individual stocks' market capitalization of equity, each representing a decile of the cross-section of stocks in a given year. Secondly, we limit our attention to SSD efficient portfolios and we test their efficiency with respect to higher order stochastic dominance. Finally, we compare all results with mean-variance efficiency frontier. In all our models we assume discrete distribution of monthly returns and no short sales.

Keywords: stochastic dominance, portfolio efficiency, mean-variance efficiency

JEL classification: D81, G11
AMS classification: 91B16, 91B30

1 Introduction

The theory of decision making under risk is one of the most appealing issues within financial mathematics. It is based on the basic economical principles, however it also exploits optimization techniques and statistical tools. Mathematical formulations of decision making problems under risk lead to stochastic programming models which are searching for the optimal solution with respect to a chosen objective (criterion) and feasibility constraints. In financial applications, they turn out to so called portfolio selection problems. These problems basically capture two fundamental principles: non-satiation and risk attitude. While the non-satiation axiom is easy to implement and generally accepted in economics and finance, the risk attitude can be understood or expressed in various ways.

The first portfolio selection problem was introduced by Markowitz (1952). The model jointly focuses on maximizing expected return and minimizing variance of the portfolio, where variance serves as a measure of risk. Consider N risky assets with returns modeled by random vector $\boldsymbol{\varrho}$. Let $\boldsymbol{\lambda} \in \mathbb{R}^M$ be a vector of weights determining the way how the initial wealth is invested. Following Markowitz (1952), short sales are allowed, that is, the set of all feasible portfolios Λ is given by condition: $\mathbf{1}'\boldsymbol{\lambda} = 1$. Consequently, the return and variance of portfolio (with weights) $\boldsymbol{\lambda}$ are $E(\boldsymbol{\varrho}'\boldsymbol{\lambda})$ and $\mathrm{var}(\boldsymbol{\varrho}'\boldsymbol{\lambda})$. Hence, Markowitz portfolio selection problem may be formulated as follows:

$$\max_{\boldsymbol{\lambda}\in\Lambda} E(\boldsymbol{\varrho}'\boldsymbol{\lambda}) - \vartheta \mathrm{var}(\boldsymbol{\varrho}'\boldsymbol{\lambda})$$

where $\vartheta \geq 0$ is a risk aversion parameter.

Alternatively, one may model the risk attitude using utility functions introduced in von Neumann and Morgenstern (1944). Contrary to the bi-objective mean-variance models, application of utility function leads to classical nonlinear programming problem - to maximize expected utility of the final wealth. Unfortunately, identifying the particular utility function of the decision maker is usually very difficult. When the utility function is not precisely known, one can consider at least some suitable, economically meaningful classes of utility functions, for example the class of all nondecreasing and concave utility

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functions U_2. In this case, the optimal solution of the maximizing expected utility problem can not be exactly found. However, one can at least compare two portfolios. If one of them gives higher or equal expected utility of final wealth (or returns) than the other one for all considered utility functions then a relation of stochastic dominance (SD) between them exists. The notion of stochastic dominance was introduced in statistics more than 50 years ago and it was firstly applied to economics and finance in Quirk and Saposnik (1962), Hadar and Russell (1969) and Hanoch and Levy (1969). Moreover, considering the particular class of utility functions one can test whether a given portfolio is efficient or not. Until recently, practical applications of stochastic dominance relations in portfolio analysis were hampered by the absence of tractable algorithms to deal with diversification across multiple assets. The pairwise comparisons criteria have only a limited use when looking for efficient portfolios with respect to SD relations. In the last decade, under assumptions of discrete distribution of returns, Post (2003), Kuosmanen (2004) and Kopa and Chovanec (2008) developed linear programming tests to analyze if a portfolio is efficient with respect to the second-order stochastic dominance (nondecreasing concave utility functions) relative to all possible portfolios formed from the considered assets. Later on, Kopa and Post (2013) introduced a general SSD efficiency test which is a generalization of all previous SSD efficiency tests what makes the theory of SSD portfolio efficiency more compact. Moreover, they presented also a new SSD portfolio efficiency test that is much less computationally demanding and applicable even for thousands of scenarios. Although the test does not identify a dominating portfolio, the computational tractability allows for further sensitivity, robustness or bootstrap techniques. The tests are formulated for the case when no short sales are allowed, that is :

$$\Lambda = \{\boldsymbol{\lambda} \in R^M | \mathbf{1}'\boldsymbol{\lambda} = 1, \ \lambda_m \geq 0, \ m = 1, 2, \ldots, M\} \tag{1}$$

however, one can easily modify the results for any nonempty bounded polytope set Λ. If only non-satiation of decision maker is assumed, that is, the set of all non-decreasing utility functions (U_1) is considered, one must distinguish between two concepts of efficiency: " FSD admissibility" and " FSD optimality". A mixed-integer programming test for FSD admissibility was proposed by Kuosmanen (2004) while the FSD optimality test was derived in Kopa and Post (2009). Recently, Post and Kopa (2013) introduced a general linear programming portfolio efficiency test for any Nth-order of stochastic dominance when $N \geq 2$. The notion of portfolio efficiency with respect to stochastic dominance criteria is related also to the other efficiency approaches, mainly, it is related to DEA models and also to mean-risk models, see Branda and Kopa (2012, 2013) for more details.

In this paper we test the efficiency of billions of portfolios using the Post and Kopa (2013) NSD portfolio efficiency test for $N = 2, 3, 4$. Contrary to Post and Kopa (2013) we test the efficiency of all portfolios from the regular grid created from 11 base assets. Moreover, we consider the shorter time period for our data than Post and Kopa (2013) in order to increase the importance of during-crises observations (scenarios). We compare the sets of efficient portfolios for $N = 2, 3, 4$ and mean-variance efficiency frontier among each others.

The rest of this paper is structured as follows. Section 2 introduces the basic notation and formulates definitions of stochastic dominance criteria. Moreover, it recalls the NSD portfolio efficiency test of Post and Kopa (2013). Section 3 presents our empirical study. It introduces the considered data from Kenneth French library, summarizes its basic descriptive statistics and presents the results. Finally, Section 4 concludes the paper with several ideas for future theoretical improvements based on bootstrap and robustness techniques.

2 Nth-order stochastic dominance

Until very recently, stochastic dominance portfolio efficiency tests (Post 2003, Kuosmanen 2004, Kopa and Chovanec 2008, Lizyayev 2012, Dupačová and Kopa 2012) were derived only with respect to class U_1 or U_2. However, in general, one can consider any subset of U_1. Let $U_N \subset U_1$ be the set of N times differentiable utility functions such that: $(-1)^k u^{(k)} \leq 0$ for all $k = 1, 2, ..., N$. As the limiting case, $U_\infty \subset U_1$ is the set of infinitely differentiable utility functions with alternating signs of the derivatives. These functions are also called completely monotonic utility functions. More details about completely monotonic utility functions can be found in Whitmore (1989) and references therein.

For each set $U_N \subset U_1$, $N = 1, 2, ...$ we define weak Nth-order stochastic dominance relation between portfolios $\boldsymbol{\lambda}$ and $\boldsymbol{\tau}$ as follows: $\boldsymbol{\lambda} \succeq_{NSD} \boldsymbol{\tau}$ if $Eu(\boldsymbol{\varrho}'\boldsymbol{\lambda}) - Eu(\boldsymbol{\varrho}'\boldsymbol{\tau}) \geq 0$ for every utility function $u \in U_N$.

If strict inequality holds true for at least one $u \in U_N$ then the relation is called Nth-order stochastic dominance.

Let $F^{(1)}_{\boldsymbol{\varrho}'\boldsymbol{\lambda}}$ be the cumulative distribution function of returns of portfolio $\boldsymbol{\lambda}$. Considering random variables with bounded support (a, b), Levy (2006) presents a necessary and sufficient condition for weak Nth-order stochastic dominance relation in terms of N times cumulated distribution functions

$$F^{(N)}_{\boldsymbol{\varrho}'\boldsymbol{\lambda}}(t) = \int_{-\infty}^{t} F^{(N-1)}_{\boldsymbol{\varrho}'\boldsymbol{\lambda}}(x)dx$$

as follows: $\boldsymbol{\lambda} \succeq_{NSD} \boldsymbol{\tau}$ if and only if $F^{(k)}_{\boldsymbol{\varrho}'\boldsymbol{\lambda}}(b) \leq F^{(k)}_{\boldsymbol{\varrho}'\boldsymbol{\tau}}(b)$, for all $k = 2, 3, ..., N-1$ and $F^{(N)}_{\boldsymbol{\varrho}'\boldsymbol{\lambda}}(x) \leq F^{(N)}_{\boldsymbol{\varrho}'\boldsymbol{\tau}}(x)$ for all $x \in \langle a, b\rangle$. Again, in the case of NSD relation, at least one strict inequality is required.

The infinite-order stochastic dominance relation (ISD) is defined as a limit case of NSD when $N \to \infty$: $\boldsymbol{\lambda} \succeq_{ISD} \boldsymbol{\tau}$ if $Eu(\boldsymbol{\varrho}'\boldsymbol{\lambda}) - Eu(\boldsymbol{\varrho}'\boldsymbol{\tau}) \geq 0$ for every utility function $u \in U_\infty$ and $\boldsymbol{\lambda} \succ_{ISD} \boldsymbol{\tau}$ if $Eu(\boldsymbol{\varrho}'\boldsymbol{\lambda}) - Eu(\boldsymbol{\varrho}'\boldsymbol{\tau}) \geq 0$ for every utility function $u \in U_\infty$ and strict inequality holds for at least one $u \in U_\infty$.

For infinite-order of stochastic dominance, Whitmore (1989) derived a necessary and sufficient condition based on the Bernstein Theorem: $\boldsymbol{\lambda} \succ_{ISD} \boldsymbol{\tau}$ if and only if

$$E\left(e^{-a\boldsymbol{\varrho}'\boldsymbol{\lambda}} - e^{-a\boldsymbol{\varrho}'\boldsymbol{\tau}}\right) \leq 0 \quad \text{for all} \quad a \geq 0.$$

The recursive nature of the definition of the N-times cumulated distribution function makes it clear that (weak) Nth-order stochastic dominance implies (weak) $\overline{N}$th-order stochastic dominance for $\overline{N} > N$. Moreover, any Nth-order stochastic dominance implies infinite-order stochastic dominance. Thistle (1993) also proved that if $\boldsymbol{\lambda} \succ_{ISD} \boldsymbol{\tau}$ then $\boldsymbol{\lambda} \succ_{NSD} \boldsymbol{\tau}$ for some finite N.

The general definition of NSD efficiency for $N \geq 2$ can be seen as an extension of SSD efficiency and, following Post and Kopa (2013), we formulate it in the "NSD optimality" form.

Definition 1. A given portfolio $\boldsymbol{\tau}$ is NSD efficient ($N \geq 2$), if there exists at least one utility function $u \in U_N$ such that $Eu(\boldsymbol{\varrho}'\boldsymbol{\tau}) - Eu(\boldsymbol{\varrho}'\boldsymbol{\lambda}) \geq 0$ for all $\boldsymbol{\lambda} \in \Lambda$ with strict inequality for at least one $\boldsymbol{\lambda} \in \Lambda$.

In the rest of the paper we will assume a discrete probability distribution of returns $\boldsymbol{\varrho}$ that is given by scenarios $\mathbf{x}^t$, $t = 1, 2, ..., T$ that occurs with probability p_t. The scenarios are collected in the following matrix:

$$X = \begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \vdots \\ \mathbf{x}^T \end{pmatrix}$$

where $\mathbf{x}^t = (x^t_1, x^t_2, \ldots, x^t_M)$ is the t-th row of matrix X.

Moreover, let $(X\boldsymbol{\tau})^{[k]}$ be the k-th smallest element among $(\mathbf{x}^1\boldsymbol{\tau}), (\mathbf{x}^2\boldsymbol{\tau}), ..., (\mathbf{x}^T\boldsymbol{\tau})$, that is, $(X\boldsymbol{\tau})^{[1]} \leq (X\boldsymbol{\tau})^{[2]} \leq \ldots \leq (X\boldsymbol{\tau})^{[T]}$.

Post and Kopa (2013) consider the following reformulation of Definition 1.

Proposition 1. *An evaluated portfolio $\boldsymbol{\tau} \in \Lambda$ is efficient in terms of N-th order stochastic dominance, $N \geq 2$ relative to all feasible portfolios $\boldsymbol{\lambda} \in \Lambda$ if it is an optimal solution of the maximizing expected utility problem for some admissible utility function $u \in U_N$, that is:*

$$\sum_{t=1}^{T} p_t u(\mathbf{x}^t\boldsymbol{\tau}) \geq \sum_{t=1}^{T} p_t u(\mathbf{x}^t\boldsymbol{\lambda}) \quad \forall \boldsymbol{\lambda} \in \Lambda \Longleftrightarrow$$

$$\sum_{t=1}^{T} p_t u'(\mathbf{x}^t\boldsymbol{\tau})(\mathbf{x}^t\boldsymbol{\tau} - x^t_j) \geq 0, \quad j = 1, 2, ..., M.$$

The proposition follows from the Karush-Kuhn-Tucker first-order condition for selecting the optimal combination of assets: $\max_{\boldsymbol{\lambda}\in\Lambda} \sum_{t=1}^{T} p_t u\left(\mathbf{x}^t\boldsymbol{\lambda}\right)$. An admissible function $u \in U_N$ is now chosen as any function from U_N which is not constant on the relevant interval $\left\langle (X\boldsymbol{\tau})^{[1]}, (X\boldsymbol{\tau})^{[T]} \right\rangle$ in order to avoid some

trivial cases. This reformulation was first introduced by Post (2003) for SSD ($N = 2$) and applies also for higher-order criteria ($N > 2$), but it does not apply for FSD ($N = 1$). Kopa and Post (2009) presented a different utility-based formulation for this case. Finally, Post and Kopa (2013) derived the following necessary and sufficient condition for NSD efficiency of a given portfolio.

Theorem 2. *Let*

$$\begin{aligned}
\theta^*(\boldsymbol{\tau}) = \min_{\beta_n, \gamma_k, \theta} \ & \theta \qquad (2)\\
\text{s.t.} \quad & \left(\mathbf{x}^t\boldsymbol{\tau} - x_j^t\right)p_r \sum_{t=1}^{T}\left(\sum_{n=1}^{N-2} n\beta_n\left(\mathbf{x}^t\boldsymbol{\tau} - \mathbf{x}^T\boldsymbol{\tau}\right)^{n-1} + \right.\\
& \left.(N-1)\sum_{k=t}^{T}\gamma_k\left(\mathbf{x}^t\boldsymbol{\tau} - \mathbf{x}^k\boldsymbol{\tau}\right)^{N-2}\right) + \theta \geq 0,\ j = 1, \cdots, M\\
& (-1)^n\beta_n \leq 0,\ n = 1, \ldots, N-2\\
& (-1)^{N-1}\gamma_k \leq 0,\ k = 1, 2, \ldots, T\\
& \sum_{t=1}^{T}\left(\sum_{n=1}^{N-2} n\beta_n\left(\mathbf{x}^t\boldsymbol{\tau} - \mathbf{x}^T\boldsymbol{\tau}\right)^{n-1} + (N-1)\sum_{k=t}^{T}\gamma_k\left(\mathbf{x}^t\boldsymbol{\tau} - \mathbf{x}^k\boldsymbol{\tau}\right)^{N-2}\right)p_r = 1.
\end{aligned}$$

A portfolio $\boldsymbol{\tau}$ is NSD efficient if and only if $\theta^*(\boldsymbol{\tau})$ given by (2) is equal to zero.

Solving problem (2) one tries to find an "optimal" utility function, that is, a function from U_N that maximizes the expected utility of returns in $\boldsymbol{\tau}$. If $\theta^*(\boldsymbol{\tau}) = 0$ then this optimal function can be reconstructed from the optimal solution of (2), see Post and Kopa (2013) for more details.

3 Empirical application

We follow Post and Kopa (2013) in taking data from the Kenneth French library. We consider a standard set of 10 active benchmark stock portfolios as the base assets. They are formed, and annually rebalanced, based on individual stocks market capitalization of equity, each representing a decile of the cross-section of stocks in a given year. Furthermore, we include the 1-month US Treasury bill as a riskless asset. We use data on monthly returns from January 2002 to December 2011 (120 months). We assume that all scenarios are equiprobable, that is, $p_t = 1/120$, $t = 1, ..., 120$.

Portfolios	Mean	St. deviation	Min	Max	Skewness	Kurtosis
1st decile	0.70	6.28	-20.46	15.05	-0.45	0.51
2nd decile	0.47	6.71	-20.10	17.61	-0.28	0.30
3rd decile	0.68	6.45	-22.07	16.21	-0.41	0.72
4th decile	0.61	6.18	-20.62	15.83	-0.39	0.80
5th decile	0.63	6.13	-20.37	17.52	-0.27	0.67
6th decile	0.74	5.50	-19.42	15.41	-0.40	0.98
7th decile	0.60	5.53	-22.45	14.32	-0.64	1.88
8th decile	0.53	5.45	-20.86	13.69	-0.59	1.42
9th decile	0.48	5.02	-21.44	14.67	-0.75	2.73
10th decile	0.12	4.38	-14.94	10.18	-0.48	0.88

Table 1 Basic data descriptive statistics

First, we construct a regular grid $\{0, 0.01, 0.02..., 0.99, 1\}^{11} \cap \Lambda$ from the base assets. Then we test SSD efficiency of each portfolio from the grid using (2) for $N = 2$. We identify 748 SSD efficient portfolios, all of them are composed from the 1st decile portfolio, the 6th decile portfolio and the riskless asset. Finally, we test NSD efficiency for $N = 3, 4$ and mean-variance efficiency of all these 748 portfolios. We find 101 of these portfolios, that are NSD efficient for $N = 3$. Perhaps surprisingly, these 101 portfolios are NSD efficient for $N = 4$ and mean-variance efficient as well. These portfolios are linear convex combinations of the 6th decile portfolio (the asset with the highest mean return) and the riskless asset. The results are summarized at Figure 1.

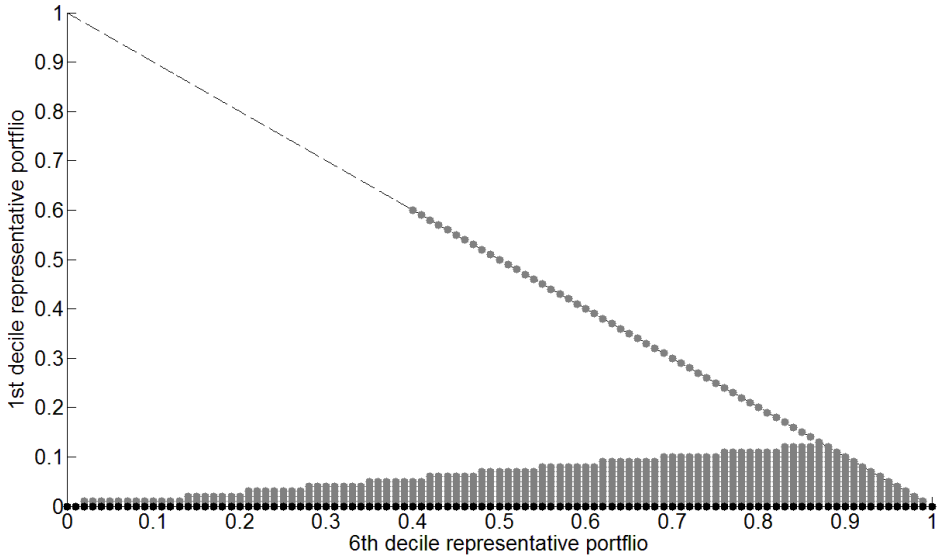

Figure 1 The figure captures a comparison of various efficiency criteria. The grey area and black dots express the set of SSD efficient portfolios. The black dots correspond to portfolios that are NSD efficient for $N = 3, 4$ and mean-variance efficient, too. Each of these portfolios consists only of the 6th decile portfolio and the riskless asset.

4 Conclusions

In this paper a portfolio efficiency with respect to various criteria was analyzed. Considering 11 base assets a regular grid was constructed. Using the general NSD efficiency test of Post and Kopa (2013), the second-order, third-order and fourth-order stochastic dominance efficiency of portfolios from the grid was tested. Then it was compared to the mean-variance efficiency frontier. It was found that 748 portfolios satisfy the conditions of SSD efficiency, but only 101 of them were also identified as third-order stochastic dominance efficient. Moreover, these 101 portfolios were classified as the fourth-order efficient and mean-variance efficient as well.

For future research, this study can be improved in various ways. For example, longer historical data can be used. In addition, one can consider the portfolio efficiency in a more robust way as it was done in Kopa (2010, 2012) or Dupačová and Kopa (2012, 2013) for the first and the second-order stochastic dominance, using contamination techniques and the worst-case approach. Alternatively, one can compare the results also with the first-order stochastic dominance efficiency set using the Kopa and Post (2009) test. Unfortunately, all these improvements would lead to more computationally demanding efficiency tests what requires much better hardware equipment than is currently available.

Acknowledgements

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Verification of the Linder Hypothesis with using the Gravity Model of International Trade

Jana Kovářová[1]

Abstract. New role of the demand in the trade theories is associated with the theory of overlapping demand by Swedish economist, Staffan Linder. The Linder hypothesis postulates that countries with similar level of income will trade more than countries with dissimilar level of income. On the contrary, the supply side Heckscher-Ohlin hypothesis implies that countries with dissimilar level of income will trade more than countries with similar level of income. The Linder hypothesis is associated with high level of the intra-industry trade (similarity of demands) and the Heckscher-Ohlin theory with high level of the inter-industry trade (different factor endowments). The subject of this paper is an empirical investigation of the Linder hypothesis with using the gravity model of international trade in manufactured goods in the case of the Czech economy with other EU member countries. Motivations for this research are two-fold. First, new information is provided on verification of the Linder hypothesis of international trade in manufactured goods between the Czech Republic and other EU member countries. Second is the application of the gravity model of international trade with using panel data in the static and dynamic form.

Keywords: Linder hypothesis, Heckscher-Ohlin hypothesis, Czech Republic, gravity model of international trade, panel data.

JEL Classification: C23 ; C50 ; F12 ; F14 ; O52.
AMS Classification: 62G05.

1 Introduction

Many trade theories emphasized the supply side in explaining trade direction. For example the Heckscher-Ohlin (HO) model which was developed by two Swedish economists, Eli Heckscher and Bertil Ohlin. In this model comparative advantage is determined by differences in endowments of factors across countries. Other supply oriented model of international trade is the Ricardian model which postulates that comparative advantage is determined by differences in technology. In 1961, Steffan Linder presented alternative view in explaining the international trade direction which is based on the similarity of the demands. Linder hypothesis of international trade departs from the neoclassical theories where supply conditions are the most important factor of trade [4]. Linder proposed that trade patterns in manufacturing are dependent on the similarity of preference among nations. He argued that countries have similar demands for manufacturing with others that have similar per capita income levels and got favorable empirical results [12].

Since Linder did not present a formal model to test his hypothesis[2], many others have tried different approaches to test it. The primary tool for testing Linder hypothesis was rank correlation analysis used for example by Sailors, et al. [22] or Greytak and McHugh [11]. These studies were heavily criticized, however, for their failure to employ regression analysis, a technique that could have controlled for the effects of distance on trade intensities [6]. However, the empirical evidence based upon the regression analysis of the Linder hypothesis was rather mixed. For example, Kennedy and McHugh [14, 15] and Qureshi et al. [20] tested the theory in terms of changes in propensities to trade against changes in income differences between two points in time to control for distance and found no support for the Linder model. Bergstrand [3] made the theoretical link between the Linder model and the gravity model specification, but found little or no evidence to support Linder's hypothesis. On the other hand, Thursby and Thursby [23] tested the Linder hypothesis using gravity model specification and they

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[2] Linder proposed a tabular form of test comparing the import propensities between similar-income countries with those for dissimilar countries to test his hypothesis. He arrayed the countries vertically and horizontally in order of per capita incomes and tried to find a tendency toward bunching of the high bilateral import propensities around the main diagonal, which would be in favor of the thesis [17].

found overwhelming support for this hypothesis. Arnon and Weinblatt [1] use a simple gravity equation model to analyze the nature of bilateral trade flows among and between the 35 developed and less developed countries. Their study provides empirical evidence supporting the comprehensive validity of the Linder's hypothesis for all country groups with both high and low income. Chow et al. [13] tested the Linder hypothesis for trade between the four original tiger economies (Hong Kong, Singapore, South Korea and Taiwan) and their major OECD markets during the 1965-1990 period. They conclude that the Linder hypothesis may provide a relatively good explanation of trade for countries above some per capita income threshold and for trade in differentiated products. McPherson et al. [18] support the Linder hypothesis for five East African developing countries (Ethiopia, Kenya, Rwanda, Sudan and Uganda). Their study is based on panel data analysis in the 1984-1992 periods. The results imply that these countries trade more intensively with others who have similar per capita income levels, as predicted Linder. Fillat-Castejon and Serrano-sanz [9] tested the verification of the Linder hypothesis in Spain using data for the 1959-1986 period find international demand to be an important determinant of trade and suggest that foreign markets can be considered an extension of the domestic market. Leitao and Faustino [16] examine the features and determinants of Portuguese intra-industry trade in the years 1995-2003 and find that differences in income levels have a positive impact on intra-industry trade. Bohman and Nilsson [5] suggested a new approach to assess the Linder hypothesis, incorporating the distribution of income within a country. They develop two different variables to capture the similarity in demand structures between two trading partners and the size of the market, constituting a market overlap. The results imply that similarity in structure of demand acts as a catalyst of trade flows between countries. This similarity is more important for the differentiated goods than homogenous goods.

The subject of this paper is an empirical investigation of the Linder hypothesis with using the gravity model of international trade in goods in the case of the Czech economy with other EU member countries. Motivations for this research are two-fold. First, new information is provided on verification of the Linder hypothesis of international trade in goods between the Czech Republic and other EU member countries. Second is the application of the gravity model of international trade with using panel data in the static and dynamic form.

2 Czech trade with EU member states

The Czech economy has changed in terms of the extent and the share of the economy openness during the transformation process. The changes in the territorial structure of the Czech foreign trade were clearly associated with the changes in the commodity structure. Requirements to succeed in the markets of industrialized countries were restructuring of the industrial production, investments in new equipment, purchases of new technologies and human resources. At the beginning of the 90th 20th century, Czech Republic exported mainly raw materials and supplies (e.g. metallurgical and steel products). The second half of the 90th years means positive turn. In the Czech exports begin dominate products with higher added value, i.e. mainly machinery and transport equipment [10]. Table 1 shows the Czech commodity structure with the EU in 2011.

Commodity group	Export		Import	
	Value (mill. CZK)	Share (%)	Value (mill. CZK)	Share (%)
Food and live animals	83 988	3.5	103 486	6.1
Beverages and tobacco	15 875	0.7	13 756	0.8
Crude materials, inedible, except fuels	72 697	3.1	39 273	2.3
Mineral fuels, lubricants and related materials	100 046	4.2	103 124	6.0
Animal and vegetable oils, fats and waxes	4 812	0.2	5 482	0.3
Chemicals and related products, n.e.s.	140 130	5.9	246 854	14.5
Manufactured goods classified chiefly by material	425 381	17.9	397 496	23.3
Machinery and transport equipment	1 271 143	53.4	636 475	37.3
Miscellaneous manufactured articles	262 958	11.0	158 742	9.3
Commodities and transactions not classified elsewhere in the SITC	3 102	0.1	2 923	0.2

Table 1 Czech commodity structure of the foreign trade with the European Union in 2011. [7]

In 2011, more than half of exports (53.4 %) to the EU countries were machinery and transport equipment (SITC 7), which represented the absolute level 1 271 143 mill. CZK. The most exported items were following: the road vehicles (412 437 million CZK, 17.33 % share of total SITC 7 exports), the electrical machinery, apparatus and appliances (225 290 million CZK, 9.47 %), the office machines and automatic data processing (199 552 million CZK, 8.38 %), the telecommunications and sound recording and reproducing (148 277 million CZK, 6.23 %) and the machinery and equipment generally used in the industry (144 488 million CZK, 6.07 %). The aggregate share of these five items in the total Czech exports to the European Union was 47.48 % [7].

Also in imports were the largest commodity group machinery and transport equipment with 37.3 % share of the total Czech imports (636 475 million CZK). The biggest amount of imported manufactured goods was in these categories: the electrical machinery, apparatus and appliances (171 758 million CZK, 10.06 %), the road vehicles (168 058 million CZK, 9.84 %) and the equipment generally used in industry (99 622 million CZK, 5.83 %) [7].

The Czech commodity structure with EU member countries points to a very high level of the intra-industry trade. The Linder hypothesis, which is based on a higher level of intra-industry trade[3], is assumed.

3 Empirical model

The gravity model represents a simple empirical tool for analyzing bilateral trade flows between geographical entities. This model originates from the Newtonian physics notion. The gravity model for trade is analogous: the trade flow between two countries (integration units, trading block etc.) is proportional to the product of each country's "economic mass", generally measured by GDP (national income) and inversely proportional to the distance between the countries respective "economic centers of gravity", generally their capitals [21]. The original gravity equation proposed by Tinbergen is as follows [2]:

$$Trade_{ij} = \alpha \cdot \frac{GDP_i \cdot GDP_j}{Dis_{ij}}, \tag{1}$$

where $Trade_{ij}$ represents the value of the bilateral trade exchange between country i and j, GDP_i and GDP_j are country i and j's respective national incomes, Dis_{ij} is the geographical distance between the countries and α is an intercept. The original equation (1) is transformed into the linear form using natural logarithm:

$$\ln(Trade_{ij}) = \alpha + \beta_1 \ln(GDP_i \cdot GDP_j) - \beta_2 \ln(Dis_{ij}) + \varepsilon_{ij}, \tag{2}$$

where $Trade_{ij}$ is the value of the bilateral trade exchange between country i and j, GDP_i and GDP_j are country i and j's respective national incomes, Dis_{ij} is the geographical distance between the countries, α is an intercept, β_1 and β_2 are the regression coefficients, ε_{ij} is the random component.

The gravity model in the form of the equation (2) is typically used to analyze cross-sectional data. Often other variables are included such as population size to reflect the possible economies of scale or set of dummy variables incorporating institutional characteristic such as economic integration arrangements, infrastructure or cultural and linguistic proximity, historical links, exchange rate, foreign direct investments, common language, common border, inland location, cultural differences and various barriers to trade.

In this study a modified gravity model with panel data structure is used. For verification the Linder hypothesis is included a variable which measures the degree of similarity between the per capita income levels of the selected countries. The estimated equation is in form:

$$\ln(Trade_{ijt}) = \alpha + \beta_1 \ln(GDP_{it} \cdot GDP_{jt}) - \beta_2 \ln(Dis_{ijt}) + \beta_3 Linder_{ijt} + \varepsilon_{ij}, \tag{3}$$

where $Trade_{ijt}$ is the value of the bilateral trade exchange between country i and j in time t, GDP_{it} and GDP_{jt} are country i and j's respective national incomes in time t, Dis_{itj} is the geographical distance between the countries in time t, $Linder_{ijt}$ is the natural logarithm of the absolute value of per capita GDP differential between country i and j in time t, α is an intercept, $\beta_1, \beta_2, \beta_3$ are the regression coefficients, ε_{ij} is the random component.

4 Data and methodology

The Linder hypothesis is tested in the case of Czech Republic (CR) international trade in manufactured goods with other member states of the European Union (EU) in the 1995-2011 periods. The dependent variable $Trade_{ijt}$

[3] The Czech share of the manufacturing intra-industry trade in the total manufacturing was 66.3 % between the 1992-1995 periods and increased by 11.1 % in the 1996-2000 periods [19].

expresses the sum of the export and import of manufactured goods of the Czech Republic with other member states of the European Union. The independent variable $GDP_{it} \cdot GDP_{jt}$ is the economic mass of the Czech Republic and one of the EU member state. According to Linder's theory, GDP of exporting country express the supply and GDP of importing country reflects the demand. It is expected a positive estimated coefficient β_1 because the bigger GDP implies the bigger trade between countries. The independent variable Dis_{itj} is the physical distance between the capital cities of the Czech Republic and one of the EU member state[4]. Longer distance means higher transportation cots thus decreasing the potential for bilateral trade flows. For this reason, it is expected a negative estimated coefficient β_2. The independent variable $Linder_{ijt}$ is the most important variable in this model. It measures the difference between per capita GDP of the Czech Republic and one of the EU member state. According to Linder hypothesis, countries with similar level of income will trade more than countries with dissimilar level of income. Therefore when two countries GDP per capita are close to each other, this income similarity variable would tend to be smaller and the dependent variable $Trade_{ijt}$ would tend to be bigger. It is expected a negative estimated coefficient β_3 to confirm the Linder hypothesis. On the contrary, the positive sign of this estimated coefficient implies that country traded in accordance of the Heckscher-Ohlin trade theory. All data were obtained from the United Nation Conference on Trade and Development (UNCTAD) in current USD. Table 2 presents the summary statistics for the variables included in the gravity model.

Variable	Observation	Mean	Std. Dev.	Min	Max	Skewness	Kurtosis
$ln(Trade_{ijt})$	442	20.41568	1.987548	15.36927	25.11724	-0.225306	2.606046
$ln(GDP_{it} \cdot GDP_{jt})$	442	50.94896	1.971154	46.78412	55.05977	-0.096112	2.259857
$ln(Dis_{itj})$	442	6.757133	0.595602	5.540871	7.721792	-0.430465	2.457037
$Linder_{ijt}$	442	9.186802	1.003350	6.567194	11.48195	-0.340264	2.396873

Table 2 Summary statistics. Self-elaboration with using EViews 7.

The gravity model is estimated with panel data structure in the static and dynamic form. The Least Squares Method (LMS) with fixed time effects is used for static estimate. The Generalized Method of Moments (GMM) is used for dynamic estimate. The lagged values of the dependent variable $Trade_{ijt}$ were chosen as instruments. The advantage of the dynamic estimate is that it takes into account the existence of the long-term contracts.

5 Empirical results

The estimation results for static form are presented in table 3.

Variable	Observation	Coefficient	t-Statistic	Probability
$ln(GDP_{it} \cdot GDP_{jt})$	442	0.847261	51.31611	0.0000
$ln(Dis_{itj})$	442	-1.563049	-38.71572	0.0000
$Linder_{ijt}$	442	-0.287804	-9.978777	0.0000

Table 3 Static results. Self-elaboration with using EViews 7.

The overall performance of this model is very good, with typically high R-squared (0.94) [5]. All explanatory variables are significant, indicating that the gravity model is appropriate in explaining CR international trade in manufactured goods with other EU member states. Presented results confirm that the bilateral trade in manufactured goods of the Czech Republic with other EU member states positively depends on their economic mass measured by GDP and negatively depends on the distances measured by distances between their capitals. The estimated coefficient can be interpreted as follows. When the variable $GDP_{it} \cdot GDP_{jt}$ increased by 1 %, the bilateral trade exchange of the manufactured goods increased by 0.84 % in the 1995-2011 period. On the contrary, when the distance increased by 1 %, the bilateral trade exchange of the manufactured goods decreased by 1.56 % in the same period. The estimated coefficient β_3 has a negative sign. This result confirms the validity of the Linder hypothesis which implies that the Czech Republic traded in manufactured goods with other EU member states in accordance with the postulates of the Linder theory. The estimated coefficient β_3 can be interpreted as follows, when the difference of the income measured by absolute difference in GDP per capita increased by 1 %, the bilateral trade exchange of the manufactured goods decreased by 0,28 %.

The estimation results for dynamic form are presented in table 4.

[4] The distance between capital cities is measured using the great circle formula, which takes into account the longitude and latitude of the capitals.

[5] The adjusted R-squared is 0.93 for static form.

Variable	Observation	Coefficient	t-Statistic	Probability
$ln(GDP_{it} \cdot GDP_{jt})$	442	0.830990	13.87685	0.0000
$ln(Dis_{itj})$	442	-2.180653	-21.07554	0.0000
$Linder_{ijt}$	442	-0.268055	-5.064925	0.0000

Table 4 Dynamic results. Self-elaboration with using EViews 7.

The dynamic results are very similar to the static results. The overall performance of this model is very good too, with high R-squared (0.91) [6]. The estimated coefficients β_1 and β_3 have almost identical value in the dynamic form as well as in the static form. The higher value of the estimated coefficient β_2 indicates greater importance of the transportation costs in the dynamic form.

Conclusion

Linder's trade theory for the first time suggested that in some areas of foreign trade, concretely trade in manufactured goods, it is not only supply but also demand that should be taken into account when explaining bilateral trade flows between countries. Linder suggested that his theory was applicable only to trade in differentiated manufactured goods in which consumer tastes and scale economies were deemed to be particularly important. Linder was pursued that trade in raw material or agricultural products can be adequately explained by the traditional theory with its emphasis on the supply of productive factors, including climate and natural resources. Linder's trade theory also provides an explanation for intra-industry trade, an important phenomenon in international trade, which depends on economies of scale and implies imperfectly competitive markets [8].

The validity of the Linder hypothesis, in the case of the Czech economy with other EU member countries in the 1995-2001 periods, was confirmed using the gravity model of international trade. The model is estimated with panel data structure in the static and dynamic form. The Least Squares Method with fixed time effects is used for static estimate. The Generalized Method of Moments is used for dynamic estimate. The lagged values of the dependent variable $Trade_{ijt}$ were chosen as instruments. The advantage of the dynamic estimate is that it takes into account the existence of the long-term contracts. The results of both estimations are similar with exception of the transaction costs which depends on trade distance. The sensitivity of the dependent variable $Trade_{ijt}$ on the distance is higher in the dynamic specification of the model.

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Application of structural models in Credit Risk

Petr Královič[1]

Abstract. In last years (especially after the financial crisis in 2008), there is a great emphasis on the correct state of the credit risk. The most commonly used method is probably the portfolio model CreditMetrics. Very sophisticated methods, based on the options theory, especially on Merton's model (1974), are structural models.
The paper focuses on the application of the structural models in credit risk. Specially, a KMV model will be used at a few Czech firms.
First there will be described the main approach of Merton models with details of KMV model. In the application part there is calculated the default probability, using this model. In conclusion, the results are evaluated.

Keywords: KMV model, credit risk, options, probability of default.

JEL Classification: G33
AMS Classification: 91G40

1 Introduction

Credit risk is the risk that the borrower doesn't pay his liabilities at the time. The credit risk of the firm is based on the computing the probability of the firm's solvency, or insolvency, i.e. default.

There are many rating agencies, such as Standard & Poor's, Moody's, etc., which focus of evaluating and computing this credit risk. One way, how to predict the corporate defaults, is using the structural models. These models apply the Merton's work [4]. In the late 1980's, KMV Corporation (founded by Oldrich Vasicek and Stephen Kealhofer) developed the KMV model, where the distance-to-default was used for the first time.

In present, many of practitioners use KMV model. For all, can be named Moody's, who bought the KMV model and Crosbie and Bohn [2] made some modifications. In academic research, for example, Bharath and Shumway [1] and Crouchy [3] are interested in Merton models.

In this paper there will be used one of the structural models. It is the most commercial successful model, called KMV model. This model determines the default probability, i.e. probability, which the firm is bankrupt.

The aim of this paper is to apply the KMV on real data from Czech economy. The first part of the paper will be devoted to description of the principles and the derivation of default probability in the KMV model. Real data from Prague stock exchange and the accounting data from firms will be applied in the second part. The last part of the paper will be the assessment of the results.

2 The Merton Model

The Merton Model was published in year 1974 for the first time [4]. This model is based on two important assumptions. First is about the debt structure. The firm has just issued a single homogenous class of bond, which has maturity at one time. So, the value of the firm at any time t ($V_{A,t}$) is the sum of equity E_t and debt D_t.

$$V_{A,t} = E_t + D_t \,. \tag{1}$$

The second basic assumption is about dynamic of the firm's assets. It is assumed to follow the geometric Brownian motion,

$$dV_A = \mu V_A dt + \sigma_A V_A dz \,, \tag{2}$$

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where V_A is the total value of the firm, μ is the rate of return on the assets, σ_A is the asset volatility and *dz* is a standard Wiener process.

It also assumes that coupon, and dividend payments and taxes are ignored.

If at time *T* the value of the firm is over debt, the bondholders will receive their debt in full and equity holders will get the rest. If the value of the firm is lower than the debt, the equity holders will receive nothing. Value of the equity can be written as the call option with the strike price equal the face value of the debt.

$$V_E = \max(V_A - D;0). \tag{3}$$

The Black-Scholes model shows that the value of the equity in the case of previous assumptions is:

$$V_E = V_A N(d_1) - e^{rT} DN(d_2), \tag{4}$$

where V_E, V_A is the value of the equity, resp. firm, *r* is the risk-free rate, *D* is the face value of the debt and $N(\cdot)$ is the cumulative standard normal distribution function, d_1 is

$$d_1 = \frac{\ln\left(\frac{V_A}{D}\right) + \left(r + \frac{\sigma_A^2}{2}\right)T}{\sigma\sqrt{T}} \tag{5}$$

and d_2 is

$$d_2 = d_1 - \sigma_A\sqrt{T}\,. \tag{6}$$

2.1 Merton's KMV model

This model has two important equations. The first one is the Black-Scholes equation (4), which shows that the value of the equity is a function of the value of firm's assets. The second one makes the relationship between the volatility of the equity and the volatility of the firm's assets. Using Ito's lemma, it can be shown that

$$\sigma_E = \left(\frac{V_A}{V_E}\right)\frac{\partial V_E}{\partial V_A}\sigma_A. \tag{7}$$

Because $\frac{\partial V_E}{\partial V_A} = N(d_1)$ (4), the volatilities of the equity and firm's assets are related by

$$\sigma_E = \left(\frac{V_A}{V_E}\right)N(d_1)\sigma_A. \tag{8}$$

In practice, the value of the equity can be observed from a stock exchange market. The volatility of the equity can be obtained by estimating the implied volatility from the option price or by using a historical approach. If we have the volatility of firm's assets and the value of the firm, the number of standard deviations that the firm is away from default. The KMV calls this as *distance-to-default (DD)*. With Black-Scholes formula, the distance-to-default is given by:

$$DD = \frac{\ln\left(\frac{V_A}{D}\right) + \left(\mu - \frac{\sigma_A^2}{2}\right)t}{\sigma_A\sqrt{t}}, \tag{9}$$

where μ is expected annual return of the firm's assets.

The probability of default, sometimes in KMV model called *expected default frequency (EDF)* is

$$P = N(-DD). \tag{10}$$

3 Application of KMV model

To apply the KMV model, three Czech firms were chosen. Chosen firms were AAA Auto, TATRA, a.s. and Unipetrol. All of these firms were in trouble in the past and that was the next reason, why there were only three firms.

Data were taken from the stock market, especially Burza cenných papírů Praha, or RM-System, which is over the counter market. For all firms data should be between years 2002 and 2011, but in case of AAA Auto, there are only data since September 2007, because it is the moment, when the company enters the Prague stock exchange. Accounting data were used too. From accounting data (balance sheet) was important face value of debt. The risk-free rate was used by Česká národní banka [7]. Time to maturity is provided as one year.

3.1 Computing model and results

Default probability of KMV model is computed by equation (10). Data were computed for every quarter of examined years. First step was calculating the volatility of the equity. It was calculated by the historical approach. The value of equity is market capitalization of the firm. With equations (4) and (8) were computed volatility and value of firm's assets. As written in the previous text, risk-free rate was used from database of Czech national bank.

A kind of trouble may be the book value of liabilities determining. Because in reality there isn't only one zero-coupon debt, but much more, KMV model is based on the assumption [2], [3], that debt is equivalent as the face value of all short-term liabilities and a half of the face value of all long-term liabilities.

If the volatilities and values of equities and firm's assets and values of liabilities are known, and with the expected time to maturity in one year, it is possible to compute the distance-to-default (9). Expected default frequency (10), which is the probability of default, is now simple to calculate. The graphic results are in the Figures 1-3.

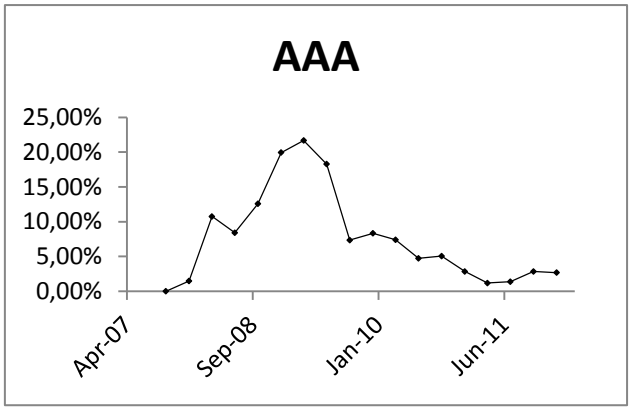

Figure 1 AAA Auto default probability

In Figure 1 is illustrated the default probability of the firm AAA Auto. It can be seen, that firm was in a bad situation between years 2008 and 2009. At this time the EDF was almost 20%. The main reason was the heavy decline of share prices. From about 50 CZK/share to about 10 CZK/share. After that, the share prices rise again, but not so much. At the present, firm announced the plan to retire the Prague stock exchange.

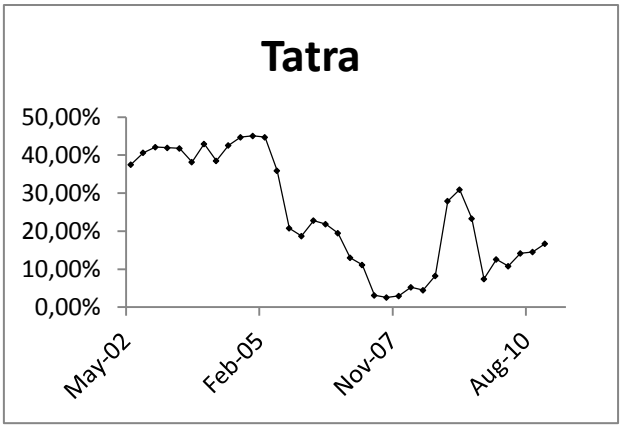

Figure 2 TATRA, a.s. default probability

The Figure 2 illustrated the evolution of the firm TATRA, a.s. In first part (until year 2005) is default probability about 40%. It is huge number. It was due to decreased production and the decline of share prices. For a long time, TATRA, a.s. was under the threat of bankruptcy. But all the time it was resisting. The results are only until year 2010, because later data were unavailable. In last 2 years the share prices still declines and in present are at historical minimum. The last information about this firm is, that it was sold to new subject, which promises a growth of the firm.

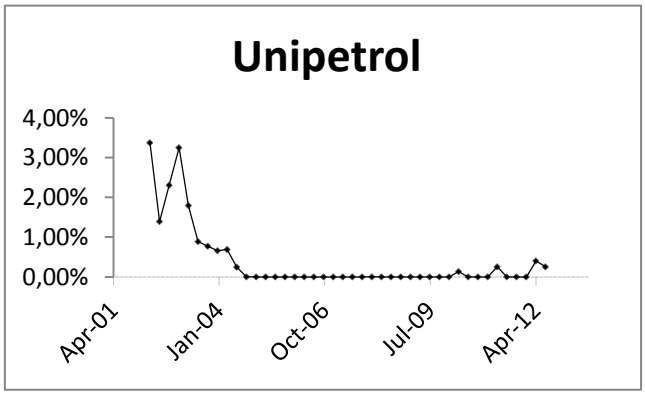

Figure 3 Unipetrol default probability

The Figure 3 illustrates the evolution of default probability of the firm Unipetrol. Until year 2005 was the probability level was between 1 – 4 %. As compared with two other firms, it is very good number. But at this time, Unipetrol was on the brink of bankruptcy. At this time a new Polish company PKN Orlen bought Unipetrol, and the growth started. Between years 2005 and 2010 was default probability basically 0%. A little bit of fluctuations were caused by economy crisis until year 2008.

4 Conclusion

The paper describes the basic idea of the Merton model, especially the Merton's KMV (or Moody's KMV). This model was applied to the real data of three Czech firms. The results, that threat of bankruptcy, which was given by poor economic or market situation, was shown at figures.

The advantage of this model is that it is based on market data, especially share prices. If data are good implemented, this model can be used for prediction the default probability.

There are two disadvantages of the model. First is the assumption of only one simple debt structure. But in real economy it isn't observed. So the results may be distorted, which is probably also the case of this paper. The second disadvantage is that the private companies have difficult valuable equity.

Last, the KMV model, how was described and with public available data, can be used also for prediction the default probability of the Czech public firms.

Acknowledgements

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Analysis of competitiveness of selected countries using AHP models

Jana Kramulová[1], Josef Jablonský[2]

Abstract. A great effort to measure and analyse competitiveness of countries and regions can be seen in recent years. If we skip the discussion about whether a country or region can even be competitive or whether these issues should be left just for companies, several competitiveness indices can be found. Different authorities and researchers use different methods as well as indicators and perform different rankings. There is no single procedure that should be applied always when comparing countries or regions from the point of view of competitiveness. It can be hardly stated, which of the measurement approaches is the most proper.
The IMD World Competitiveness Online Database works with more than 300 indicators aggregating them subsequently in the Overall Competitiveness Ranking that evaluates 59 world countries in the year 2012. We compare in the paper these results with results from AHP methods using the same subgroups of indicators. The overall competitiveness is measured on the basis of AHP model with absolute measurement and application of weight expert estimation. The differences in both methods are being discussed and differences in results are analysed.

Keywords: competitiveness, AHP model with absolute measurement, countries, weight expert estimate.

JEL Classification: C44
AMS Classification: 90B50

1 Introduction

Scientists as well as politicians often compare various indicators of selected countries and/or regions. They compare e.g. economic performance, level of environmental degradation or unemployment. It is in this sense often spoken about competitiveness. Competitiveness is mainly connected with firms and industries and famous Porter's competitiveness diamond (Porter [12]). Successful firms usually concentrate not far from universities and other firms (suppliers etc.) to gain comparative advantages. This also contributes to faster diffusion of innovations. Important is also rivalry within the countries among their regions.

The process of considering company's competitiveness led step by step to assessments of regional competitiveness. Barkley [2] points out the quantity issue of different national and regional indices. Various approaches usually differ in number of indicators as well as in number of compared regions/nations. Barkley [2] also discusses the usefulness of such indices and rankings. Also Czesaný [4] considers the disadvantages of using composite indicators, especially the possibility of conclusions' simplification when inadequate composite indicator construction or problematic selection of partial indicators and determination of their weights. On the other hand Czesaný [4] sees also advantages, e.g. ability to summarize multiple perspectives, which can be among the main reasons, why competitiveness composite indicators are so popular.

Further problem connected with regional competitiveness, apart from composite indicators difficulties, can be seen straight in the regional competitiveness definition. Kitson et al. [11] describe regional competitiveness as an elusive concept which is very complicated to measure. Gardiner et al. [5] align regional competitiveness with success on export market. Their main focus is then on regional productivity. Boschma [3] is unlike Viturka [15] convinced that region can be assessed as a sum of companies located in its area. Therefore according to Boschma [3] "*successful region is just lucky to host more successful firms on average, and it does not have to be assumed that regions (like firms) act for this to happen*". On the other hand Viturka [15] states that costs and revenues (and subsequently competitiveness) of firms are (significantly) influenced by quality of local public administration and its decisions.

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Despite aforementioned obstacles many composite indicators evaluating competitiveness of regions at almost any level (nations, regions, cities etc.) exist. We decided to compare in the paper rankings of IMD World Competitiveness Online Database with results from AHP method while using the same subgroups of indicators. The overall competitiveness is measured on the basis of AHP model with absolute measurement and application of weight expert estimation. The differences in both methods are being discussed as well as differences in results.

The paper is divided into several parts. After the theoretical background and overview of data set the computations and results are presented and discussed. Finally the conclusions are drawn.

2 Theoretical background

2.1 Data

As it was already stated, many competitiveness composite indicators exist. The main ones are according to [15] Global Competitiveness Index [14] published yearly by World Economic Forum (hereinafter WEF) in Global Competitiveness Report or data from World Competitiveness Yearbook (WCY) published yearly by Institute for Management and Development (hereinafter IMD). Among others are e.g. Regional Competitiveness Index calculated for EU (Annoni and Kozovska [1]) or countries (Czech Republic has RCICZ, methodology can be found in [6]), or UK Competitiveness Index (Huggins and Thompson [8], Huggins [7]) and many others. We compared results of both WEF and EU indices with IMD approach resulting in the fact, that all of them give very similar rankings (correlation coefficients were very high). Due to limited space we chose just one of them, i.e. IMD approach. However, the methodology introduced below could be applied to the other indices as well.

We decided to attract our attention to national level and cover into analysis indicators chosen by IMD World Competitiveness Online. We did not want to build up a totally new indicator composed from different "new" indicators but on the contrary we tried to compare results obtained from the database with results computed by using other methods while using the same data. The aim of this paper is to assess if the method selection has an impact on ranking of countries in relation to competitiveness issue.

IMD World Competitiveness Online database [9] contains information on 59 world countries (in the year 2012). According to methodology the overall competitiveness is divided into four groups and these are further again divided as shown in Figure 1. 329 indicators are covered in analysis, as Economic Performance contains 78 indicators, Government Efficiency 70, Business Efficiency 67 and Infrastructure 114.

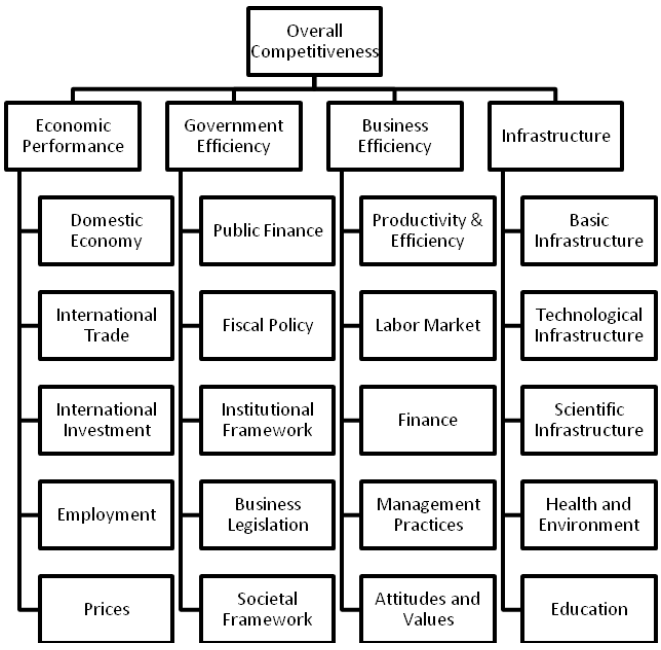

Figure 1 Hierarchy of groups and sub-criteria for evaluation of overall competitiveness according to IMD

2.2 Analytic Hierarchy Process

Competitiveness forms a typical example (another one can be sustainability, see [10]) of multiple criteria decision making (MCDM) problem. An ideal tool for analysis of such MCDM problem is the Analytic Hierarchy

Process (AHP) developed by Saaty [13]. The decision problem is divided into several partly independent hierarchical levels that can be analysed separately. AHP hierarchy usually consists of the following levels:

- overall objective of the decision problem – in our case the overall competitiveness of selected countries,
- criteria used for the evaluation – in our case 4 groups, each of them further divided into 5 sub-criteria (i.e. two hierarchical levels as shown in Figure 1),
- alternatives are usually at the lowest level of the hierarchy – in our case 59 world countries.

The principle of AHP method consists in the division of overall priorities from the topmost level of the hierarchy into the underling levels according to the decision maker's preferences. The decision maker expresses his preferences by comparing the importance of the elements at the given level with respect to an element of the preceding level. In standard AHP models the decision maker's judgments on elements at a given level with respect to an element at the level above are organized into paired comparison matrices. The judgments are estimates of the preference between two elements of the lower level with respect to the element at the level above. Let us denote the paired comparison matrix (1)

$$\mathbf{A} = \left\{ a_{ij} \middle| a_{ji} = \frac{1}{a_{ij}}, a_{ij} > 0, i, j = 1,2,\ldots,k \right\}, \tag{1}$$

where k is the number of elements in the particular comparison set of the lower level. Saaty [13] proposes to use a_{ij} integers in the range 1 through 9 to express preference, where 1 means that the i-th and the j-th element are equally important and 9 means that the i-th element is absolutely more important than the j-th element. The local priorities are derived by solving the following eigenvector problem (2):

$$\begin{aligned} &\mathbf{A}.\mathbf{v} = \lambda_{max}\mathbf{v}, \\ &\sum_{i=1}^{k} v_i = 1, \end{aligned} \tag{2}$$

where λ_{max} is the largest eigenvalue of **A** and **v** is the normalised right eigenvector belonging to λ_{max}. The eigenvector problem (2) is not always easily solvable. That is why several approximation methods can be used. Among them logarithmic least square method is the most popular. In this case the local priorities v_i are derived as the geometric average of all elements in the i-th row of the matrix **A** and then they are normalised:

$$\begin{aligned} v_i' &= \left(\prod_{j=1}^{k} a_{ij} \right)^{\frac{1}{k}}, i = 1,2,\ldots,k, \\ v_i &= \frac{v_i'}{\sum_{i=1}^{k} v_i'}, i = 1,2,\ldots,k. \end{aligned} \tag{3}$$

3 Computations and results

3.1 Our AHP model for competitiveness evaluation

Our AHP model for evaluation of competitiveness of selected countries is presented in Figure 2. As shown below the models consist of 5 levels. The first defines the main objective of the decision problem – evaluation of overall competitiveness. The next level contains four groups of criteria overall competitiveness consists of – we suppose that all of them have the same importance. That is why pairwise comparisons are not used for deriving priorities at this level and all the priorities of four main groups are $p_1 = p_2 = p_3 = p_4 = 0.25$. The priorities of these four main groups are further divided into underling level containing directly sub-criteria (each group is divided into 5 sub-criteria). The priorities of these sub-criteria (i.e. final criteria) are denoted as v_1, v_2, …, v_{20}, where index corresponds to each sub-criteria listed in Figure 1. As in case of the main groups the priorities of sub-criteria are derived by the pairwise comparisons according to their importance with respect to the main groups (preceding level). Of course the condition that the sum of priorities of the elements of an underling level equals to the priority of the element of the preceding level must hold.

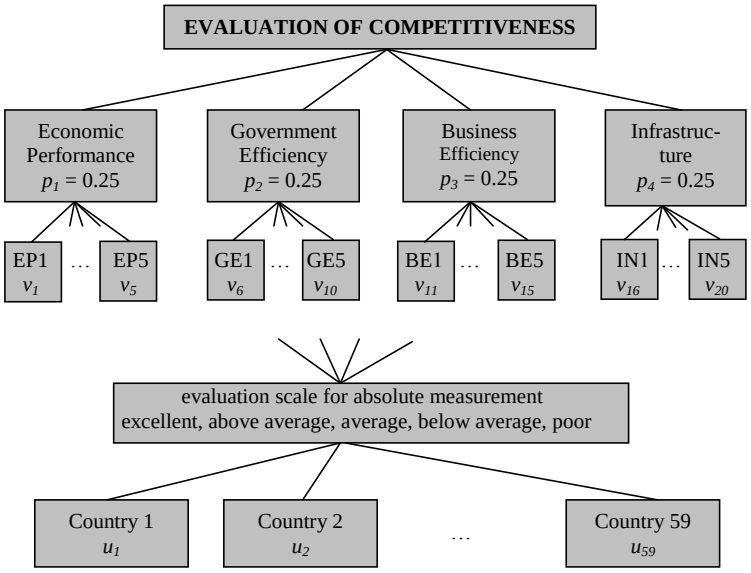

Figure 2 AHP model for evaluation of overall competitiveness

AHP offers two ways of measurements and deriving priorities – relative and absolute. The first uses standard pairwise comparisons. In case of a high number of alternatives or other elements at the level of the hierarchy the relative measurement cannot be used. This is the case of our model, in which the number of alternatives is very high (59 world countries). Subsequently, the number of pairwise comparisons would be vast and it would be technically almost impossible to do them. Therefore absolute measurement was used. It consists in evaluation of elements of the bottom level of the hierarchy (usually alternatives) by their assignment to one of the elements of the evaluation scale. Each of the elements of this scale has its numerical judgment given either directly by decision maker or by pairwise comparisons of elements of the evaluation scale. For simplicity let us suppose that there is the same five-elements evaluation scale with the identical numerical judgment q_i, i = 1,2,…,5 for all criteria of the model. Table 1 presents how the judgments q_i are derived by pairwise comparisons. Consistency index (hereinafter C.I.) should be in all pairwise comparison matrices lower than 0.1, which is fulfilled (C.I. equals 0.059).

Absolute measurement	E	AA	A	BA	P	q_i	C.I.
Excellent	1	3	5	7	9	0.5128	0.059
Above average	1/3	1	3	5	7	0.2615	
Average	1/5	1/3	1	3	5	0.1290	
Below average	1/7	1/5	1/3	1	3	0.0634	
Poor	1/9	1/7	1/5	1/3	1	0.0333	

Table 1 Judgment of elements of the scale with absolute measurement

Let us finally denote the value of the i-th alternative by criterion j as x_{ij}, i=1, 2, …, 59, j =1, 2, …, 20. According to criterion values the alternatives are evaluated by one of the elements of the evaluation scale and the values x_{ij} are replaced by one of the numerical judgment of the evaluation scale – they can be denoted as y_{ij}. Final utility of evaluated alternatives (countries) is simple weighted average of y_{ij} values:

$$u_i = \sum_{j}^{20} v_j y_{ij}, i = 1,2,\ldots,59. \tag{4}$$

The alternatives can be ranked by their utilities u_i as shown in equation (4).

3.2 Computational procedure and results

In the first step of application of the AHP model the weights of all 20 sub-criteria are derived. Table 2 contains pairwise comparison matrix for sub-criteria weights deriving of Economic Performance group. Consistency

index of this matrix equals 0.03, again fulfilling the condition mentioned above. The same procedure (i.e. pair-wise comparison matrix) is applied for remaining three groups – Government Efficiency, Business Efficiency and Infrastructure. All comparisons were object of expert estimation. Due to lack of space Table 3 shows just the resulting weights, not the whole matrices for the other groups. Corresponding consistency indices are again lower than 0.1, namely 0.053 for Government Efficiency, 0.042 for Business Efficiency and 0.026 for Infrastructure.

Economic Performance	DE	IT	II	E	P	v_i	C.I.
Domestic Economy	1	1/3	5	3	1/2	0.1677	0.03
International Trade	3	1	8	6	2	0.4378	
International Investment	1/5	1/8	1	1/3	1/7	0.0372	
Employment	1/3	1/6	3	1	1/5	0.0722	
Prices	2	1/2	7	5	1	0.2852	

Table 2 Weights of sub-criteria of Economic Performance

Government Efficiency	v_i	Business Efficiency	v_i	Infrastructure	v_i
Public Finance	0.0598	**Productivity & Efficiency**	0.4460	**Basic Infrastructure**	0.5098
Fiscal Policy	0.1276	**Labour Market**	0.0816	**Technological Infrastructure**	0.0521
Institutional Framework	0.5079	**Finance**	0.0561	**Scientific Infrastructure**	0.2055
Business Legislation	0.2623	**Management Practices**	0.1690	**Health and Environment**	0.1163
Societal Framework	0.0423	**Attitudes and Values**	0.2472	**Education**	0.1163

Table 3 Weights and consistency indices of sub-criteria of remaining three groups

As all the priorities of four main groups are the same ($p_1 = p_2 = p_3 = p_4 = 0.25$), we decided not to normalise the given weights v_i according to the weight of the preceding node now, but at the very end of computational procedure.

Subsequently, after obtaining weights for all 20 sub-criteria four quintiles were computed to allow matching of the scale with absolute measurement. Afterwards values of four main groups were computed, equation (5) serves as an example for group Economic Performance (5):

$$u_i^{EP} = \sum_{j}^{5} v_j y_{ij}, i = 1,2,\ldots,59. \tag{5}$$

Country	IMD value	Rank IMD	AHP u_i^{EP}	Rank AHP	Rank difference
Germany	77,51	5	0,4028	3	2
Greece	10,79	58	0,0551	59	1
Hong Kong	79,16	4	0,3761	8	4
Hungary	47,31	35	0,1850	33	2

Table 4 Four countries example of rank comparison

As is shown in Table 4, we were able to compare in each main group results of original IMD values and u_i^{EP} values computed by AHP as well as to assess rank differences among both methods in each country. Computing of utility value of four main groups made us possible to finalise the procedure and compute the overall competitiveness ranking of countries u_i by simple average of the four components.

4 Discussion and conclusion

Due to lack of space we will just discuss part of the results obtained. Table 5 shows first 5 and last 5 countries according to both methodologies. We computed the overall competitiveness of all 59 countries, ranked them and then compared with results of IMD methodology. There were slight differences, but the average rank difference equals 3.02, what we see as a very good result. The greatest differences were observed in cases of Bulgaria (the greatest positive shift: +13 positions from 54th to 41st position) and Portugal and USA (the greatest negative shift: –8 positions from 41st to 49th and from 2nd to 10th position). While in case of Bulgaria the main discrepancy between both approaches was in group of Economic Performance (+35), in case of Portugal and USA it was in

group of Infrastructure (–9 and –10). Generally the biggest differences were inside group of Economic Performance, on the other hand the lowest were in case of Infrastructure group. It would be useful to discuss why the results of AHP and IMD approaches vary in different intensities for diverse indicators. Deeper analysis of these relations will be included in the follow-up publications, as here is a limited space for the presentation of comprehensive rankings.

First 5	IMD	AHP	Last 5	IMD	AHP
1	Hong Kong	Hong Kong	**55**	Argentina	Colombia
2	USA	Singapore	**56**	Ukraine	Romania
3	Switzerland	Sweden	**57**	Croatia	Ukraine
4	Singapore	Switzerland	**58**	Greece	Greece
5	Sweden	Germany	**59**	Venezuela	Venezuela

Table 5 First 5 and last 5 countries in both methodologies (IMD and AHP) rankings

As a conclusion, our case study demonstrated that AHP method with absolute measurement represents a simple, fast and useful tool even for a big amount of alternatives. Presented case study also shows that results of this procedure correspond with ranking determined by more sophisticated method.

Acknowledgements

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Explaining the differences in on-trade prices of beer in Europe

Ondřej Krčál [1]

Abstract. The goal of this paper is to explain the differences in prices of beer in the hospitality sector in Europe. Using cross-section data for 27 European countries, the paper finds that a rise in labor costs and a reduction in on-trade per capita consumption and number of beers served per employee in the hospitality sector increases the price of beer and the value added in the hospitality sector. Moreover, these variables explain a large proportion of the variance in prices and the value added in the European data. The empirical results are consistent with the predictions of an extension of the Salop circular city model, which is used as a model of beer consumption in the hospitality sector.

Keywords: price, draft beer, circular city, Salop, Europe.

JEL classification: L10, L81, R30

1 Introduction

On-trade prices of beer differ widely in Europe. For 10 euro, consumers are able to buy approximately 1.5 beers in a pub in Norway, 3 beers in Germany, 5 beers in Slovenia, and more than 10 beers in the Czech Republic. On the other hand, prices of other products, such as groceries, garments or electronics, are relatively similar in all European countries. Even prices of other products sold in the hospitality sector, like coffee or accommodation, tend to differ less.

The aim of this paper is to explain differences in on-trade prices of beer in Europe. Although the question is of interest even for people outside of the economic profession, there is no academic work, to my knowledge, explaining the differences in on-trade prices of beer. Most of the papers studying beer market follow different goals, see e.g. [8] for a study of divestiture of brewer-owned public houses in the UK, or [5] for a survey of literature studying the demand for beer. For an application of this literature in competition economics, see [6]. In the study with a goal similar to this paper, Culbertson & Bradford [1] explore the factors responsible for the inter-state differences in off-trade (retail) prices of beer in the US. They find that "a substantial proportion of the variance can be explained by demand pressure, transportation costs, and the price of substitutes, as well as taxes and the presence of legally-mandated exclusive territories governing the wholesale distribution of beer." ([1], p. 275)

This paper finds that on-trade prices of beer in Europe are increasing in labor costs and decreasing in on-trade consumption and beer turnover per employee in the hospitality sector and that a large proportion of the variance in on-trade prices of beer can be attributed to these factors. The effects of these factors are apparent in the pattern of beer prices across Europe. For instance, beer in Ireland or France is more expensive than beer in the Czech Republic or Estonia probably because of differences in labor costs. While an hour of labor in accommodation and food services in France and Ireland costs 21.7 and 15.76 euro respectively, the cost of labor in the Czech Republic or Estonia is approximately 5.3 euro. However, beer in Estonia is still more than twice as expensive as in the Czech Republic. Here, higher Estonian prices can be explained by lower yearly per-capita consumption and beer turnover in the hospitality sector (8 liters per capita, 7.050 liters of beer per employee) compared to the Czech Republic (80 liters per capita, 25.882 liters of beer per employee).

The rest of the paper is organized as follows. Section 2 presents an extension of the Salop circular city model. Section 3 introduces the data, discusses the empirical strategy used, and presents the results of the empirical analysis. Finally, Section 4 concludes.

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2 A model of beer consumption in the hospitality sector

In this section, I present an extension of the Salop model with endogenous number of firms ([7]). The Salop model seems to be an appropriate model of beer consumption in the hospitality sector which is characterized by spacial product differentiation and low entry costs.

In the circular city model, N pubs serve identical product with quasi-fixed costs F and constant marginal costs c. The profit of pub i is

$$\pi_i(q_i) = \begin{cases} (p_i - c)q_i - F & \text{for } q_i > 0 \\ 0 & \text{for } q_i = 0, \end{cases}$$

where p_i is the price of beer in pub i and q_i is the quantity of beer sold by pub i. The city is inhabited by consumers who are uniformly distributed on a circle with unit circumference. Each consumer visits a pub v times per year, buys a total number of b liters of beer, $b_v = b/v$ liters on each visit. Consumers go to the pub that minimizes their total spending on beer plus total transportation costs $bp_i + v\tau x_i$, where τ represents transportation costs per unit of distance and x_i is the distance to pub i.

The model is a two stage game. In the first stage, N pubs enter the market. In the second stage, they choose profit-maximizing prices. I solve the model using backward induction.

In a symmetric equilibrium in the second stage, the distance between N pubs will be $1/N$. Assuming that pubs $i+1$ and $i-1$ charge the same equilibrium price p^*, the consumer that is indifferent as to whether she buys from pub i or $i+1$ lives at a distance $\hat{x}_i$ from pub i. The distance $\hat{x}_i$ is determined by $bp_i + v\tau\hat{x}_i = bp^* + v\tau(1/N - \hat{x}_i)$. Substituting $v = b/b_v$, the demand function of pub i is

$$q_i(p_i, p^*) = 2b\hat{x}_i = \frac{bb_v(p^* - p_i)}{\tau} + \frac{b}{N}.$$

Pub i chooses the price p_i for which

$$\frac{\partial \pi_i(p_i, p^*)}{\partial p_i} = \frac{bb_v(p^* - 2p_i + c)}{\tau} + \frac{b}{N} = 0.$$

In a symmetric equilibrium in the second stage, the price of firm i is

$$p_i(N) = p^*(N) = c + \frac{\tau}{b_v N}.$$

In the first stage, firms know profit-maximizing prices for different numbers of firms N. Additional firms are willing to enter the market until

$$\pi_i(p^*) = (p^*(N^*) - c)\frac{b}{N^*} - F = \frac{b\tau}{b_v N^{*2}} - F = 0.$$

The equilibrium number of firms in the industry is $N^* = \sqrt{\frac{b\tau}{b_v F}}$. The equilibrium price is

$$p^*(N^*) = c + \frac{\tau}{b_v N^*} = c + \sqrt{\frac{F\tau}{b_v b}}. \tag{1}$$

The model predicts the following effects of individual variables on the equilibrium price of beer:

- Marginal costs c enter the price of beer directly because pubs earn zero profits and a rise in marginal costs has no effect on the equilibrium number of pubs in the market.
- Quasi-fixed costs F increase the price of beer as prices charged by zero-profit pubs have to reflect costs. However, prices increase at a diminishing rate because quasi-fixed costs F reduce the equilibrium number of pubs N^* in the market, so that each pub serves a larger market.
- Transportation costs per unit of distance τ increase the price of beer because they reduce competition in the market. They increase prices at a diminishing rate because they also raise the equilibrium number of pubs N^*. The number of beers per visit b_v has the opposite effect since it reduces total transportation costs of consumers.
- The number beers b reduces the price of beer at a diminishing rate because it increases the equilibrium number of pubs N^* (and consequently competition in the market) at a diminishing rate.

3 Empirical Analysis

This section provides some evidence on the determinants of the price of beer using cross-section data from 27 European countries. First, I introduce the data and empirical methodology. Then I present the results of the empirical analysis and discuss the main findings.

3.1 Data and methodology

The data is compiled from two main sources: most variables are calculated from the 2008 data published in the Ernst & Young study "The Contribution Made by Beer to the European Economy", edition 2009 (see [2]), and the 2008 labor cost data taken from the Eurostat pocketbook "Labour Market Statistics" (see [4], p. 90). The data in both sources are available for the following 27 European countries: Austria, Bulgaria, Cyprus, Czech Republic, Denmark, France, Estonia, Finland, Germany, Greece, Hungary, Ireland, Italy, Latvia, Lithuania, Luxembourg, Malta, the Netherlands, Norway, Poland, Portugal, Romania, Slovakia, Slovenia, Spain, Sweden, and United Kingdom.

In the empirical analysis I use two dependent variables:

- Price of beer in the hospitality sector net of taxes (euro/liter) is calculated as

$$\frac{\text{on-trade price (1 liter)}}{1 + \text{VAT rate}} - \text{excise tax},$$

  where the excise tax is calculated as total excise revenues on beer divided by total consumption in liters. All values are taken from [2] except for the 2008 VAT rates of Italy (20%) and United Kingdom (17.5%)[1] which are not listed in [2] and were therefore found on the internet.

- Value added per liter of beer in the hospitality sector (euro) is calculated as value added due to sale of beer in the hospitality industry divided by total consumption in hospitality (total consumption in 2008 times the share of consumption in hospitality). All values are taken from [2].

The dependent variables are regressed against the following independent variables:

- Labor cost (euro) is the 2008 hourly labor cost in accommodation and food services activities taken from [4].

- On-trade consumption per capita (liters) is calculated as total consumption in hospitality (total consumption in 2008 times the share of consumption in hospitality) taken from [2] divided by 2008 population found on the Eurostat webpage.[2]

- Beer turnover per employee and year in the hospitality sector (liters) calculated as total consumption in hospitality (2008 total consumption in liters times the share of consumption in hospitality) divided by the employment in the hospitality sector due to beer sales. All values are taken from [2].

I made an additional adjustment in the data. Since the price of a liter of beer in Estonia (4.5 euro) in [2] is lower than the calculated value added in the hospitality sector (4.7 euro), I replaced the data on the share of on-trade consumption and employment in the hospitality sector for Estonia by the data from 2011 edition of the Ernst & Young study [3], which results in a more realistic value added in the hospitality sector (2.6 Euro). I also tried to omit Estonia from the data. The results without Estonia are similar to the results presented in the following section. Table 1 presents summary statistics for the adjusted data.

The regression equations presented in the following section should reflect the predictions of the model presented in Section 2.

- Labor cost is likely to enter both marginal and quasi-fixed costs. A rise in marginal costs increases the price of beer linearly and a rise in quasi-fixed costs at a diminishing rate. The parameter should be positive and, if part of labor cost is quasi-fixed, a rise in labor cost should increase the price of beer or value added in the hospitality sector at a diminishing rate.

[1] The VAT rate was 17.5% till 1 December 2008, when it was reduced to 15%.

[2] http://epp.eurostat.ec.europa.eu/tgm/table.do?tab=table&language=en&pcode=tps00001&tableSelection=1&footnotes=yes&labeling=labels&plugin=1

- Beer turnover per employee and year in the hospitality sector reduces labor cost per liter of beer. Hence, a rise in turnover should reduce the price of beer at a diminishing rate. Beer turnover may be also a proxy for the number of beers per visit b_v. The intuition is as follows. The higher the consumption per visit of a pub, the higher is likely to be consumption in a given pub, and the higher tends to be the number of beers sold per employee, at least in small pubs with a low number of staff. For instance, productivity of an employee who is alone in a pub is determined by the consumption of beer in the pub, because there has to be at least one person present all the time. Hence, a rise in turnover reduces the price of beer at a diminishing rate, too.
- On-trade consumption per capita corresponds to the theoretical parameter of beer consumption b. The parameter should be negative and a rise in on-trade consumption per capita should reduce the price of beer or value added in the hospitality sector at a diminishing rate.

Variable	Mean	Median	Minimum	Maximum
On-trade price of beer net of taxes (euro/liter)	4.50355	4.72189	0.884490	9.27491
Value added in the hospitality sector (euro/liter)	2.12149	2.46733	0.342291	4.68165
Labor cost (euro/hour)	11.7811	11.3900	1.55000	25.8000
On-trade consumption per capita (liters/year)	30.2714	26.6636	5.46151	81.4110
Beer turnover per employee (liters/year)	9726.59	8838.12	5442.57	25882.5

Table 1 Summary Statistics

3.2 Results

Table 2 presents OLS regressions with the price of beer net of taxes as a dependent variable. Model I does not include beer turnover per employee. Model II with log of beer turnover has lower Schwarz, Akaike, and Hannan–Quinn information criteria than the same model with turnover and turnover squared. Both models with labor cost and labor cost squared have lower information criteria than the models with square root of labor cost, and the models with the log of labor cost are misspecified according to the RESET test (squares and cubes). Similarly, both models with the log of on-trade consumption have lower information criteria than the models with on-trade consumption and on-trade consumption squared.

	I	II
constant	2.168^{**}	14.73^{**}
	(0.9039)	(4.468)
labor cost	0.6475^{**}	0.6258^{**}
	(0.1146)	(0.1003)
labor cost squared	-0.01315^{**}	-0.01278^{**}
	(0.004359)	(0.003809)
log of on-trade consumption	-0.8894^{**}	-0.5092^{*}
	(0.2859)	(0.2830)
log of beer turnover per employee		-1.490^{**}
		(0.5217)
n	27	27
$\bar{R}^2$	0.8616	0.8944
ℓ	-34.06	-29.81

Standard errors in parentheses
* indicates significance at the 10 percent level
** indicates significance at the 5 percent level

Table 2 The OLS estimates for the dependent variable on-trade price of beer net of taxes

The estimates in Table 2 appear to be consistent with the predictions of the theoretical model. In both models, the labor cost increases the price of beer at a diminishing rate, and on-trade consumption per capita reduces the price of beer at a diminishing rate too. In model II, turnover reduces the price of beer at a diminishing rate. Beer turnover per employee in model II reduces significance of the log of on-trade consumption because correlation between beer turnover and on-trade consumption is relatively high (the correlation coefficient is 0.578). Additionally, all specifications explain more than 86% of the variation in the price of beer ($\bar{R}^2 > 0.86$).

Table 3 studies the effect of different variables on the value added per liter of beer in hospitality. Again, Model I does not include turnover. However instead of the labor cost and labor cost squared it uses the square root of labor cost. The model with the labor cost and labor cost squared gives also reasonable results but has higher information criteria. Model II adds the log of beer turnover per employee (the same equation with turnover and turnover squared has higher information criteria). The parameter of the log of on-trade consumption in model II is negative but not statistically significant. Model III omits the log of on-trade consumption. Finally, model IV uses the labor cost and labor cost squared instead of the square root of labor cost. In this specification, both the log of on-trade consumption per capita and beer turnover per employee are significantly lower than zero.

	I	II	III	IV
constant	0.1317	6.444**	7.733**	1.295**
	(0.4900)	(2.226)	(2.129)	(0.4161)
log of on-trade consumption	−0.3870**	−0.1980		−0.2627*
	(0.1242)	(0.1269)		(0.1514)
square root of labor cost	0.9881**	0.9499**	0.9413**	
	(0.08210)	(0.07304)	(0.07498)	
log of beer turnover per employee		−0.7450**	−0.9524**	
		(0.2578)	(0.2274)	
labor cost				0.2557**
				(0.05336)
labor cost squared				−0.004373**
				(0.002012)
beer turnover per employee				−5.457e-05**
				(2.374e-05)
n	27	27	27	27
$\bar{R}^2$	0.8557	0.8895	0.8829	0.8772
ℓ	−15.63	−11.45	−12.81	−12.27

Standard errors in parentheses
* indicates significance at the 10 percent level
** indicates significance at the 5 percent level

Table 3 The OLS estimates for the dependent variable value added per liter of beer

Again, the findings in Table 3 seem to support predictions of the theoretical model. Labor cost increases the value added per employee at a diminishing rate in all four models. Moreover, the data support the square root specification predicted directly by the theoretical model. Better support for the square root specification compared to the regressions presented in Table 2 might be due to the fact that the value added per employee does not include part the cost of beer production. Hence marginal costs c in equation (1) are lower, which makes the relationship between the labor cost and value added more square-root shaped. The on-trade consumption per capita reduces the value added in models I, II and III at a diminishing rate. However, it is less or not at all significant in models that include the beer turnover per employee. Similarly, the turnover per employee reduces the value added at a diminishing rate in models II and III and at a constant rate in model IV.

I also test the classical assumptions of the OLS regression. In all models, I do not reject normality of residuals using Doornik-Hansen test, Shapiro-Wilk W-test, Lilliefors test, and Jarque-Bera test at the 5% level. Similarly, neither Breusch-Pagan test nor White's test reject homoskedasticity at the 5% level. Furthermore, the RESET test (squares and cubes) in both models does not indicate incorrect specification at the 5% level. Finally, there may be endogeneity in both models since the price of beer or value added per liter of beer may affect the on-trade consumption and turnover per employee. However, the cross-section differences in these variables seem to be mainly due to different cultural norms and drinking habits in these countries. This view is supported by the fact that there is no statistically significant relationship between off-trade (retail) prices of beer and off-trade or on-trade consumption of beer per capita or turnover per employee. Similarly, there is no statistically significant relationship between the share of on-trade consumption and the difference between on-trade and off-trade prices or the ratio of off-trade to on-trade prices (the data on off-trade prices and consumption are taken from [2]).

4 Conclusion

The goal of the paper is to explain the differences in on-trade prices of beer in Europe. Using cross-section data from 27 European countries, the paper finds that more than 86% of the variance in the price of beer net of taxes is due to differences labor costs, on-trade consumption per-capita and beer turnover per employee in the hospitality sector. Specifically, it finds that a rise in labor costs and a reduction in consumption per capita and beer turnover increase the price of beer. Moreover, it finds that all the variables affect prices at a diminishing rate. The findings hold also if value added in the hospitality sector is used instead of prices. These findings are consistent with the predictions of the extended Salop model of circular city which seems to be appropriate for studying the determinants of beer prices in the hospitality sector.

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A proper fuzzification of Saaty's scale and an improved method for computing fuzzy weights in fuzzy AHP

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Abstract. A modification to the fuzzified Saaty's scale for designing a multiplicative fuzzy pairwise comparison matrix and an improved method for computing fuzzy weights of elements from a multiplicative fuzzy pairwise comparison matrix will be proposed. For simplicity of explanation triangular fuzzy numbers will be used in this paper. In many cases of the fuzzified AHP method that can be found in the literature, the fuzzification of the Saaty's scale is not appropriate. Therefore, a new fuzzification of the Saaty's scale will be suggested. Afterwards, an improvement of the formulas proposed by Buckley [4] and Pan and Yuan [9] will be suggested for computing fuzzy weights from a multiplicative fuzzy pairwise comparison matrix. In the new algorithm the reciprocity condition for the pairwise comparison matrix is taken into account in each step of the computation. The fuzzy weights resulting from the proposed fuzzification of the Saaty's AHP are less uncertain than those calculated by means of Buckley's algorithms.

Keywords: fuzzy pairwise comparison matrix, fuzzified Saaty's scale, triangular fuzzy number, fuzzy AHP

JEL classification: C44
AMS classification: 90C15

1 Introduction

The Analytic hierarchy process (AHP) is a methodology developed by T. L. Saaty in 1970s [10] for coping with unstructured complex problems using a hierarchical structure and pairwise comparisons. It is based on a construction of pairwise comparison matrices of alternatives against each criterion and a pairwise comparison matrix of criteria against the overall goal of the decision making problem. In both cases, the Saaty's 5-point scale or extended 9-point scale are used for making pairwise comparisons. Afterwards, weights of criteria and alternatives are computed from the pairwise comparison matrices and finally global priority weights of all alternatives are obtained by agregation.

After the AHP was introduced in 1970s many ways of fuzzification of the method were proposed. Van Laarhoven and Pedrycz [12] were first who applied fuzzy theory to the AHP. Buckley [4] initiated trapezoidal fuzzy numbers to express the decision maker's preferences and fuzzified geometric mean method for obtaining weights from the Saaty's pairwise comparison matrix. After then many other methods for obtaining fuzzy weights from fuzzy pairwise comparison matrices were introduced. However, as was shown in [3] the geometric mean is the only solution to the problem satisfying logical conditions. That is why we focus on this method and its fuzzification. For correct computation of fuzzy weights of elements also properly fuzzified Saaty's scale is essential. Therefore, we suggest also a modification to the fuzzification of the Saaty's scale.

This paper is organized as follows: In Section 2, the fuzzification of the Saaty's scale will be discussed. Section 3 will be focused on the fuzzification of the geometric mean method for obtaining weights from multiplicative pairwise comparison matrices. Numerical example of the computation of fuzzy weights from fuzzy pairwise comparison matrix will be given afterwards. Finally, the conclusion we will made in

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Section 4.

2 Fuzzification of the Saaty's scale

In this section the Saaty's scale used in the AHP will be remined. Then, fuzzifications of the Saaty's scale proposed by various authors will be described, lack of these fuzzifications will be shown and then more appropriate fuzzification of the Saaty's scale will be proposed.

Saaty [10] proposed the fundamental 5-point and extended 9-point scales of intensities of preferences to compare two elements in one level of the hierarchy with respect to an element from the upper level. Making comparisons of every two elements in one level of hierarchy with respect to an element from the upper level of the hierarchy a pairwise comparison matrix is constructed. Let us denote such a pairwise comparison matrix $A = \{a_{ij}\}_{i,j=1}^{n}$, where n is a number of elements $x_1, x_2, \ldots, x_n$ in one level of hierarchy which are compared and a_{ij} expresses how many times more important is the element x_i over the element x_j. When the element x_i is $a_{ij}-$times more important than the element x_j then the element x_j takes only $\frac{1}{a_{ij}}-$th of the importance of the element x_i. Therefore, the pairwise comparison matrix has to be reciprocal, i.e. $a_{ij} = \frac{1}{a_{ji}}$, where $i, j = 1, \ldots, n$.

To capture the vagueness in the decision maker's preferences the Saaty's scales are fuzzified. For the fuzzification triangular fuzzy numbers are usually used. Therefore we will use them in this article too.

A triangular fuzzy number $\tilde{c}$ is a fuzzy number whose membership function is uniquely determined by a triplet of significant values $c_1 \leq c_2 \leq c_3$ in the following way:

$$\tilde{c}\,(x) = \begin{cases} \frac{x-c_l}{c_2-c_1}, & c_1 \leq x \leq c_2, \\ \frac{c_3-x}{c_3-c_2}, & c_2 < x \leq c_3, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

For a triangular fuzzy number $\tilde{c}$ whose membership function is given by (1), the notation $\tilde{c} = (c_1, c_2, c_3)$ will be used hereafter.The real numbers c_1, c_2, and c_3 will be called the lower, the middle, and the upper significant values of the triangular fuzzy number $\tilde{c}$, respectively.

Some examples of the fuzzification of the Saaty's scale proposed in the literature are shown in Table 1 and Table 2. Using the fuzzified Saaty's scale fuzzy pairwise comparison matrices are constructed. In the following, a Saaty's fuzzy pairwise comparison matrix is a matrix $\tilde{A} = \{\tilde{a}_{ij}\}_{i,j=1}^{n}$ whose elements $\tilde{a}_{ij}, i \neq j$, are triangular fuzzy numbers from the fuzzified Saaty's scale in case that the element x_i is more important than the element x_j or their reciprocal in case that the element x_j is more important than the element x_i. On the main diagonal of the Saaty's fuzzy pairwise comparison matrix there are always real number 1 as we compare one element with itself. Therefore, there is no fuzziness in this comparison.

Let us denote that the reciprocal of the triangular fuzzy number is not a triangular fuzzy number. However, for the sake of simplicity it is usually aproximated by a triangular fuzzy number. In the following, the reciprocal of a triangular fuzzy number $\tilde{c} = (c_1, c_2, c_3)$ will be defined as a triangular fuzzy number $\frac{1}{\tilde{c}} = \left(\frac{1}{c_3}, \frac{1}{c_2}, \frac{1}{c_1}\right)$.

Fuzzy	Membership functions used in:			Linguistic
number	[2]	[5]	[7]	term
$\widetilde{1}$	$(1,1,3)$	$(1,1,3)$	$(1,1,2)$	equal importance
$\widetilde{3}$	$(1,3,5)$	$(1,3,5)$	$(2,3,4)$	moderate importance
$\widetilde{5}$	$(3,5,7)$	$(3,5,7)$	$(4,5,6)$	strong importance
$\widetilde{7}$	$(5,7,9)$	$(5,7,9)$	$(6,7,8)$	very strong importance
$\widetilde{9}$	$(7,9,11)$	$(7,9,9)$	$(8,9,9)$	extreme importance

Table 1 Examples of the fuzzified 5-point scale applied in the literature.

The fuzzifications of the Saaty's scale mentioned above are not suitable for constructing fuzzy pairwise comparison matrices. The problem is caused by the fuzzy number $\tilde{1}$ which means that two elements are

Fuzzy	Membership functions used in:			Linguistic
number	[1]	[6]	[11]	term
$\widetilde{1}$	$(1,1,1)$	$(1,1,1)$	$(1,1,1)$	equal importance
$\widetilde{2}$	$(1,2,3)$	$(1,2,3)$	$(2-x^*,2,2+x)$	
$\widetilde{3}$	$(2,3,4)$	$(2,3,4)$	$(3-x,3,3+x)$	moderate importance
$\widetilde{4}$	$(3,4,5)$	$(3,4,5)$	$(4-x,4,4+x)$	
$\widetilde{5}$	$(4,5,6)$	$(4,5,6)$	$(5-x,5,5+x)$	strong importance
$\widetilde{6}$	$(5,6,7)$	$(5,6,7)$	$(6-x,6,6+x)$	
$\widetilde{7}$	$(6,7,8)$	$(6,7,8)$	$(7-x,7,7+x)$	very strong importance
$\widetilde{8}$	$(7,8,9)$	$(7,8,9)$	$(8-x,8,8+x)$	
$\widetilde{9}$	$(9,9,9)$	$(8,9,10)$	$(8,9,9)$	extreme importance

$^*x$ is a fuzzification factor

Table 2 Examples of the fuzzified 9-point scale applied in the literature.

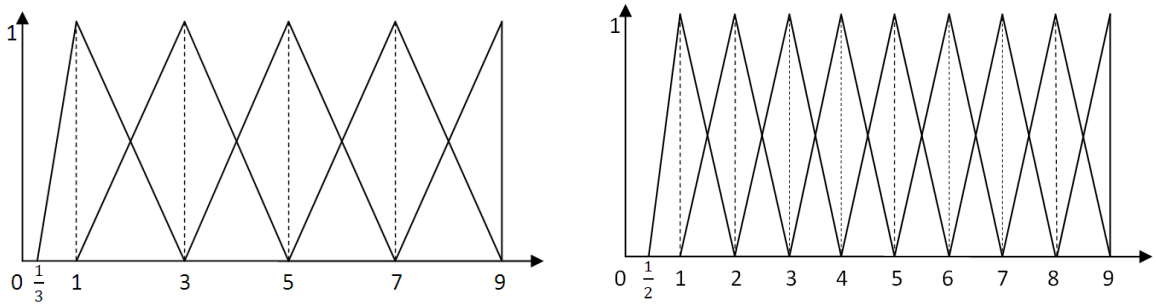

Figure 1 Suitable fuzzified 5-point end extended 9-point scales.

of the same importance. In Table 2, the fuzzy number $\tilde{1}$ is defined as a real number $(1,1,1)$. However, $\tilde{1}$ should contain some fuzziness as we want to deal with vagueness in meanings. In Table 1, on the other hand, the fuzzy number $\tilde{1}$ is defined as a triangular fuzzy number $(1,1,c)$, where $c=2$, resp. $c=3$. Let us assume an example in which two criteria, let say $\mathcal{K}_1$ and $\mathcal{K}_2$, are of the same importance. Then in the fuzzy pairwise comparison matrix of criteria there will be $\tilde{a}_{12}=(1,1,c)$. And because the fuzzy pairwise comparison matrix has to be reciprocal, it follows that $\tilde{a}_{21}=\frac{1}{\tilde{a}_{12}}=\frac{1}{(1,1,c)}=\left(\frac{1}{c},1,1\right)$. However, when an expert says that the criterion $\mathcal{K}_1$ is of the same importance as the criterion $\mathcal{K}_2$, then also the criterion $\mathcal{K}_2$ should be of the same importance as the criterion $\mathcal{K}_1$. Therefore, it should hold that $\tilde{a}_{12}=\tilde{a}_{21}$. According to this consideration we should define the fuzzy number $\tilde{1}$ in such a way that the equation $\tilde{1}=\frac{1}{\tilde{1}}$ holds. Therefore, it follows that it should be defined as $\tilde{1}=\left(\frac{1}{c},1,c\right)$. The vagueness of meaning of other linguistic terms from the fuzzified Saaty's scale is understood so that the meaning of the linguistic term ranges between two neigbouring values of the scale and the typical numerical value corresponding to the linguistic term is the value in the core of the corresponding fuzzy number. The properly fuzzified 5-point and extended 9-point scales are shown in Figure 1.

The elements of the fuzzified Saaty's scales shown in Figure 1 together with their reciprocal form a special structure called *fuzzy scale* on the interval $\left[\frac{1}{9},9\right]$. Generally, the fuzzy numbers $\tilde{c}_1,\ldots,\tilde{c}_n$ defined on the interval $[a,b]$, $a,b\in\mathbb{R}$ form a fuzzy scale if $\sum_{i=1}^{n}\tilde{c}_i(x)=1,\forall x\in[a,b]$, and if the fuzzy numbers are numbered in conformity with their linear ordering, i.e. $\tilde{c}_1<\cdots<\tilde{c}_n$.

3 Fuzzification of the geometric mean method

In this section the fuzzification of the geometric mean method for obtaining fuzzy weights of elements from the Saaty's fuzzy pairwise comparison matrix will be discussed. The approaches of Buckley, Pan and Yuan will be described, their lacks will be pointed out and then improved formulas for computing

fuzzy weights will be proposed.

As was already mentioned in Introduction, the geometric mean method was fuzzified by Buckley in [4]. The formulas for computing fuzzy weight $\tilde{w}_i = (w_{i1}, w_{i2}, w_{i3})$ of the $i-$th element, where $i = 1, \ldots, n$, from the fuzzy pairwise comparison matrix $\tilde{A} = \{\tilde{a}_{ij}\}_{i,j=1}^{n}$ proposed by Buckley are as follow:

$$w_{i1} = \frac{\sqrt[n]{\prod_{j=1}^{n} a_{ij1}}}{\sum_{k=1}^{n} \sqrt[n]{\prod_{j=1}^{n} a_{kj3}}}, \tag{2}$$

$$w_{i2} = \frac{\sqrt[n]{\prod_{j=1}^{n} a_{ij2}}}{\sum_{k=1}^{n} \sqrt[n]{\prod_{j=1}^{n} a_{kj2}}}, \tag{3}$$

$$w_{i3} = \frac{\sqrt[n]{\prod_{j=1}^{n} a_{ij3}}}{\sum_{k=1}^{n} \sqrt[n]{\prod_{j=1}^{n} a_{kj1}}}, \tag{4}$$

where $\tilde{a}_{ij} = (a_{ij1}, a_{ij2}, a_{ij3})$. In general, these fuzzy weights are not constrained to the interval $[0, 1]$. In that case, Buckley [4] suggests to multiply the fuzzy weights by a normalizing constant so that the supports of all fuzzy weights are in the interval $[0, 1]$. In that case the sum of the middle values of the fuzzy weights is not equal to one. However, Buckey did not proceeded correctly during the construction of the formulas (2), (3) and (4). At first, he computed geometric means of the rows of the Saaty's fuzzy pairwise comparison matrix and then computed their sum. Thus he obtained two triangular fuzzy numbers which he then devided. However, he did not take into account that for the normalization the extention principle has to by applied within the whole formula at one time.

Unlike Buckley, Pan and Yuan [9] realized the problem and applied the extention principle on the whole formula for computing normalized fuzzy weights:

$$w_{i1} = \frac{\sqrt[n]{\prod_{j=1}^{n} a_{ij1}}}{\sqrt[n]{\prod_{j=1}^{n} a_{ij1}} + \sum_{k=1, k\neq i}^{n} \sqrt[n]{\prod_{j=1}^{n} a_{kj3}}}, \tag{5}$$

$$w_{i2} = \frac{\sqrt[n]{\prod_{j=1}^{n} a_{ij2}}}{\sum_{k=1}^{n} \sqrt[n]{\prod_{j=1}^{n} a_{kj2}}}, \tag{6}$$

$$w_{i3} = \frac{\sqrt[n]{\prod_{j=1}^{n} a_{ij3}}}{\sqrt[n]{\prod_{j=1}^{n} a_{ij3}} + \sum_{k=1, k\neq i}^{n} \sqrt[n]{\prod_{j=1}^{n} a_{kj1}}}. \tag{7}$$

Fuzzy weights obtained from the Saaty's fuzzy pairwise comparison matrix using the formulas (5), (6) and (7) are constrained to the interval $[0, 1]$ and the sum of the middle significant values of the fuzzy weights is equal to one. However, in these formulas the condition of reciprocity of the fuzzy pairwise comparison matrix $\tilde{A}$ is still not taken into account. Therefore, with respect to the reciprocity the significant values shoud be computed according to these formulas:

$$w_{i1} = \min \left\{ \frac{\sqrt[n]{\prod_{j=1}^{n} a_{ij}}}{\sum_{k=1}^{n} \sqrt[n]{\prod_{j=1}^{n} a_{kj}}} \; ; \; a_{kj} \in [a_{kj1}, a_{kj3}], \; k, j = 1, \ldots, n, \; a_{kj} = \frac{1}{a_{jk}} \right\}, \tag{8}$$

$$w_{i2} = \frac{\sqrt[n]{\prod_{j=1}^{n} a_{ij2}}}{\sum_{k=1}^{n} \sqrt[n]{\prod_{j=1}^{n} a_{kj2}}}, \tag{9}$$

$$w_{i3} = \max \left\{ \frac{\sqrt[n]{\prod_{j=1}^{n} a_{ij}}}{\sum_{k=1}^{n} \sqrt[n]{\prod_{j=1}^{n} a_{kj}}} \; ; \; a_{kj} \in [a_{kj1}, a_{kj3}], \; k, j = 1, \ldots, n, \; a_{kj} = \frac{1}{a_{jk}} \right\}. \tag{10}$$

Let us remark that the formulas (2), (6) and (9) for computing middle significant values of the fuzzy weights are the same.

The algorithm for the compution of fuzzy weights according to the formulas (8) and (10) is described in [8]. The supports of the fuzzy weights computed using the formulas (8) and (10) are in the interval $[0,1]$ and the sum of the middle significant values is equal to one. Moreover, the supports of the fuzzy weights are narrower than the supports of the fuzzy weights computed using the formulas (2), (3) and (4), resp. (5), (6) and (7) because we took into account the condition of the reciprocity of the matrices used in the formulas.

Let us remark that the fuzzy weights obtained from the Saaty's fuzzy pairwise comparison matrix are not triangular fuzzy numbers. We work only with their $3-$point representation. However, the representation of real fuzzy weights by triangular fuzzy numbers is sufficient in real applications.

In the following example we will compute fuzzy weights of the criteria using the formulas proposed by Buckley, Pan and Yuan and by us to compare the results.

Example 1. Let us assume the fuzzy pairwise comparison matrix of the 6 criteria from [8]:

$$\begin{array}{c} \\ \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \\ \mathcal{K}_4 \\ \mathcal{K}_5 \\ \mathcal{K}_6 \end{array} \begin{array}{c} \begin{array}{cccccc} \mathcal{K}_1 & \mathcal{K}_2 & \mathcal{K}_3 & \mathcal{K}_4 & \mathcal{K}_5 & \mathcal{K}_6 \end{array} \\ \begin{pmatrix} 1 & (1,3,5) & (3,5,7) & (3,5,7) & (3,5,7) & (5,7,9) \\ \left(\frac{1}{5},\frac{1}{3},1\right) & 1 & (3,5,7) & (3,5,7) & (3,5,7) & (5,7,9) \\ \left(\frac{1}{7},\frac{1}{5},\frac{1}{3}\right) & \left(\frac{1}{7},\frac{1}{5},\frac{1}{3}\right) & 1 & \left(\frac{1}{3},1,3\right) & \left(\frac{1}{3},1,3\right) & (3,5,7) \\ \left(\frac{1}{7},\frac{1}{5},\frac{1}{3}\right) & \left(\frac{1}{7},\frac{1}{5},\frac{1}{3}\right) & \left(\frac{1}{3},1,3\right) & 1 & \left(\frac{1}{3},1,3\right) & (1,3,5) \\ \left(\frac{1}{7},\frac{1}{5},\frac{1}{3}\right) & \left(\frac{1}{7},\frac{1}{5},\frac{1}{3}\right) & \left(\frac{1}{3},1,3\right) & \left(\frac{1}{3},1,3\right) & 1 & (1,3,5) \\ \left(\frac{1}{9},\frac{1}{7},\frac{1}{5}\right) & \left(\frac{1}{9},\frac{1}{7},\frac{1}{5}\right) & \left(\frac{1}{7},\frac{1}{5},\frac{1}{3}\right) & \left(\frac{1}{5},\frac{1}{3},1\right) & \left(\frac{1}{5},\frac{1}{3},1\right) & 1 \end{pmatrix} \end{array}. \tag{11}$$

The fuzzy weights of the criteria computed using the formulas proposed by Buckley, Pan and Yuan and by us are given in Table 3. As we can see from the table, the fuzzy weights computed using the formulas (8) and (10) have the smallest vagueness. This was reached by the fact that we applied the extention principle correctly and took into account the condition of the reciprocity of the matrices used in the formulas.

	Fuzzy weights of the criteria using the formulas		
	(2), (3), (4),	(5), (6), (7)	(8), (6), (10)
$\widetilde{w}_1$	$(0.1704, 0.4252, 0.9309)$	$(0.2144, 0.4252, 0.6171)$	$(0.2578, 0.4252, 0.5489)$
$\widetilde{w}_2$	$(0.1303, 0.2948, 0.7118)$	$(0.1545, 0.2948, 0.5126)$	$(0.1837, 0.2948, 0.4516)$
$\widetilde{w}_3$	$(0.0328, 0.0875, 0.2580)$	$(0.0353, 0.0875, 0.2193)$	$(0.0425, 0.0875, 0.1795)$
$\widetilde{w}_4$	$(0.0273, 0.0804, 0.2439)$	$(0.0294, 0.0804, 0.2074)$	$(0.0353, 0.0804, 0.1748)$
$\widetilde{w}_5$	$(0.0273, 0.0804, 0.2439)$	$(0.0294, 0.0804, 0.2074)$	$(0.0353, 0.0804, 0.1748)$
$\widetilde{w}_6$	$(0.0153, 0.0317, 0.0908)$	$(0.0156, 0.0317, 0.0863)$	$(0.0197, 0.0317, 0.0673)$

Table 3 Fuzzy weights of the criteria.

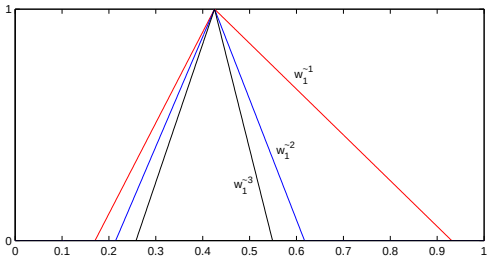

Figure 2 The fuzzy weight of the criterion $\mathcal{K}_1$.

For illustration, fuzzy weights of the criterion $\mathcal{K}_1$ computed using the different formulas are given in Figure 2. The fuzzy weight of the criterion obtained from the formulas (2), (3), (4), resp. (5), (7), resp. (8), (10) is represented by the fuzzy number $\widetilde{w}_1^1$, resp. $\widetilde{w}_1^2$, resp. $\widetilde{w}_1^3$.

4 Conclusion

A proper fuzzification of the Saaty's scale and the fuzzification of the geometric mean method for computing fuzzy weights from a multiplicative fuzzy pairwise comparison matrix were introduced. Incorrect definitions of the fuzzy number $\tilde{1}$ were rectified and a clear explanation of the other elements of the fuzzified Saaty's scale was given. The Saaty's fuzzy pairwise comparison matrix constructed from the fuzzified Saaty's scale was defined then.

Afterwards the fuzzification of the geometric mean method proposed by Buckley [4] and an improvement of the formulas proposed by Pan and Yuan [9] were described. The lacks of these formulas were mentioned and new more adequate formulas (8) and (10) for computing fuzzy weights from a multiplicative fuzzy pairwise comparison matrix were introduced. These formulas are based on the extention principle and take into account the condition of the reciprocity of pairwise comparison matrices. The resulting fuzzy weights computed from the formulas (8), (9) and (10) are then less vague than the fuzzy weights computed from the formulas (2), (3), (4), resp. (5), (6) and (7).

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VaR backtesting results under different volatility models

Aleš Kresta[1]

Abstract. Correct risk estimation is nowadays very important and no less challenging part of financial management. It is particularly crucial for financial institutions and their regulatory authorities. The simplest case is the risk estimation of investment into stock market index. Different models can be applied for risk estimation in terms of Value at Risk. These models differ mainly in the stochastic process, which price (respectively returns) time series are assumed to follow. For all the models the accurate prediction of returns' volatility is fundamental. Although it was empirically shown that the returns of financial assets are heteroscedastic, for simple models the homoscedasticity is usually assumed. More precise models take the heteroscedasticy into account. The paper examines the effect of different volatility models application on accuracy of VaR estimation for investment into stock market index. The accuracy is back-tested and the backtesting results are compared. The comparison of the results is made by means of Kupiec's unconditional test and Christoffersen conditional test, i.e. the number of exceptions and their independency in time are tested.

Keywords: Backtesting, Value at Risk, risk management.

JEL Classification: G17
AMS Classification: 91B84

1 Introduction

In order to estimate the risk of financial investments, it is necessary to model the future evolution of returns. This is clearly difficult and no less interesting part of financial management. For calculation of risk (in terms of Value at Risk) we can recognize generally three groups of methods: variance-covariance method, (filtered) historical simulation and Monte Carlo simulation. While it is crucial for all the models to estimate the future variance correctly, we can distinguish also the methods which assume the volatility to be constant over time and methods modeling variance over time. The second group of methods should give more accurate results. Both type models were tested for example by Alexander and Sheedy [1].

Majority of the recent papers published on risk estimation topic is also focused on the dependence modeling (i.e. the authors assume the portfolio composed of more than one asset). We can mention for instance papers published by Huang et al. [8], Ignatieva and Platen [11] or Kresta and Tichý [13]. In these papers the accuracy of models was measured by the backtesting procedure (i.e. the quantity of cases in which the observed loss exceeded the estimated Value at Risk was observed). Statistical tests due to Kupiec [14] and Christoffersen [10] were utilized mostly in these papers, but there were not given attention to the length of period utilized for parameters estimation.

It is obvious that, if the modeling of volatility is inaccurate for a single asset (one-dimensional problem), increasing the dimensionality of the problem (incorporating the dependence to it) would not improve the results (accuracy of the joint model). Hence, we focus solely on the volatility modeling and neglect the dependence issues. Thus, we assume simple one dimensional time series (i.e. the investment into only one asset), which allow us to work with the longer time series with still reasonable requirements on computational time. In this paper we examine the impact of the length of period utilized for parameters estimation on the volatility modeling accuracy and thus backtesting results.

The goal of the paper is to back-test the different volatility models, concretely GARCH model and GJR model with Gaussian and Student innovations, on the chosen time series. Utilized time series is more than 50 years long time series of American stock index S&P 500.

The paper is organized as follows. Applied volatility models are defined in the next section. Then, Value at Risk and method of its backtesting are described. In the last section, utilized dataset is described and backtesting results are presented.

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2 Volatility models

Volatility models have become important tool in time series analysis, particularly in financial applications. Engle [6] observed that, although the future value of many financial time series is unpredictable, there is a clustering in volatility. He proposed autoregressive conditional heteroskedasticity (ARCH) process, which has been later expanded to generalized autoregressive conditional heteroskedasticity (GARCH) model by Bollerslev [4]. Further extension assumed in this paper is asymmetric GJR model proposed by Glosten, Jagannathan and Runkle [7]. There are also other volatility models such as IGARCH, FIGARCH, GARCH-M, EGARCH, etc. For their description see e.g. [2].

For all models the conditional mean will also be assumed. Thus the models of time series $\{x_t\}_{t=1}^{N}$ will be of general form as follows,

$$x_t = \mu_0 + \sum_{i=1}^{R} \mu_i \cdot x_{t-i} + \sigma_t \cdot \tilde{\varepsilon}_t , \tag{1}$$

$$\tilde{\varepsilon}_t \sim N(0,1) \quad or \quad \tilde{\varepsilon}_t \sim t_\nu(0,1), \tag{2}$$

where μ_0 is unconditional mean of the series, μ_i are autocorrelation coefficients for lag 1 up to R, σ_t is modeled standard deviation (volatility) and $\tilde{\varepsilon}_t$ is a random number from chosen probability distribution. In the paper two probability distributions are assumed: Gaussian distribution (normal distribution, henceforth n) and Student distribution (henceforth t).

2.1 GARCH model

The GARCH model [4] was proposed as the extension of ARCH model in order to avoid problematic parameters estimation, when there are many of them. The model takes the following form,

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{P} \alpha_i \cdot \sigma_{t-i}^2 + \sum_{j=1}^{Q} \beta_j \cdot \varepsilon_{t-j}^2 , \tag{3}$$

where α_0, α_i and β_j are parameters needed to be estimated. The positive variance is assured if $\alpha_0 > 0$, $\alpha_i \geq 0 \; \forall i$ and $\beta_j \geq 0 \; \forall j$. The model is stationary if $\sum_{i=1}^{P} \alpha_i + \sum_{j=1}^{Q} \beta_j < 1$.

2.2 GJR model

It was shown (firstly by Black [3]) that there is usually different impact of the positive and negative shocks on the volatility. GJR model, proposed by by Glosten, Jagannathan and Runkle, takes this into account. It is similar to the GARCH model (3), but if the previous innovation was negative (dummy variable i_{t-j}) the impact on volatility is bigger (by the parameter γ_j). The model takes the following form,

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{P} \alpha_i \cdot \sigma_{t-i}^2 + \sum_{j=1}^{Q} \beta_j \cdot \varepsilon_{t-j}^2 + \sum_{j=1}^{Q} \gamma_j \cdot i_{t-j} \cdot \varepsilon_{t-j}^2 , \tag{4}$$

where i_{t-j} is a dummy variable which equals to one when innovation ε_{t-j} is negative and null otherwise. Variables α_0, α_i, β_j and γ_j are parameters needed to be estimated. The positive variance is assured if $\alpha_0 > 0$, $\alpha_i \geq 0 \; \forall i$, $\beta_j \geq 0 \; \forall j$ and $\beta_j + \gamma_j \geq 0 \; \forall j$ and model is stationary if $\sum_{i=1}^{P} \alpha_i + \sum_{j=1}^{Q} \beta_j + \frac{1}{2} \sum_{j=1}^{Q} \gamma_i < 1$.

3 VaR and backtesting procedure

Value at Risk (VaR) is nowadays commonly accepted measure of the risk. If we assume a random variable X – the profit from asset / portfolio with the (un)known distribution function F_X , VaR at a given probability level α is the maximum loss which will occur in $1-\alpha$ cases (confidence level),

$$VaR_\alpha(X) = \sup\{x \in R : F_X(-x) \geq \alpha\}. \tag{5}$$

VaR is usually estimated for one day ahead period and then (if needed) recalculated for longer periods. Mostly utilized values of α are 15%, 5%, 1% and 0.5%. For further explanation of VaR concept and methods (or models) utilized for its estimation see e.g. [5, 12].

There are three basic approaches to VaR estimation – (i) analytical formula utilizing parametrical probability distribution function, (ii) stochastic (Monte Carlo) simulation that estimates the quantile of a given distribution numerically, and (iii) (filtered) historical simulation that relates VaR estimation to the quantile obtained from historical observations. For all the models the accurate prediction of returns' volatility is fundamental. In this paper we assume that the financial returns can be modeled by the AR-GARCH and AR-GJR processes, which were described in the previous section. When parameters of these processes are estimated, VaR can be calculated as follows,

$$VaR_{\alpha,t+1} := VaR_{\alpha}(x_{t+1}) = \hat{\mu}_0 + \sum_{i=1}^{R} \hat{\mu}_i \cdot x_{t-i+1} + \hat{\sigma}_{t+1} \cdot q_{\alpha}, \tag{6}$$

where $\hat{\mu}_0$ and $\hat{\mu}_i$ are estimated coefficients of conditional mean, $\hat{\sigma}_{t+1}$ is estimated standard deviation by one of the models defined in previous section and q_α is an appropriate quantile of the innovations distribution (assumed are Gaussian and Student distributions).

By means of backtesting procedure the model is verified. This procedure is based on the comparison of the risk estimated at time t for time $t+1$ with the true loss observed at time $t+1$. Within the backtesting procedure on a given time series the following two situations can arise – the loss is higher or lower than its estimation,

$$I_t = \begin{cases} 1 & if\ r_t < -VaR_t \\ 0 & if\ r_t \geq -VaR_t \end{cases}. \tag{7}$$

While the former case is denoted by 1 as an exception, the latter one is denoted by zero. If the model is accurate, than roughly $(1-\alpha)\cdot n$ exceptions (where n is the length of the data set utilized for backtesting) should be experienced. Bigger quantity of exceptions means, that the model underestimates the risk and vice versa. For further details see [9, 15].

Mostly applied statistical tests are due to Kupiec [14] and Christoffersen [10]. Kupiec's test (henceforth K-test) is derived from a relative amount of exceptions, i.e. whether their quantity is from the statistical point of view different from the assumption. The null hypothesis is that the observer probability of exception occurring is equal to the assumed. A given likelihood ratio on the basis of χ^2 probability distribution with one degree of freedom is formulated as follows:

$$LR = -2\ln\left[\frac{\pi_{ex}^{n_1}\left(1-\pi_{ex}\right)^{n_0}}{\pi_{obs}^{n_1}\left(1-\pi_{obs}\right)^{n_0}}\right], \tag{8}$$

where π_{ex} is expected probability of exception occurring, π_{obs} is observed probability of exception occurring, n_0 is the number of zeros and n_1 is the number of ones (exceptions). The Kupiec's test takes into account only the quantity of exceptions.

By contrast, in order to assess whether the exceptions are distributed equally in time, i.e. without any dependence (autocorrelation), we should define the time lag first. Therefore, we replace the original sequence by a new one, where 01, 00, 11 or 10 are recorded. The null hypothesis is that the probability of exception occurring is independent on the information whether the exception has occurred also previous day. Then we have the likelihood ratio as follows (C-test):

$$LR = -2\ln\left[\frac{\pi_{obs}^{n_1}\left(1-\pi_{obs}\right)^{n_0}}{\pi_{01}^{n_{01}}\left(1-\pi_{01}\right)^{n_{00}}\pi_{11}^{n_{11}}\left(1-\pi_{11}\right)^{n_{10}}}\right], \tag{9}$$

where $\pi_{ij} = \Pr\left(I_t = j \middle| I_{t-1} = i\right)$ and $\pi_{obs} = \dfrac{n_{01}+n_{11}}{n_{00}+n_{01}+n_{10}+n_{11}}$. This test statistic has χ^2 probability distribution with one degree of freedom.

4 Results

Particular models described in the previous section will be applied on data series of log-returns calculated from adjusted closing prices of S&P 500 index over the period January 3, 1950 to December 31, 2012.[2] The length of utilized series of log-returns is 15,850 observations. The distribution of log-returns in time and evolution of the adjusted closing price is depicted in Figure 1.

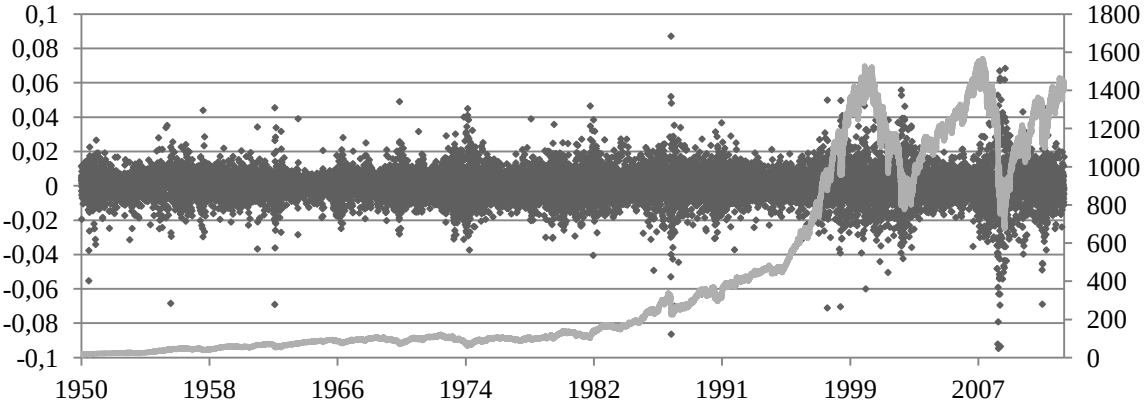

Figure 1 Distribution of log-returns (black, left axis) and evolution of the adjusted closing prices (gray, right axis) of S&P 500 index

It can be observed that although the most of the returns are in interval <-2%, 2%>, the log-returns around ±4% are not exceptional. There is even a quantity of losses around 7%. This corresponds with the high value of the kurtosis (30.683) of the log-returns. It is also interesting that there are more profits than losses (the median is 0.046%), but the average loss is bigger than the average profit (the mean is 0.028%, skewness is -1.301). Moreover, from time to time, there can be observed periods with significantly higher volatility than in the rest of the series. The periods with increased volatility is mostly characterized by the decline of the index value (e.g. 1962, 1987-1988, 1998, 200-2003, 2007-2009). From this observation we can make an assumption that GJR model should better fit the data and thus provide better VaR estimation.

Parameters of all four models were firstly estimated on the whole dataset. For all four assumed models, the first two autoregressive coefficients and only first order of coefficients for volatility equation were found statistically significant (thus we further assume only the 2-1-1 models). According to both the values of log-likelihood function (LLF) as well as Bayesian information criteria (BIC)[3] the models with the Student distribution are superior to those with Gaussian distribution and GJR models are superior to GARCH models (also GARCH-t is superior to GJR-n). While the GJR model is the generalization of GARCH model, the unrestricted GJR model can be tested against restricted GARCH model by means of likelihood ratio test. Both GJR-n and GJR-t are statistically accepted compared to GARCH-n and GARCH-t respectively (p-values of both tests are close to 0). This suggests that GJR model should perform better.

The models (each with $R=2$, $P=1$, $Q=1$) were back-tested with different periods utilized for parameters estimation and the ratio of observed quantity of exceptions to the expected quantity were recorded. Different probability levels are assumed: 0.5% and 15% (required by legal regulations for insurance companies); 1% (required by legal regulations for banks); 5% (original methodology proposed by JP Morgan); 20% and 25% for better clarification. The results are depicted on Figure 2. The ratio should be ideally equal to one or at least close to one (as we want the quantity of observed exceptions to be approximately equal to the expected quantity). As can be seen, for GARCH model (both with Gaussian and Student innovations) the period utilized for parameters estimation is not important and model is applicable only for α bigger or equal to 5% (for $\alpha = 1\%$ the ratio is oscillating around 1.25 and for $\alpha = 0.5\%$ it is oscillating around 1.9, i.e. the observed quantities of exceptions are 1.25 and 1.9 times higher). Model GJR-n provides even worse results than GARCH models: for $\alpha = 1\%$ the ratio converges to two; for $\alpha = 0.5\%$ the ratio converges to 2.5; for $\alpha = 5\%$ the ratio is clearly bigger than 1. The best results are obtained by application of GJR-t model: for $\alpha = 1\%$ and $\alpha = 0.5\%$ the ratios converge to

[2] The dataset was obtained from finance.yahoo.com webserver.

[3] Bayesian information criteria (BIC) is computed as $\log(n) \cdot k - 2 \cdot LLF$, where n is the number of observations, k is the quantity of parameters to be estimated and LLF is the value of log-likelihood function.

1; however for α bigger or equal to 5% the ratios are slightly lower than 1 (it is in interval 0.9-1.0); the curves are smoother than for other models, i.e. the results are not sensitive to small changes in the length of estimation period. From the graph we can see that it is better to utilize longer period for parameters estimation (in our case we further assume 250 days).

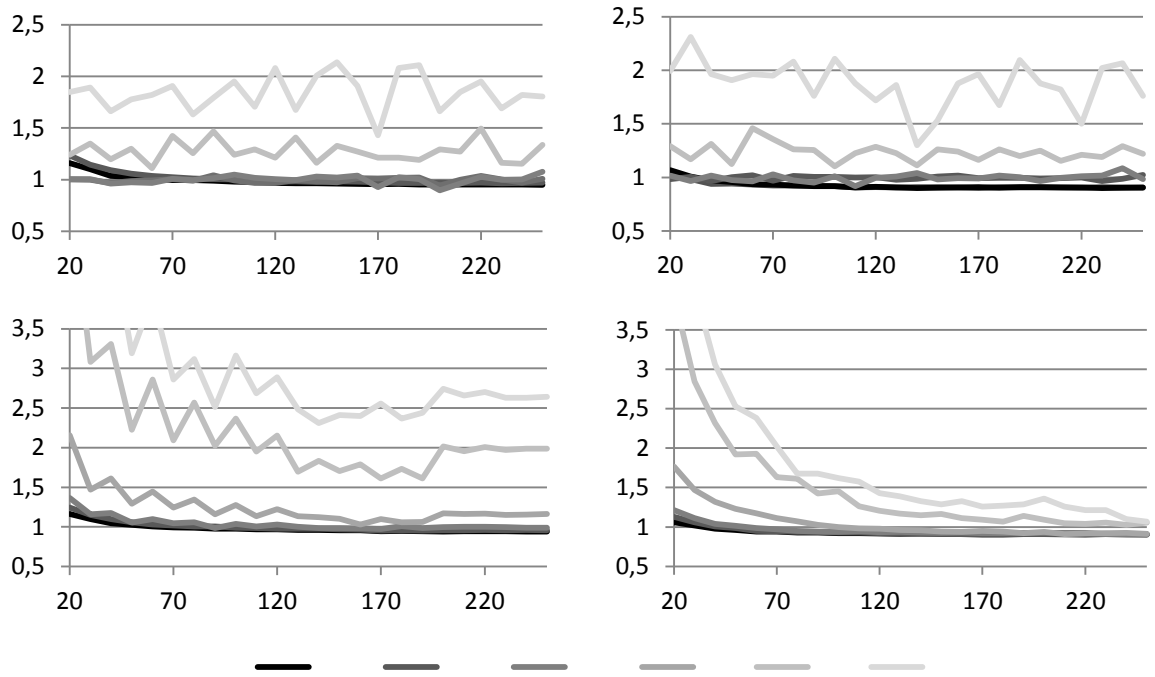

Figure 2 The ratios of observed and expected quantities of exceptions for different periods utilized for parameters estimation (from 20 days till 250 days with the step of 10 days) and different models (from top GARCH-n, GARCH-t, GJR-n, GJR-t)

From the Figure 1 we can see that for $\alpha = 15\%$ and $\alpha = 5\%$ the best model is GARCH-n model. There are not big changes in results for different periods utilized for parameter estimation. The backtesting results of the model for 30 days period are depicted in Table 1. There are shown assumed and observed number of exceptions as well as the p-values of statistical tests defined in previous section. The p-value is a measure of probability of exceptions (to be more specific: the quantity of exceptions in K-test; their distribution over time for C-test) occurring by the chance, assuming the null hypothesis of the test to be true. In statistics, the values 5% or 1% are usually assumed as the critical levels. As can be seen for $\alpha = 15\%$ and $\alpha = 5\%$ the model perform well – the quantity of exceptions is really close to the expected quantity (p-values are even higher than 90%). Also the bunching of exceptions can be rejected (C test). On the other hand, the model performs poorly (the quantity of exceptions is too high) for $\alpha = 1\%$ and $\alpha = 0.5\%$.

	15%	5%	1%	0.5%
Assumed quantity of exceptions	2077.35	692.45	138.49	69.25
Quantity of observed exceptions	2076.00	695.00	187.00	131.00
P-value of K-test	97.44%	92.08%	0.01%	0.00%
P-value of C-test	43.41%	98.30%	76.85%	52.60%

Table 1 Backtesting results for AR(2)-GARCH(1,1)-n model estimated from 30 days

In Table 2 the backtesting results of AR-GJR-t model are shown (period utilized for parameters estimation is 250 days). This model is acceptable for $\alpha = 1\%$ and $\alpha = 0.5\%$ – the quantities of exceptions are close to the assumed and bunching of exceptions can be rejected. On the other hand, the risk for $\alpha = 15\%$ and $\alpha = 5\%$ is overestimated. For both models we can reject the dependence of exceptions for all α.

	15%	5%	1%	0.5%
Assumed quantity of exceptions	2077.35	692.45	138.49	69.25
Quantity of observed exceptions	1873.00	633.00	146.00	74.00
P-value of K-test	0.00%	1.87%	52.49%	57.10%
P-value of C-test	81.61%	83.65%	29.16%	6.74%

Table 2 Backtesting results for AR(2)-GJR(1,1)-t model estimated from 250 days

5 Conclusion

Unexpectedly high decreases in the prices of financial assets and swift changes in volatility are challenging task for any risk model. In this article we utilized the parametric model for Value at Risk estimation based on Gaussian and Student distributions with conditional mean and variance. For modeling the conditional variance we assumed GARCH and GJR models and we examined the influence of the periods utilized for parameters estimation on backtesting results. It was found out that for Value at Risk estimation on probability levels $\alpha = 15\%$ and $\alpha = 5\%$ for S&P 500 index the best model is GARCH model with Gaussian innovations. For Value at Risk estimation on probability levels $\alpha = 1\%$ or $\alpha = 0.5\%$ and for investment into S&P 500 index the most accurate model was GJR model with Student innovations. For this model, in this particular case longer periods for parameters estimation should be applied.

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Mixed-correlated ARFIMA processes for power-law cross-correlations

Ladislav Krištoufek [1]

Abstract. We introduce a general framework of the Mixed-correlated ARFIMA (MC-ARFIMA) processes which allows for various specifications of univariate and bivariate long-term memory. Apart from a standard case when $H_{xy} = \frac{1}{2}(H_x + H_y)$, MC-ARFIMA also allows for processes with $H_{xy} < \frac{1}{2}(H_x + H_y)$ but also for long-range correlated processes which are either short-range cross-correlated or simply correlated. The major contribution of MC-ARFIMA lays in the fact that the processes have well-defined asymptotic properties for H_x, H_y and H_{xy}, which are derived in the paper, so that the processes can be used in simulation studies comparing various estimators of the bivariate Hurst exponent H_{xy}. Moreover, the framework allows for modeling of processes which are found to have $H_{xy} < \frac{1}{2}(H_x + H_y)$.

Keywords: power-law cross-correlations, long-term memory, econophysics

1 Introduction

Studying long-range correlations has become a stable part of financial econometrics and econophysics in recent years. Long-term memory has been studied for stock indices, bonds, exchange rates, commodities, interest rates and others [2, 8, 6]. Most recently, a focus has been put on analysis of long-range (power-law) cross-correlations as an addition to more traditional research of power-law auto-correlations. Similarly to the power-law autocorrelation case, we assume that a pair of time series is long-range cross-correlated if their cross-correlation function $\rho_{xy}(k)$ follows an asymptotic power law so that $\rho_{xy}(k) \propto k^{2H_{xy}-2}$ as $k \to +\infty$ where H_{xy} is the bivariate Hurst exponent. For $H_{xy} > 0.5$, we have cross-persistent processes which possess non-zero cross-correlations even for very long lags. Contrary to the univariate case, we can have both positively and negatively cross-persistent process with $H_{xy} > 0.5$ due to different properties of the auto- and cross-correlation functions.

As the financial time series provide sufficient number of observations and they have been shown to possess long-range auto-correlations (volatility, traded volume, signs of changes and absolute returns to name the most frequently studied ones), these also create an appropriate setting for the power-law cross-correlations analysis [17, 23, 9, 13]. For estimation of the bivariate Hurst exponent H_{xy}, several estimators have been proposed. Podobnik & Stanley [19] propose the detrended cross-correlation analysis (DCCA) as a bivariate generalization of the detrended fluctuation analysis (DFA) [16]. Zhou [25] then generalizes the method to the multifractal setting with the multifractal detrended cross-correlations analysis (MF-DXA). Kristoufek [11] generalizes the height-height correlation analysis [4, 5] and the generalized Hurst exponent approach [7] into the multifractal height cross-correlation analysis (MF-HXA). The detrending moving average (DMA) [1] is generalized by He & Chen [10] forming the detrended moving-average cross-correlation analysis (DMCA). The most recent estimator from the time domain is proposed by Wang *et al.* [24] – multifractal cross-correlation analysis based in statistical moments (MFSMXA). However, relatively little attention has been given to processes which can be characterized as cross-persistent with a specific bivariate Hurst exponent.

Several processes that possess such long-term correlations have been proposed in the literature. The most frequently discussed and applied ones are the multivariate generalizations of the well-established fractionally integrated ARMA processes (usually labeled as FARIMA and ARFIMA) – VARFIMA or MVARFIMA processes [14] – and fractional Gaussian noise processes or fractional Brownian motions,

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which are their integrated version (these are labeled as fGn and fBm in the literature, respectively). The construction of the multivariate ARFIMA process implies that the bivariate Hurst exponent is the average of the separate Hurst exponents [14]. The same property holds for the fractional Brownian motion [3]. The long-range cross-correlations thus simply arise from the specification of these processes.

Lobato [12] and then in some detail Sela & Hurvich [21] discuss two types of fractionally integrated models – VARFI and FIVAR. VARFI is a vector autoregressive model with fractionally integrated innovations (or error terms), and FIVAR consists of fractionally integrated processes with innovations that come from a VAR model. Nielsen [15] discusses the case of the fractional cointegration in the bivariate long-term memory setting and shows that the coherence of the processes is equal to unity which implies that the bivariate Hurst exponent is the same as the separate Hurst exponents and so again is equal to their average.

All of the above mentioned processes possibly yield cross-persistence but only with the bivariate Hurst exponent equal to the average of the Hurst exponents of the separate processes. There are only two studies which propose models with H_{xy} different from $\frac{1}{2}(H_x + H_y)$. Sela & Hurvich [22] propose an anti-cointegration model, which is in fact a linear combination of ARFIMA processes with a subset of innovations (but not all pairs) being identical across the two processes. The model allows to control the separate Hurst exponents as well as the bivariate Hurst exponent as long as it is lower or equal to the average of the separate parameters. Podobnik *et al.* [18] introduce two-component ARFIMA processes, which are based on ARFIMA-like mixing of two processes. Unfortunately, the authors neither provide any clue how to control the bivariate parameter H_{xy} nor is it evident whether the processes are even stationary.

In this paper, we introduce a new kind of a bivariate process which we call the mixed-correlated ARFIMA process. The process allows to control for the separate and bivariate Hurst exponents as long as the bivariate one is not higher than the average of the separate ones, and additionally allows for short-range dependence as well.

2 Mixed-correlated ARFIMA framework

We start with a general framework of a bivariate series where each of the series consists of a linear combination of two ARFIMA(0,d,0) processes so that

$$\begin{aligned} x_t &= \alpha \sum_{n=0}^{+\infty} a_n(d_1)\varepsilon_{1,t-n} + \beta \sum_{n=0}^{+\infty} a_n(d_2)\varepsilon_{2,t-n} \\ y_t &= \gamma \sum_{n=0}^{+\infty} a_n(d_3)\varepsilon_{3,t-n} + \delta \sum_{n=0}^{+\infty} a_n(d_4)\varepsilon_{4,t-n}. \end{aligned} \tag{1}$$

Innovations are characterized by

$$\begin{aligned} \langle \varepsilon_{i,t} \rangle &= 0 \text{ for } i = 1,2,3,4 \\ \langle \varepsilon_{i,t}^2 \rangle &= \sigma_{\varepsilon_i}^2 \text{ for } i = 1,2,3,4 \\ \langle \varepsilon_{i,t}\varepsilon_{j,t-n} \rangle &= 0 \text{ for } n \neq 0 \text{ and } i,j = 1,2,3,4 \\ \langle \varepsilon_{i,t}\varepsilon_{j,t} \rangle &= \sigma_{ij} \text{ for } i,j = 1,2,3,4 \text{ and } i \neq j. \end{aligned} \tag{2}$$

In words, we have two processes and each one is a linear combination of two long-range correlated processes with possibly correlated innovations. Note that the separate long-term memory parameters d_1, d_2, d_3, d_4 can vary or be the same. We call the set of processes $\{x_t\}$ and $\{y_t\}$ as the mixed-correlated ARFIMA processes (MC-ARFIMA). As MC-ARFIMA is a new kind of process not discussed in the literature, even though these can be seen as a generalization of the anti-cointegration model of Sela & Hurvich [22], we shortly discuss its stationarity. For the wide-sense stationarity, it suffices to state that both $\{x_t\}$ and $\{y_t\}$ are linear combinations of two ARFIMA(0,d,0) processes with correlated innovations which are wide-sense stationary so that MC-ARFIMA processes are stationary as long as $0 \leq d_1, d_2, d_3, d_4 < 0.5$ [20]. Evidently, we have $\langle x_t \rangle = \langle y_t \rangle = 0$ and both processes have finite variance, i.e. $\langle x_t^2 \rangle \equiv \sigma_x^2 < +\infty$ and $\langle y_t^2 \rangle \equiv \sigma_y^2 < +\infty$ since the separate ARFIMA(0,d,0) processes have zero means and finite variances.

As ARFIMA(0,d,0) processes are long-range correlated, their linear combination is also long-range correlated. The higher d will dominate in the linear combination so that process $\{x_t\}$ is integrated of order $\max(d_1, d_2)$ and $\{y_t\}$ of order $\max(d_3, d_4)$. The separate processes are thus wide-sense stationary.

To show that $\{x_t\}$ and $\{y_t\}$ are also jointly wide-sense stationary, we need to show that $\rho_{xy}(k)$ does not depend on t. It can be easily shown that the cross-correlation function is dependent only on the parameters d_1, d_2, d_3, d_4, α, β, γ, δ and σ_{ij} (with $i, j = 1, 2, 3, 4$) and the processes $\{x_t\}$ and $\{y_t\}$ are thus also jointly wide-sense stationary. Based on the cross-correlation structure, the cross-power spectrum can be written as

$$\begin{aligned} f_{xy}(\lambda) = \frac{\alpha\gamma\sigma_{13}}{2\pi}\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty} a_k(d_1)a_l(d_3)\exp(i(k-l)\lambda) + \frac{\alpha\delta\sigma_{14}}{2\pi}\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty} a_k(d_1)a_l(d_4)\exp(i(k-l)\lambda) + \\ \frac{\beta\gamma\sigma_{23}}{2\pi}\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty} a_k(d_2)a_l(d_3)\exp(i(k-l)\lambda) + \frac{\beta\delta\sigma_{24}}{2\pi}\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty} a_k(d_2)a_l(d_4)\exp(i(k-l)\lambda) = \\ \frac{1}{2\pi}\Big[\alpha\gamma\sigma_{13}\left(1-\exp(i\lambda)\right)^{-d_1}\left(1-\exp(-i\lambda)\right)^{-d_3} + \alpha\delta\sigma_{14}\left(1-\exp(i\lambda)\right)^{-d_1}\left(1-\exp(-i\lambda)\right)^{-d_4} + \\ \beta\gamma\sigma_{23}\left(1-\exp(i\lambda)\right)^{-d_2}\left(1-\exp(-i\lambda)\right)^{-d_3} + \beta\delta\sigma_{24}\left(1-\exp(i\lambda)\right)^{-d_2}\left(1-\exp(-i\lambda)\right)^{-d_4}\Big]. \end{aligned} \quad (3)$$

Using the inverse Fourier transform and the Dirac delta function, we get

$$\begin{aligned} \rho_{xy}(n) = \frac{\alpha\gamma\sigma_{13}}{\sigma_x\sigma_y}\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty} a_k(d_1)a_l(d_3)\delta(n+k-l) + \frac{\alpha\delta\sigma_{14}}{\sigma_x\sigma_y}\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty} a_k(d_1)a_l(d_4)\delta(n+k-l) + \\ \frac{\beta\gamma\sigma_{23}}{\sigma_x\sigma_y}\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty} a_k(d_2)a_l(d_3)\delta(n+k-l) + \frac{\beta\delta\sigma_{24}}{\sigma_x\sigma_y}\sum_{k=0}^{+\infty}\sum_{l=0}^{+\infty} a_k(d_2)a_l(d_4)\delta(n+k-l) = \\ \frac{\alpha\gamma\sigma_{13}}{\sigma_x\sigma_y}\underbrace{\sum_{k=0}^{+\infty} a_k(d_1)a_{n+k}(d_3)}_{\approx\int_0^{+\infty} k^{d_1-1}(n+k)^{d_3-1}dk\propto n^{d_1+d_3-1}} + \frac{\alpha\delta\sigma_{14}}{\sigma_x\sigma_y}\underbrace{\sum_{k=0}^{+\infty} a_k(d_1)a_{n+k}(d_4)}_{\approx\int_0^{+\infty} k^{d_1-1}(n+k)^{d_4-1}dk\propto n^{d_1+d_4-1}} + \\ \frac{\beta\gamma\sigma_{23}}{\sigma_x\sigma_y}\underbrace{\sum_{k=0}^{+\infty} a_k(d_2)a_{n+k}(d_3)}_{\approx\int_0^{+\infty} k^{d_2-1}(n+k)^{d_3-1}dk\propto n^{d_2+d_3-1}} + \frac{\beta\delta\sigma_{24}}{\sigma_x\sigma_y}\underbrace{\sum_{k=0}^{+\infty} a_k(d_2)a_{n+k}(d_4)}_{\approx\int_0^{+\infty} k^{d_2-1}(n+k)^{d_4-1}dk\propto n^{d_2+d_4-1}}. \end{aligned} \quad (4)$$

The results are obtained by using the Stirling's approximation and by approximating the infinite sum by the definite integrals. As we are interested in the asymptotic case $n \to +\infty$, the scaling of $\rho_{xy}(n)$ will be dominated by the highest exponent. This leads us to several interesting settings.

Firstly, let's have $\alpha, \beta, \gamma, \delta \neq 0$ and $\sigma_{ij} \neq 0$ for all $i, j = 1, 2, 3, 4$. Labeling $H_i = d_i + 0.5$ for $i = 1, 2, 3, 4$, we have

$$\begin{aligned} H_x &= \max(H_1, H_2) \\ H_y &= \max(H_3, H_4) \end{aligned} \quad (5)$$

$$H_{xy} = \frac{\max(H_1+H_2, H_1+H_4, H_2+H_3, H_2+H_4)}{2} = \frac{\max(H_1, H_2)+\max(H_3, H_4)}{2} = \frac{H_x+H_y}{2}. \quad (6)$$

Therefore, if the innovations are correlated without restriction, we arrive at $H_{xy} = \frac{1}{2}(H_x + H_y)$.

Secondly, let's again have $\alpha, \beta, \gamma, \delta \neq 0$ and without loss on generality, let's have $\max(H_1, H_2) = H_1$ and $\max(H_3, H_4) = H_4$ so that $\{x_t\}$ is integrated of order d_1 (with $H_x = 0.5 + d_1$) and $\{y_t\}$ of order d_4 (with $H_y = 0.5 + d_4$). Moreover, assume that $\sigma_{23} = \sigma_{32} \neq 0$ and all the other covariances are equal to zero. From Eq. 4, this implies

$$H_{xy} = \frac{H_2+H_3}{2} \leq \frac{H_x+H_y}{2} = \frac{H_1+H_4}{2} = \frac{\max(H_1, H_2)+\max(H_3, H_4)}{2}. \quad (7)$$

The equality holds only if $H_1 = H_2$ and $H_3 = H_4$. For the other cases, it implies that the bivariate Hurst exponent H_{xy} is not equal to the average of the univariate Hurst exponents H_x and H_y while still

showing long-range cross-correlations, i.e. without $H_{xy} = 0.5$. Processes $\{x_t\}$ and $\{y_t\}$ are thus long-range cross-correlated but possess the power-law coherency in a similar manner as the anti-cointegration model of Sela & Hurvich [22].

Apart from the framework introduced above, we can slightly adjust the setting in the following way:

$$x_t = \alpha \sum_{n=0}^{+\infty} a_n(d_1)\varepsilon_{1,t-n} + \beta\varepsilon_{2,t}$$
$$y_t = \gamma\varepsilon_{3,t} + \delta \sum_{n=0}^{+\infty} a_n(d_4)\varepsilon_{4,t-n}. \tag{8}$$

We thus again have $\{x_t\}$ with the long-term memory parameter d_1 (with $H_x = 0.5 + d_1$) and $\{y_t\}$ with d_4 (with $H_y = 0.5 + d_4$) but $H_{xy} = 0.5$ since it can be easily shown that $\rho_{xy}(0) = \frac{\sigma_{23}}{\sigma_x\sigma_y}$ and $\rho_{xy}(k) = 0$ for $k \neq 0$. Therefore, we have two long-range dependent processes which are correlated but not cross-correlated.

In a similar manner, let's have the same assumptions about correlations and parameters $\alpha, \beta, \gamma, \delta$ as in the previous two cases (only innovations ε_2 and ε_3 are correlated) but let's adjust the model specification to

$$x_t = \alpha \sum_{n=0}^{+\infty} a_n(d_1)\varepsilon_{1,t-n} + \beta \sum_{n=0}^{+\infty} \theta_2^n\varepsilon_{2,t-n}$$
$$y_t = \gamma \sum_{n=0}^{+\infty} \theta_3^n\varepsilon_{3,t-n} + \delta \sum_{n=0}^{+\infty} a_n(d_4)\varepsilon_{4,t-n}. \tag{9}$$

Processes $\{x_t\}$ and $\{y_t\}$ are thus linear combinations of ARFIMA(0,d,0) and AR(1) processes. In this case, we again have $\{x_t\}$ with memory d_1 (and $H_x = 0.5 + d_1$) and $\{y_t\}$ with memory d_4 (and $H_y = 0.5 + d_4$). And as the only non-zero correlation between innovations is $\sigma_{23} = \sigma_{32}$, we have $H_{xy} = 0.5$ as the cross-correlations quickly vanish to zero. We thus have two long-range correlated processes $\{x_t\}$ and $\{y_t\}$, which are only short-range cross-correlated.

The MC-ARFIMA framework thus provides quite a wide range of possible model specifications yielding long-range cross-correlated processes which can be either long-range cross-correlated (with or without power law coherency), short-range cross-correlated, pairwise correlated or uncorrelated. Generally, the framework allows for even more possible specifications. In the next section, we present several simulated processes[1] based on specifications developed in this section.

3 Illustrative examples

As Model 1, we use the specification of Eq. 1 with $\alpha = \delta = 0.2$, $\beta = \gamma = 1$, $d_1 = d_4 = 0.4$, $d_2 = d_3 = 0.3$, $\sigma_i^2 = 1$ for $i = 1, 2, 3, 4$ and $\sigma_{23} = 0.9$ with $T = 10000$. We thus obtain two long-range correlated processes with $H_x = H_y = 0.9$ which are also long-range cross-correlated with $H_{xy} = 0.8$. The values of parameters $\alpha, \beta, \gamma, \delta$ are selected to highlight the cross-persistence. The cross-correlation function between simulated processes is shown in Fig. 1. The cross-correlations evidently follow a very slow decay.

As Model 2, we use the specification given in Eq. 9 with $\alpha = \beta = \gamma = \delta = 1$, $d_1 = d_4 = 0.4$, $\theta_2 = \theta_3 = 0.8$, $\sigma_i^2 = 1$ for $i = 1, 2, 3, 4$ and $\sigma_{23} = 0.9$ with $T = 10000$. Model 2 thus represents two processes which are long-range correlated but only short-range cross-correlated. A rapid (exponential) decay of cross-correlation function is presented in Fig. 1. Note that even for a rather strong short-term memory parameter $\theta = 0.8$, the cross-correlations vanish quickly and after lag 15, the cross-correlations are close to zero. The exponential decay to insignificant cross-correlation values is more evident from the cross-correlation function.

As Model 3, we use the specification given in Eq. 8 with $\alpha = \beta = \gamma = \delta = 1$, $d_1 = d_4 = 0.4$, $\sigma_i^2 = 1$ for $i = 1, 2, 3, 4$ and $\sigma_{23} = 0.9$ with $T = 10000$. Model 3 thus represents two processes which are long-range correlated but only correlated (not cross-correlated). Form of the cross-correlation function is presented

[1] R-project codes for MC-ARFIMA are available at http://staff.utia.cas.cz/kristoufek/Ladislav_Kristoufek/Codes.html.

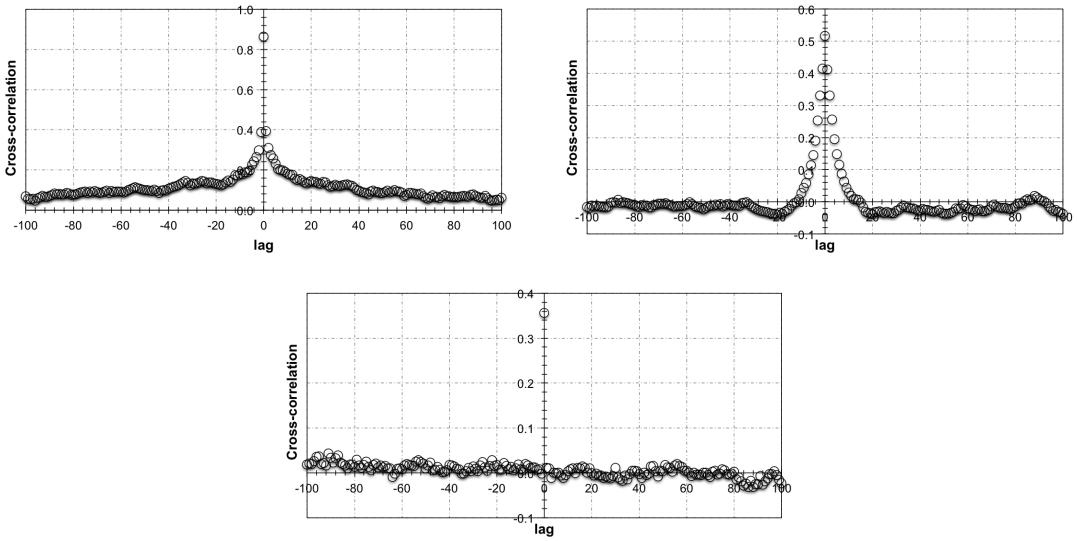

Figure 1: **Cross-correlation functions of Model 1, Model 2 and Model 3.** Cross-correlation function shows a very slow decay for both negative and positive lags for Model 1 (top left), exponential decay for Model 2 (top right) and correlation only at lag 0 for Model 3 (bottom).

in Fig. 1. We observe that indeed the cross-correlation function shows non-zero values only for the zeroth lag which is evident also from the scatter plots for specific lags.

4 Conclusions

Analysis of slowly decaying cross-correlations has recently become a widely and frequently discussed topic in the econophysics literature. However, only little attention has been put on the actual processes that can produce the bivariate Hurst exponent which is different from the average of the separate (univariate) Hurst exponents of the analyzed processes. In this paper, we have introduced a rather general framework of the Mixed-correlated ARFIMA (MC-ARFIMA) processes which allows for various specifications. Apart from a standard case when $H_{xy} = \frac{1}{2}(H_x + H_y)$, it also allows for processes with $H_{xy} < \frac{1}{2}(H_x + H_y)$ but also for long-range correlated processes which are either short-range cross-correlated or simply correlated. The major contribution of MC-ARFIMA lays in the fact that the processes have well-defined asymptotic properties for H_x, H_y and H_{xy} so that the processes can be used in simulation studies comparing various estimators of the bivariate Hurst exponent H_{xy}. Moreover, the framework allows for modeling of processes which are found to have $H_{xy} < \frac{1}{2}(H_x + H_y)$.

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Measuring capital market efficiency: Long-term memory, fractal dimension and approximate entropy

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Abstract. We utilize long-term memory, fractal dimension and approximate entropy as input variables for the Efficiency Index [Kristoufek & Vosvrda (2013), Physica A 392]. This way, we are able to comment on stock market efficiency after controlling for different types of inefficiencies. Applying the methodology on 38 stock market indices across the world, we find that the most efficient markets are situated in the Eurozone (the Netherlands, France and Germany) and the least efficient ones in the Latin America (Venezuela and Chile).

Keywords: capital market efficiency, long-range dependence, fractal dimension, approximate entropy

1 Introduction

Efficient markets hypothesis (EMH) is one of the cornerstones of the modern financial economics. Since its introduction in 1960s [10, 24, 11], EMH has been a controversial topic. Nonetheless, the theory still remains a stable part of the classical financial economics. Regardless of its definition via a random walk [10] or a martingale [24], the main idea of EMH is that risk-adjusted returns cannot be systematically predicted and there can be no long-term profits above the market profits assuming the same risk. The EMH definition is also tightly connected with a notion of rational homogenous agents and Gaussian distribution of returns. Both these assumptions have been widely disregarded in the literature [6].

There are several papers ranking various financial markets with respect to their efficiency. Research group around Di Matteo [8, 9, 7] show that the correlations structure of various assets (stocks, exchange rates and interest rates) is connected to the development of the specific countries and stock markets. In the series of papers, Cajueiro & Tabak [3, 4, 2, 5] study the relationship between the long-term memory parameter H and development stages of the countries' economy. Both groups find interesting results connecting persistent (long-term correlated) behavior to the least developed markets but also anti-persistent behavior for the most developed ones. Lim [20] investigates how the ranking of stock markets based on Hurst exponent evolves in time and reports that the behavior can be quite erratic. Zunino *et al.* [28] utilize entropy to rank stock markets to show that the emergent/developing markets are indeed less efficient than the developed ones. Even though the ranking is provided in these studies, the type of memory taken into consideration (either long-term memory or entropy/complexity) is limited and treated separately. In this paper, we utilize the Efficiency Index proposed by Kristoufek & Vosvrda [19] incorporating long-term memory, fractal dimension and entropy to control for various types of correlations and complexity using a single measure. Basing the definition of the market efficiency simply on no correlation structure, we can state the expected values of long-term memory, fractal dimension and entropy to construct an efficiency measure based on a distance from the efficient market state. The procedure is then applied on 38 stock indices from different parts of the world and we show that the most efficient markets are indeed the most developed ones – the Western European markets and the US markets – and the least efficient ones are situated in the Latin America and South-East Asia.

The paper is structured as follows. In Section 2, we provide brief description of used methodology focusing on long-term memory, fractal dimension, entropy and efficiency measure. Section 3 introduces the dataset and describes the results. Section 4 concludes.

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2 Methodology

2.1 Long-term memory

Long-term memory (long-range dependence) is usually characterized in time domain by a power-law decay of autocorrelation function and in frequency domain by a power-law divergence of spectrum close to the origin. More specifically, the autocorrelation function $\rho(k)$ with lag k of a long-range correlated process decays as $\rho(k) \propto k^{2H-2}$ for $k \to +\infty$, and the spectrum $f(\lambda)$ with frequency λ of a long-range correlated process diverges as $f(\lambda) \propto \lambda^{1-2H}$ for $\lambda \to 0+$. The characteristic parameter of the long-term memory Hurst exponent H ranges between $0 \leq H < 1$ for stationary processes. The breaking value of 0.5 indicates no long-term memory so that the autocorrelations decay rapidly (exponentially). For $H > 0.5$, the series is persistent with strong positive correlations characteristic by a trend-like behavior while still remaining stationary. For $H < 0.5$, the series is anti-persistent and it switches the direction of increments more frequently than a random process does. As the inputs to the Efficiency Index, we utilize two estimators from the frequency domain – the local Whittle [23] and GPH estimators [13] – which are more appropriate for rather short financial series with a possible weak short-term memory [25, 26], which can easily bias the time domain estimators [27, 1, 17]. Moreover, the frequency domain estimators have well-defined asymptotic properties and the selected two are consistent and asymptotically normal estimators.

2.2 Fractal dimension

Fractal dimension D is a measure of roughness of the series and can be taken as a measure of local memory of the series [19]. For a univariate series, it holds that $1 < D \leq 2$. For self-similar processes, the fractal dimension is connected to the long-term memory of the series so that $D + H = 2$. This can be attributed to a perfect reflection of a local behavior (fractal dimension) to a global behavior (long-term memory). However, the relation usually does not hold perfectly for the financial series so that both D and H give different insights on the dynamics of the series. In general, $D = 1.5$ holds for a random series with no local trending or no local anti-correlations. For a low fractal dimension $D < 1.5$, the series is locally less rough and thus resembles a local persistence. Reversely, a high fractal dimension $D > 1.5$ is characteristic for rougher series with local anti-persistence. For purposes of the Efficiency Index, we utilize Hall-Wood and Genton estimators [14, 15, 16, 12].

2.3 Approximate entropy

Entropy can be taken as a measure of complexity of the system. The systems with high entropy can be characterized by no information and are thus random and reversely, the series with low entropy can be seen as deterministic [22]. The efficient market can be then seen as the one with maximum entropy and the lower the entropy, the less efficient the market is. For purposes of the Efficiency Index, we need an entropy measure which is bounded. Therefore, we utilize the approximate entropy introduced by Pincus [21] which is bounded between 0 (completely deterministic behavior) and 1 (completely random behavior).

2.4 Capital market efficiency measure

According to Kristoufek & Vosvrda [18, 19], the Efficiency Index (EI) is defined as

$$EI = \sqrt{\sum_{i=1}^{n} \left(\frac{\widehat{M_i} - M_i^*}{R_i} \right)^2},$$

where M_i is the ith measure of efficiency, $\widehat{M_i}$ is an estimate of the ith measure, M_i^* is an expected value of the ith measure for the efficient market and R_i is a range of the ith measure. In words, the efficiency measure is simply defined as a distance from the efficient market specification based on various measures of the market efficiency. In our case, we consider three measures of market efficiency – Hurst exponent H with an expected value of 0.5 for the efficient market ($M_H^* = 0.5$), fractal dimension D with an expected

value of 1.5 ($M_D^* = 1.5$) and the approximate entropy with an expected value of 1 ($M_{AE}^* = 1$). The estimate of Hurst exponent is taken as an average of estimates based on GPH and the local Whittle estimators. The estimate of the fractal dimension is again taken as an average of the estimates based on the Hall-Wood and Genton methods. For the approximate entropy, we utilize the estimate described in the corresponding section. However, as the approximate entropy ranges between 0 (for completely deterministic market) and 1 (for random series), we need to rescale its effect, i.e. we use $R_{AE} = 2$ for the approximate entropy and $R_H = R_D = 1$ for the other two measures so that the maximum deviation from the efficient market value is the same for all measures.

3 Application and discussion

We analyze 38 stock indices from various locations – the complete list is given in Tab. 1 – between January 2000 and August 2011. Various phases of the market behavior – DotCom bubble, bursting of the bubble, stable growth of 2003-2007 and the current financial crisis – are thus covered in the analyzed period. The indices cover stock markets in both North and Latin Americas, Western and Eastern Europe, Asia and Oceania so that markets at various levels of development are included in the study. The logarithmic returns are asymptotically stationary (according to the KPSS test), leptokurtic and returns of majority of the indices are negatively skewed (the results are available upon request).

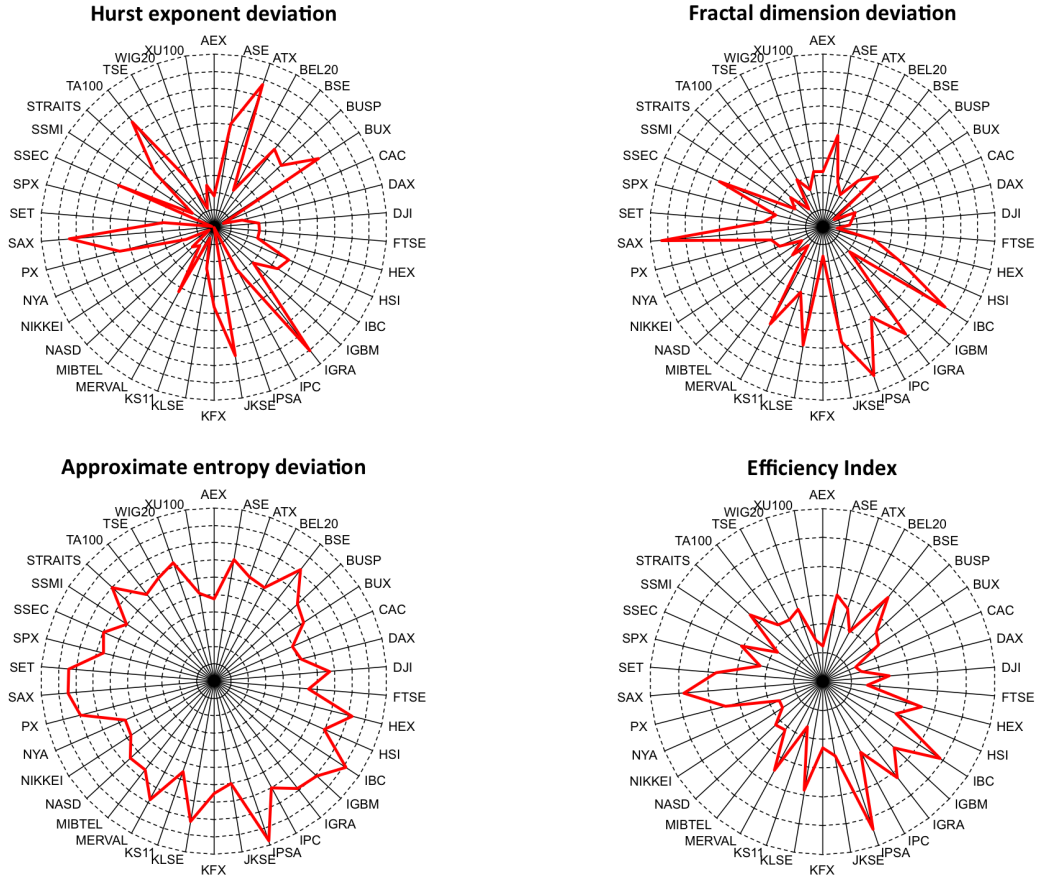

Figure 1: *Hurst exponent, fractal dimension, approximate entropy and efficiency index for analyzed indices.*

Let us now turn to the results. In Fig. 1, all the results are summarized graphically. For the utilized three measures – Hurst exponent, fractal dimension and approximate entropy – we present the absolute deviations from the expected values of the efficient market for comparison. For the Hurst exponent estimates, we observe huge diversity – between practically zero (for IPSA of Chile) and 0.18 (for Peruvian IGRA). Interestingly, for some of the most developed markets, we observe Hurst exponents well below 0.5 (Tab. 1 gives the specific estimates) which is, however, in hand with results of other authors [9, 7]. The results for the fractal dimension again vary strongly across the stock indices. The highest deviation

is observed for the Slovakian SAX (0.19) and the lowest for the FTSE of the UK (0.02). In Tab. 1, we observe that apart from FTSE, all the other stock indices possess the fractal dimension below 1.5 which indicates that the indices are locally persistent, i.e. in some periods, the indices experience significant positively autocorrelated behavior which is well in hand with expectations about the herding behavior during critical events. The approximate entropy estimates are more stable across indices compared to the previous two cases. The highest deviation from the expected value for the efficient market is observed for the Chilean IPSA (0.98) and the lowest for the Dutch AEX (0.48). Evidently, all the analyzed stock indices are highly complex as the approximate entropy is far from the ideal (efficient market) value of 1 and such complexity is not sufficiently covered by the other two applied measures. The inclusion of the approximate entropy into the Efficiency Index thus proves its worth.

Table 1: Ranked stock indices according to the Efficiency Index

Index	Country	Hurst exponent	Fractal dimension	Approximate entropy	Efficiency index
AEX	Netherlands	0.5358	1.4356	0.5246	0.0619
CAC	France	0.5118	1.4592	0.5059	0.0628
DAX	Germany	0.5334	1.4646	0.4807	0.0698
XU100	Turkey	0.5493	1.4350	0.4870	0.0724
FTSE	UK	0.4470	1.5171	0.4500	0.0787
NYA	USA	0.5348	1.4457	0.4418	0.0821
NIKKEI	Japan	0.5063	1.4716	0.4285	0.0825
KS11	South Korea	0.5137	1.4204	0.4473	0.0829
SSMI	Switzerland	0.5297	1.4617	0.3983	0.0929
BEL20	Belgium	0.5481	1.4574	0.3869	0.0981
MIBTEL	Italy	0.5267	1.4728	0.3525	0.1063
NASD	USA	0.5340	1.4526	0.3428	0.1114
SPX	USA	0.5026	1.4437	0.3405	0.1119
KFX	Denmark	0.5927	1.4665	0.3516	0.1148
DJI	USA	0.4477	1.4685	0.3284	0.1165
BUX	Hungary	0.6448	1.4844	0.3811	0.1170
TSE	Canada	0.5626	1.4375	0.3272	0.1210
TA100	Israel	0.6536	1.4739	0.3648	0.1251
BUSP	Brazil	0.6055	1.4142	0.3435	0.1262
JKSE	Indonesia	0.6505	1.3657	0.3986	0.1311
WIG20	Poland	0.5232	1.4545	0.2790	0.1326
ATX	Austria	0.6744	1.4455	0.3669	0.1336
HSI	Hong-Kong	0.5945	1.4033	0.3033	0.1396
IPC	Mexico	0.5550	1.3817	0.2991	0.1398
ASE	Greece	0.6210	1.3926	0.2911	0.1518
SSEC	China	0.6205	1.3698	0.3019	0.1533
IGBM	Spain	0.5615	1.4581	0.1912	0.1691
STRAITS	Singapore	0.5937	1.4500	0.2027	0.1702
PX	Czech Rep	0.6124	1.4386	0.2053	0.1743
MERVAL	Argentina	0.5850	1.3729	0.2225	0.1745
HEX	Finland	0.5524	1.4385	0.1747	0.1768
BSE	India	0.6139	1.4313	0.1842	0.1841
SET	Thailand	0.5591	1.4311	0.1590	0.1851
KLSE	Malaysia	0.5489	1.3620	0.1773	0.1906
IGRA	Peru	0.6806	1.3435	0.2160	0.2108
SAX	Slovakia	0.6673	1.3132	0.1534	0.2421
IBC	Venezuela	0.5881	1.3308	0.0890	0.2439
IPSA	Chile	0.4997	1.3187	0.0239	0.2711

Putting the estimates of the three measures together, we get the Efficiency Index which is also graphically presented in 1. For the ranking of indices according to their efficiency, we present Tab. 1. The most efficient stock market turns out to be the Dutch AEX closely followed by the French CAC and the German DAX. We can observe that the most efficient markets are usually the EU (or rather Eurozone) countries followed by the US markets and other developed markets from the rest of the world – Japanese NIKKEI, Korean KS11, Swiss SSMI. The least efficient part of the ranking is dominated by the Asian

and the Latin American countries. At the very end, we have the Slovakian SAX, Venezuelan IBC and Chilean IPSA. The efficiency of the stock markets is thus strongly geographically determined which is connected to the stage of development of the specific markets.

4 Conclusions

We have utilized long-term memory, fractal dimension and approximate entropy as input variables for the Efficiency Index [19]. This way, we are able to comment on stock market efficiency after controlling for different types of inefficiencies. Applying the methodology on 38 stock market indices across the world, we find that the most efficient markets are situated in the Eurozone (the Netherlands, France and Germany) and the least efficient ones in the Latin America (Venezuela and Chile). The Efficiency Index thus well corresponds to the expectation that the stock market efficiency is connected to the development of the market.

Acknowledgements

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Portfolio competitions and rationality

Aleš Antonín Kuběna[1], Martin Šmíd[2]

Abstract. We study investment competitions in which the players with highest achieved returns are rewarded by fixed prizes. We show that, under realistic assumptions, a game the participants play lacks a pure equilibrium and that the "max-min" solution of the game lies in one of the extremal points of the feasible set, namely in the one having maximal probability that the portfolio return falls into its normal cone. We analyse empirically a portfolio competition held recently by the Czech portal "lidovky.cz"; we find that the majority of people do not behave according to the game-theoretic conclusions. Consequently, searching for factors influencing a choice of particular stocks, we find that that the only significant determinant of the choice is a size of the stock's issuer.

Keywords: portfolio competition, game theory, behavioural finance

JEL classification: C7, D03
AMS classification: 91B99

1 Introduction

With the public availability of the Internet, various investment competitions started to be held, usually with the following rules: each player obtains a virtual sum of money which he has to divide into several (real-life) financial assets. After a pre-determined time, gains of the players are evaluated (according to the real-life prices) and a selected number of the best players are rewarded by monetary prizes. If several participants achieve the same evaluation, the prize(s) divide(s) equally.

Whether the organizers realize it or not, those games are far from being a simulation of a real-life investment; the main reason for this is the fact that the objectives of "players" in real life differ from those in the game. In particular, while the actual return is simultaneously the gain in real life, which forces a risk averse individual to diversify (see [1]), only the best returns bring positive gains in the competition which, as shown in Section 2 of the present paper, makes even a risk-averse participant to take positions which are the most risky ones from the point of view of portfolio selection theory. In particular, the only portfolios getting a positive max-min gain are those lying in extremal points of the feasible set.

Analysing an actual portfolio game held by Czech internet portal "lidovky.cz", however, we found that people do not behave according to such a conclusion. As shown in Section 3, only 16.8% of participants chose portfolios lying in extremal points.

Section 4 tries to give alternative explanations of the player's behaviour. It is shown that, out of several fundamental and technical-analysis indicators, the only significant factor of a stock's selection is the size of the stock's issuer.

Even if all the game theoretic results and the method of our subsequent analysis are rather straightforward, we regard our work as original because, to our best knowledge, there is no other paper analysing this type of competition.

The paper is concluded by Section 5.

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2 Game Theoretic Approach

Denote $R \in \mathbb{R}^n$ a random vector of asset returns, possibly discounted by a deterministic risk free rate r_0, having an absolutely continuous joint distribution such that

$$\operatorname{supp}(R) = (-1, \infty)^n.$$

We assume that the set of feasible actions of the players is defined as

$$S = \{\pi \in \mathbb{R}^n : \gamma \leq 1'\pi \leq 1, 0 \leq \pi_i \leq \alpha, 1 \leq i \leq n\}$$

where α and γ are some constants. In the definition above, π stands for a vector fractions of the initial sum invested into the individual assets.

We assume the competitors to be risk averse, the i-th one having a strictly increasing utility function u_i. For simplicity, we assume that (the participants act as if) there is only single prize which implies that the motility of the i-th player is

$$v_i = \mathbb{E}(u_i(Z_i))$$

where Z_i is a gain of the player given by

$$Z_i \quad = Z_i(\pi_1, \ldots, \pi_m) = \quad \begin{cases} \frac{1}{k_i} & \text{if } R \in \Gamma_i \\ 0 & \text{otherwise} \end{cases}$$

Here

- $\Gamma_i = \Gamma_i(\pi_1, \ldots, \pi_m) := \{r : \pi_i' r > \pi_j' r, j \notin K_i\}$
- $K_i = \{1 \leq j \leq m : \pi_j' R = \pi_i' R\}$,
- $k_i = |K_i|$
- $\pi_1, \pi_2, \ldots, \pi_m$ are the strategies (portfolios) of individual players.

Remark 1. The vector $(Z_1, \ldots, Z_m)$ is uniquely defined by $\rho = \frac{R}{|R|}$ a.s. where the support of ρ is the unit sphere.

In the present paper, assume all the strategies be the pure ones, i.e. deterministic. Then

Proposition 1. *The set*

$$K_i = \{1 \leq j \leq m : \pi_j' R = \pi_i' R\}$$

is a.s. deterministic.

Proof. Because $\mathbb{P}[w'S = 0] = 0$ for any absolutely continuous random vector S, any deterministic vector $w \neq 0$ and a constant c, we have

$$e_{i,j} := 1\{\pi_j' R = \pi_i' R\} = 1\{(\pi_j - \pi_i)' R = 0\} \overset{a.s.}{=} 1\{\pi_j = \pi_i\}$$

i.e., is deterministic, so it has to be

$$K_i = \{j : e_{i,j} = 1\}.$$

□

Corollary 1. $v_i = u_i(1/k_i)p_i$ where $p_i = \mathbb{P}(R \in \Gamma_i)$.

The following Proposition shows that some portfolios would never win regardless of the distribution of R

Proposition 2. *If π_i is not an extremal point of $C := \operatorname{conv}(\pi_1, \pi_2, \ldots, \pi_{\mathrm{m}})$ then $Z_i \equiv 0$ a.s.*

Proof. Assume WLOG that the first k strategies are extremal points of C. From the basic convex analysis (see [2]) we have that $\pi_i = \sum_{j=1}^{k} \lambda_j \pi_j$ where $\lambda_j \geq 0$. For Z_i to be positive, there should exists at least one possible return value r such that $\lambda_j r'(\pi_i - \pi_j) > 0$ for all $j \leq k$ giving $r'0 > 0$ by summing over all j. □

The following result says that the best max-min strategy is to take the most "advantegeous" corner of S; however, no equilibrium in pure strategies exists whenever there do not exist a group of stocks strongly outperforming the rest.

Theorem 3. *Denote $E = (e_1, \ldots, e_r)$ the set of extremal points of S and put*

$$\sigma_i = \mathbb{P}(\rho \in N_S(e_i))$$

where

$$N_S(e) = \{r : r'(\pi - e) \leq 0 \text{ for all } \pi \in S\}$$

is a normal cone.
(i) If $m \geq n + 2$ then

$$\max_{\pi_i} \min_{\pi_j, j \neq i} v_i = 0$$

whenever $\pi_i \notin E$.
(ii)

$$\max_{\pi_i} \min_{\pi_j, j \neq i} v_i \geq u_i(\frac{1}{m})\sigma_i$$

whenever $\pi_i \in E$.
(iii) Denote $I_j = \left\lfloor \frac{1}{\alpha} \right\rfloor$. If there is a player, say the i-th one, such for each $j \geq 1$ there exist $j_1, j_2, \ldots, j_{I_i+1}$, differing from j fulfilling

$$\mathbb{P}(R_j \geq R_{j_k}) > \frac{u_i(\frac{1}{m})}{u_i(1)}, \qquad 1 \leq k \leq I_j \tag{1}$$

then there exists no symmetric equilibrium in pure strategies.

Before proving the Theorem note that the RHS of (1) goes to zero with the growing number of participants.

Proof. (i) Assume $\pi_1 \notin E$. Then, by basic convex analysis, there exist $\pi_2, \ldots, \pi_{n+2} \in E$ such that π_1 lies in their convex hull so (ii) is implied by Proposition 2.
(ii) If $\pi_1 \in E$, then clearly $N_S(\pi_1) \subseteq \Gamma_1$ for any $\pi_2, \ldots, \pi_m$ so

$$v_i = u_i(1/k_i)p_i \geq u_i(1/m)\sigma_i.$$

(iii) Let $\pi \in S$ be an equilibrium. Let j be one of its non-zero components and let j_k fulfil (1) so that $\pi_{j_k} < \alpha$ (such j_k has to exist because at the weight of at most I_j components may equal to α). Consider portfolio $\tilde{\pi} = (\pi_1, \pi_2, \ldots \pi_j - s, \ldots \pi_{j_k} + s, \ldots, \pi_n)$ where s is small enough for $\tilde{\pi}$ to be feasible. However, if the i-th player holds portfolio $\tilde{\pi}$ and the rest of players hold π then it will be

$$p_i = \mathbb{P}(\tilde{\pi}R > \pi R) = \mathbb{P}(s(R_i - R_{j_k}) > 0) > \frac{u_i(\frac{1}{m})}{u_i(1)}$$

hence the expected utility given $\tilde{\pi}$ (equal to $p_i u_i(1)$) would be greater than that given the equilibrium (being $u_i(1/m)$).

□

Summarizing: if one wants to be sure with a positive expected gain, he has to choose one of the extremal points as his strategy. If, in addition, there are no significant leaders among stocks and/or there is a large number of participants, then no such or another point is a pure equilibrium, i.e., possible common strategy.

3 Empirical Evidence

In order to verify whether actual people behave according to the game theoretic conclusions, we analysed a portfolio competition held by Czech news internet portal "lidovky.cz" this year. The competition started in April and is supposed to end in July. According to the rules, its participants could split a virtual million Czech crowns among 27 stocks listed in Table 3, and a (fictitious) bank account yielding 0.4% p.a. The three participants with the highest value of their virtual portfolios, measured on July 9, are

promised to obtain 30.000, 20.000, and 10.000 Czech crowns, respectively. If there were more participants with the highest value of their portfolios then the prize would be divided equally.[1] The upper limit α of an investment asset is 40% for stocks, 50% for the bank account, respectively. The rules also say that at least 10% could be invested into a single stock if it is invested into it which, however, was violated by 6 portfolios for unknown reasons.[2]

The data we used come from the internet site of the competition `http://portfolio.lidovky.cz` and a subsequent preprocessing by a special software written in `C++` by us and by a free OCR program `gocr`. As the text recognition appeared to be inaccurate, several consistency checks were performed and, subsequently, manual correction were made; nevertheless, it is still possible that there are minor errors left in data caused by an inaccurate OCR recognition, which may be, however, regarded as noise if the data is analysed statistically .

There was as much as 2699 portfolios competing in the game. Even if it is highly probable that some players created multiple identities to increase their chances, we neglect this suspicion as we have no means to identify those cases.

There is 9828 extremal points of a feasible set in total,[3] 345 of which were occupied by portfolios of 453 (16.8%) participants (the most popular being portfolio CETV 40%, NWR 40%, ORCO 20% which was used 8 times). In other words, only 16.8% of players behaved "rationally" in the sense of Theorem 3. Out of remaining (non-extremal) portfolios, 975 was dominated in the sense of Proposition 2, having no chance for the first prize given the configuration portfolios of the other players. We used Iredundancy problem algorithm to determine which portfolios were dominated (see [3], Chp. 19 for details).

As the participants could optionally publish their gender and age, which was actually done by 2163 of them, 1559 of them, respectively, we tested for a correlation of a type of strategy chosen (possible types being an extremal point, a non-extremal not dominated point and a non-extremal dominated point) with these values. However, no significant results have been found here.

These facts lead us to a conclusion that people did not behave according to game theory given that only pure strategies are assumed.

4 Alternative Explanation.

Opposed to a rational approach, a hypothesis of purely random choice of portfolio, i.e., that the portfolios are chosen from the uniform distribution on the feasible set, suggests itself. This hypothesis, however, is falsified by the fact that SCHHV was never chosen because the probability that some stock has zero weight in all the 2699 portfolios is less than 0.02.

There could be many potential factors possibly influencing the choice of the stocks. In the present introductory paper, we restricted ourselves to considering selected data concerning the individual stocks published by the Prague stock exchange on their website, in particular to

- price-earning ratio (P/E),
- market capitalization (the monetary value of the issued shares), measuring the size of the firm,
- long-time trend (the ratio of the price of the stock at the time of the game's start and the average of the highest and the lowest price form the last year) and
- the short-time trend (the ratio of an OLS trend, computed from observations of the price from the five weeks preceding the competition, and the current price of the stock).

Note that, while the first two factors belong to fundamental analysis, the latter are more technical-analysis ones. In order to discover the dependency of a particular stock's choice on those factors, we run the logistic regression with the relative frequency as a dependent variable and the four mentioned factors

[1] It is, however, not said what would happen in case of equality on the second and/or the third place.

[2] We neglect these lower bounds in our theoretical analysis in Section 2 as they bring non-convexity of the feasible set which consequently complicates the treatment.

[3] Note that this number depends only on the number of stocks

Code	Name	p	a	MC	P/E	long	short
AAA	AAA Auto Group N.V.	0.17	3.0	1.565	5.98	0.043	0.002
CETV	CE Media Enterprises Ltd.	0.15	3.2	5.797	0	8.753	−0.037
ČEZ	ČEZ, a.s.	0.50	12.2	304.502	7.44	−0.163	−0.011
EFORU	E4U a.s.	0.04	0.7	0.167	10.56	0.022	−0.001
ENCHE	ENERGOCHEMICA SE	0.06	0.9	3.810	0	0.002	0.000
ENRGA	Energoaqua, a.s.	0.08	1.3	1.185	8.92	0.056	0.000
ERSTE	Erste Group Bank AG	0.42	8.5	223.957	0	0.108	−0.021
FOREG	Fortuna Entertainment Group N.V.	0.37	7.5	5.096	15.21	0.105	0.020
JIP	VET ASSETS a.s.	0.04	0.7	0.010	0	−0.052	0.000
KB	Komerční banka, a.s.	0.43	8.3	144.095	15.53	0.038	0.008
LAZJA	Jáchymov Property Management, a.s.	0.03	0.4	0.477	89.91	−0.025	0.000
NWR	New World Resources Plc	0.22	4.7	17.877	0	−0.318	−0.019
OCELH	OCEL HOLDING SE	0.09	1.5	3.798	0	0.000	0.000
ORCO	Orco Property Group S.A.	0.18	3.7	5.899	0.12	−0.148	−0.014
PEGAS	PEGAS NONWOVENS SA	0.26	5.2	4.661	12.92	0.064	−0.009
PM ČR	Philip Morris ČR a.s.	0.43	9.2	22.302	12.74	0.051	0.001
PRSLU	Pražské služby, a.s.	0.05	0.9	0.795	18.09	−0.066	0.000
PVT	RMS Mezzanine, a.s.	0.03	0.6	1.225	0	0.036	0.032
SCHHV	SPOLEK PRO CHEM.A HUT.VÝR.,a.s	0.00	0.0	775.763	0	0.000	0.000
SMPLY	Severomoravská plynárenská, a.s.	0.12	2.0	13.251	17.05	0.007	0.000
TEL. O2	Telefónica Czech Republic, a.s.	0.35	6.9	93.245	10.67	−0.181	−0.019
TMR	Tatry mountain resort, a.s.	0.16	3.3	7.814	0	0.029	0.004
TOMA	TOMA, a.s.	0.08	1.2	1.006	7.8	0.010	−0.001
UNI	UNIPETROL, a.s.	0.26	4.7	31.190	0	0.003	0.000
VCPLY	Východočeská plynárenská,a.s.	0.09	1.6	6.657	13.94	−0.037	−0.018
VGP	VGP NV	0.02	0.4	6.504	19.35	0.000	0.000
VIG	VIENNA INSURANCE GROUP	0.23	4.1	123.110	12.93	0.104	0.000

Table 1: Menu of stocks: p - frequency of choice, a - average weight (in %), MC - market capitalization, P/E - price to earning ration, *long* - long term trend, *short* - short term trend.

	Coefficient	Std. Error	t-ratio	p-value
const	−2.02345	0.238733	−8.4758	0.0000***
MC	8.09090e–06	2.55727e–06	3.1639	0.0047***
P/E	−0.0145792	0.0103617	−1.4070	0.1740
long	−0.0203306	0.125300	−0.1623	0.8727
short	−12.5787	16.7717	−0.7500	0.4616

Sum squared resid	16.92489	S.E. of regression	0.897746
R^2	0.462091	Adjusted R^2	0.359632
$F(4, 21)$	4.510016	P-value(F)	0.008704
Log-likelihood	−31.31136	Akaike criterion	72.62272
Schwarz criterion	78.91320	Hannan–Quinn	74.43415

Table 2: Result of logistic regression $p = 1/(1 + e^{-(1,MC,P/E,long,short)'\beta})$

as independent ones.[4] The results, shown by table 3, clearly show that only the market capitalization comes out as significant.[5]

5 Conclusion

We analysed a rather general case of a portfolio competition. As the behaviour of players in an actual game of this type appeared to be inconsistent with the from the game-theoretical point of view, we tried to give a simple behavioural explanation: in particular, we found that - out of four factors - players take only the size of the stock's issuer into account when constructing their portfolios.

Acknowledgements

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[4] Even if there is a dependency between choices of the particular stocks by the individuals, we omit it relying on the law of large numbers promising a closeness of the relative frequencies with the probabilities.

[5] We excluded SCHHV from the regression as it is clear from its data that the stocks is not traded.

Empirical copula for missing observations

Adam Kubetta [1]

Abstract. Empirical copula (EC) is a rank-based method for estimating the dependence structure of a random vector, which is convenient when someone wants to estimate non-parametrically the whole distribution of the random vector in two separated stages: first the margins and then the dependence structure fully represented by a copula. The discrete nature of an EC can be successively smoothed using Bernstein polynomials approximation to obtain differentiable estimate of underlying copula or its density. In practice, however, some components of some observations are often missing. In this case the marginal distributions of individual vector components can be estimated without any changes using all available information, whilst the second step is not so straightforward. One can employ only complete observations to capture the dependence but then the mapping of estimated copula on individual marginal quantiles is not reflected appropriately. In the article the author suggests to generalize the classical EC to be applicable also on incomplete observations and further shows that Bernstein approximation based on this generalized EC is only slightly modified and all its important attributes remain unchanged.

Keywords: Bernstein approximation, empirical copula, missing observations

JEL classification: C44
AMS classification: 90C15

1 Introduction

A motivation for this study was the task to estimate the measure called CoVaR [1] for Czech banks. As the classical measure VaR describes the critical value of one specific subject's future loss, the CoVaR measures the same value under some conditions on other subject's losses. When some heavy-tailed losses of certain subjects are considered, the unconditional VaR can significantly change for subjects with losses, which are correlated. Therefore CoVaR can be used to compare the sensitivity of individual subjects to a financial contagion.

Although originally the method was considered to be applied on quite dense public market data, here only a sparse data (monthly reports to the regulator) are considered, because the majority of the Czech banking sector is not publicly traded. That is why the fully empirical estimate can not be used. Moreover, all subjects have history of different lengths so that the periods of their coexistence are even shorter. The reports in times when all subjects submitted the data are called complete observations in this text. For complete datasets the method based on quantile regression, for example, mentioned in [1] seems reasonable. Anyway, traditional approaches usually do not consider partly incomplete datasets and therefore can be applied indeed only on those complete observations.

Due to the mentioned data characteristics the choice of an approach via copulas seems natural as it can divide the task on separable steps that can be easily adjusted. In the first step the distributions of marginal losses for individual subjects is estimated beforehand using all information in the data. In the second step, which is studied in detail in this text, the dependence structure of margins is captured and as explained further, this step can be also adjusted on incomplete data. Finally, the CoVaR can be, more or less, directly calculated from the full distribution estimate obtained from the previous steps.

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2 Notation

In the text $d \in \mathcal{N}$ always denotes the dimension of the underlying random vector $\mathbf{X} = \{\mathrm{X}_1, ..., \mathrm{X}_d\}$ with random components $\mathrm{X}_i \in \mathcal{R}$, $i \in \mathbb{i} = \{1, ..., d\}$. All references to d-dimensional objects are abbreviated to d-objects, for example d-vector $\mathbf{X}$, etc. The observed sample of length n from $\mathbf{X}$, denoted $\mathbb{x} = \{\mathbf{x}_1, ..., \mathbf{x}_n\}$, determines the series of its d-ranks $\mathbb{r} = \{\mathbf{r}_1, ..., \mathbf{r}_n\}$. Values $\mathbf{r}_k \in \mathcal{N}^d$, for $k \in \mathbb{k} = \{1, ..., n\}$, say that the k^{th} observation $\mathbf{x}_k \in \mathcal{R}^d$ is $(\mathbf{r}_k)_i^{\text{th}}$ (according to the increasing magnitude) when comparing i^{th} coordinates of all observations. For simplicity, ties are not considered here or decided randomly. On the other hand, any i^{th} coordinate of any k^{th} observation $\mathbf{x}_k$ might be missing, which is denoted $(\mathbf{x}_k)_i = x_{k,i} = \square$. Technically, whole vectors can be unavailable, $\{\square, ..., \square\}$.

Any arithmetic operation a applied on $\square$ returns $a(\square) = \square$ so that a rank corresponding to incomplete observation is also incomplete. Missing components are matched with ranks with the highest orders and n_i denotes the number of those observations having the i^{th} component available. The vector of n_i is denoted $\mathbf{n} = \{n_1, ..., n_d\}$.

Vectors taking values only from d-dimensional unit interval $\mathcal{I}^d = [0, 1] \times ... \times [0, 1]$ are distinguished from those from $\mathcal{R}^d$ by using letters $\mathbf{u}$ or $\mathbf{w}$ instead of $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$. The binary operators of product and division applied on vectors are considered component-wise, the product is then denoted $*$ to avoid misinterpreting as the scalar (dot) product.

Complete operator $\mathcal{C}$ applied on observations (ranks) returns the set of only those observations (ranks), which are complete, and similarly $\mathcal{C}$ applied on indices $\mathcal{C}(1, ..., n)$ returns only those referring to complete observations (ranks). For example, indices of incomplete observations can be expressed as $\mathbb{k} \backslash \mathcal{C}(\mathbb{k})$.

Vector inequalities $\mathbf{x} \leq \mathbf{y}$ means that $x_i \leq y_i$ for each $i \in \mathbb{i}$ and λ is the Lebesgue measure on $\mathcal{R}^d$.

3 Brief overview of copula

This section sums up the necessary essentials regarding copulas, which can be all found in [3].

Definition 1 (Copula)**.** For any $d \in \mathcal{N}$, d-copula is a function $C : \mathcal{I}^d \to \mathcal{I}$ that satisfies the following conditions:
(C1) $C(u_1, ..., u_d) = 0$ whenever $u_i = 0$ for at least one index $i \in \mathbb{i}$
(C2) $C(u_1, ..., u_d) = u_i$ whenever $u_j = 1$ for each index $j \neq i$
(C3) C is d-increasing, i.e. the volume of an arbitrary d-interval $[\mathbf{x}, \mathbf{y}] = [x_1, y_1] \times ... \times [x_d, y_d] \subset \mathcal{I}^d$ measured by C is non-negative, or symbolically

$$\int_{\mathcal{I}^d} \mathbb{1}(\mathbf{u} \in [\mathbf{x}, \mathbf{y}]) \mathrm{d}C(\mathbf{u}) = \sum_{\mathbf{z} \in \{x_1, y_1\} \times ... \times \{x_d, y_d\}} (-1)^{\mathbb{1}(x_1 = z_1) + ... + \mathbb{1}(x_d = z_d)} C(\mathbf{z}) \geq 0$$

Copulas are particularly important in the probability theory due to the following result by Sklar.

Theorem 1 (Sklar)**.** *For each random d-vector $\mathbf{X}$ with d-CDF F and marginal CDFs $F_1, ..., F_d$ there is a d-copula $C_{\mathbf{X}}$ such that $F(\mathbf{x}) = C_{\mathbf{X}}(F_1(x_1), ..., F_d(x_d))$ and this copula is unique on the domain specified by the Cartesian product of ranges $F_1(\mathcal{R}) \times ... \times F_d(\mathcal{R})$.*

Remark 1. Note that the range of a *continuous* random variable CDF is always the whole unit interval $\mathcal{I}$, thus the d-copula associated with a continuous random d-vector is always unique on the whole $\mathcal{I}^d$. Moreover, copula can then be interpreted as d-CDF of $\mathbf{U} = \{F_1(\mathrm{X}_1), ..., F_d(\mathrm{X}_d)\} \in \mathcal{I}^d$, i.e. the random d-vector $\mathbf{X}$ transformed by its margins. Since the transformation of a continuous random variable specified by CDF (of the variable) is always uniformly distributed on $\mathcal{I}$, the random d-vector $\mathbf{U}$ has indeed uniformly distributed margins.

4 Empirical Copula

The definition of classical EC (see definition 3) can be also found in [3]. However, the author finds useful to clarify its meaning beforehand, so that a reader can get an insight into the way how this definition is further generalized. Other definitions in this section are proposed by the author and relate only to a notation.

CDF of a random d-vector $\mathbf{X}$ is defined as the following function $F_{\mathbf{X}} : \mathcal{R}^d \to \mathcal{I}$.

$$F(\mathbf{x}) = F_{\mathbf{X}}(\mathbf{x}) = \mathbb{P}(\mathbf{X} \leq \mathbf{x}) = \mathbb{P}(\{\mathrm{X}_1, ..., \mathrm{X}_d\} \leq \mathbf{x})$$

This function can be estimated based on complete observations $\{\mathbf{x}_1, ..., \mathbf{x}_n\}$ of the d-vector $\mathbf{X}$ by the *empirical* d-CDF, which is the following function $\hat{F}_{\{\mathbf{x}_1,...,\mathbf{x}_n\}} : \mathcal{R}^d \to \mathcal{I}$.

$$\hat{F}(\mathbf{x}) = \hat{F}_{\{\mathbf{x}_1,...,\mathbf{x}_n\}}(\mathbf{x}) = \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}(\mathbf{x}_k \leq \mathbf{x}) = \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}(\{x_{k,1}, ..., x_{k,d}\} \leq \mathbf{x})$$

The d-copula corresponding to a random d-vector $\mathbf{X}$ having marginal CDFs $F_1, ..., F_d$ can be also written as the function $C_{\mathbf{X}} : \mathcal{I}^d \to \mathcal{I}$.

$$C(\mathbf{u}) = C_{\mathbf{X}}(\mathbf{u}) = \mathbb{P}(\{F_1(\mathrm{X}_1), ..., F_d(\mathrm{X}_d)\} \leq \mathbf{u})$$

Analogically, one can estimate this function based on a complete sample $\{\mathbf{x}_1, ..., \mathbf{x}_n\}$ as

$$\hat{C}(\mathbf{u}) = \hat{C}_{\{\mathbf{x}_1,...,\mathbf{x}_n\}}(\mathbf{u}) = \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}(\{F_1(x_{k,1}), ..., F_d(x_{k,d})\} \leq \mathbf{u})$$

However, this estimate depends on marginal CDFs $F_1, ..., F_d$, which are not observed. Plugging empirical CDFs $\hat{F}_1, ..., \hat{F}_d$ to the previous formula (in the place of $F_1, ..., F_d$ respectively) yields the estimate called empirical copula. After substituting, the expression can be further simplified, because $n\hat{F}_i((\mathbf{x}_k)_i) = n\hat{F}_i(x_{k,i})$ is nothing else than the rank $(\mathbf{r}_k)_i$ for each $k \in \mathbb{k}$ and $i \in \mathbb{i}$.

Definition 2 ($\mathbf{n}$-lattice, elementary interval)**.** For a given $\mathbf{n} \in \mathcal{N}^d$ the $\mathbf{n}$-lattice $\mathcal{L}(\mathbf{n})$ is a collection of coordinates from $\mathcal{I}^d$ created as the following Cartesian product

$$\mathcal{L}(\mathbf{n}) = \left\{\frac{1}{n_1}, ..., \frac{n_1}{n_1}\right\} \times ... \times \left\{\frac{1}{n_d}, ..., \frac{n_d}{n_d}\right\}$$

Each point from $\mathcal{L}(\mathbf{n})$ can be addressed by a d-index $\mathbf{i} \in \{1, ..., n_1\} \times ... \times \{1, ..., n_d\}$ as $\mathbf{i}/\mathbf{n}$ as well as its neighbourhood $\mathbf{i}^{\text{th}}$ elementary interval

$$\Xi_{\mathbf{i}/\mathbf{n}} = \Xi_{\mathbf{i}/\mathbf{n},\mathbf{n}} = \left[\frac{(\mathbf{i}-1)}{\mathbf{n}}, \frac{\mathbf{i}}{\mathbf{n}}\right] = \left[(\mathbf{i}-1) * \left\{\frac{1}{n_1}, ..., \frac{1}{n_d}\right\}, \mathbf{i} * \left\{\frac{1}{n_1}, ..., \frac{1}{n_d}\right\}\right]$$

Definition 3 (Classical empirical copula)**.** Empirical d-copula of a random d-vector $\mathbf{X}$ with marginal CDFs $F_1, ..., F_d$ based on a complete random sample $\{\mathbf{x}_1, ..., \mathbf{x}_n\}$ is defined for $\mathbf{u} \in \mathcal{L}(\{n, ..., n\})$ as

$$\mathrm{EC}(\mathbf{u}) = \mathrm{EC}_{\{\mathbf{x}_1,...,\mathbf{x}_n\}}(\mathbf{u}) = \frac{1}{n}\sum_{k=1}^{n} \mathbb{1}(\mathbf{r}_k \leq n\mathbf{u})$$

The domain of EC is restricted on the lattice $\mathcal{L}(\{n, ..., n\})$ only in order to avoid technical difficulties, because the traditional definition requires copula to have continuous margins. There are some approaches that extends the EC domain to $\mathcal{I}^d$ in such a way that EC complies with the definition of copula. They differ in allocating the probability mass over each $\mathbf{i}^{\text{th}}$ elementary interval $\Xi_{\mathbf{i}/\mathbf{n}}$.

Remark 2 (EC approximation)**.** The uniform allocation over elementary intervals gives the so called *checkerboard approximation*, others are e.g. check-min or shuffle-of-min approximations, see [2, 4]. However, further study is focused mainly on capturing the copula on the lattice $\mathcal{L}(\mathbf{n})$.

Definition 4 ($\mathbf{x}$-manifold)**.** For given $\mathbf{x} \in \mathcal{R}^d$ that may contain missing components the $\mathbf{x}$-manifold $\mathcal{M}(\mathbf{x})$ is defined as

$$\mathcal{M}(\mathbf{x}) = \left\{\mathbf{y} \in \mathcal{R}^d : y_i = x_i \text{ for all such } i \text{ that } x_i \neq \square\right\}$$

For any complete observation $\mathbf{x}$ holds $\mathcal{M}(\mathbf{x}) = \mathbf{x}$. When $\mathbf{x}$ contains one missing component at i^{th} position, $\mathcal{M}(\mathbf{x})$ is a line parallel to i^{th} axis, when $\mathbf{x}$ contains two missing components at positions i and j, $\mathcal{M}(\mathbf{x})$ is a plane parallel to both i^{th} and j^{th} axis, etc.

Definition 5 (**u**-space)**.** For given $\mathbf{n} \in \mathcal{N}^d$ and $\mathbf{u} \in \mathcal{L}(\mathbf{n})$ that may contain missing components the **u**-space $\Xi_{\mathbf{n}}(\mathbf{u}) \subset \mathcal{I}^d$ is defined as

$$\Xi(\mathbf{u}) = \Xi_{\mathbf{n}}(\mathbf{u}) = \bigcup_{\mathbf{w} \in \mathcal{M}(\mathbf{u}) \cap \mathcal{L}(\mathbf{n})} \Xi_{\mathbf{w},\mathbf{n}}$$

For any complete vector $\mathbf{u} \in \mathcal{L}(\mathbf{n})$ the $\Xi_{\mathbf{n}}(\mathbf{u})$ coincides with the elementary interval $\Xi_{\mathbf{u}/\mathbf{n}}$.

Proposal 1 (Extension of empirical copula)**.** For given $\mathbf{n} \in \mathcal{N}^d$ the function $\mathrm{EC} : \mathcal{L}(\mathbf{n}) \to \mathcal{I}$ is called empirical copula if EC complies with the following requirements.
(EC1) EC is a d-CDF on the lattice $\mathcal{L}(\mathbf{n})$
(EC2) For $\mathbf{U} \sim \mathrm{EC}$ holds $\mathbb{P}(\mathbf{U} \in \cup_{\mathbf{r} \in \mathbb{r}} \Xi_{\mathbf{n}}(\mathbf{r}/\mathbf{n})) = 1$
(EC3) For $\mathbf{U} \sim \mathrm{EC}$ holds $\mathbb{P}(\mathbf{U} \in \Xi_{\mathbf{n}}(\mathbf{r}_k/\mathbf{n}) | \mathbf{U} \in \cup_{\mathbf{r} \in \mathcal{C}(\mathbb{r})} \Xi_{\mathbf{n}}(\mathbf{r}/\mathbf{n}) = 1/n$ for all $k \in \mathcal{C}(\mathbb{k})$

The first condition is clear. The second says that the probability mass described by an empirical copula can not be allocated outside the collection of **u**-spaces associated with rescaled ranks of observations. The third condition means that all complete observations (as well as their rescaled ranks) are weighted equally and proportionally to incomplete ones.

Remark 3. Note that in higher dimensions ($d \geq 3$) there are more possibilities how to construct an EC.

The series of pictures in Figure 1 shows the process of creating an empirical copula (here for $d = 3$) by the author's algorithm, which can be described here only roughly due to limited scope of the article.

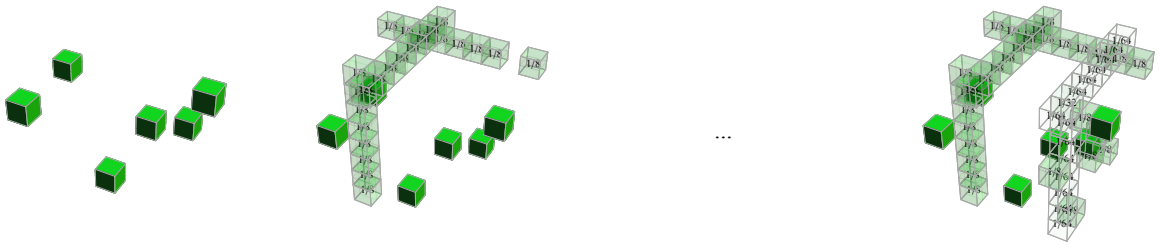

Figure 1 Finding an EC. The final density-plot of an EC for $d = 3$ can be seen on the right.

In the beginning the probability mass corresponding to all complete observations is located to adequate elementary intervals in advance. This is the first and also the last step for complete datasets and can be seen on the left. Then a certain ratio is associated with each incomplete observation based on a mutual spatial relation of observations. After that follows a multi-level process of projecting the mass already located onto elementary intervals that has not been matched with any amount yet. The probability mass once matched to an elementary interval is never changed, which requires to consider the order of projecting carefully. The final result can be seen as the checkerboard empirical copula density after multiplying each amount by $n_1 ... n_d$. Each result can be easily checked, whether it has uniform margins.

Although the purpose of this text is not to study the asymptotics, the following theorem states some trivial sufficient conditions for convergence of EC to C.

Theorem 2 (Empirical copula convergence)**.** *Let $m \in \mathcal{N}$ be fixed number. If a random sample $\{\mathbf{x}_1, ..., \mathbf{x}_n\}$ taken from a random vector $\mathbf{X}$ with a copula $C_{\mathbf{X}}$ that have a density with respect to Lebesgue measure λ on $\mathcal{I}^d$ contains for arbitrary $n \in \mathcal{N}$ no more than m incomplete observations, then $\mathrm{EC}_{\{\mathbf{x}_1,...,\mathbf{x}_n\}}$ converges uniformly to $C_{\mathbf{x}}$ on $\mathcal{I}^d$ when $n \to \infty$.*

Proof. The result follows from the facts that $\mathrm{EC} \to C$ for complete observations (uniform convergence of CDF) and that the probability that $\mathbf{X} \in \cup_{k \in \mathbb{k} \setminus \mathcal{C}(\mathbb{k})} \Xi_{\mathbf{n}}(\mathbf{r}_k/\mathbf{n})$ converges to 0, because

$$\lambda \left(\cup_{k \in \mathbb{k} \setminus \mathcal{C}(\mathbb{k})} \Xi_{\mathbf{n}}(\mathbf{r}_k/\mathbf{n}) \right) \to 0$$

as $n \to \infty$ and $C_{\mathbf{X}}$-measure is absolutely continuous with respect to λ. □

5 Bernstein approximation of a copula

An option, how to extend a copula specified only on $\mathcal{L}(\mathbf{n})$ to the whole $\mathcal{I}^d$, is to use some approximation (see Remark 2). The approximation described in this section, studied e.g. in [4] (only for complete datasets), is based on polynomials and thus returns a smooth function on $\mathcal{I}^d$.

For $n \in \mathcal{N}$, $k = 0, ..., n$ and $u \in \mathcal{I}$ a *Bernstein polynomial* $b_{n,k}(u)$ (or k^{th} Bernstein basis function of degree n at u) is defined as the following expression

$$b_{n,k}(u) = \binom{n}{k} u^k (1-u)^{n-k}$$

In order to simplify further notation in this text, let $b_{n,k}(u) = 0$ for other indices then those specified above and let $\mathbf{b}_{\mathbf{n},\mathbf{k}}(\mathbf{u}) = b_{n_1,k_1}(u_1)...b_{n_d,k_d}(u_d)$ for $\mathbf{n} \in \mathcal{N}^d$, $\mathbf{k} \in \{0, ..., n_1\} \times ... \times \{0, ..., n_d\}$ and $\mathbf{u} \in \mathcal{I}^d$ denote the object that might be called *the product Bernstein polynomial*.

Proposal 2 (Bernstein approximation of copula)**.** Bernstein approximation of a copula C can be expressed as

$$B_{\mathbf{n}}(C)(\mathbf{u}) = \sum_{\mathbf{w} \in \mathcal{L}(\mathbf{n})} C(\mathbf{w}) \mathbf{b}_{\mathbf{n},\mathbf{n}*\mathbf{w}}(\mathbf{u})$$

Remark 4. Note that the classical Bernstein approximation of a copula, described e.g. in [4], is a special case of the proposal above, where $\mathbf{n} = \{n, ..., n\}$. Be aware that the Bernstein approximation is not an interpolation, i.e. $\mathbf{u} \in \mathcal{L}(\mathbf{n}) \Rightarrow B_{\mathbf{n}}(C)(\mathbf{u}) = C(\mathbf{u})$ generally does not hold.

Theorem 3 (Bernstein approximation attributes)**.** *For an arbitrary continuous copula C the following statements are true.*
(BA1) $B_{\mathbf{n}}(C) \to C$ *uniformly on* $\mathcal{I}^d$ *when* $\mathbf{n} \to \{\infty, ..., \infty\}$
(BA2) $B_{\mathbf{n}}(C)$ *is a copula for any* $\mathbf{n} \in \mathcal{N}^d$

Proof. (BA1) is a special case of a known result from analysis [5]. (BA2) can be verified analogically to the classical case, see sketch of proof in [4], by showing that $B_{\mathbf{n}}(C)$ complies with (C1), (C2) and (C3).

The Bernstein polynomial $b_{n,k}(0) = 0$ for arbitrary indices $0 < k \leq n$ and also $\mathbf{n} * \mathbf{w} \geq \mathbf{0}$ for any $\mathbf{n} \in \mathcal{L}(\mathbf{n})$, hence $\mathbf{b}_{\mathbf{n},\mathbf{n}*\mathbf{w}}(\mathbf{u}) = 0$ for all $\mathbf{w} \in \mathcal{L}(\mathbf{n})$ whenever $u_k = 0$ for at least one index k, which implies that $B_{\mathbf{n}}(C)$ satisfies (C1). Similarly, $b_{n,k}(1) = 0$ for indices $0 < k < n$ and $b_{n,k}(1) = 1$ for $0 < k = n$ and generally also holds $\Sigma_{k=1,...,n}\ k\ b_{n,k}(u)/n = u$ from which follows that $B_{\mathbf{n}}(C)$ satisfies also (C2), because

$$B_{\mathbf{n}}(C)(\{1, ..., 1, u_i, 1, ..., 1\}) = \sum_{\mathbf{w} \in \mathcal{L}(\mathbf{n}) \cap \mathcal{M}(\{1,...,1,\square,1,...,1\})} C(\{1, ..., 1, w_i, 1, ..., 1\}) b_{n_i,n_i w_i}(u_i) = u_i$$

The d-derivative of $B_{\mathbf{n}}(C)$ can be simplified considering the rule $\partial_u b_{n,k}(u) = n\,(b_{n-1,k-1}(u) - b_{n-1,k}(u))$ for $n \in \mathcal{N}$ and $k = 0, ..., n$ to the following form

$$D(B_{\mathbf{n}}(C))(\mathbf{u}) = n_1...n_d \sum_{\mathbf{w} \in \mathcal{L}(\mathbf{n})} \left(C(\mathbf{w}) - C\left(\mathbf{w} - \tfrac{1}{\mathbf{n}}\right)\right) \mathbf{b}_{\mathbf{n}-1,\mathbf{n}*\mathbf{w}-1}(\mathbf{n})$$

The expression is obviously non-negative and can be seen as the density of $B_{\mathbf{n}}(C)$-measure on $\mathcal{I}^d$ so the condition (C3) on $B_{\mathbf{n}}(C)$ is verified. □

The estimate $B_{\mathbf{n}}(C)$ takes values of C only at the lattice $\mathcal{L}(\mathbf{n})$, and therefore C can be successively replaced by an empirical copula EC when estimating it. Therefore one can define the Bernstein Copula estimate as an object depending only on a sample $\{\mathbf{x}_1, ..., \mathbf{x}_n\}$ as

$$\mathrm{BC}_{\mathbf{n}}(\mathbf{x}) = B_{\mathbf{n}}\left(\mathrm{EC}_{\{\mathbf{x}_1,...,\mathbf{x}_n\}}\right)(\mathbf{x}) = \sum_{\mathbf{u} \in \mathcal{L}(\mathbf{n})} \mathrm{EC}_{\{\mathbf{x}_1,...,\mathbf{x}_n\}}(\mathbf{u}) \mathbf{b}_{\mathbf{n},\mathbf{n}*\mathbf{u}}(\mathbf{x})$$

Theorem 4 (Bernstein Copula estimate convergence)**.** *For $n \to \infty$ the Bernstein Copula estimate calculated based on a complete random sample $\{\mathbf{x}_1, ..., \mathbf{x}_n\}$ drawn from the underlying random vector $\mathbf{X}$ having a continuous copula $C_{\mathbf{X}}$ converges to this copula $C_{\mathbf{X}}$ uniformly on $\mathcal{I}^d$.*

If the sample is incomplete but meets all the requirements from Theorem 2 on empirical copula convergence, then the statement is still valid. The result follows immediately from composing two uniform convergences in Theorems 2 and 3.

6 Comparison

In figure 2 some different approaches to a 2-vector density estimation are compared on an artificial data. All use the same kernel estimates of margins. The first approach constructs a MLE of the vector density expressed through the AMH parametric copula, where of course only complete data can be used in the maximization. The second approach uses the classical EC constructed again only from complete observations. The last approach employing the proposed extended EC can process all available data.

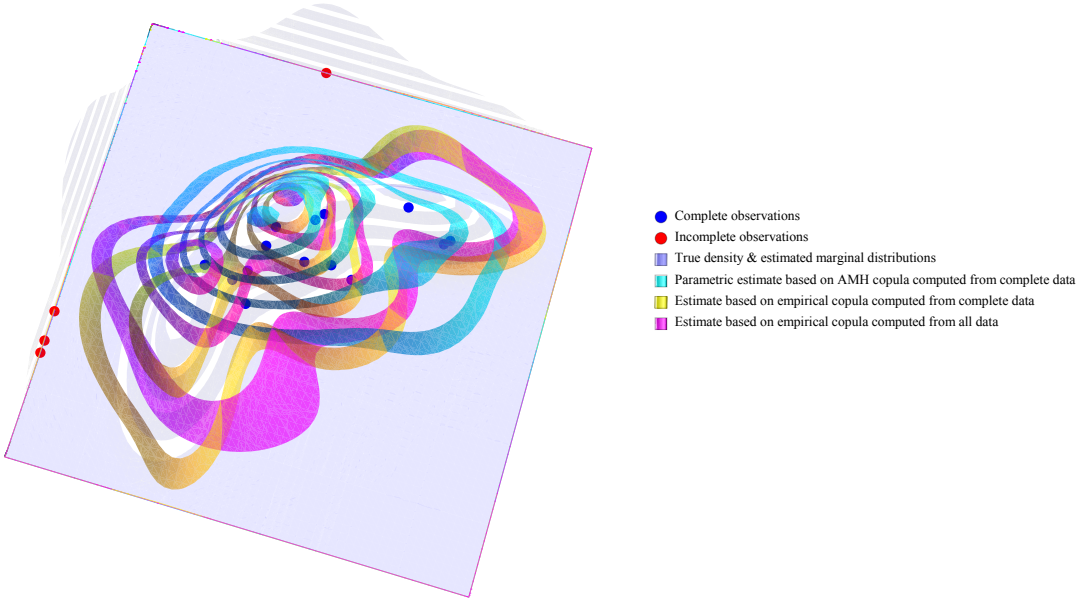

Figure 2 Comparison of random vector estimates.

7 Conclusion

The proposed extension of the empirical copula introduces the opportunity to estimate the dependence structure of a random vector based on observations, which do not have to be necessarily complete. When all observed data are complete, the extended algorithm coincides with the classical one. The empirical approach leaves open the choice of method for estimating the random vector margins, because the empirical copula is a rank based function of a random sample.

Acknowledgements

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The Synchronization of Financial and Trade Integration of the EU10 Countries by Using the Method of Classical and Dynamic Correlation

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Abstract. Financial and trade integration are interconnected processes and should not be assessed separately. Using a data sample of EU10 member countries over the period 1993-2012 we assess the relationship between financial and trade integration in the EU10 member countries. As the basic measurement of the dependence between indicators we use the method classical correlation. Because the sample size is small and with respect to the annual character of the data, we additionally calculate the moving correlation to evaluate behaving dependence in the time. Consequently we apply the method of dynamic correlation which provides a detailed view to the structure of dependency also from the frequency point of view. The result of the dynamic correlation is used for validation of the results obtained from the previous methodological steps. The empirical evidence of the dependence is validated across all three mentioned approaches. We conclude that there is a strong linear association between financial and trade integration in the pre-crisis period (1993-2007) in the EU10 countries. However, this relationship was seriously weakened by the financial crisis (2008-2010). We also found out, that the progress in financial integration was smaller compared to the progress of trade integration in the EU10 countries.

Keywords: financial integration, foreign trade, international investment position, correlation analysis.

JEL Classification: C23, C36, E44, F36, F42
AMS Classification: 62P20, 62M10, 91B84

1 Introduction

The integration of financial markets contributes to the overall integration and economic growth by removing the exchange rate risk and the barriers and frictions in cross-border capital movement. This allows the capital to be allocated more efficiently (Baele *et al*. [3]). Financial integration is an important factor in increasing the efficiency of a financial system and lowering the costs for business as well as for consumers. However, the process of financial integration of the past decade was associated with an unprecedented accumulation of risks and the national and supranational financial regulation and supervisory practices lagged behind the highly integrated, fast expanding and sophisticated financial sector (European Commission [5]).

Foreign trade (i.e. trade integration) is an important factor influencing financial integration. Lane and Milesi-Ferreti [14], [15] mention several important linkages between foreign trade and trade with foreign assets and liabilities. Firstly, the high volume of trade with goods and services evokes the corresponding financial transactions. Foreign direct investments had a great impact on the external balance of the "new" EU member countries from Central and Eastern Europe; large trade deficits originating from the transformation process were compensated by investment inflows (i.e. by increasing financial integration). International trade and international financial flows are thus able to equilibrate the balance of payment. Secondly, a high share of bilateral trade linkages between countries leads to the willingness of economic agents to increase the number of financial transactions with these countries. Investors have a better knowledge of foreign companies from these countries and are thus more prone to buying the shares of these companies (the "familiarity effect"). Thirdly, a high degree of trade openness of a country reflects the liberal approach of macroeconomic policy authorities not only in the area of foreign trade, but also in the area of cross-border capital flows.

The aim of the paper is to assess the relationship between financial and trade integration in the EU10 member countries over the period 1993-2012. We use quantity-based measures of financial integration derived from

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the countries' international investment positions with a view to the foreign trade and the method of correlation analysis (including the moving correlation method). The text is structured as follows.

1.1 Previous empirical research

Lane and Milesi-Ferretti [13] created a methodology to produce a unique data set containing an estimation of foreign assets and liabilities for a large set of industrial and developing countries. This is one of the first attempts to study the foreign assets and liabilities. In another paper, Lane and Milesi-Ferretti [15] examine the cross-country and time-series variation in the size of international balance sheets. They study the relation between foreign assets and liabilities on one side and a set of various regressors (GDP per capita, trade openness, external liberalisation, financial depth, stock market capitalisation, privatisation revenues etc.) on the other side. They find that international trade and stock market capitalisation are the two most important variables influencing international balance sheets. This study was updated in Lane and Milesi-Ferretti [17].

Kose *et al*. [10] focus on cross-country trade and financial linkages and produce a comprehensive analysis of the roles of both trade and financial integration in driving the growth-volatility relationship. They conclude that both trade and financial integration significantly weaken the negative association between output volatility and growth. Kose *et al*. [11] analyse the impact of selected macroeconomic variables (the depth of financial markets, trade openness, real GDP per capita, macroeconomic policies stability, institutional quality, and the regulation of an economy) on a country's financial openness (the sum of financial assets and liabilities relative to nominal GDP). They conclude that foreign direct investments and cross-border flows of equity securities are safer for the economy than cross-border flows of debt securities especially in the case of a low level of a country's financial openness and quality of institutions.

According to Rusek [18] and Spiegel [20], a common currency fosters the foreign trade of the euro area countries (the "euro effect"). Whereas foreign trade requires external financing, trade integration intensifies financial integration. Sebnem *et al*. [19] investigate the underlying channels of the "euro effect" on financial integration, i.e. the elimination of the currency risk among euro area countries, various financial sector legislative-regulatory reforms or increased goods trade. They find that the impact of this effect on financial integration is primarily driven by eliminating the currency risk. While financial and trade integration are highly correlated processes, trade in goods does not play a role in explaining the positive effect of euro on financial integration.

Aviat and Coeurdacier [2] explore the complementarity between bilateral trade in goods and bilateral asset holdings in a simultaneous gravity equations framework. According to results, trade in goods and trade in assets are closely related. They find a very robust and significant effect of trade on financial asset holdings and this causality runs in both ways; however, the impact of asset holdings on trade in goods is smaller. Kucerova [12] confirms the same results by using the simultaneous equations model. Aizenman and Noy [1] also study the endogenous determination of financial and trade openness. They construct a theoretical framework leading to two-way feedbacks between financial and trade openness and then identify these feedbacks empirically. They find that countries cannot choose the degree of financial openness independently of their degree of trade openness.

1.2 Methods

In order to quantify co-movements between time series, we use the method of *dynamic correlation* according to Croux et al. [4]. It measures the similarity of the frequency components of two time series *y* and *z* and can be defined as:

$$\rho_{yz}(\omega) = \frac{C_{yz}(\omega)}{\sqrt{S_z(\omega) S_y(\omega)}}, \tag{1}$$

where *Cyz* is a co-spectrum (the real part of the cross-spectrum) and *Sy*, *Sz* are the individual spectra of time series *y* and *z* for frequencies ω. The dynamic correlation values belong to the interval from -1 to +1. Integrating the equation (1) in the frequency band from ω_1 to ω_2 a dynamic correlation coefficient arises:

$$\rho_{yz}(\omega_1, \omega_2) = \frac{\int_{\omega_1}^{\omega_2} C_{yz}(\omega) d\omega}{\sqrt{\int_{\omega_1}^{\omega_2} S_z(\omega) d\omega \int_{\omega_1}^{\omega_2} S_y(\omega) d\omega}} \tag{2}$$

which evaluates the common behaviour of two time series in the given band of frequencies. For ω_1=0, ω_2 =π the integration is done over the whole defined frequency range and thus the dynamic correlation coefficient corresponds to the classical correlation coefficient (Fidrmuc et al. [6]).

1.3 Data

The yearly data 1993-2012 used to calculate the measures of financial integration are from the International Monetary Fund International Financial Statistics online database, specifically a category called the international investment position (IMF [8]). Incomplete data for some countries and some years have been completed from the on-line database External Wealth of Nations Mark II (Lane and Milesi-Ferretti [16]). Data concerning nominal exports and imports (in USD) are also extracted from the on-line database IMF IFS. Data concerning nominal GDP (in USD) are extracted from the on-line database IMF World Economic Outlook (WEO) Database, version April 2013 (IMF [9]). Each variable defined below (IFI, GI, GEQ, TRADE, etc.) was calculated as the average of corresponding values for 10 representative countries of EU, namely Bulgaria, the Czech Republic, Estonia, Hungary, Latvia, Lithuania, Poland, Romania, Slovakia, Slovenia.

Foreign assets include several categories: foreign assets, foreign direct investment abroad, portfolio investment equity securities, portfolio investment debt securities, financial derivatives, other investment/sectors, reserve assets. Foreign liabilities assets include these categories: foreign direct investment in the economy, portfolio investment equity securities, portfolio investment debt securities, financial derivatives, other investment/sectors.

The variable IFI_{it} is an average indicator of financial integration. It is *a quantity-based measure* of financial integration. This indicator is constructed as follows:

$$IFI_{it} = \frac{\left(FA_{it} + FL_{it}\right)}{GDP_{it}}, \tag{3}$$

where FA_{it} is the stock of total foreign assets of country i in time t, FL_{it} is the stock of total financial liabilities of country i in time t and GDP_{it} is the nominal GDP of country i in time t. Absolute levels of any variable do not reflect the size of the economy properly, can be misleading and are thus not convenient for direct comparisons of different countries. Therefore, it is better to adjust the *IFI* indicator by including the nominal GDP of countries in order to take into account the size of the economy and reveal the true differences in the level of financial integration. In our empirical analysis we use also separated indicators *FA* and *FL*.

The second average measure is *the investment-based measure* of financial integration (*GI*): it contains only foreign direct investments and portfolio investments (equity and debt securities). The other categories were dropped from this measure because they are either volatile (other investments) or time series are not long enough (financial derivatives). The construction of this adjusted measure is as follows:

$$GI_{it} = \frac{\left(FDIA_{it} + FDIL_{it} + PEQA_{it} + PEQL_{it} + PDEA_{it} + PDEL_{it}\right)}{GDP_{it}} \tag{4}$$

where $FDIA_{it}$ is the stock of foreign direct investment assets of country i abroad, $FDIL_{it}$ is the stock of foreign direct investment liabilities of the rest of the world in country i, $PEQA_{it}$ is the stock of portfolio equity assets of country i abroad, $PEQL_{it}$ the stock of portfolio equity liabilities in country i, $PDEA_{it}$ the stock of portfolio debt assets of country i abroad, and $PDEL_{it}$ is the stock of portfolio debt liabilities in a country i.

A third possible average measure of financial market integration – *the equity-based measure* of financial integration (*GEQ*) – is based solely on the equity cross-holdings – that is, flows of portfolio equity and foreign direct investments. As international trade in debt instruments can be sometimes influenced by special factors, it was omitted in this indicator:

$$GEQ_{it} = \frac{\left(FDIA_{it} + FDIL_{it} + PEQA_{it} + PEQL_{it}\right)}{GDP_{it}} \tag{5}$$

Trade openness is expressed by using the *TRADE* indicator and this indicator of trade integration is constructed as follows:

$$TRADE_{it} = \frac{\left(EX_{it} + IM_{it}\right)}{GDP_{it}}, \tag{6}$$

where EX_{it} is the total sum of exports of country i in time t, IM_{it} is the total sum of imports of country i in time t and GDP_{it} is the nominal GDP of country i in time t. The higher the value of this indicator, the higher the country's trade openness is.

1.4 Correlation Analysis

In the first step of empirical analysis we calculate classical (static) correlation coefficients for the annual data 1993-2012. In the second step the dynamic correlations were calculated according eq. (2) for ω_1=0, ω_2 =π. Correlated variables are denoted as bold in the tab. 1. Tab. 1 (lower left triangle) shows a significant (p-value=0,000 for all coefficients) positive dependence between trade and other measures of financial integration measured by static correlation. When calculating the dynamic correlation over the whole frequency range (Tab. 1, upper right triangle), we should get the same results as in the case of the classical correlation. Minor changes are due to the computational precision of the software Matlab R2011b. of calculation.

	TRADE	**IFI**	**GI**	**GEQ**		**TRADE**	**FDI**	**PEQ**	**PDE**
TRADE	1	0,8287	0,8252	0,8168	**TRADE**	1	0,8034	0,8369	0,8715
IFI	0,8270	1	0,9739	0,9802	**FDI**	0,8013	1	0,8183	0,9412
GI	0,8233	0,9736	1	0,9975	**PEQ**	0,8350	0,8160	1	0,8830
GEQ	0,8148	0,9799	0,9975	1	**PDE**	0,8696	0,9410	0,8815	1

Table 1 Correlation coefficients: static and dynamic correlation

Source: own calculation
Note: Lower left triangle contains correlation coefficient; upper right triangle presents dynamic correlation.

Unfortunately, classical correlation does not provide a detailed view of the structure of dependence. Therefore, we proceed with the calculation of the moving correlation (Tab. 2). Moving correlation is calculated as classical static correlation, but on the moving time window. We establish the moving part of the size of 10 observations, we start with time window 1993-2002 and we move per one observation, i.e. 1994-2003, 1995-2005 etc. Denote the sample size is n=20.

TRADE	**1993-2002**	**1994-2003**	**1995-2004**	**1996-2005**	**1997-2006**	**1998-2007**	**1999-2008**	**2000-2009**	**2001-2010**	**2002-2011**	**2003-2012**
	1	2	3	4	5	6	7	8	9	10	11
						Correlation coefficient					
IFI	0,8786	0,8997	0,9228	0,9102	0,9468	0,9253	0,9163	0,3730	0,3805	0,3254	0,3847
GI	0,6314	0,8836	0,9018	0,8811	0,9355	0,9269	0,9086	0,3739	0,3808	0,3324	0,4008
GEQ	0,6598	0,8845	0,9052	0,8992	0,9402	0,9196	0,8913	0,3887	0,3931	0,3453	0,3941
						p-value					
IFI	0,0008	0,0004	0,0001	0,0003	0,0000	0,0001	0,0002	0,2884	0,2780	0,3588	0,2723
GI	0,0502	0,0007	0,0004	0,0008	0,0001	0,0001	0,0003	0,2872	0,2776	0,3481	0,2510
GEQ	0,0379	0,0007	0,0003	0,0004	0,0001	0,0002	0,0005	0,2669	0,2611	0,3285	0,2598
						Correlation coefficient					
FDI	0,6409	0,8839	0,9039	0,8983	0,9363	0,9182	0,8709	0,3523	0,3596	0,3140	0,3729
PEQ	0,6635	0,6769	0,7596	0,8060	0,8726	0,9001	0,7835	0,6569	0,6532	0,5835	0,4866
PDE	0,6430	0,8648	0,8921	0,9015	0,9241	0,9375	0,9299	0,5494	0,5307	0,4666	0,4948
						p-value					
FDI	0,0459	0,0007	0,0003	0,0004	0,0001	0,0002	0,0010	0,3180	0,3075	0,3769	0,2886
PEQ	0,0365	0,0316	0,0108	0,0049	0,0010	0,0004	0,0073	0,0391	0,0406	0,0766	0,1538
PDE	0,0449	0,0012	0,0005	0,0004	0,0001	0,0001	0,0001	0,1000	0,1145	0,1740	0,1460

Table 2 Correlation coefficients: moving version of static correlation

Source: own calculation

In the Tab. 2, we can see the statistically significant dependence till the year 2008. After that, a moving part containing values after 2008 indicates a substantial decrease in the level of correlation caused by the financial crisis, which began in 2007 and has turned into economic and debt crisis. There is only one indicator (*PEQ*, containing shares, stocks and depository receipts) showing a significant dependence with trade after year 2008. In the case of *PDE*, the drop in correlation coefficients was not so large, but the results are not as significant as in the case of the *PEQ*. FDI experienced the worst drop in correlation.

Additional detailed view of the structure of correlation provides the dynamic correlation (equation (1)). We present the development of dynamic correlation between *TRADE* and the selected indicators in relation to different frequencies in Fig. 1a (*FDI*, *PEQ*, *PDE*) and Fig. 2a (*IFI*, *GI*, *GEQ*). For a better illustration, we present moving correlation figures too; in order to compare the results of these two methods, see Fig. 1b and Fig. 2b.

According to the results, there is the same tendency in the correlation in the both groups of indicators (*FDI*, *PEQ*, *PDE* and *IFI*, *GI*, *GEQ*) within the period 2000-2009.

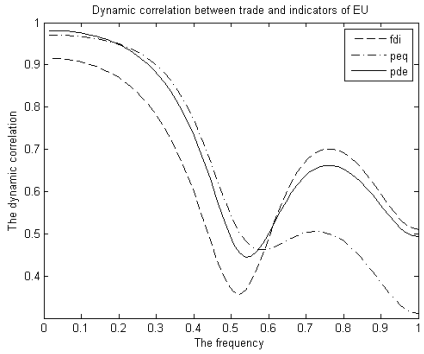

Figure 1a Dynamic correlation curves

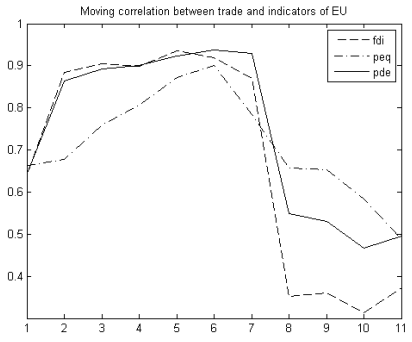

Figure 1b Moving correlation curves

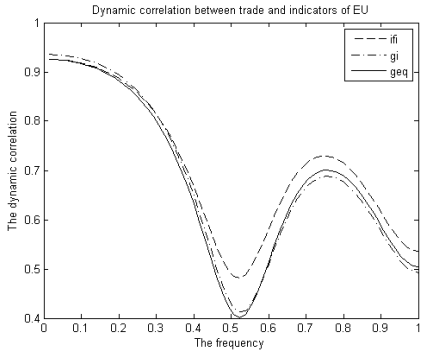

Figure 2a Dynamic correlation curves

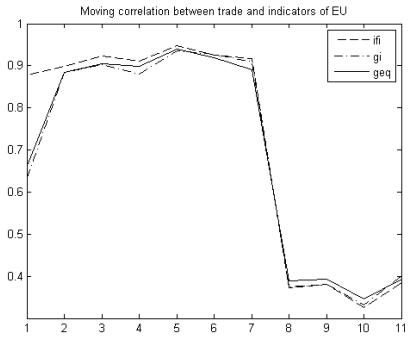

Figure 2b Moving correlation curves

In Tab. 1, we present the dynamic correlation over the whole frequency range (equation (2)). The variability of all dynamic correlation curves is presented in Tab. 3. In the case of dependencies between *TRADE* and the selected financial indicators (Fig. 1a), we see the volatility of the dynamic correlation curve, which varies for different lengths of cyclic components (frequencies). For all indicators, there is a high correlation with the indicator of trade in long cycles and economic cycles (low frequency value on the *x*-axis, i.e. the frequency value on the *x*-axis ranges from 0.001-0.33), the correlation is close to 0.8. Dynamic correlations corresponding to short cycles also achieve high levels in the case of the *FDI* and *PDE* indicators (higher frequencies, i.e. 0.65-0.85). This similar tendency is apparent in the Fig. 2a for the *IFI*, *GI* and *GEQ* indicators. Contrary to the Fig. 1a, all the indicators have the same development tendency.

	TRADE	**IFI**	**GI**	**GEQ**		**TRADE**	**FDI**	**PEQ**	**PDE**
TRADE	0%	2,008%	2,804%	2,671%	**TRADE**	0%	2,867%	5,507%	3,423%
IFI	2,008%	0%	0,169%	0,157%	**FDI**	2,867%	0%	3,279%	0,0745%
GI	2,804%	0,169%	0%	0,0002%	**PEQ**	5,507%	3,279%	0%	1,735%
GEQ	2,671%	0,157%	0,0002%	0%	**PDE**	3,423%	0,075%	1,735%	0%

Table 3 Variability of dynamic correlation

Source: own calculation

1.5 Conclusion

The process of monetary integration in Europe has to a great extent influenced the European financial markets because a single currency requires well-integrated financial markets. Financial integration is a substantial condition for introducing a single currency. However, foreign trade (i.e. trade integration) is an important factor influencing financial integration. This paper analysed the relationship between financial and trade integration in the EU10 member countries over the period 1993-2012. We used quantity-based measures of financial integration derived from the countries' international investment positions with a view to foreign trade by using various the methods of classical and dynamic correlation.

We concluded that financial and trade integration has been deepening since 1993. Both integration processes were broken by the world financial crises (the process of financial integration in 2008 and the process of trade integration in 2009). The deepening integration trend was restored one year later in both cases. We also conclude that there is a linear association between financial and trade integration in the pre-crisis period (1993–2008) in the EU10 countries. However, this relationship was seriously weakened by the financial crisis. These integration processes are interconnected, i.e. the more the countries trade the more financially integrated they are. It confirms a strong relationship between the two main balance-of-payment components: the current account and the financial account.

Acknowledgements

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Pay-to-Bid Auctions: Evidence from Czech Data

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Abstract. This paper examines properties of pay-to-bid auctions. Pay-to-bid auctions represent a relatively new type of all-pay auctions in which participants have to pay a fee in order to increase the purchase price by a small fixed bidding increment. The player who submits the last bid wins the right to buy the auctioned item at the current purchase price, which is normally substantially lower than the value of the item. While most of the existing studies examine the data on the international server Swoopo.com, we present a unique dataset retrieved from the Czech auction site Bonus.cz. In the paper, we introduce econometric models that explain the total number of bids and seller's revenue using data from the period between July 2009 and April 2010 where the basic properties of the auctions correspond to the stylized facts found in the US data. We compare the results with predictions of the model of pay-to-bid auctions introduced by [3]. We find that the empirical evidence is consistent with theoretical predictions and suggests risk-seeking preferences of bidders.

Keywords: internet auctions, bidding fees, risk-seeking preferences.

JEL Classification: D44

AMS Classification: 62J05, 91A80

1 Introduction

Pay-to-bid auctions have been introduced as a new type of internet auctions relatively recently. Former leading provider of these auctions Swoopo.com, operating in Germany, Austria, the US, the UK, Canada and Spain, entered the market only in 2005. Pay-to-bid auctions quickly managed to raise interest of shoppers, media and academics with the number of websites soaring over one hundred in the US alone and monthly number of unique visitors reaching 16% of the traffic held by eBay, the undisputed leader in online auctions [3].

While the pay-to-bid auctions offered on the internet may differ in details, the basic concept remains the same across the websites. The auction site (or seller) offers a single item in an auction, for example a new model of mobile phone. The initial price is zero and there is a timer counting down from 30 seconds. If a player places a bid, the price of the item increases by a fixed amount and the timer is set at 30 seconds again. Players pay a bidding fee for each bid they make, usually in the form of pre-paid tokens. If nobody bids before the timer reaches zero, the person who made the last bid wins the right to buy the item at the current purchase price.

Using the data from the US version of Swoopo, the existing studies provide interesting stylized facts about pay-to-bid auctions. For instance, [1] reports that while most auctions end with a loss for the seller, the rest more than compensates for the losses. In fact, the average revenue from an auction reaches 150% of the retail price of the item on the auction. Moreover, the main part of seller's revenue comes from the bidding fees, not from the final purchase prices winners pay for the items. There are several game-theoretical models explaining the stylized facts ([1], [2], [3]). All papers note that the basic model of fully rational, risk-neutral players in an environment with full information is unable to explain the empirical data. However, they differ in the assumptions they use to improve the fit of their models. For example, [1] introduces the sunk-cost fallacy, and [2] assumes asymmetric information among players. In this paper, we compare the empirical evidence from the Czech auction server Bonus.cz with predictions of the model presented in [3], which is able to explain the profit of the auction sites using the basic environment of utility-maximizing players by adding risk-seeking preferences.

In the rest of the paper, we proceed as follows. In Section 2, we describe our dataset retrieved from Bonus.cz and provide basic descriptive statistics. In Section 3, we present the intuition behind the theoretical explanation in [3] and summarize predictions of the model. In Section 4, we introduce the econometric models, present the results and compare the empirical findings with theoretical predictions. Finally, we conclude in Section 5.

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2 Data

The full dataset contains information about more than 55 000 auctions from the auction site Bonus.cz completed in the period from July 2009 to April 2012. Our software robot collected all information about the completed auctions provided on the site – the name of the item, retail price, purchase price, bidding increment, number of winner's bids, and special rules used in the auction. Bonus offered ten different special rules, such as shorter countdown times, limited number of bids by a single player, the right to sell the item back to the site for 60% of its retail price or the right to sell the item for the full price in bidding tokens. There were auctions with zero, one, or more than one special rules applied.

Using the collected data, we calculated two additional variables: the total number of bids equal to the final purchase price divided by the bidding increment and the seller's revenue equal to the total number of bids times the sum of bidding increment and price of a token. The price of a token itself had to be calculated from the available data. Although the basic price of the token on the site was set to 8 CZK, the actual average price was somewhat lower due to the fact that tokens could also be bought in packs with free tokens included. Unfortunately, the data on the number of paid and free tokens were available only for the winners of the auctions. Therefore, we calculated the average price of a token (approx. equal to 6.1 CZK) using the percentage of free bids which was calculated from the data on free and paid tokens used by the winners.

Furthermore, we divided all auctioned items into the following 13 categories covering the basic range of consumer products and services:

- culture – books, movies, magazines, etc.,
- free time activities – vouchers (e.g. for horseback riding, sauna, parachuting) and tickets,
- gold – actual gold or virtual gold, which is equivalent to a monetary prize,
- electronics – computers, tablets, televisions, phones, cameras, etc.,
- food and drinks – mostly luxury items such as chocolate pralines, expensive coffee or bottles of alcohol, often in the form of gift baskets,
- appliances – washing machines, coffee machines, vacuum cleaners, microwave ovens, etc.,
- toys and sport equipment – Lego, Barbie dolls, sports equipment, slot car tracks, etc.,
- clothing and accessories – mostly clothing, bags and handbags,
- video games – mostly game consoles (Xbox, Playstation, Nintendo) and games for all platforms,
- cosmetics – mostly gift packages including creams, gels, shampoos,
- phone credit – phone credit of different nominal values,
- bidding tokens,
- and others.

Moreover, we divided the data in two periods according to the size of the bidding increment. The bidding increments were exclusively 1 CZK or 0.1 CZK in the first period (from July 2009 to April 2010) and mostly 0.01 CZK in the second period (from April 2010 to April 2012). The periods also differed in bidding behavior of buyers and in profitability of the auction site. While the average profitability of the auctions in the first period was positive and relatively high, which is consistent with the evidence from the US data (see [1] or [3]), it was negative in the second period. The reasons for this change remain a matter of speculation and are beyond the scope of this study. However, the decreasing trend of auction profitability is noticeable already during the first period. This suggests that changes in the parameters of the auctions, most importantly the introduction of auctions with the bidding increment of 0.01 CZK, were a reaction to the fall in profitability, and not its cause. Because of the abnormal results of the auctions in second period, we restrict our study only to the first period. The total number of observations in the restricted dataset is 13 817.

The stylized facts about the pay-to-bid auctions in the first period are as follows. The average purchase price is 258 CZK, which is only 6.3% of the average retail price. Hence, the purchase price paid by the winner covers only a small part of the retail price. Adding the costs of bidding tokens increases the average amount paid by the winners to 899 CZK which amounts to 21.8% of the average retail price. Moreover, while most auctions resulted in net loss for the seller (the median profit is –354 CZK), the average profit was positive (1 035 CZK, markup of 27.2%). The median and average number of bids in an auction was 268 and 802, respectively, showing a skewed distribution with a lot of auctions ending relatively early and a long tail of auctions with high number of bids.

3 The Platt et al.'s model of pay-to-bid auctions

The game-theoretical model introduced in [3] is ideal for our analysis because it offers clearly defined predictions that can be tested on the aggregated auction data. Furthermore, its theoretical parameters correspond to the variables in our dataset. The model assumes a common valuation of the auctioned item, known to all potential bidders. This allows us to isolate an important aspect of the game - the players are willing to bid (and pay the bidding fee) while knowing that the other players are willing to pay for the item more than the current purchase price and can overbid them in the following period. Their bid is therefore similar to a bet that no other player will bid after them. Because the auctioned items are new and normally available from traditional retailers, the

common valuation seems a reasonable assumption. In the rest of the section, we will present a basic setup of the game, the intuition behind the equilibrium outcome and predictions that will be tested on our data.

The game contains a constant number of players watching the auction. The auction is divided into periods which end after the timer runs out or after someone places a bid. In each period, all players who are not currently winning simultaneously decide whether to place a bid. If only one player decides to place a bid, she pays the bidding fee and enters the next period as the current winner and the purchase price increases by the bidding increment. If there are more players placing a bid, the player who pays the bidding fee and becomes the current winner is randomly selected out of these players. If no player places a bid, the auction ends. The current winner pays the purchase price and receives the item.

In this game, the so-called *indifference condition* is used to guarantee a reasonable equilibrium outcome. It means that in all periods, players have the same expected utility from bidding as well as from abstaining from bidding. They use individual mixed strategies with certain probabilities of bidding that guarantee aggregate probabilities of another bid being placed in the current period, such that players in the previous period are exactly indifferent about placing their bids. While there are other symmetric equilibria, they produce "degenerate" outcomes, meaning that auctions end without any bids or exactly after one bid. If players decided to use mixed strategies with higher probabilities of bidding in period q, they would also cease to be indifferent about their bids in the previous period $q - 1$. Due to a higher probability of being outbid in the next period, abstaining from bidding in period $q - 1$ would become a strictly dominant strategy. Consequently, placing a bid would become a strictly dominant strategy in period $q - 2$ as the winner in that period could not be outbid. This effect would propagate back to the first period, causing the "degenerate" outcome. Similar outcomes arise also for other deviations from the strategy leading to indifference between placing and not placing the bid.

Hence we consider only individual strategies that are consistent with the indifference condition. These strategies result in unique aggregate probability of the q^{th} bid being placed, conditional on the $q - 1^{\text{st}}$ bid having already been placed. We will also assume that players follow the same strategy in the first period even though they would not be forced to do so by the indifference condition because there are no previous periods.

The authors find symmetric subgame perfect equilibria for two cases of players' utility functions - risk-neutral bidders and bidders with risk preferences. Let's consider the situation with risk-neutral players first. The players have the utility function $u(w) = w$, where w is the wealth of the player which depends on the initial endowment of the player, on the bidding decision of the player and on the outcome of the auction. Using the indifference condition, the probability density function for unconditional probability of an auction ending exactly after q bids is given by

$$f(q) = \left(1 - \mu_{q+1}\right) \prod_{j=1}^{q} \mu_j = \frac{b}{v - s \cdot q} \prod_{j=1}^{q} \left(1 - \frac{b}{v - s \cdot (j-1)}\right) \tag{1}$$

where q is the number of bids, b is the bidding fee, v is the value of the item, s is the bidding increment and μ_q is the aggregate probability of the q^{th} bid occurring. Because of the form of the utility function, the probability density function $f(q)$ is not a function of the unobservable endowment of players. The unconditional probability of the auction ending $f(q)$ is decreasing in the number of bids q. The expected number of bids is increasing in the value of the item v and decreasing in the bidding fee b and bidding increment s. The authors also show that expected revenue from this auction is equal to the item's valuation v and is independent from the bidding fee b and bidding increment s.

Next, we describe the more general situation in which preferences of players are represented by the CARA utility function $u(w) = (1/\alpha)(1 - \mathrm{e})^{-\alpha w}$, where α is the absolute risk aversion parameter. Positive values of α indicate risk-averse preferences and negative values indicate risk-seeking preferences. The advantage of the functional form used is that the unobservable initial endowment is not present in the final probability density function, which is given by

$$f(q) \equiv \left(1 - \mu_{q+1}\right) \prod_{j=1}^{q} \mu_j = \frac{e^{\alpha b} - 1}{e^{\alpha b} - e^{\alpha(b + sq - v)}} \prod_{j=1}^{q} \left(\frac{1 - e^{\alpha\left[b + s(j-1) - v\right]}}{e^{\alpha b} - e^{\alpha\left[b + s(j-1) - v\right]}}\right). \tag{2}$$

As for risk neutral players, the function is decreasing in number of bids q. The expected number of bids is increasing in the value of the item v and decreasing in the bidding fee b, bidding increment s and risk-aversion parameter α. A reduction in α leading to more risk-seeking preferences increases the expected number of bids, as players are willing to accept lower monetary payoffs and more adverse probabilities of winnings. The expected revenue from the auction cannot be analytically derived in this case. However, the effect of changes in parameters can be found using numerical simulations. The authors in [3] show that for risk-seeking players, the expected revenue increases in bidding fees b and decreases in bidding increment s and in risk-aversion parameter α. If players are more risk-seeking, they have additional utility from the risk which the seller is able to extract as revenue thanks to the indifference condition. A rise in the bidding fee b or a reduction in the bidding increment s cause higher variance in outcomes which contributes to higher utility of risk-seeking players. Conversely, a rise

in bidding fee b and a reduction in the bidding increment s reduces the expected revenue for risk-averse players. Furthermore, our simulations showed that the expected revenue is increasing in the value of the item v. Predictions of the model are summarized in Table 1.

	Players' risk preferences		
	risk neutral	**risk-seeking**	**risk-averse**
Expected number of bids E($f(q)$)	*increasing* in v *decreasing* in b and s	*increasing* in v *decreasing* in b, s, and α	*increasing* in v *decreasing* in b, s, and α
Expected revenue E(rev)	*increasing* in v (E(rev) = v)	*increasing* in v and b *decreasing* in s and α	*increasing* in v *and* s *decreasing* in b and α

Table 1 Predictions of the model

4 Empirical analysis

In this section, we present econometric models with two explained variables: number of bids (in log) and seller's revenue (in log of CZK). Because there is no theoretical foundation concerning the exact form of the estimated equations, we use several specifications and examine the estimated effects of the variables with respect to their stability across all specifications. Both explained variables were estimated as depending on the auctioned item's retail price, dummy variables for the value of bidding increment, dummy variables for special auction types and dummy variables for item categories. Some specifications also include the item's price to the power of 2 and to the power of 3, log of the price or interaction of the price with the dummy for the bidding increment value. The model specifications are described in Table 2. We estimated the models on the data from the first period (with the bidding increments of 1 and 0.1 CZK) using the ordinary least squares method with heteroskedasticity-consistent standard errors. Variables with high p-values were sequentially eliminated until the resulting model consisted only of the variables significant at least at the 0.10 level.

specification	**retail price**	**interaction**	**dummies**
base	basic	-	types, categories
log	log of	-	types, categories
exponential	second power, third power	-	types, categories
interaction	basic	crown * price	types, categories
combined	second power, third power	crown * price	types, categories

Table 2 Estimated specifications

Predictions of the theoretical model [3] are summarized in Table 2. Specifically, the model predicts that the expected number of bids increases in retail price, which measures the value of the item v, and decreases in bidding increment s. Hence the effect of retail the price on the number of bids should be positive and the effect of the dummy variable for the increment 1 CZK should be negative. These effects hold regardless of risk preferences of bidders. Furthermore, the model predicts a positive effect of retail price on seller's revenue. The bidding increment s has no effect on expected revenue for risk-neutral preferences, positive effect for risk-averse preferences and negative effect for risk-seeking preferences.

Table 3 presents summary of the results. For each variable and model, we show a total number of specifications for which the estimated parameter is positive, negative and not statistically significant at the 10% level [positive / negative / not significant][3]. If the results are consistent at least across three different specifications and there is no major conflict, we conclude that a variable has either positive (+) or negative impact (–). Otherwise, results remain inconclusive (0).

[3] For the sake of easy orientation, we present results of various parameterizations using just two variables - *retail price* and *increment 1 CZK*. In some of the estimations, the influence of price is not strictly positive throughout all possible values. However, the price range where the influence is negative is limited to very low or very high prices and only a handful of items fall into this category.

	model	
variable	number of bids	seller's revenue
retail price	+ [5/0/0]	+ [5/0/0]
increment 1 CZK	- [0/5/0]	- [0/5/0]
some tokens returned after the end of auction	+ [5/0/0]	+ [5/0/0]
right to purchase the item with a price cut	+ [5/0/0]	+ [5/0/0]
limited number of bids by a single player	- [0/5/0]	- [0/5/0]
progressively shortening timer	- [0/4/1]	0 [0/2/3]
shorter timer	0 [1/0/4]	0 [1/0/4]
right to sell the item back	0 [0/0/5]	0 [0/0/5]
appliances	+ [5/0/0]	+ [5/0/0]
toys and sport equipment	+ [5/0/0]	+ [5/0/0]
bidding tokens	+ [5/0/0]	+ [5/0/0]
electronics	+ [5/0/0]	+ [3/0/2]
culture	- [0/5/0]	- [0/5/0]
free time activities	- [0/5/0]	- [0/5/0]
video games	- [0/3/2]	- [0/3/2]
gold	0 [2/1/2]	0 [2/3/0]
food and drinks	0 [2/0/3]	0 [2/0/3]
clothing and accessories	0 [0/1/4]	0 [0/1/4]
cosmetics	0 [2/0/3]	0 [2/0/3]
phone credit	0 [2/0/3]	0 [2/0/3]
average adjusted R^2	0.35	0.35

Table 3 Estimation results

The results show the dominant influence of a variable: + (positive), – (negative), 0 (inconclusive); total number of specifications where variable parameter estimation is [positive / negative / not significant].

The results presented in Table 3 show a positive effect of retail price on the number of bids and seller's revenue and a negative effect of bidding increment on the number of bids. This is consistent with the model predictions regardless of players' risk preferences. On the other hand, the predicted effect of the bidding increment depends on risk preferences of players. A negative effect of bidding increment is consistent with predictions of the model with risk-seeking players. Hence our findings suggest that risk-seeking bidders were dominant in the online auctions on Bonus. This result is supported by Platt at el. [3] who find that players participating in US version of Swoopo had predominantly risk-seeking preferences as well.

Moreover, Table 3 provides interesting findings concerning the effects of special auction types. If some tokens are returned after the end of auction or the winner can purchase the item with a price cut, players tend to bid more, which in turn generates higher revenue for the seller. Similarly, if the number of bids by a single player is limited, the average number of bids and the revenue of the seller are lower. Hence, the results for special auction types seem to be consistent with the assumption of rational players. Furthermore, there seem to be categories of auctioned items which are generally more popular and able to attract more bids and the revenue the seller earned on these items was higher compared to the category others. This is especially true for appliances, toys and sport equipment, bidding tokens and electronics. On the other hand, the items in categories free time activities, culture and video games attracted rather less bids and generated relatively low seller's revenues.

5 Conclusion

In this paper, we analyze a unique dataset recovered from the Czech auction site Bonus.cz. In the period from July 2009 to April 2010, the data reveals similar pattern of profitability from individual auctions to that reported by other studies using US data from Swoopo.com (see [1] and [3]). It shows that a majority of the auctions ended with a low number of bids and a loss for the auction site. However, a relatively low number of auctions attracted a large number of bids, so that the overall average profit for the seller was positive. Using the data for the period from July 2009 to April 2010, we estimate the effect of retail price, bidding increment and dummy variables for special types of the auction and categories of the items on number of bids and revenue of the seller. We find that a rise in retail price and a reduction bidding increment increases the number of bids. Furthermore, we find the same effect of both variables on revenue of the auction site. We compare the results with the predictions of the

model with rational players under perfect information in [3]. We find that the basic version of model with risk-neutral players is incapable of explaining our data. However, after incorporating risk-seeking preferences into the model, the predictions correspond perfectly to the empirical results. Hence, our analysis suggests that the bidding pattern on Bonus.cz is consistent with the optimal choices of utility-maximizing consumer under perfect information and risk-seeking preferences.

Acknowledgements

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Economic Analysis of Czech Regions by Econometric Models, MCDM and DEA

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Abstract. The Czech statistical office regularly publishes economic data about Czech regions. Based on these data we would like to analyze selected economic indices, such as income of regions, amount of grants and subsidies transferred to other subjects etc., and to model them using by econometric models. We try to use equilibrium models and to interpret obtained results. With respect to econometric results we would like to compare Czech regions with each other. The economics of regions can be evaluated also by multicriteria decision making (MCDM) analysis or using data envelopment analysis (DEA) models and we would like to compare econometric results also with MCDM and DEA ones.

Keywords: Czech Regions, economic indicators, budgets, DEA, multicriteria, regression

JEL Classification: C01, C30, C44, C67
AMS Classification: 62J05, 62J12, 62P20, 90B50, 90C29, 91B06

1 Introduction

Since 2000, the Czech Republic is divided into thirteen regions – Central Bohemian Region, South Bohemian Region, Plzeň Region, Karlovy Vary Region, Ústí nad Labem Region, Liberec Region, Hradec Králové Region, Pardubice Region, Vysočina Region, South Moravian Region, Olomouc Region, Moravian-Silesian Region, Zlín Region – and Prague, the Capital City (see figure 1). Each region has its own elected Regional Assembly and hetman. In Prague, their powers are executed by the city council and the mayor.

Figure 1 Czech regions [4]

In this paper we would like to analyze selected data from regional budgets and also from the Czech statistical office and compare all fourteen Czech regions with each other. For our analysis we used official budgets of regions from years 2005 – 2012 available from official web pages of Czech regions. Relevant data from these budgets are income of region (in thousands of CZK), common expenditures (mainly transferred grants and subsidies, in thousands of CZK), capital expenditures (mainly investments, in thousands of CZK) and expenditures of regional assembly (in thousands of CZK). We also use data taken from the web pages of Czech Statistical Office [4], especially unemployment rate (in percentage), ratio of economic activity (in percentage), average wage (in CZK), average age (in years) and the number of free workplaces.

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We assume that region with higher income is more effective than the one with lower budget. Higher income indicates higher amount of grants and subsidies transferred to the economic subjects. Also the region with higher amount of investments is evaluated better as well as region with lower unemployment rate and higher economic activity. Of course we would like region with young people so the lower average age is desirable and higher average wage is also wanted.

For comparison of regions it is possible to use various kinds of techniques and methods such as statistical methods, econometric models, data envelopment analysis (DEA) models, or multicriteria decision making (MCDM) models, especially multicriteria evaluation of alternatives (MCEA). The aim of this article is to compare the Czech regions from the economic point of view via econometric model (chapter 2), MCEA (chapter 4) and DEA models (chapter 3) and also compare the results of this methods (chapter 5).

2 Econometric model

For analysis of relationships between described economic indicators we used the simple econometric model of simultaneous equations. This model consists of two equations in following form:

$$C_t = C_0 + cY_t + u_t, \tag{1}$$

$$Y_t = C_t + I_t + G_t, \tag{2}$$

where C_t denotes consumption (in the case of this paper amount of expenditures) in the time t, Y_t denotes income of region, I_t denotes investments and G_t denotes expenditures of regional assembly. Symbols C_0 and c are parameters of model, C_0 denotes autonomous consumption and c denotes marginal propensity to consume. It is obvious that $c = \frac{\partial C_t}{\partial Y_t}$ and so this parameter can be interpreted as the increase in consumption with respect to increase of one unit of income.

The previous model can be transformed into two independent equations in the form:

$$C_t = \frac{C_0}{1-c} + \frac{c}{1-c} I_t + \frac{c}{1-c} G_t + v_t, \tag{3}$$

$$Y_t = \frac{C_0}{1-c} + \frac{1}{1-c} I_t + \frac{1}{1-c} G_t + v_t, \tag{4}$$

and the parameters of these equations can be simply estimated by ordinary least squares (or generalized least squares) [7]. We applied this theory on regional data and we obtained data in table 1.

Region		*c*	*c(std err)*	*p-value*	***Rank***
1.	Prague, the Capital City	0.668544	0.028304	6.19E-08	**10**
2.	Central Bohemian	0.628461	0.035494	4.52E-07	**11**
3.	South Bohemian	0.829656	0.018657	7.60E-10	**5**
4.	Plzeň	0.498379	0.037294	3.08E-06	**14**
5.	Karlovy Vary	0.887146	0.016535	2.05E-10	**1**
6.	Ústí nad Labem	0.829405	0.015272	1.88E-10	**6**
7.	Liberec	0.853120	0.018474	5.84E-10	**3**
8.	Hradec Králové	0.740908	0.016342	6.64E-10	**7**
9.	Pardubice	0.577770	0.026701	1.14E-07	**13**
10.	Vysočina	0.668813	0.084357	4.20E-03	**9**
11.	South Moravian	0.729998	0.024013	1.07E-08	**8**
12.	Olomouc	0.850651	0.024865	4.73E-09	**4**
13.	Moravian-Silesian	0.613880	0.018846	6.65E-09	**12**
14.	Zlín	0.864907	0.008427	2.20E-12	**2**

Table 1 Marginal propensity to consume

Note that for all regions parameter c is statistically significant at 1% significance level and parameter C_0 is nonsignificant at the same level. As consumption in our model includes mainly grants and subsidies intended for transfer to other subjects, marginal propensity to consume denotes how many percentage of regional income is

transferred in the form of grants and subsidies. It is clear that higher value of parameter *c* is better for people and the rank with respect to this parameter is in the last column of table 1.

We can see that the highest ratio of grants and subsidies in budget has Karlovy Vary Region, the second one is Zlín Region and the third is Liberec Region. Region Prague, the Capital City placed tenth place. It is given by large expenditures of town council and huge investments. The last region is Plzeň with really low consumption.

For information we analyzed also dependency of regional incomes on other economic factors introduced in introduction (average of years 2005 – 2012). For 10% level of significance regional income (Y_t) is dependent on constant, consumption (C_t), investments (I_t) and economic activity (A_t) in the form:

$$Y_t = -54.48 + 0.66\,C_t + 0.96\,I_t + 0.97A_t + e_t \tag{5}$$

and for 5% and 1% level of significance the model has this form:

$$Y_t = 1.30 + 0.83\,C_t + 0.98\,I_t + e_t. \tag{6}$$

Both models are statistically significant at 1% level and remaining variables are insignificant in these models.

3 Data envelopment analysis model

In the second step of our analysis we used the data envelopment analysis (DEA) not only to compare the regions but also to find which one is efficient. DEA belongs to the operational research methods, especially to the linear programming models, that have been used many times in private or public sector to evaluate the performances of many different kinds of entities (countries, regions, enterprises, schools, hospitals, insurance companies, military units etc.) engaged in many different kinds of activities in many different contexts [2]. These entities must have identical inputs and outputs to measure the efficiency from the same parameters.

DEA models are widely used in comparison of countries, regions or districts from various points of view. Melecky and Stanickova [9] compared evaluated the performance of the four Visegrad countries and their NUTS 2 regions. Friebel and Friebelova [5] measured life quality in 14 Southwest Czech districts (LAU 1) by DEA using 4 inputs and 1 output. These articles are aimed at the measurement of competitiveness by economic and social characteristics or measuring of the life quality.

The basic idea of DEA models consists in estimation of an efficient frontier that defines production possibility set of the problem. Based on the set of available decision making units (DMUs) DEA estimates so-called efficient frontier, and projects all DMUs onto this frontier. If a DMU lies on the frontier, it is referred to as an efficient unit, otherwise inefficient. DEA models can be oriented to inputs or outputs. In the case of input oriented models we assume fixed level of outputs (CCR-I), the output oriented model assumes fixed level of inputs and maximize level of outputs with respect to given inputs (CCR-O) [1]. These models are used if we assume constant return to scale. In the case of variable return to scale we work with BCC (Banker, Charnes, Cooper) models. The review and detailed information about DEA models can be found in [2]. The basic idea for the efficiency calculation is to maximize the rate of weighted sum of outputs divided by weighted sum of inputs. For example the model transformed (Charnes-Cooper transformation) into the linear programming form can be defined as follows (CCR-I):

$$\begin{aligned}
&\text{Maximize } z = \sum_{i=1}^{r} u_i y_{iq} \\
&\text{Subject to: } \sum_{i=1}^{r} u_i y_{ik} \le \sum_{j=1}^{m} v_j x_{jk}\ ,\ k = 1, 2, \ldots, n \\
&\sum_{j=1}^{m} v_j x_{jq} = 1 \\
&u_i \ge 0\ ,\ i=1,2,\ldots,m \\
&v_j \ge 0\ ,\ j=1,2,\ldots,r,
\end{aligned} \tag{7}$$

where *q* represents the evaluated DMU, y_{rj} are known outputs, x_{ij} are known inputs of the *j*th DMU, u_r and v_j are the variable weights to be determined by the solution of this problem. The efficient unit U_q lies on the efficient frontier in case that the optimal efficiency (calculated by the model) $z = 1$. The inefficient units have *z* lower than 1 (in CCR-I model) [1]. In some situations the outputs cannot be influenced or changed (like average age) – so the model is transformed into the nondiscreditory model where the weighted output is decrease by these uncontrollable outputs (models UCCR-I and UBCC-I).

In our case we compared 14 regions with respect to available data. The models had five inputs (unemployment rate, economic activity, average wage, average age and free workplaces per capita) and four outputs (marginal propensity to consume, income per capita, consumption per capita and investments per capita). The DEA model assumes inputs are minimized and outputs are maximized for efficient units. In this case we had to transform input data to minimize them. The results (objective values) obtained after transformation from DEA models are in table 2. Note that the last column (*AAD*) displays average absolute deviation from efficient frontier. Zero values denote regions those are efficient in all six models.

From the data in table 2 we can easily conclude that five regions are efficient. There are Prague, the Capital City, South Bohemian, Karlovy Vary, Liberec and Zlín Region – these are efficient in the case of all six DEA models. The absolutely inefficient is Moravian-Silesian Region with average absolute deviation more than 20%.

Region	*CCR-I*	*CCR-O*	*BCC-I*	*BCC-O*	*UCCR-I*	*UBCC-I*	***AAD***
Prague, the Capital City	1	1	1	1	1	1	***0***
Central Bohemian	0.8144	1.2279	1	1	0.7843	1	***0.1049***
South Bohemian	1	1	1	1	1	1	***0***
Plzeň	0.7047	1.419	1	1	0.6695	1	***0.1741***
Karlovy Vary	1	1	1	1	1	1	***0***
Ústí nad Labem	0.9778	1.0226	1	1	0.9154	1	***0.0216***
Liberec	1	1	1	1	1	1	***0***
Hradec Králové	0.8816	1.1343	0.982	1.1245	0.8684	0.9145	***0.1021***
Pardubice	0.7687	1.301	1	1	0.7132	1	***0.1365***
Vysočina	0.8128	1.2303	0.9965	1.1520	0.7512	0.9313	***0.1484***
South Moravian	0.8530	1.1724	0.9793	1.1612	0.8049	0.9063	***0.1317***
Olomouc	0.9588	1.0430	0.9840	1.0334	0.8977	0.8993	***0.0561***
Moravian-Silesian	0.7633	1.3101	0.9892	1.3073	0.6955	0.9085	***0.2101***
Zlín	1	1	1	1	1	1	***0***

Table 2 DEA results

4 Multicriteria evaluation of alternative model

In the third step of our analysis we used the multicriteria evaluation of alternative (MCEA). MCEA belongs to the category of discrete multi-criteria decision making models where all the alternatives (a_1, a_2, …, a_p) and criteria (f_1, f_2, …, f_k) are known. To solve this kind of model it is necessary to know the preferences of the decision maker. These preferences can be described by aspiration levels (or requirements), criteria order or by the weights of the criteria. For our analysis we used three methods – WSA, TOPSIS and ELECTREE III. that need only decision matrix **Y** and weights of criteria as inputs [5]. WSA (Weighted Sum Approach) sorts the alternatives based on the values of their utility functions which in this case are assumed to be linear. It requires the information about the weights of the criteria. Higher value of utility means better alternative.

The basic concept of the TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) method is that the best alternative should have the shortest distance from the ideal alternative and the farthest from the basal alternative. The method is also able to rank the alternatives using the relative index of distance of the alternatives from the basal alternative. Higher relative index of distance means better alternative. The user must supply only the information about the weights of criteria

ELECTRE III. method uses pairwise comparison of the alternatives and summarizes the weights of the criteria where the alternative is better than the other. According to the strength of the preference the final indifference classes are made. It is necessary that all the alternatives are nondominated (so there is no alternative that is better in at least one criterion and no worse in any other) [5].

For analysis we use the same criteria as in DEA model and weight of each criterion was equal (v_j = 1/9, j = 1, …, 9). We obtained the results displayed in table 3 (note that all alternatives are nondominated).

Column *Utility* displays weighted sum gained by WSA and *Rank WSA* presents order with respect to maximal utility. Similarly, *R.R.D.* column displays the relative ratio of distance to the basal alternative and *Rank TOPSIS* presents order with respect to maximal relative ratio of this distance. Column *Rank ELECTRE III.* presents order with respect to ELECTRE classes and *Average rank* is the mean of three previous ranks. *Total rank* sorts regions with respect to average rank.

MCEA with equal weights places South Bohemian Region at the first place before Prague, the capital city and Ústí nad Labem Region. Central Bohemian Region is the fourth and Karlovy Vary Region has the fifth place. The last ones are Pardubice Region together with South Moravian Region.

Region	*Utility*	*Rank WSA*	*R. R. D.*	*Rank TOPSIS*	*Rank ELECTRE III.*	*Average rank*	***Total rank***
Prague, the Capital City	0.71530	1	0.71081	1	5	2.3	**2**
Central Bohemian	0.42405	5	0.34276	4	3	4.0	**4**
South Bohemian	0.49468	3	0.40463	2	1	2.0	**1**
Plzeň	0.25370	13	0.24751	13	8	11.3	**12**
Karlovy Vary	0.50629	2	0.34048	5	6	4.3	**5**
Ústí nad Labem	0.44655	4	0.36677	3	2	3.0	**3**
Liberec	0.37764	7	0.29446	8	10	8.3	**8**
Hradec Králové	0.32561	10	0.27495	10	10	10.0	**10**
Pardubice	0.24963	14	0.25215	12	12	12.7	**13**
Vysočina	0.38033	6	0.32939	7	4	5.7	**6**
South Moravian	0.29332	12	0.24053	14	12	12.7	**13**
Olomouc	0.33361	9	0.27070	11	12	10.7	**11**
Moravian-Silesian	0.32052	11	0.33479	6	7	8.0	**7**
Zlín	0.35227	8	0.27497	9	9	8.7	**9**

Table 3 Multicriteria evaluation of alternatives – results

5 Results

Now we can compare results of all three parts. Table 4 summarizes the previous ranks.

Region		*Econometric rank*	*DEA rank*	*MCEA rank*	***Total rank***
1.	Prague, the Capital City	10	1	2	**5**
2.	Central Bohemian	11	9	4	**8**
3.	South Bohemian	5	1	1	**1**
4.	Plzeň	14	13	12	**14**
5.	Karlovy Vary	1	1	5	**1**
6.	Ústí nad Labem	6	6	3	**6**
7.	Liberec	3	1	8	**3**
8.	Hradec Králové	7	8	10	**9**
9.	Pardubice	13	11	13	**13**
10.	Vysočina	9	12	6	**10**
11.	South Moravian	8	10	13	**11**
12.	Olomouc	4	7	11	**7**
13.	Moravian-Silesian	12	14	7	**12**
14.	Zlín	2	1	9	**3**

Table 4 Recapitulation of results

Econometric and DEA models provide similar results expected Region Prague, the Capital City. With reference to marginal propensity to consume Prague, the Capital City places the tenth rank however DEA and MCEA place it at first two positions. This fact is clear from budget point of view. Prague, the Capital City has large income that divides between grants, investments and also expenditures of regional assembly. These expenditures

are much higher than in the case of others regions and so ratio of grants is smaller. In DEA and also MCEA models the other economic indicators play the role and so marginal propensity to consume is not so important with respect to income, investments and grants per capita where Prague, the Capital City places the first rank, as well as unemployment rate, economic activity, average wage and free workplaces.

The last column of table 4 denotes the rank of region with respect to average rank of all three approaches. Note that regions efficient according to DEA models are placed at first five places. First two places can be divided between South Bohemian and Karlovy Vary Regions, next two places are allocated to Liberec Region and Zlín Regions. Surprisingly Prague, the Capital City is placed at the fifth position. There follow inefficient regions (according to DEA model) and the last one is Plzeň Region.

6 Conclusion

The aim of this contribution was to analyze the economic position of the fourteen regions of the Czech Republic by different methods to set the order of the regions and also to compare the results of the methods. At first econometric model of simultaneous equations was used, next DEA models were calculated and finally mucltricriteria evaluation of alternatives was applied. It is clear that DEA models are not able always to create the order of the alternatives but the results are close to the econometric model (except of Prague but this region is always completely different than the others because of the capital city). The MCEA methods do not cover the relations among the criteria and so these results are more nuanced. As for the regions we can see that Prague do not need to be the best region, as Karlovy Vary starts to be better together with South Bohemian region, Liberec and Zlín.

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Comparison of selected municipalities of Vysocina region from the economic activity and traffic accessibility point of view using DEA model

Martina Kuncová[1], Jana Sekničková[2], Libuše Měrtlová[3], Hana Vojáčková[4]

Abstract. Traffic accessibility can be the key factor for economic activity, employment and unemployment for each municipality, district region. It is influenced by various factors like number of kilometers of roads, highways and railroads and also by travel times and number of public transportation lines or distance from the district center or from the center of the region. This article describes the analyses of selected municipalities from the economic activity and traffic accessibility point of view using DEA models. DEA (Data Envelopment Analysis) models are usually used to find the relative efficiency among homogenous units according to selected criteria (inputs and outputs). In this article we try to compare municipalities to find the problematic places.

Keywords: DEA models, Vysocina Region, traffic accessibility, economic activity

JEL Classification: C44, C67
AMS Classification: 90B50, 90B90

1 Introduction

Vysocina region lies in the center of the Czech Republic between Bohemia and Moravia. This region belongs to the agricultural and industrial ones. The unemployment rate is usually below the republic average but it differs from the districts inside the region. On the other hand the average wages are lower than in other regions. The regional disparities can be seen especially in the socio-economic indicators. If we take into account the GDP per capita and unemployment rate we can see that there are also big differences between the centers of the region and in the municipalities. Those which lie far from the industrial centers or from the highways have lower GDP and also higher unemployment. These facts cause the higher demand on social benefits, the decrease of inhabitants in those areas and the rising pressure for people to travel to the far-away centers. For travelling the traffic accessibility is crucial. The measurement of traffic accessibility is not easy because it is influenced by many factors such as number of kilometers of roads, highways and railroads and also by travel times and number of public transportation lines or distance from the district center or from the center of the region. In this paper we would like to compare the selected areas in Vysocina region according to the traffic accessibility conditions and economic activity represented by unemployment and number of businesses (companies and entrepreneurs).

For comparison of regions or municipalities it is possible to use various kinds of techniques and methods such as multi-criteria evaluation of alternatives, statistical methods, econometric models or DEA models. In this first phase we have decided for the data envelopment analysis (DEA) not only to compare the selected areas but also to find which one is efficient. DEA belongs to the operational research methods, especially to the linear programming models, that have been used many times in private or public sector to evaluate the performances of many different kinds of entities (countries, regions, enterprises, schools, hospitals, insurance companies, military units etc.) engaged in many different kinds of activities in many different contexts [2]. These entities must have identical inputs and outputs to measure the efficiency from the same parameters.

DEA models are widely used in comparison of countries, regions or districts from various points of view. Melecky and Stanickova [9] compared evaluated the performance of the four Visegrad countries and their NUTS

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2 regions. Friebel and Friebelova [5] measured life quality in 14 Southwest Czech districts (LAU 1) by DEA using 4 inputs and 1 output. These articles are aimed at the measurement of competitiveness by economic and social characteristics or measuring of the life quality. If we talk about the efficiency of traffic and transportation system it is harder to find a comparison of regions or municipalities because of the lack of data. The studies of traffic accessibility concentrate usually on the small area and they are created under the support of grants as the process of data collection is long and expensive. A big study of the accessibility of the employers in the district Bruntal (Czech Republic) was made under the grant of the Grant Agency of the Czech Republic [6]. Authors analyzed the situation of 302 municipalities on the basis of the public and private transport accessibility, time and kilometers distance from the main employers. For the comparison they used network and statistical analysis. One study using DEA model for evaluation of traffic and transportation system was made in China [11]. This study uses fixed assets investment and quantity of employee as inputs and passengers volume, passengers' transportation volume, freights volume and freight turnover volume as outputs to evaluate the development of traffic and transportation in China's 31 province.

In the beginning of our analysis of the traffic accessibility in the Vysocina region we decided to compare selected areas from the traffic accessibility and unemployment point of view. As there are only 5 districts in Vysocina region it is not enough for the DEA model. But it is not possible to obtain the relevant data for all 704 municipalities and so we analyze the municipalities with extended competence called in Czech ORP (15 in Vysocina region). These municipalities lie between the NUTS IV (LAU 1) – districts (5 in Vysocina region), and NUTS V (LAU 2) – municipalities (704 in Vysocina region). Figure 1 shows the Vysocina region and the areas that belong to each municipality with extended competence (so all ORP's).

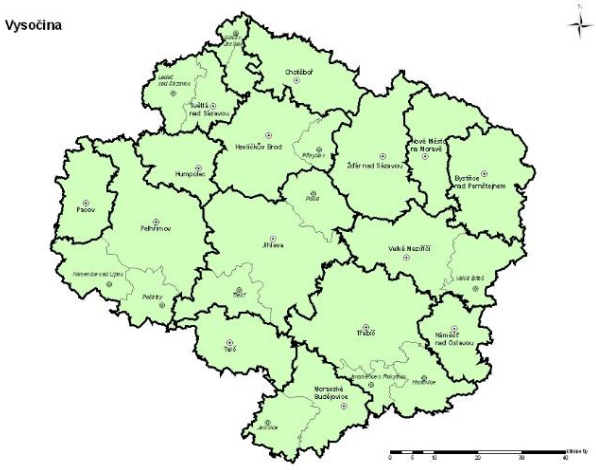

Figure 1 Vysocina region and 15 areas with municipalities with extended competences (ORP = obec s rozšířenou působností) [4]

2 Data and methods

The basic idea of DEA models consists in estimation of an efficient frontier that defines production possibility set of the problem. Based on the set of available decision making units (DMUs) DEA estimates so-called efficient frontier, and projects all DMUs onto this frontier. If a DMU lies on the frontier, it is referred to as an efficient unit, otherwise inefficient. DEA also provides efficiency scores and virtual units for inefficient DMUs. Reference units are hypothetical units on the efficient frontier, which can be regarded as target units for inefficient units. DEA models can be oriented to inputs or outputs. In the case of input oriented models we assume fixed level of outputs (CCR-I), the output oriented model assumes fixed level of inputs and maximize level of outputs with respect to given inputs (CCR-O) [1]. These models are used if we assume constant return to scale. In the case of variable return to scale we work with BCC (Banker, Charnes, Cooper) models. The review and detailed information about DEA models can be found in [2] and [3]. The basic idea for the efficiency calculation is to maximize the rate of weighted sum of outputs divided by weighted sum of inputs. For example the model transformed (Charnes-Cooper transformation) into the linear programming form can be defined as follows (CCR-I):

$$\text{Maximize } z = \sum_{i=1}^{r} u_i y_{iq}$$

$$\text{Subject to: } \sum_{i=1}^{r} u_i y_{ik} \le \sum_{j=1}^{m} v_j x_{jk} \text{ , } k = 1, 2, \ldots, n \qquad (1)$$

$$\sum_{j=1}^{m} v_j x_{jq} = 1$$

$u_i \geq 0$, i=1,2,…,m

$v_j \geq 0$, j=1,2,…,r,

where q represents the evaluated DMU, y_{rj} are known outputs, x_{ij} are known inputs of the jth DMU, u_r and v_j are the variable weights to be determined by the solution of this problem. The efficient unit U_q lies on the efficient frontier in case that the optimal efficiency (calculated by the model) $z = 1$. The inefficient units have z lower than 1 (in CCR-I model) [1].

The aim of DEA is to separate the DMUs into efficient and inefficient ones according to the defined inputs and outputs. DEA models are based on the fact that it is possible to find the efficient frontier formed by combinations of inputs and outputs of some DMUs. The units lying on the frontier are considered as efficient and the remaining ones as inefficient. Their efficiency score is measured as a distance from the efficient frontier [2]. The number of DMUs should be high enough because if we have few units and a lot of inputs and outputs, all units are considered to be efficient. It is not necessary to have a lot of criteria especially when we would like to describe the situation in a graph and show the efficient frontier.

Typical CCR and BCC models suppose it is possible to change inputs and outputs so as the inefficient DMU could be efficient. Sometimes it is necessary to use inputs or outputs that are given and cannot be change. In this situation the models with uncontrollable or nondiscretionary inputs or outputs have to be applied. The model stays the same as in (1) only in the objective function the weighted sum of outputs is decreased by the weighted sum of nondiscretionary inputs [2].

In our case we compare 15 municipalities with extended competence (Figure 1). According to the available data taken from the web pages of Czech Statistical Office [4] and web pages with regional characteristics [10], [8] we have decided to analyze the relationship between the traffic accessibility and the economic activity. The traffic accessibility can be described by many different factors. We have chosen the distance from the town Jihlava (the county seat) from the ORP, the average distance of the municipalities from its ORP (calculated from all the municipalities that belongs to the ORP) and the possibility to reach one of these places. The distances are measured in kilometers or in minutes (as a time to reach the town), the possibility to reach is expressed by the number (or average number) of lines (bus or train together) – it means how many times it is possible to go from a municipality to the ORP or Jihlava (or vice versa) daily. The outputs should characterize the economic activity, so the first one is unemployment and the second one number of all types of businesses in the ORP area. As in the usual DEA model the outputs are supposed to be maximized (and inputs minimized) we have transferred the unemployment percentage into the employment (100 – unemployment percentage). Also the number of lines can be view to be maximized or minimized. Firstly we take into account the regional authority point of view and so it is better to minimize it (because of the cost connected with the public transportation). Afterwards we change this criterion to be output just to maximize it (people point of view). We have thought also to include the number of economically active persons but this criterion is highly correlated (correlation coefficient 0.997) with the number of businesses and so it in not good to have them both in a model. In the last step we have added also the number of applicants for a job as an input. The first model uses these inputs and outputs:

Inputs:

- Distance from Jihlava in kilometers
- Time to reach Jihlava in minutes
- Number of lines to Jihlava
- Average distance to the ORP in kilometers
- Average time to reach ORP in minutes
- Average number of lines to ORP

Outputs:

- Unemployment (change into Employment)
- Number of businesses

As you can see we have 4 inputs that can be characterized as nondiscretionary – these are the distances in kms or time in minutes (because it is hard or impossible to reduce the distance) so we also try the models with nondiscretionary inputs.

3 Results

When we would like to create a graph with the efficient frontier we can use only 1 input and 2 outputs or 2 inputs and 1 output. When we consider both outputs and number of lines to ORP as an input we can see that there are only 2 efficient ORPs – Jihlava and Humpolec. It can be also influenced by the fact that these two ORPs are towns that are very close to the biggest motorway D1 and that is why a lot of buses goes there. On the other hand Velke Mezirici is also close to D1 but it is very far from the efficient frontier. It is because in Jihlava and Humpolec the unemployment rate is smaller (they are industrial centers) than in Velke Mezirici and Bystrice nad Pernstejnem or Trebic where the unemployment rate is higher than 10 %.

As Jihlava is the biggest town of the Vysocina region a lot of people work there. If we compare the relationship between the employment as an output and the distance of the ORP from Jihlava and number of lines to Jihlava as 2 inputs (Figure 2) Jihlava must be efficient but if we erase it then Havlickuv Brod would lie on the efficient frontier with Pelhrimov, Svetla nad Sazavou and Pacov. First two units are district towns and so the number of lines to Jihlava is large, next two units have very low unemployment. Inefficient ORP Telc has higher unemployment rate but also very high number of lines to Jihlava in comparison with the distance. It is because this town is registered on the UNESCO's List of World Cultural Heritage sites. As we cannot change the distance, the only possibility how can Telc became efficient is to increase employment (or decrease unemployment). It is clear that this recommendation is easy to say but hard to put into practice. The result is not influenced by other criteria that are important (such as number of businesses, number of free working places, wages, .. – but some of them are unavailable for the municipalities). The next unefficient Namest nad Oslavou has the highest unemployment rate in Vysocina region (14,1 %).

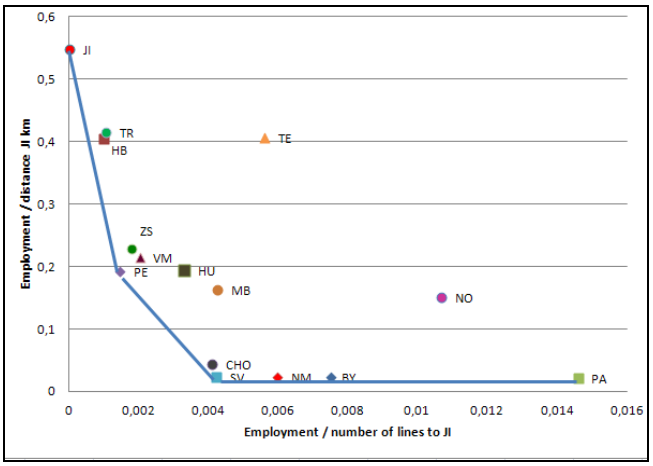

Figure 2 Efficient frontier for 2 inputs and 1 output

For the complete results we have used DEA Excel Solver [7]. We have mentioned above that 4 of our 6 inputs are nondiscretionary and that is why we have calculated not only CCR and BCC models but also its alternative with this specifications (nondiscretionary CCR model = ND-CCR, nondiscretionary BCC model = ND-BCC). The first two models were calculated as output oriented because outputs can be better changed than inputs, the second two models are input oriented. All the results are in the Table 1. From the 15 ORPs nearly half of them is efficient in all models – Humpolec, Chotebor, Jihlava, Pacov, Pelhrimov, Svetla nad Sazavou and Velke Mezirici. In all these municipality areas the unemployment rate is lower than 9.9 % whereas from the inefficient ORPs only Havlickuv Brod and Zdar nad Sazavou has the unemployment rate lower than 10 %.

These results take into account the regional authority point of view (with minimization number of lines). As we have calculated output oriented model the Table 2 shows possible changes of 2 outputs for the inefficient DMUs to be efficient. It has been mentioned that in reality it is not easy to decrease the unemployment rate but we can see that in the areas of Moravske Budejovice, Namest nad Oslavou and Trebic the unemployment should decrease for more than 4 % which is today unreal. The extreme change of number of businesses should happen in Namest nad Oslavou to be efficient (or rapidly decrease the number of lines to Jihlava) – so we see that this ORP is the most inefficient.

The number of lines can be seen also from the usual people's point of view – so we have tried to change the model so as the number of lines has started to be two new outputs. In this case BCC output oriented model has figured out only 4 inefficient units: Bystrice nad Pernstejnem, Chotebor, Nove Mesto na Morave and Zdar nad Sazavou. CCR output oriented model adds Havlickuv Brod, Pelhrimov and Trebic. Table 3 describes the possi-

ble increase of the number of lines for the DMUs to be efficient. It is clear that for the places with low number of lines to Jihlava the increase should be very large but it is because of the other criteria not only because of the low number (we put into the table also Pacov which is efficient with low number of lines). This analysis tells us that there are some localities far from Jihlava (Bystrice nad Pernstejnem, Nove Mesto na Morave, Chotebor) where it is better to increase the number of lines to its ORP than to Jihlava (as it is more expensive) or try to increase number of businesses (and number of entrepreneurs) to be comparable with other localities in Vysocina region. As a last step we have added the number of applicants for a job as a new input but the results has stayed nearly the same as in the previous analysis.

DMUs (ORP) / Model	CCR-O	BCC-O	ND-CCR	ND-BCC
Bystrice nad Pernstejnem (BY)	0.96906	0.96906	0.96906	1
Havlickuv Brod (HB)	0.98481	0.99148	0.98481	0.98634
Humpolec (HU)	1	1	1	1
Chotebor (CHO)	1	1	1	1
Jihlava (JI)	1	1	1	1
Moravske Budejovice (MB)	0.89364	0.92677	0.64587	0.69496
Namest nad Oslavou (NO)	0.95292	0.95292	0.85447	0.90909
Nove Mesto na Morave (NM)	0.98320	0.98320	0.98320	1
Pacov (PA)	1	1	1	1
Pelhrimov (PE)	1	1	1	1
Svetla nad Sazavou (SV)	1	1	1	1
Telc (TE)	0,99648	1	0.99352	1
Trebic (TR)	0.94032	0.94536	0.92310	0.92364
Velke Mezirici (VM)	1	1	1	1
Zdar nad Sazavou (ZS)	0.9623	0.99084	0.9623	0.97344

Table 1 Results of the DEA models using 6 inputs and 2 outputs.

Outputs / DMUs	BY	HB	MB	NO	NM	TE	TR	ZS
Real unemployment	12.24	8.57	13.88	14.1	10.92	11.27	13.1	8.39
Virtual unemployment	9.44	7.79	7.08	9.85	9.39	11.27	8.06	7.55
% change	2.8	0.78	6.8	4.25	1.53	0	5.04	0.84
Real Number of businesses	3730	10339	4423	2500	3764	2759	15448	9295
Virtual number of businesses	3849	10428	4773	4084	3828	2759	16341	9381
% change	3.19	0.86	7.9	63.36	1.71	0	5.78	0.92

Table 2 Possible changes of outputs according to CCR-O model

Outputs / DMUs	BY	CHO	NM	PA	ZS
Real number of lines to Jihlava	2	4	2	2	21
Virtual number of lines to Jihlava	16.9	15.8	7.8	2	21.4
% change	745	295	290	0	1.9
Real number of lines to ORP	14	10	11	12	12
Virtual number of lines to ORP	14,2	10,2	12.4	12	13
% change	1.4	2	12.7	0	8.33

Table 3 Possible changes of outputs according to CCR-O model with number of lines as output

This analysis can be influence by the fact that we have a lot of inputs and some of them can be correlated. The correlation matrix showed us that the highest correlation (90%) is between the distance to Jihlava and time to reach Jihlava (which can be expected) but the high correlation is also between the number of lines to ORP and time to reach ORP (86%) which is interesting. So we have tested some other models with lower number of inputs and outputs. If we take into account the municipality point of view that would like to minimize number of

lines (to Jihlava and ORP, 2 inputs) and maximize employment and number of businesses (2 outputs) the results are nearly the same like in first model only Velke Mezirici starts to be inefficient. So we can say that in this model the results and number of efficient units are not influenced so much by correlated inputs. When we change the point of view to respect the inhabitants requirements (so the number of lines starts to be 2 outputs that should be maximize and we have taken 3 inputs: unemployment, time to reach Jihlava and time to reach ORP that should be minimize), we have different results in CCR and BCC models. CCR (input and output) models are more strict and showed only 4 efficient areas – Jihlava, Moravske Budejovice, Pacov, Pelhrimov, whilst the BCC models has had 8 efficient – it has added 4 other areas (Humpolec, Namest nad Oslavou, Svetla nad Sazavou, Telc). When we compare results of all models we can see that 4 areas are always inefficient: Bystrice nad Pernstejnem, Nove Mesto na Morave, Trebic, Zdar nad Sazavou. From the inhabitants point of view also Chotebor belongs to inefficient areas whereas from the municipality perspective Havlickuv Brod, Moravske Budejovice, Namest nad Oslavou and Telc do not belong to efficient units.

4 Conclusion

The aim of this contribution was to compare the relative relationship between the factors connected with the traffic accessibility of the selected municipalities and its economic activity. According to the results from DEA models we can say that in Vysocina region there exist efficient municipalities with extended competences (especially those near to D1 highway) and inefficient ones where the situation could be better by changing some parameters. One possibility is to decrease the number of lines to Jihlava in the far-away ORPs and increase the number of lines to the center of its ORP to increase the business activity and decrease the unemployment rate. The other part shows that for some areas it would be better to increase the number of lines to Jihlava and ORP for the same effect. We can also see that there are some municipalities which are always inefficient no matter what point of view is taken into account. The results can be used by the Regional authority in the process of employment subsidy allocation and for the support of expansion by new businesses in inefficient areas. The research will continue when more information will be available.

Acknowledgements

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Markets, social networks, endogenous preferences, and opinion leaders[1]

Michal Kvasničkа[2]

Abstract. This paper studies the impact of opinion leaders ("stars") and their fans on equilibrium market prices within the Bell's model (JEBO 2002). The simulation shows that 1) the model may not converge when the opinion leader consumes the good that is extremely scarce—it can create infinite cycles in her fans' preferences; 2) the preferences may not be completely polarized in the same situation—the agents with non-polarized preferences prevent the cycles; 3) while the agents in the Bell's model consume only the more abundant good when the other good is extremely scarce, the presence of the opinion leader eliminates this when she consumes the scarce good, and 4) the presence of the opinion leader and her fans can sometimes surprisingly lower the price of the good that the opinion leader consumes.

Keywords: endogenous preferences, market, social network, opinion leaders, agent-based simulation

JEL classification: D83, D51, D85
AMS classification: 68U20, 91D30, 91B69

1 Introduction

The seminal agent-based study on endogenous preferences and how they are influenced by interactions within a social network is A. M. Bell's paper "Locally interdependent preferences in a general equilibrium environment" [1] which explored the endogenous preference adaptation for a grid social network. The robustness of her conclusions was later tested by Kvasnička [2] who showed that some of Bell's original conclusions do not apply for other symmetric network structures. The present paper enhances the Bell's model in another way: it studies the impact of an opinion leader ("a star") on human preferences within the framework of Bells model. It introduces a new kind of agent ("an opinion leader" or "a star") and a second social network, the asymmetric star network of her fans. By means of agent-based simulations it investigates how the properties of the fans network (number of agents in the network, their concentration, and the strength of the opinion leader's impact on one fan) affect the structure of the model equilibrium and the relative price of the two goods. It is the first step toward a quantitative modeling of an important marketing problem how opinion-leaders (e.g. pop stars) can be used to enhance the demand for one of competing products in environments where fashion matters.

2 Model

The model enhances Bell's model of "exchange economy", see [1]. There are two kinds of agents: "ordinary people" and one "opinion leader". All ordinary agents consume two comparable kinds of goods (e.g. white and black t-shirts). The opinion leader consumes only good 1 (the white t-shirts). In every period, all agents get an initial endowment of each good which they exchange with each other in the centralized market at the market clearing price. The ordinary agents' preferences evolve over time: each ordinary agent increases her preference for the good that has been recently more popular (i.e. more consumed) in her neighborhood. Her neighborhood consists of three parts: 1) the agent herself, 2) her eight closest agents, and 3) the opinion leader, if the agent is her fan. The opinion leader's preferences do not change in time.

More formally, there are N agents: one opinion leader indexed $i = 1$ and $N - 1$ ordinary agents indexed $i = 2, \ldots, N$. In every period, each agent gets the same endowment: e_1 units of good 1 and e_2 units of good 2 ($e_1, e_2 > 0$,

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and $e_1 + e_2 = 100$). Agent i then demands x_{i1} units of good 1 and x_{i2} units of good 2 to maximize her one-period Cobb-Douglas utility function subject to the constraint given by her endowment, i.e.

$$\max_{x_{i1},x_{i2}} x_{i1}^{a_{it}} x_{i2}^{1-a_{it}} \qquad \text{s.t.} \quad p_1 x_{i1} + p_2 x_{i2} = p_1 e_1 + p_2 e_2, \tag{1}$$

where p_1 and p_2 are the prices of good 1 and good 2 respectively, and a_{it} is agent i's relative preference for good 1 at time t. The initial value of the preference parameter a_{i0} is drawn independently for each ordinary agent $i > 1$ from the continuous uniform distribution $U(0, 1)$; the opinion leader's preference $a_{1t} = 1$ for each t.

Agent i's demand for the two goods is then

$$x_{i1}(p_1, p_2) = a_{it}\left(e_1 + \frac{p_2}{p_1}e_2\right), \qquad x_{i2}(p_1, p_2) = (1 - a_{it})\frac{p_1}{p_2}\left(e_1 + \frac{p_2}{p_1}e_2\right). \tag{2}$$

Since the total endowment is given, the market clearing relative price of good 1 in terms of good 2 is

$$\frac{p_1^*}{p_2^*} = \frac{e_2 \sum_{\forall j} a_{jt}}{e_1 \sum_{\forall j}(1 - a_{jt})}. \tag{3}$$

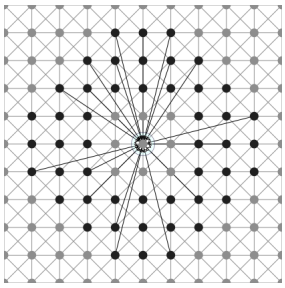
The relationships among the agents are defined through two kinds of social networks: one network defining friends, and the other defining fans of the opinion leader. Each social network is represented by a graph, in which agents are vertices and their relationships are edges (connections). Agent i has relationship to agent j in graph G if i is connected with an edge to j; we then write $i \sim j \in G$. Both social networks are created independently for each simulation and are fixed within the simulation. For an example see Figure 1.

The first social network portrays friendship, and is equivalent to the Bell's original social network, see [1]. It is represented by an undirected graph G_1, which is a grid network on a toroid; see [6] for its definition. (Since the graph G_1 is undirected, the friendship relationship is symmetric, i.e. $i \sim j \in G_1 \Leftrightarrow j \sim i \in G_1$.) One can see the network as a cellular automaton on a lattice where each agent is represented by a cell, the set of the agent's friends consists of the cell's Moore neighborhood, and the lattice edges are connected to each other in such a way that an agent at the very top of the lattice is at the same time located also at the very bottom of the lattice, and so on. Let us define index function $n(i) = \{j : j = i \vee j \sim i \in G_1\}$ as the set of indices of agent i's friends and the index of the agent i herself.

The second social network portrays the relationships of the fans to the opinion leader. This network is represented by a directed star graph G_2; see [6] for its definition. In this network, the fans are connected to the opinion leader in such a way that each fan cares about the consumption of the opinion leader but not about the consumption of the other fans; the opinion leader does not care about the consumption of her fans. (The relation in directed networks is neither reflexive, nor symmetric, i.e. $1 \sim 1 \notin G_2$, and $i \sim j \in G_2$ does not imply $j \sim i \in G_2$.) The fans are selected from among a subset A of the ordinary agents that are not the opinion leader's friends. Let $\rho = |A|/(N - 9)$ denote the share of the agents that can be selected as opinion leader's fans on the total population of all ordinary agents except the opinion leader's friends. Let $\pi \leq \rho$ denote the share of the actual fans on the total population of the ordinary agents that are not the opinion leader's friends. The network G_2 is constructed this way: 1) construct the set A that contains indexes of all ordinary agents that are not the opinion leader's friends that are located within the smallest circle around the opinion leader that includes at least $\rho(N - 9) + 9$ agents (I assume that the agents are distributed equidistantly in the friendship space, as on the lattice); 2) construct the set B as $\pi(N - 9)$ randomly chosen indexes from the set A, and 3) connect the agents in B to the opinion leader in G_2, i.e. set $j \sim 1 \in G_2$ for $\forall j \in B$. Notice that the parameter ρ together with π determines how concentrated within the whole population the fans are. For instance, $\pi = 0.1$ and $\rho = 1$ means that there are few fans and they are sparsely scattered within the whole population of agents. On the other hand, $\pi = 0.3$ and $\rho = 0.3$ means that all the fans are concentrated around the opinion leader—the 30 % of opinion leader's closest non-friend agents are her fan, and there are no fans farther from the opinion leader. Let us define Boolean function f such that $f(i) = 1$ if $i \sim 1 \in G_2$, and $f(i) = 0$ otherwise.

Agent $i > 1$'s neighborhood thus consists of the agent herself, her friends (i.e. the agents with indexes in $n(i)$), and the opinion leader if the agent is her friend (i.e. $f(i) = 1$). After observing the consumption in her neighborhood, each ordinary agent $i > 1$ adjusts her preferences in such a way that she increases the preference for the good that is consumed more in her neighborhood. Specifically, agent $i > 1$ sets the future value of her preference parameter $a_{i,t+1}$ at

$$a_{i,t+1} = a_{it} + r\left(\frac{\sum_{j\in n(i)} x_{j1} + sf(i)x_{11}}{\sum_{j\in n(i)} x_{j1} + sf(i)x_{11} + \sum_{j\in n(i)} x_{j2} + sf(i)x_{12}} - 0.5\right) \quad \text{for } i > 1, \tag{4}$$



The dots denote the agents, the lines denote the edges of the social networks. The agents on the very top wrap over to the very bottom and the agents at the right edge wrap over to the left edge.
The encircled dot in the middle of the figure is the opinion leader.
The black dots are the agents that can be selected as fans, i.e. agents with indexes $i \in A$.
The short gray vertical, horizontal, and diagonal lines are the vertices of the friends network G_1.
The long black lines are the vertices of the fans network G_2.

Figure 1 An example of the social networks for $N = 100$, $\pi = 0.2$, and $\rho = 0.5$.

where the adjustment parameter $r \in (0, 1)$ regulates the speed of the preference adjustment and s is the strength of impact of the opinion leader's consumption on a fan. Setting $s > 1$ allows us to study the cases where the fans value the consumption of the opinion leader higher then the consumption of their friends.

The evolution of agents' preferences and consumption is simulated in the agent-based computational fashion (for introduction to it, see e.g. [4]). First, the model is initialized: N agents are created, assigned their random initial preference $a_{i,0}$ drawn from $U(0, 1)$, and organized in the social networks G_1; then one agent is selected as the opinion leader and her preference is changed to $a_{1,0} = 1$; then the social network G_2 is created. The simulation than proceeds in discrete steps repeated until the model converges (i.e. the agents' preferences change no more), or the maximal amount of steps is reached. In each step, 1) the market clearing relative price p_1^*/p_2^* is calculated for the agents' current preferences (equation 3), 2) each agent's equilibrium consumption is calculated (equation 2), and 3) each agent $i > 1$'s preference is adjusted (equation 4).

3 Results of simulations

The model has been simulated for $N = 100, 900$, and 2 500 agents, the concentration of fans $\rho = 0.1, 0.2, \ldots, 1$, the share of fans $\pi = 0.1, 0.2, \ldots, 0.7$ (only combinations with $\pi \leq \rho$ were simulated), and the strength of opinion leader's impact $s = 1, 2, 3$. The adjustment constant r was set to 0.5 as in [1]. Three variants of the model were simulated for each random seed: 1) the "grid" model, i.e. Bell's original model without the opinion leader and her fans where agent 1 was treated as an ordinary agent, 2) the "stubborn" model, i.e. an intermediate model where agent 1 had the preference parameter $a_{1t} = 1$ at each t but had no fans (i.e. G_2 was not present), and 3) the "leader" model, i.e. the full model described above. The three variants can be straightforwardly compared because only the initialization is stochastic (the rest of the simulation is deterministic), and the order of steps in the initialization secures that the common part of each two variants is the same for each initial random seed. The model was simulated one hundred times for each feasible combination of parameters and each variant. The total number of the simulation runs was 573 300 triples. The maximal amount of simulation steps was set to 5 000. The model was simulated in NetLogo 5.0.4 [5] and the results were analyzed in R [3]. The web interface of the model is available at http://www.econ.muni.cz/~qasar/english/models.html.

3.1 Simulation outcomes

Bell [1] discusses two kinds of simulation outcomes on the grid friends network G_1. In both, the system converges, each good is consumed by some agents, and fewer agents consume the scarcer good than the more abundant one. The difference between the outcomes lies in the agents' equilibrium preferences. In the first case, each agent specializes in consumption of only one kind of good (i.e. $a_{it} \in \{0, 1\}$, which Bell calls "polarized" preferences), while in the second case, at least some agents consume both goods (i.e. have $a_{it} \in (0, 1)$). Bell claims that the second outcome is unstable, and hence cannot occur in a simulation [1, p. 321]. In the polarized state, the agents with the same preference are clustered together (for an example of the clusters, see Figure 3, panel a). The clusters arise because no ordinary agent can keep its polarized preference unless she is surrounded by a sufficient number of other agents with the same preference. The reason why more agents specialize in consumption of the more abundant good is the negative feedback provided by the market: when the number of agents consuming a good is out of proportion to the good endowment, the relative price of the goods is unequal; the consumers of the cheaper good consume more, which motivates some consumers of the other good to change their preferences. In the limit, this market feedback presses the relative price of the goods to unity. However, there is also the "bandwagon" effect: the higher the proportion of consumers of one good, the higher the probability that an agent is surrounded

by the consumers of this good, and hence that she switches her preference to this good. The bandwagon effect rises demand for the more abundant good, and hence presses the relative price of the goods away from unity. See [2] for a more detailed analysis.

Kvasnička [2] showed that there may be a richer set of outcomes with a general social network: 1) the simulation may not converge, 2) it may converge but some agents can remain non-polarized, 3) the simulation can converge, all agents can be polarized but all of them consume only the more abundant good, and 4) the simulation converges, all agents are polarized and both kinds of good are consumed by some agents, which is the Bell's standard outcome. All these four states occurred also in the present simulation, though the first three cases occurred only with extreme proportions of the endowment: either $e_1 \geq 95$ and $e_2 \leq 5$, or $e_1 \leq 5$ and $e_2 \geq 95$. The standard logistic regression was not suitable here for the detection of the combinations of treatments for which these outcomes were realized because the model behavior is highly non-linear; decision trees constructed by R packages rpart, rpart.plot and Rattle [7] were used instead.

The state 1, the non-convergence, is caused by fans network G_2—the grid and stubborn models always converge. The model with the opinion leader did not converge in 22 737 out of 573 300 runs (i.e. about 3.97 % of runs). Most non-convergent runs happened when $e_2 = 99$ (i.e. $e_1 = 1$), $N = 900$ or 2 500, and $\pi \geq 30$ %. For details, see the decision tree in Figure 2. When the model does not converge, it is because it cycles. Let us show the mechanism on an example. Assume the model with $N = 2500$ agents, endowments $e_1 = 1$ and $e_2 = 99$, $\pi = 0.2$, $\rho = 0.3$ and $s = 2$. Because of the bandwagon effect, the model soon reaches the state where only the opinion leader consumes the scarce good 1 and all other agents consume only the abundant good 2. The opinion leader then consumes 2 500 units of good 1, while each other agent consumes only about 99 units of good 2. Thus according to equation (4), all fans increase their taste for good 1 in the next step. This rises the demand for good 1, which rises its relative price, and the consumption of all agents who consume good 1 declines. In the following step, the fans' preferences for the good drop according to equation (4) again. The opinion leader is again the only agent consuming the scarce good 1, and the new cycle begins.

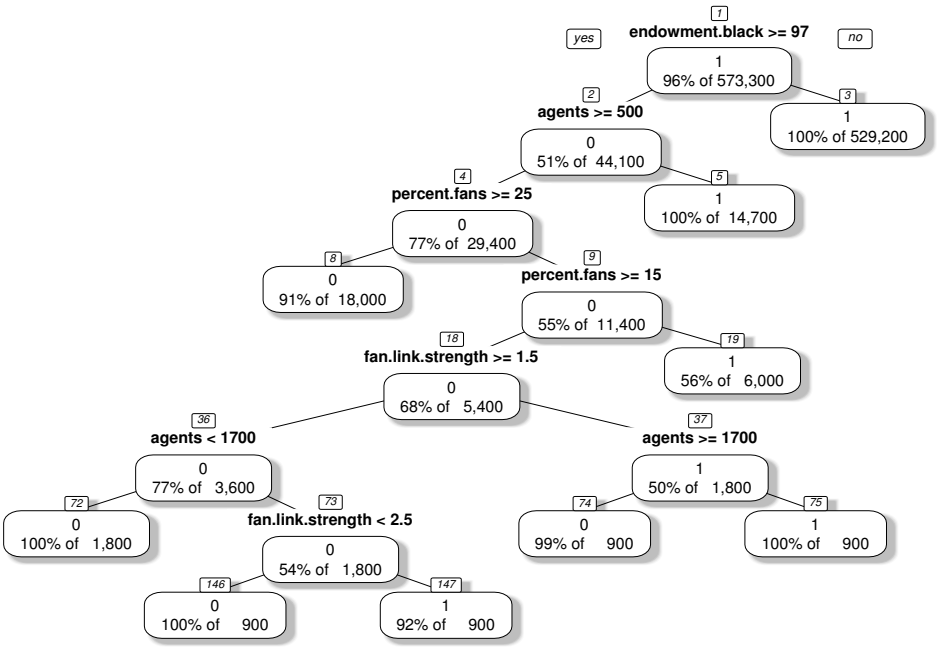


Figure 2 Decision tree showing when the leader model converged. Convergence is denoted by 1, N is denoted as agents, e_2 as endowment.black, π as percent.fans, and s as fan.link.strength.

The state 2, where the model converges but some agents have non-polarized preferences, occurs in 4 930 runs out of the 550 563 converged runs (i.e. 0.9 % of runs). It only occurs in the leader model when $N = 100$, $e_1 = 5$ and $e_2 = 95$, and $s \geq 2$. Usually, there were one to four non-polarized agents (the fans) that advanced the relative price of the scare good 1 in such a way that the opinion leader's consumption of good 1 was low enough, so that the other fans were not tempted to change their preferences. This balancing problem obviously becomes more delicate when there are more agents, and the state was not observed in data for higher N.

The state 3, where the model converges, all agents' preference are polarized but only the more abundant good is consumed because of the bandwagon effect, occurred in the grid model when the endowment was extremely unequal. It always occurred in the grid model when $e_j = 99$ (and $e_{-j} = 1$), and often also when $e_j = 95$ and $N = 100$. Good 1 was solely consumed in 31 650 and good 2 in 58 653 runs out of 545 633 converged polarized runs totally (together 16.6 % of the converged polarized runs). The presence of a stubborn consumer of good 1 (whether she had a fans network or not) eliminated this state when the scarce good was good 1—it was at least consumed by agent 1 herself. When the scarce good was good 2, then the presence of a stubborn consumer of good 1 slightly rose the number of runs when this happened: the state occurred in 58 946 runs in the stubborn model and in 59 015 runs in the leader model.

3.2 Opinion leader's impact on relative price

To study the impact of the opinion leader' presence on the relative price of good 1, we will confine to the converged polarized runs when both goods were consumed in positive quantities. It may seem intuitive that the presence of the opinion leader that always consumes good 1 and her fans that can follow her in her consumption should rise the demand for good 1, and hence its price. However, it is not this simple. In fact, the inclusion of the opinion leader and her fans increased the relative price of good 1 in 360 910 runs out of 454 703 runs in view (i.e. in 79.4 % runs), did not change it in 35 973 (7.9 %) runs, and decreased it in 57 820 (12.7 %) runs. It is instructive to decompose the total effect to the impact of the opinion leader herself (the change from the grid to the corresponding stubborn model) and the impact of the fans network (the change from the stubborn model to the corresponding leader model). The detailed decomposition is presented in Table 1.

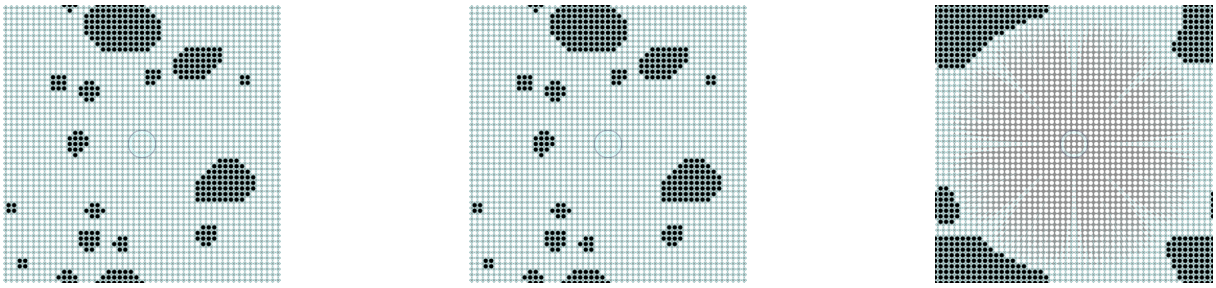
	total rise			total no change			total decline		
	decline	no change	rise	decline	no change	rise	decline	no change	rise
decline	0	0	94 343	0	0	4 988	7 842	3 298	6 189
no change	0	0	123 588	0	27 790	0	31 465	0	0
rise	10 477	8 885	123 617	3 195	0	0	9 026	0	0

Table 1 Number of converged polarized runs where both goods were consumed decomposed. Each block corresponds to a total change in the relative price; e.g. the block denoted "total rise" includes the runs where the relative price of good 1 rose with transition from the grid model to the leader model. Within the blocks, the rows correspond to a change of the relative price when one agent was assigned the stubborn preference for good 1 (the transition from the grid to the stubborn model). The columns correspond a change of the relative price when the stubborn agent was given the fans network G_2 (the transition from the stubborn to the leader model). Each number is the number of the corresponding runs; e.g. 4 988 is the number of runs where the relative price declined when the stubborn consumer of good 1 was introduced, it rose again when the stubborn agent was given a network of fans, and the two effects had precisely the same magnitude with the opposite sign, so that the relative price was the same in the leader model as in the corresponding grid model. (Note that the relative price can take only values from a discrete set because of equation (3), polarized preferences, and discrete number of agents.

When one agent in the grid network was assigned the stubborn preference for good 1 (i.e. the grid model was changed into the stubborn model), the relative price of good 1 rose in 34 % runs, did not change in 40 % runs, and decreased in 26 % runs. The reason is that when one agent in the grid network is given a stubborn preference for good 1, the clusters of consumers may evolve differently, and the number of the consumers of each good can randomly rise, decline, or stay the same (as the transition from panel a to panel b in Figure 3). The relative price of good 1 declines more often when the number of agents is high and the endowment e_2 is high. For instance, the relative price of good 1 declines in 53 % runs when one stubborn consumer of good 1 is added to the grid network of $N = 2500$ agents with endowment $e_2 = 90$, but it declines in only 9 % runs with $N = 100$ and $e_2 = 10$.

When the fans network G_2 was added (i.e. the stubborn model was changed into the leader model), the relative price of good 2 rose in 77 % runs, did not change in 9 % runs, and declined in 14 % runs. The relative price declined more often when the endowment of good 1 was higher then the endowment of good 2 (72 % of this relative price declines happened when $e_2 \leq 40$), and when the number of agents was high (80 % of the relative price declines happened when $N \geq 900$). The reason is again that the clustering evolves differently when the fans network G_2 is added. As the clusters evolve differently, the number of the consumers of each good can randomly rise, decline, or stay the same, and the relative price changes accordingly.

The most interesting case of the decline of the relative price of good 1 caused by introduction of the fans



(a) grid model (b) stubborn model (c) leader model

Figure 3 An example of the outcome of the simulation where the relative price of good 1 consumed by the opinion leader decreased when because of the network G_2 of her fans. Here $N = 2500$, $e_2 = 20$, $\pi = \rho = 0.7$, and $s = 3$. The opinion leader or corresponding agent has been moved to the center and denoted by the circle. The black dots correspond to the agents with preference $a_{it} = 0$, the white dots to the agents with preference $a_{it} = 1$.

network G_2 happens when the opinion leader's fans are sufficiently concentrated, i.e. π/ρ is close to unity. An example can be seen in Figure 3. The introduction of the fans network G_2 (the transition from the panel b to the panel c) cleanses the surrounding of the opinion leader, and creates a great cluster of consumers of good 1 which rises the demand for good 1, and hence its relative price. The side effect is that consumers of good 2 often create a smaller number of bigger clusters, which allows more agents to keep the preference for good 2. This second effect lowers the relative price of goods 1. Quite often the second effect is stronger than the former one, and the relative price of good 1 declines.

4 Conclusions

The simulation shows that the presence of an opinion leader and a network of her fans within Bell's model modifies Bell's results: 1) The model may not converge because the presence of the opinion leader that stubbornly consumes a good that is extremely scarce can create infinite cycles in her fans' preferences. 2) The preferences need not be completely polarized in the same situation—the agents with non-polarized preferences can eliminate the cycles. 3) While the agents in the Bell's model consume only the more abundant good when the other good is extremely scarce, the presence of the opinion leader can eliminate this effect when she consumes the scarce good, or slightly enhance it when she consumes the abundant good. 4) The most surprising effect is that the presence of the opinion leader and her fans can sometimes lower the demand, and hence the price of the good that the opinion leader consumes. This suggests that demand manipulation through opinion-leaders may be risky even when the leader succeeds in rising demand of her fans.

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Payoff size variation problem in simple reinforcement learning algorithms[1]

Michal Kvasničk[2]

Abstract. This paper shows that the speed of the reinforcement learning depends on the size of the payoffs, at least when all payoffs are positive. When the speed of learning is too fast, the agents tend to learn to play the actions which they randomly chosen in the first rounds of the learning process. The compositions of the agents' strategies then on the aggregate level resembles the initial individual agent's mixed strategy. This may create artificial effects in the simulations where the size of payoffs depend on the model treatments because the speed of learning cannot be tuned in.

Keywords: reinforcement learning, agent-based simulation, economic experiments, voluntary provision of public goods

JEL classification: C92, D83, H41
AMS classification: 68U, 68W, 91B, 91E

1 Introduction

Standard economic models based on optimizing omniscient agents and instantly attained equilibria are not able to explain many observed phenomena. For instance, they often fail to explain the outcomes of laboratory experiments with human subjects. That is why there are attempts to create alternative formal explanatory models. One approach to this goal is Agent-based computational economics (for a general overview see [14]). The ACE models populated with boundedly rational agents are able to predict not only the eventual equilibrium but also the adjusting process leading to it. Their structure makes them especially useful for modeling the behavior observed in the experiments, see [6]. Since the agents in these models are only boundedly rational they usually have to learn how to act from the feedback provided by their model environment. Thus the learning process constitutes an important part of the models (for a general overview of the learning algorithms used in the ACE model see [4]).

In this paper, we will explore one overlooked property of one of the most often used learning algorithms: the simple reinforcement learning (we use the adjective "simple" to distinguish the reinforcement learning algorithms used in ACE from the more complex algorithms used in the field of artificial intelligence, see [12].) We will claim that the speed of the simple reinforcement learning depends on the size of the payoffs, at least when all payoffs are positive. When the speed of learning is too fast, the agents tend to learn to play the actions which they randomly chosen in the first rounds of the learning process. The compositions of the agents' strategies then on the aggregate level resembles the initial individual agent's mixed strategy. This is no problem in most simulations because the modeler has tools to tune in the speed of learning. However, this may create strange effects in the simulations where the size of payoffs depends on the model treatments—then the speed of learning varies with the treatment, and hence cannot be fine-tuned.

Several parts of the claim has been previously known. Sutton and Barto claim without any proof that the proper setting of the algorithm depends on the size of the payoffs [12, p. 31] and that the eventual mixed strategy may be biased by its prior [12, p. 35]. Fudenberg and Levine claim (also without any proof or elaboration) that there is a positive probability that the algorithm converges to a state "where an inferior choice is played with probability 1" [7, p. 73]. The closest to this paper came Bell [3] who inquired the influence of the initial prior on the algorithm's outcomes with a constant payoff. As far as we know, the influence of the variable payoff size on the outcomes of the algorithm has never been explored.

The rest of the paper proceeds like this: The second section describes the reinforcement learning algorithm and shows the claim analytically in the simplest possible case of one agent playing against a deterministic automaton.

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The third section provides a more complex example: a simulated version of the provision of public goods experiment where the treatment is the number of agents. It shows that the agents learn to contribute more when the number of agents in the game is higher. The fourth section compares the reinforcement learning to the replicator dynamics and then discusses various ways to solve the problem. It discusses also one more possibility, namely that the discussed effect is not only a computational artifact but a true property of human learning.

2 Speed of reinforcement learning: an analytical example

The basic idea behind the reinforcement learning is that the rewarded behavior is strengthened and the punished behavior is weakened. More specifically, it is assumed that an agent (human or animal) can choose an action from a known discrete set of actions. She has a mixed strategy, i.e. chooses each action with some probability. After choosing an action, the agent earns a payoff that depends on the chosen action and the state of the environment (which can include actions chosen by other agents). After observing the payoff associated with the chosen action, the agent updates her mixed strategy. The probability that the chosen action would be played in the future is increased in proportion to the payoff (it is decreased if the payoff was negative). The future probabilities of all other actions are changed accordingly so that the probabilities of all action sum to unity. The reinforcement learning is classified as "non-conscious" learning (see [4]) because the algorithm does not explicitly model the agent's cognitive reflection. Specifically, it models neither the agent's belief about the state of her environment, nor her belief about the strategies of other agents. Nonetheless, the reinforcement learning was successfully used in modeling the situations where the agents learned consciously (such as in laboratory experiments with human subjects) and where the situation was strategic (e.g. in games), see [4, p. 939]. Very often it produced better predictions of human subjects' behavior in standard games then game theory, see [6, p. 1003].

There are many variants of the simple reinforcement learning algorithm but the differences between them are minor: alternative variants differ only in the speed of learning (it may be constant or slowing down in time) and in the inertia after a change of the environment, see [4, p. 905]. For our exposition, we use the variant of the algorithm taken from [1]. It works like this: The learning proceeds in discrete rounds. In each round t, agent i chooses an action j with probability p_{ij}^t which is calculated from her propensities R_{ij}^t to play the action j as

$$p_{ij}^t = \frac{e^{\lambda R_{ij}^t}}{\sum\limits_{\forall k} e^{\lambda R_{ik}^t}}. \tag{1}$$

After observing the payoff, the propensity to play each action j is updated as

$$R_{ij}^{t+1} = qR_{ij}^t + I_{ij}^t \pi_i^t, \tag{2}$$

where π_i^t is the payoff agent i gained in round t and $I_{ij}^t = 1$ if the action j was chosen in the round t, and $I_{ij}^t = 0$ otherwise. There are three parameters: the initial propensities R_{ij}^1, the "forgetting" parameter $q \in (0, 1]$, and the "focus" parameter $\lambda \geq 0$. The forgetting parameter q allows the agent to change her mind if her environment (and hence payoffs) changed; it also secures the numerical stability of the algorithm. The initial propensities R_{ij}^1 determine the agent's initial mixed strategy; e.g. $R_{i1}^1 = R_{i2}^1 = \ldots = R_{im}^t$ means that in the first round agent i chooses each of her m actions with probability $1/m$. The focus parameter λ is said to determine "the extent to which the agent focuses on choices with higher values of R_{ij}^t" [1, p. 211]. If $\lambda = 0$, then each action is chosen with the same probability. As λ rises, the higher and higher probability is attached to the action with the highest propensity; it is chosen with probability equal to 1 in the limit.

The three parameters together with the size of the payoffs determine the speed of an agent's learning. Intuitively, the speed of learning means how fast the agent's mixed strategy degenerates to the choice of one pure action. This can be measured as the speed with which the entropy of the agent's mixed strategy decreases in time. Entropy E_i^t of agent i's mixed strategy in round t is defined as

$$E_i^t = -\sum_{k=1}^{m} p_{ik}^t \log_m p_{ik}^t \tag{3}$$

where the base of the logarithm is the number m of agent i's actions. Notice that $E_i^t \in [0, 1]$ reaches its maximum when agent i plays each action with the same probability $1/m$, and its minimum when she plays one action with probability 1 and the other actions with probability 0. It has an intermediate value for other mixed strategies and decreases as the agent focuses. We claim that 1) the speed of learning depends beside others also on the payoff

size, and 2) if the speed of learning is too high, the chosen pure action need not to be the right one, i.e. the action with the highest payoff. From some speed of learning, the ex ante probability that the right action would be chosen decreases in the speed of learning and converges to its probability in the initial mixed strategy.

We will show my claim first in the simplest possible environment. Since even this case is not fully analytically tractable, we will discuss analytically only the speed of learning between the first and second round, and the corresponding ex ante probability that the right action is reinforced. The setting is like this: Let us assume a simulation where one agent plays against a deterministic automaton. The agent has two actions, a_1 and a_2. The payoffs of these actions are deterministic, $s\pi_1 > 0$ and $s\pi_2 > s\pi_1$ respectively where s is a "size" of the payoffs (let us say π_1 is normalized to 1). Let us further assume that the agent's initial propensities to play a_1 and a_2 are $R^1_{11} = R^1_{12} = r$ respectively, i.e. she plays each action with probability $1/2$ in the first round.

We have to explore two probabilities: the unconditional probability Ep^2_{12} that the right action a_2 is chosen in the second round and the probability p^- that the action randomly chosen in the first round is chosen also in the second round. Let us start with Ep^t_{12}. If the action a_1 is chosen in the first round, the agent sets the future propensities $R^2_{11} = qr + s\pi_1$ and $R^2_{12} = qr$. Hence, the conditional probability that she chooses action a_2 in the second round when she has chosen a_1 in the first round is then $p^2_{12}|a_1 = e^{\lambda qr}/(e^{\lambda qr+\lambda s\pi_1} + e^{\lambda qr}) = 1/(1 + e^{\lambda s\pi_1})$. On the other hand, if the action a_2 is chosen in the first round, the agent sets $R^2_{11} = qr$ and $R^2_{12} = qr + s\pi_2$. Then the conditional probability that she chooses action a_2 in the second round when she has chosen a_2 in the first round is $p^2_{12}|a_2 = e^{\lambda qr+\lambda s\pi_2}/(e^{\lambda qr} + e^{\lambda qr+\lambda s\pi_2}) = e^{\lambda s\pi_2}/(1 + e^{\lambda s\pi_2})$. Since each action is chosen with probability equal to $1/2$ in the first round, the unconditional probability Ep^2_{12} is

$$Ep^2_{12} = 1/2p^2_{12}|a_1 + 1/2p^2_{12}|a_2 = \left(\frac{e^{\lambda s\pi_2}}{1 + e^{\lambda s\pi_2}} + \frac{1}{1 + e^{\lambda s\pi_1}}\right)/2. \tag{4}$$

The probability that the same action is chosen in both rounds is

$$p^- = 1/2(1 - p^2_{12}|a_1) + 1/2p^2_{12}|a_2 = \left(\frac{e^{\lambda s\pi_1}}{1 + e^{\lambda s\pi_1}} + \frac{e^{\lambda s\pi_2}}{1 + e^{\lambda s\pi_2}}\right)/2. \tag{5}$$

The inspection of the equations (4) and (5) shows the following properties: First, from the equation (5) we can see that the probability p^- that the same action is played in the second round as in the first round is rising in the focus parameter λ, in the average size of the payoffs s, and in their product λs. If $\lambda s = 0$, then $p^- = 1/2$. As the product λs rises, the probability that the action chosen randomly in the first round is played again in the second round monotonically rises and converges to unity, i.e. $\lim_{\lambda s\to\infty} p^- = 1$. Notice that in our case, the agent's mixed strategy entropy E^t_i decreases as the probability that any action is chosen rises above $1/2$ (and the probability that the other action is chosen decreases below $1/2$). That means that the speed of learning between the first and the second round increases in λs.

Second, from equation (4) we can see that the unconditional probability Ep^2_{12} that the right action a_2 is chosen in the second round is increasing in π_2 and decreasing in π_1. In other words, the higher the difference in payoffs between the actions, the higher ex ante probability that the agent will choose the right action in the second round.

Third, from the same equation we can also see that the unconditional probability $Ep^2_{12} = p^1_{12} = 1/2$ in two cases: 1) when $\lambda s = 0$, i.e. when there is no learning at all and the agent chooses her actions independently in each round, each action with the same probability; 2) in the limit when λs is high ($\lim_{\lambda s\to\infty} Ep^2_{12} = 1/2$). In this case, the agent chooses in the second round the same action as in the first round (see the first point above), i.e. there is no further learning since the agent is locked in the previously randomly chosen action. The $Ep^2_{12} = 1/2$ because each action is chosen with this probability in the first round. Notice also that the agent's (with probability $1/2$ inefficient) action is locked, and the agent cannot change her mind: she chooses the previously chosen action in each round, i.e. the other action is never tried and the propensity to play it decreases to zero while the action randomly chosen in the first round is reinforced forever.

Fourth, $dEp^2_{12}/d(\lambda s) > 0$ for $\lambda s = 0$. This together with $Ep^2_{12} = p^1_{12} = 1/2$ for $\lambda s = 0$ and $\lim_{\lambda s\to\infty} Ep^2_{12} = 1/2$ implies that there is a level l such that $dEp^2_{12}/d(\lambda s) < 0$ for any $\lambda s > l$, i.e. the probability that the dominant action a_2 is chosen in the second round decreases in λs, i.e. it decreases with the speed of learning.

We can summarize it this way: If λs is too small, the agents learn too slowly. If the product is too high, the agents learns so fast that he may learn to play a dominated action. Obviously, this is no problem in most simulations—the modeler simply has to set the focus parameter λ properly: the higher s, the lower λ. This way the learning speed may be calibrated for instance to fit the convergence speed observed in an experiment. (However, it might then be difficult to interpret the parameters q and λ as behavioral.) However, there is one overlooked instance

where this is indeed a problem: in simulations in which the size of payoffs depend on the treatment, i.e. it changes within the simulation. Then the speed of the learning can change within the simulation and unexpected things can happen. We will provide an example of such a simulation in the following section.

3 Case study: voluntary provision of public goods

In this section, we will describe an agent-based computational model of the simplest version of the voluntary provision of public goods experiment. The experiment consists of T discrete rounds. There are $N > 2$ players, the same in all rounds. In each round, each player is given some endowment w. She can contribute part of the endowment to a public good and save the rest for herself. Agent i's payoff is $\pi_i^t = (w - c_i^t) + M \sum_{\forall k} c_k^t$ in the round t, where c_i^t is agent i's contribution to the public good in round t and $M \in (0, 1)$ is the payoff of the public good. Notice that the only dominant (and hence equilibrium) action of every agent (both in each round and in the whole finitely repeated game) is $c_i^t = 0$. However, in the typical situation $M > 1/N$, the socially optimal action is $c_i^t = w$.

The stylized facts on results of the experiment can be found in [2, 8, 9]. They show that most people follow neither their dominant, nor their socially optimal strategy, but contribute somewhere in-between. The typical average contribution is about one half in the first round and it decreases in the time, however not to the zero. Since the game theory is not predictive here and since we know that most agent change their actions in time, it seems plausible that they learn how to play the game. Hence the experiment is a natural candidate for an agent-based computational simulation trying to explain the agents' behavior.

We will use the version of the experiment taken from [13] for the simulation. Here, the return on the public good $M = 0.5$, the endowment $w = 40$ and the set of actions is limited to three actions: $a_1 : c_i^t = 40$, $a_2 : c_i^t = 20$, and $a_3 : c_i^t = 0$. The only treatment in the simulation (not in the experiment [13]) is the number of the agents, $N = 4, 7, 10, 20, 40$, and 100. Notice, that the structure of the game implies that an increase in the number of agents rises the size of payoffs for $c_i^t > 0$. The parameters of the reinforcement learning algorithm were casually calibrated in such a way that they follow the stylized fact with the typical number of agents, $N = 7$. The forgetting parameter $q = 0.9$, the focus parameter $\lambda = 0.005$, and initial propensities $R_{ij}^1 = 0$ for each agent i's each strategy j. The model was simulated for 21 rounds, which is more than enough for a comparison with data from any experiment. Each simulation was repeated one thousand times. The model was simulated and the resulting data were analyzed in R [11].

The results of the simulations are summarized in Table 1. In both its panels, the columns denote the number of agents in the game and the rows denote rounds. The right panel shows that the average entropy of agents' mixed strategies (averaged over the agents and the simulations). It can be clearly seen that the agents learn much faster when there are more agents, and hence the size of payoffs is higher. For instance, when $N = 4$, there is some entropy in the agents' mixed strategies even after twenty rounds, i.e. the agents are still learning even after twenty rounds. On the other hand, when $N = 20$, the learning process has stopped after 13 round, and when $N = 100$, the learning has stopped after only two rounds.

The left panel of Table 1 shows the average contributions to the public good in percents of the endowment w (averaged over the agents and the simulations). This can be seen a measure of how fast (and if at all) the agents learn to play their dominant free riding strategy a_3. The more agents learned to play the dominant strategy, the less they contribute, and hence the lower is the average contribution. In the first round, the expected value of the average percentage contribution is $1/2$, which is given by the initial mixed strategy $(1/3, 1/3, 1/3)$. The average contribution in later rounds is rising in N for $N \geq 7$. This is clearly an artifact of the learning algorithm since there is nothing in the agents' preferences or in the structure of the game that could cause it. The reason is that the learning is faster when the number N of agents in the game is higher. The faster the learning, the higher probability that the agents learn a wrong action, and hence contribute more. The extreme case happens when $N = 100$. Then the agents learns their actions almost instantly—they learn to play the action they randomly chose in the first round. The reason is simple. The expected total contribution of 100 agents is $100/3 \times 0 + 100/3 \times 20 + 100/3 \times 40 = 2000$; an agent's payoff is then about 1000 (we can neglect the agent's individual private savings here). The agent then sets the propensity of the randomly chosen action to $R_{ij}^2 = qR_{ij}^1 + \lambda\pi_i^1 \doteq 0.005 \times 1000 = 5$. The probability that the same action would be chosen in the second round is then about $e^5/(2 + e^5) \doteq 99$ %. The eventual expected value of the average contribution is then the same as the initial one: $1/2$. It is because each agent learned to play always the action she randomly chose in the first round, and each agent chose each action initially with probability equal to $1/3$. Now, one third agents play the pure action a_1, one third the pure action a_2, and one third the pure action a_3.

When the number of agents in the game is lower, the learning is slower, and the entropy of the agents' mixed strategies drops to zero later. The longer learning allows the agents to learn better action. However, not all agents

(a) average percentage contribution to the public good

$t \setminus N$	4	7	10	20	40	100
1	0.509	0.499	0.495	0.499	0.501	0.502
2	0.499	0.493	0.490	0.492	0.498	0.501
3	0.483	0.492	0.492	0.488	0.495	0.502
4	0.490	0.479	0.482	0.488	0.495	0.502
5	0.490	0.478	0.473	0.484	0.494	0.502
6	0.461	0.474	0.475	0.481	0.494	0.502
7	0.456	0.458	0.466	0.479	0.494	0.502
8	0.459	0.453	0.458	0.478	0.494	0.502
9	0.460	0.450	0.450	0.476	0.494	0.502
10	0.442	0.451	0.452	0.479	0.494	0.502
11	0.438	0.437	0.443	0.477	0.494	0.502
12	0.435	0.431	0.438	0.475	0.494	0.502
13	0.428	0.428	0.441	0.475	0.494	0.502
14	0.422	0.429	0.438	0.476	0.494	0.502
15	0.416	0.425	0.434	0.476	0.494	0.502
16	0.403	0.415	0.430	0.477	0.494	0.502
17	0.427	0.408	0.433	0.476	0.494	0.502
18	0.397	0.406	0.427	0.476	0.494	0.502
19	0.400	0.403	0.426	0.477	0.494	0.502
20	0.397	0.396	0.421	0.476	0.494	0.502
21	0.404	0.392	0.424	0.476	0.494	0.502

(b) average entropy of agents' mixed strategies

$t \setminus N$	4	7	10	20	40	100
1	1.000	1.000	1.000	1.000	1.000	1.000
2	0.964	0.924	0.865	0.610	0.199	0.002
3	0.926	0.838	0.728	0.344	0.045	0.000
4	0.887	0.752	0.598	0.185	0.009	0.000
5	0.842	0.665	0.476	0.096	0.002	0.000
6	0.802	0.582	0.383	0.050	0.001	0.000
7	0.757	0.511	0.306	0.028	0.000	0.000
8	0.712	0.448	0.249	0.016	0.000	0.000
9	0.671	0.388	0.200	0.008	0.000	0.000
10	0.633	0.337	0.160	0.005	0.000	0.000
11	0.597	0.296	0.127	0.003	0.000	0.000
12	0.563	0.268	0.102	0.001	0.000	0.000
13	0.533	0.240	0.083	0.001	0.000	0.000
14	0.508	0.218	0.069	0.000	0.000	0.000
15	0.485	0.192	0.057	0.000	0.000	0.000
16	0.461	0.174	0.049	0.000	0.000	0.000
17	0.435	0.157	0.041	0.000	0.000	0.000
18	0.419	0.145	0.035	0.000	0.000	0.000
19	0.403	0.135	0.030	0.000	0.000	0.000
20	0.388	0.125	0.027	0.000	0.000	0.000
21	0.374	0.116	0.025	0.000	0.000	0.000

Table 1 The left table shows the average contribution to the public goods in percents for a given number of agents (columns) in a given round (rows). The right table shows the average entropy of the agents' mixed strategies for a given number of agents (columns) in a given round (rows). Each value is averaged over one thousand simulations.

need to learn their dominant action. For instance, Table 1 shows that with $N = 20$, the agents' actions are locked after the 13th round but the agents still contribute in average 47.6 % of their endowment (instead of zero, as predicted by their dominant strategy).

4 Discussion

We have shown that the speed of reinforcement learning depends beside others on the size of payoffs. The higher payoffs, the faster learning. If the learning is too fast, the agents can learn to play the action they have randomly chosen in the first round. In such a case, the final distribution of the agents' pure strategies resembles the initial one agent's mixed strategy. When the modeler cannot control for this effect, unexpected results can happen. In the contribution to the public good simulation in the previous section, the result was that the average contribution rose with the number of agents in the game.

It might be interesting to compare the result with the replicator dynamics. Miller [10] showed for a similar version of the experiment that the more agents in the game, the slower the agents learn their dominant free riding actions through the replicator dynamics. He also showed that it is because the more agents imply the higher payoffs. However, there are two key differences between the replicator dynamics and reinforcement learning. First, in the replicator dynamics, the slow learning to play the right action is the result of the slow learning while in the reinforcement learning it is the result of too fast learning. Second, in the replicator dynamics, the agents eventually learn to play their dominant actions while in the reinforcement learning they can get stuck with suboptimal actions.

Although the problem discussed in this paper has been overlooked so far, there exists enhancements of the algorithm that can overcome it. The lock-in effect can possibly be alleviated by introduction of experimentation (I^t_{ij} in equation (2) is redefined so that $I^t_{ij} = 1$ when action j has been chosen, or $I^t_{ij} = \delta$ otherwise, where $\delta \in [0, 1]$) or by introduction of fictive payoffs to the non-chosen actions as in EWA (the term $I^t_{ij}\pi^t_i$ in equation (2) is redefined so that I^t_{ij} is as in experimentation and π^t_i is replaced by the payoff that the agent would obtain if she has chosen action j and the other agents have not changed their actions). However, neither of these techniques can solve the

problem completely if the payoffs are high enough since δ should be typically small.

The influence of the payoffs' size can be completely eliminated by substituting the gross payoffs determined by the game with some form of net payoffs, i.e. a transformation of the gross payoffs. There are many ways how to calculate the net payoffs. For instance, one can subtract an aspiration level from the gross payoffs, or calculate the fictive payoffs $\tilde{\pi}_{ij}^t$ and then subtract the $\max(0, \min_{\forall j}(\tilde{\pi}_{ij}^t)$ from the gross payoffs. Both these techniques can eliminate the variation in the speed of leaning if used properly. However, some modelers can object to it because it may be incompatible with learning from the feedback. For instance, in his seminal paper, Cross [5, p. 247] claims: "It should be stressed that the concept of opportunity cost has no place in this analysis either. The theory of opportunity cost is derived from the maximization hypothesis, and its introduction would be inconsistent with the action-taking orientation of this paper."

Finally, it should be mentioned that there is one more possibility, namely that the variation of the speed of learning with the payoff size and the resulting effects are not just an computational artifact but a property of human (and animal) learning. Indeed, one of the stylized facts of the contribution to public goods experiments is that the average contribution does rise with the number of human subjects in the game, see [2, 8, 9]. When the agent-based model of learning uses net payoffs (as e.g. in [2]), the fact must be explained by other-caring preferences. The variation in the speed of reinforcement learning then offers an alternative (and perhaps easier) explanation. Clearly, more research on how people truly learn is needed.

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Time series representation and appropriate estimation

Petr Lachout[1]

Abstract. We present a discussion on time series models, particularly, on ARMA and ARIMA model. Considering non-uniqueness of the model representation, we propose to select an appropriate model optimizing simultaneously two criteria: "fit" and "model complexity". These criteria are discordant. Therefore, we have to consider their suitable combination; e.g. AIC or BIC criterion. Our contribution is in larger set of competitive models than monographs suggest.

Keywords: ARMA process, ARIMA process, AIC criterion, algebra of polynomials.

JEL classification: C22
AMS classification: 91B84; 62M10

1 Introduction

Time series analysis is developed under background of the Hilbert space L_2 and the calculus of polynomials in the backward shift operator, see e.g. [1], [2], [3], [4], [5]. This theoretical description and consequent results possess relevant impact to practice. Modern analysis of financial and economic data can be hardly made without this fruitful theory.

Our realization is focused to time series models based on equations expressed by means of polynomials in the backward shift operator. Let us mention such models like AR, MA, ARMA, ARIMA, ARCH, GARCH, etc. Any such equation can be divided by a polynomial with non-zero roots. Obtained equation is equivalent with former one because the division is reversible. Multiplication with the same polynomial is giving the former equation. Thus, the theoretical model can be written in several equivalent forms. Unfortunately, the forms are theoretically equivalent, but, different in practice. Each of them can result in a different estimation of the model.

In the paper we presents the realization for the model ARMA and consequently for the model ARIMA. For them, we suggest family of competitive models which is larger than model family proposed in literature; see e.g. [1], [2], [3], [4], [5]. We select among models using AIC criterion. The proposed procedure is demonstrated on a data set.

2 Description of the idea for ARMA process

Let us recall a definition of the ARMA process.

Definition 1. A random process $(\mathsf{Y}_t, t \in \mathbb{N})$ is called $\mathsf{ARMA}(\mathsf{p},\mathsf{q})$ if it fulfills:

1. It is stationary.
2. It is causal.
3. It fulfills an equation

$$\forall t \in \mathbb{N} \qquad \Phi(\mathsf{B})\,\mathsf{Y}_t = \Theta(\mathsf{B})\,\mathsf{Z}_t, \tag{1}$$

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where

$$\Phi(z) = 1 + \phi_1 z + \phi_2 z^2 + \cdots + \phi_{\mathsf{p}} z^{\mathsf{p}},$$
$$\Theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_{\mathsf{q}} z^{\mathsf{q}}$$

are polynomials, $\mathsf{p}, \mathsf{q} \in \mathbb{N}_0$, B is the backward shift operator acting from $\mathbb{R}^{\mathbb{N}}$ to $\mathbb{R}^{\mathbb{N}}$, and, Z is a White Noise with zero mean and positive variance σ_{Z}^2.

To guarantee uniqueness of polynomials, it is assumed that the polynomials Φ, Θ are possessing no common root.

Let us recall that autoregression, moving averages and uncorrelated sequences (i.i.d. is a particular uncorrelated sequence) belong among ARMA processes. Particularly, $\mathsf{AR}(\mathsf{p}) = \mathsf{ARMA}(\mathsf{p}, 0)$, $\mathsf{MA}(\mathsf{q}) = \mathsf{ARMA}(0, \mathsf{q})$, and, $\mathsf{ARMA}(0, 0)$ means uncorrelated sequence.

Because, the ARMA process is stationary and causal, all roots of Φ must be outside of the unit circle; i.e. their norm is larger then 1. Moreover, the process can be written as an infinite sum

$$\mathsf{Y}_t = \frac{\Theta}{\Phi}(\mathsf{B})\mathsf{Z}_t = \Gamma(\mathsf{B})\,\mathsf{Z}_t = \sum_{\tau=0}^{+\infty} \gamma_\tau \mathsf{Z}_{t-\tau}, \tag{2}$$

where

$$\Gamma(z) = \frac{\Theta}{\Phi}(z) = \sum_{\tau=0}^{+\infty} \gamma_\tau z^\tau,\ \gamma_0 = 1$$

and $\frac{\Theta}{\Phi}$ denotes division of the polynomial Θ by the polynomial Φ.

The equality (1) can be rewritten in many equivalent forms. It is because, the formula can be divided by any polynomial with roots outside of the unit circle without any loss of information.

Let us consider a polynomial

$$\mathsf{Q}(z) = q_0 + q_1 z + q_2 z^2 + \cdots + q_\kappa z^\kappa$$

with roots outside of the unit circle. For simplicity we also require $q_0 = 1$.

Dividing the equality (1) by Q we are receiving its equivalent form

$$\forall t \in \mathbb{N} \qquad \widetilde{\Phi}(\mathsf{B}|\mathsf{Q}, +\infty)\,\mathsf{Y}_t = \widetilde{\Theta}(\mathsf{B}|\mathsf{Q}, +\infty)\,\mathsf{Z}_t, \tag{3}$$

where

$$\widetilde{\Phi}(z|\mathsf{Q}, +\infty) = \frac{\Phi}{\mathsf{Q}}(z) = \sum_{\tau=0}^{+\infty} \widetilde{\phi}_\tau z^\tau,\ \widetilde{\phi}_0 = 1,$$
$$\widetilde{\Theta}(z|\mathsf{Q}, +\infty) = \frac{\Theta}{\mathsf{Q}}(z) = \sum_{\tau=0}^{+\infty} \widetilde{\theta}_\tau z^\tau,\ \widetilde{\theta}_0 = 1.$$

Seeking for a process $\mathsf{ARMA}(\widetilde{\mathsf{p}}, \widetilde{\mathsf{q}})$, we actually allow an inaccuracy in the equation

$$\forall t \in \mathbb{N} \qquad \widetilde{\Phi}(\mathsf{B}|\mathsf{Q}, \widetilde{\mathsf{p}})\,\mathsf{Y}_t = \widetilde{\Theta}(\mathsf{B}|\mathsf{Q}, \widetilde{\mathsf{q}})\,\mathsf{Z}_t + \varepsilon_t, \tag{4}$$

where

$$\widetilde{\Phi}(z|\mathsf{Q}, \widetilde{\mathsf{p}}) = \sum_{\tau=0}^{\widetilde{\mathsf{p}}} \widetilde{\phi}_\tau z^\tau,\ \widetilde{\Theta}(z|\mathsf{Q}, \widetilde{\mathsf{q}}) = \sum_{\tau=0}^{\widetilde{\mathsf{q}}} \widetilde{\theta}_\tau z^\tau,$$
$$\varepsilon_t = \sum_{\tau=\widetilde{\mathsf{q}}+1}^{+\infty} \widetilde{\theta}_\tau \mathsf{Z}_{t-\tau} - \sum_{\tau=\widetilde{\mathsf{p}}+1}^{+\infty} \widetilde{\phi}_\tau \mathsf{Y}_{t-\tau}.$$

Taking $\widetilde{\mathsf{p}}, \widetilde{\mathsf{q}} \in \mathbb{N}_0$ large enough, the variance of the inaccuracy ε_t becomes to be smaller than a prescribed level. Hence, we would need enormous number of observation to be able to distinguish between models

$\mathsf{ARMA}(\mathsf{p},\mathsf{q})$ and $\mathsf{ARMA}(\widetilde{\mathsf{p}},\widetilde{\mathsf{q}})$. Thus, we receive several competitive models for an observed time series. Fortunately, these model will differ in the number of active coefficients. Thus, we can select appropriate model using simultaneously two criteria: "fit" and "model complexity". To be able to decide, we have to accept a convenient combination of these two criteria. Several such combinations are used in practice, e.g. criterion AIC or criterion BIC. The approach is suggested in monographs; e.g. [1], [2], [3], [4], [5].

Standard suggestion is to consider a set of competitive models $\mathsf{ARMA}(\widetilde{\mathsf{p}},\widetilde{\mathsf{q}})$, $0 \le \widetilde{\mathsf{p}} \le \widehat{\mathsf{p}}$, $0 \le \widetilde{\mathsf{q}} \le \widehat{\mathsf{q}}$, or, $\mathsf{ARMA}(\widetilde{\mathsf{p}},\widetilde{\mathsf{q}})$, $0 \le \widetilde{\mathsf{p}}$, $0 \le \widetilde{\mathsf{q}}$, $\widetilde{\mathsf{p}} + \widetilde{\mathsf{q}} \le \Delta$, where $\widehat{\mathsf{p}},\widehat{\mathsf{q}} \in \mathbb{N}$ (resp. $\Delta \in \mathbb{N}$) are properly chosen.

We intend to employ more of equivalent models. For that, we suggest to incorporate also the observation that some coefficients in competitive models can vanish. For such a model, number of coefficients decrease and the criterion can decrease, also. Our suggestion is to consider a set of competitive models $\mathsf{ARMA}(\widehat{\mathsf{p}},\widehat{\mathsf{q}})$, $\phi_i = 0$, $i \in I$, $\theta_j = 0$, $j \in J$ for each selection $I \subset \{1,2,\ldots,\widehat{\mathsf{p}}\}$, $J \subset \{1,2,\ldots,\widehat{\mathsf{q}}\}$.

Similar idea can be implemented for estimation of ARIMA model. Since, a random process $(\mathsf{Y}_t, t \in \mathbb{N})$ is $\mathsf{ARIMA}(\mathsf{p},\mathsf{d},\mathsf{q})$ if and only if random process $(\Delta^{\mathsf{d}}\mathsf{Y}_t, t \in \mathbb{N})$ fulfills $\mathsf{ARMA}(\mathsf{p},\mathsf{q})$. Recall that Δ^{d} denotes the d-difference, i.e. $\Delta^0\mathsf{Y}_t = \mathsf{Y}_t$, $\Delta^{\mathsf{d}+1}\mathsf{Y}_t = \Delta^{\mathsf{d}}\mathsf{Y}_{t+1} - \Delta^{\mathsf{d}}\mathsf{Y}_t$.

Hence for ARIMA process, we suggest to consider a set of competitive models $\mathsf{ARIMA}(\widehat{\mathsf{p}},\mathsf{d},\widehat{\mathsf{q}})$, $\phi_i = 0$, $i \in I$, $\theta_j = 0$, $j \in J$ for each selection $I \subset \{1,2,\ldots,\widehat{\mathsf{p}}\}$, $J \subset \{1,2,\ldots,\widehat{\mathsf{q}}\}$.

Moreover, if we hesitate between ARMA and ARIMA we can join both cases and, for example, consider competitive models $\mathsf{ARIMA}(\widehat{\mathsf{p}},\mathsf{d},\widehat{\mathsf{q}})$, $\phi_i = 0$, $i \in I$, $\theta_j = 0$, $j \in J$ for each selection $I \subset \{1,2,\ldots,\widehat{\mathsf{p}}\}$, $J \subset \{1,2,\ldots,\widehat{\mathsf{q}}\}$, $\mathsf{d} \in \{0,1\}$.

3 Illustrative example

Let us consider daily closing prices of major European stock indices, 1991-1998. Particularly, we concentrate to the stock index CAC (France). The data are freely available, e.g. in the software package R under the name "EuStockMarkets".

In the sequel, we will analyze CAC index in years 1991-1998. Numerical computations are done using the software package R.

Extracting a trend and a seasonal part we receive a time series possessing stationary features. This time series we try to fit with an ARMA model. We consider a set of competitive models $\mathsf{ARMA}(3,3)$, $\phi_i = 0$, $i \in I$, $\theta_j = 0$, $j \in J$ for each selection $I \subset \{1,2,3\}$, $J \subset \{1,2,3\}$. Thus, we receive $2^6 = 64$ competitive models. Now, we use AIC criterion to order them. Ordered values of the AIC criterion look like:

13376.15	13377.40	13380.20	13380.70	13380.87	13381.39	13381.44	13381.61
13381.63	13381.73	13381.95	13382.27	13382.41	13382.65	13382.66	13382.69
13382.72	13382.72	13382.80	13382.84	13382.84	13382.85	13383.02	13383.21
13383.23	13383.26	13383.31	13383.31	13383.33	13383.34	13383.38	13383.49
13383.62	13384.23	13384.23	13384.27	13384.61	13384.79	13384.99	13385.11
13385.18	13385.24	14251.00	14386.76	14437.97	14439.51	14439.85	14440.96
14507.44	14683.33	14806.87	14984.58	15024.94	15026.94	15925.37	15964.66
16024.95	16153.41	16300.86	17532.96	NA	NA	NA	NA

The best model according to AIC criterion differs to the second one at 1.25. Then the difference between the second and the third one is 2.80. Hence AIC criterion is slowly increasing to the 42^{th} model. Then AIC criterion jumps to the 43^{th} model for 865.76. After that AIC criterion is visibly increasing and the last four models are inconvenient because they are giving estimate forming a polynomial with a root in the unit circle.

Let us introduce the first twenty models:

	AIC		choice phi		choice theta
1	13376.147	...	• • •	...	• • 0
2	13377.403	...	• • •	...	• • •
3	13380.199	...	0 • •	...	• • 0
4	13380.695	...	• 0 0	...	• 0 •
5	13380.868	...	• • 0	...	0 0 •
6	13381.392	...	• • 0	...	0 0 0
7	13381.442	...	• 0 0	...	• 0 0
8	13381.609	...	• • 0	...	• 0 •
9	13381.631	...	• 0 0	...	0 0 •
10	13381.727	...	0 • •	...	• • •
11	13381.950	...	• 0 •	...	0 0 •
12	13382.266	...	• 0 0	...	0 0 0
13	13382.413	...	• 0 •	...	0 0 0
14	13382.653	...	• 0 0	...	• • •
15	13382.662	...	• 0 •	...	• 0 •
16	13382.689	...	0 • •	...	• 0 0
17	13382.716	...	• • 0	...	• • •
18	13382.724	...	• 0 •	...	0 • •
19	13382.803	...	• • •	...	• 0 •
20	13382.839	...	• • 0	...	• • 0

Symbol '•' means no restriction on the coefficient and '0' means that the coefficient is fixed as zero.

The models leads to estimation of their coefficients. These estimates are:

		$\widehat{\phi}_1$	$\widehat{\phi}_2$	$\widehat{\phi}_3$		$\widehat{\theta}_1$	$\widehat{\theta}_2$	$\widehat{\theta}_3$
1	...	0.53562	-0.54279	-0.88112	...	1.53368	0.95138	0.00000
2	...	0.54808	-0.54175	-0.88902	...	1.56477	0.99573	0.02450
3	...	-0.00000	-0.57452	-0.34447	...	1.02026	0.42784	0.00000
4	...	-0.96819	-0.00000	-0.00000	...	0.045494	0.000000	-0.044267
5	...	-1.012549	0.042983	-0.000000	...	0.000000	0.000000	-0.042714
6	...	-1.010695	0.043764	-0.000000	...	0	0	0
7	...	-0.96555	-0.00000	-0.00000	...	0.044348	0.000000	0.000000
8	...	-0.52697	-0.42874	-0.00000	...	0.477771	0.000000	-0.058177
9	...	-0.97089	-0.00000	-0.00000	...	0.000000	0.000000	-0.043754
10	...	-0.00000	-0.56611	-0.35537	...	1.013480	0.418849	-0.018818
11	...	-0.992952	-0.000000	0.023576	...	0.00000	0.00000	-0.04228
12	...	-0.96833	-0.00000	-0.00000	...	0	0	0
13	...	-0.991445	-0.000000	0.024749	...	0	0	0
14	...	-0.96778	-0.00000	-0.00000	...	0.0461018	0.0053509	-0.0434824
15	...	-0.9726982	-0.0000000	0.0045586	...	0.041231	0.000000	-0.043731
16	...	-0.000000	-0.963770	0.030197	...	1.0005	0.0000	0.0000
17	...	-0.44877	-0.50116	-0.00000	...	0.564738	0.030783	-0.044593
18	...	-1.01341	-0.00000	0.04262	...	0.000000	-0.039433	-0.045252
19	...	-0.492517	-0.486769	0.026708	...	0.520576	0.000000	-0.047138
20	...	-0.26757	-0.67149	-0.00000	...	0.74677	0.06496	0.00000

The coefficients estimates are similar for the first and the second model. The consequent models are giving estimates which are very different.

4 Conclusion

Statistical features of an ARMA process are determined by coefficients $\phi_1, \phi_2, \ldots, \phi_{\mathsf{p}}$, $\theta_1, \theta_2, \ldots, \theta_{\mathsf{q}}$, p, q. Allowing inaccuracy, we immediately receive several different models which are describing the same process up to a given uncertainty.

Presented numerical example is showing this phenomenon on a real data set. Let us list some models together with their order in ordering given by AIC criterion:

- $\mathsf{ARMA}(3,2)$ is the best model for the data set,
- $\mathsf{ARMA}(3,3)$ is the second best,
- $\mathsf{ARMA}(3,2)$ with $\phi_1 = 0$ is the third best,
- $\mathsf{ARMA}(1,3)$ with $\theta_2 = 0$ is the 4^{th},
- $\mathsf{ARMA}(2,3)$ with $\theta_1 = \theta_2 = 0$ is the 5^{th},
- $\mathsf{AR}(2)$ is the 6^{th},
- $\mathsf{ARMA}(1,1)$ is the 7^{th},
- $\mathsf{ARMA}(3,3)$ with $\phi_1 = 0$ is the 10^{th},
- $\mathsf{AR}(1)$ is the 12^{th},
- $\mathsf{ARMA}(1,3)$ is the 14^{th}.

Comparing estimated acf (autocorrelation function) and pacf (partial autocorrelation function), one can use a rough rule suggested in monographs, see e.g. [1], [2], [3], [4], [5]. Since estimate of pacf possesses only two significant values for lag 1 and 2, the rule leads to the $\mathsf{AR}(2)$. In the ordering given by AIC criterion, this model is the 6^{th} best model for our data.

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Similarity measures for response patterns on dichotomously scored items

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Abstract. Response patterns from test, questionnaires, or other kind of evaluation instruments can give important information about latent variables which cannot be directly observed, such as the "ability" of the respondent individuals. The study of similarity between response patterns leads to a classification of the respondents that can explain the latent variable under a new perspective. There are several known similarity measures, mostly developed in fields such as the bibliometric and scientometric research, which are based on co-occurrences between "objects", and satisfy certain properties which unfortunately don't suit to our problem. Taking inspiration from these measures, we propose a class of similarity measures for response patterns on dichotomously scored items. This study is meant as an example of applying these measures within the item response theory framework, and as a critical assessment of the usefulness of this approach. Our method focuses on the position of respondents in the multidimensional space spanned by the aggregated subject-subject co-occurrences. A number of choices/parameters can be left to the user, but we provide default options reflecting our preferences.

Keywords: similarity, map, Rasch model

JEL Classification: C44
AMS Classification: 90C15

1 Introduction

In the context of the Item Response Theory (IRT), a probabilistic "response model" refers to the issue of the measurement of a quantitative latent (unobservable) trait θ, given a set of k items used to measure that latent trait. In particular, in IRT, dichotomous response models define the probability of a score $x_{vi} = 1$ (to be interpreted as a correct answer), or $x_{vi} = 0$ (to be interpreted as a wrong answer), where v designates the individual, with $v = 1,2,...,n$, and i the item, with $i = 1, 2, ...,k$. The mathematical form of the general item response model is $p(x_{vi}) = P(x_{vi} \mid \theta_v, \beta_i)$, where the probability function p depends on a person parameter θ and on an item parameter β. Person and/or item parameters are possibly vector valued. Nevertheless, in their most simple form, they are scalar valued – the parameter β_i representing the item "difficulty", and the parameter θ_v representing the "ability" of respondent. The Rasch model (RM) [6] is a well-known instance of IRT response model, with scalar valued person and item parameters. The RM can be written:

$$p(x_{vi}) = P(X_{vi} = x_{vi} \mid \theta_v, \beta_i) = \frac{exp\{x_{vi}(\theta_v - \beta_i)\}}{1 + exp(\theta_v - \beta_i)}$$

It is worth noting that the difficulty parameter is on the same metric as the latent trait. As can be seen, the response model is based on a two dimensional data matrix (x_{vi}). For each respondent, there will be a row vector $x_v = (x_{v1},...,x_{vk})$ of item responses of length k. Let us refer to the vector x_v as a "response pattern". Individual v also corresponds to a total score (so called raw score) r_v, which is the sum of the elements of x_v. Similarly, we will denote s_i the sum $\sum_v x_{vi}$ (so called item marginal sum). In the sequel, we will assume, without loss of generality, that $s_1 \geq s_2 \geq \ldots \geq s_k$. A remarkable property of the RM is that the maximum likelihood estimates of person and item parameters are co-monotone with the corresponding marginal sums, respectively, r_v and s_i [2]. This corresponds to the intuitive notion that the ability of person v increases when his/her raw score increases. In particular, under the RM, all the persons obtaining the same raw score will receive the same estimated ability. Otherwise said, for a given a raw score the estimated abilities of a RM are not dependent on the response pat-

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terns. For a fixed raw score r_v, $0 \le r_v \le k$, the most probable response pattern is given by $x_{vi} = 1$ for every $i \le r_v$ and 0 otherwise (which is called *perfect Guttman* pattern). Correspondingly, the most improbable pattern is given by $x_{vi} = 0$ for every $i \le k - r_v$ and 1 otherwise (most *aberrant* response pattern).

The aim of this paper is to find an appropriate index to measure the "similarity" between individuals, based on the "similarity" between their corresponding response patterns. This could be useful to get an interpretation of the data (maybe) different from the traditional one, only based on the parameter estimates and the total scores.

In scientometric research, similarity measures are known, which are based on co-occurrence data. For example, co-citation data can be used to study the relations among authors or journals, co-authorship data can be used to study scientific cooperation. Unfortunately none of the known similarity measures [7] can be applied to response patterns, since they seem to be unsuitable for our particular situation (see section 3). Our idea is to integrate information from co-occurrence with information from the item marginal sums. For each pattern we define a particular corresponding distribution, based on the item marginal sums s_i, and we call it "conformity" distribution. Then, we propose a general class of similarity measures for response patterns, based on co-occurrences but also on the "dissimilarity" between these distributions.

Since some of the results we base on come from majorization theory, in section 2 we provide a small summary of the main concepts that may be useful to the reader.

2 Majorization ordering and generalizations

Majorization [5] is a pre-order on vectors used in mathematics to decide whether the components of a vector are more (or less) "spread out" or more (or less) "equal" than the components of another vector. Functions of the vectors that are consistent with this pre-order are called Schur-convex. Many statistical "dissimilarity" or "divergence" measures belongs to this class of functions. Majorization and Schur-convexity can be generalized from vectors to measurable functions [1].

Definition 1. *Let f be a μ-measurable function in a set I (where μ is a positive and finite measure on I). We define the increasing re-ordering of f with respect to μ as:*

$$f_{\uparrow}^{(\mu)}(t) = inf\{z: \nu_z \ge t\}, \quad t \in [0, \mu(I)],$$

where $\nu_z = \mu(B_z)$, $B_z = \{y \in I: f(y) \le z\}$.

Definition 2. *Let f, g be μ-integrable and μ-measurable functions on I ($f, g \in L^1(I,\mu)$). We say that g (μ)-majorizes f and write $f \prec_{(\mu)} g$ if:*

1. $\int_0^z f_{\uparrow}^{(\mu)}(t)dt \ge \int_0^z g_{\uparrow}^{(\mu)}(t)dt$, $\forall z \in [0, \mu(I)]$.
2. $\int_I f d\mu = \int_I g d\mu$.

When conditions for majorization are not completely satisfied, there are also weak definitions of majorization. In particular, we will need the following definition.

Definition 3. *Let $f, g \in L^1(I,\mu)$. We say that g weakly (μ)-majorizes f from above and write $f \prec_{(\mu)}^{w} g$ when only condition* 1) *of "Definition 2" is satisfied.*

A generalization of Karamata's theorem [4] allows us to characterize a class of functionals which are consistent with strong and weak (μ)-majorization. This result can be summarized by the following theorem.

Theorem 1. (μ)*-majorization theorem*
Let $f, g \in L^1(I,\mu)$.

1. $f \prec_{(\mu)} g$ *if and only if:*

$$\int_I \varphi(f)\, d\mu \le \int_I \varphi(g)\, d\mu \text{, for all convex functions } \varphi.$$

2. $f \prec_{(\mu)}^{w} g$ *if and only if:*

$$\int_I \varphi(f)\, d\mu \le \int_I \varphi(g)\, d\mu \text{, for all decreasing convex functions } \varphi .$$

(Proof: see [1]).

The theorem says that the functional $\int_I \varphi(f)\, d\mu$ is consistent with strong (μ)-majorization (φ convex) or weak (μ)-majorization from above (φ convex and decreasing).

3 Similarity between response patterns

As a general rule, we assume that similarity between patterns is a function of the number of co-occurences in the patterns. Co-occurrence, in our case, should be seen as the number of times when individual *v* and individual *w* give the same response, right or wrong, to the items. In order to define a similarity measure, say $S(x_v, x_w)$, between two response patterns, x_v and x_w, we assume that information from co-occurrence data should be "weighted" by information from the item marginal sums.

Let $\bar{x}_v$, with elements $\bar{x}_{vi} = 1 - x_{vi}$, $v = 1,2,...,n$, the reverse-scored pattern version (or "dual") of x_v. Then, we can obtain the number of co-occurrences c_{vw} between individual *v* and individual *w* by:

$$c_{vw} = x_v' \cdot x_w + \bar{x}_v' \cdot \bar{x}_w .$$

Anyway, the number of co-occurrences alone is a too rough measure, and is not enough to decide whether a pair of response patterns are similar or not. Consider, for example, the following response patterns (from a $n \times k$ data matrix, with $n > 5$):

1) x_1 = (1,0,0,…,0,0,0)
2) x_2 = (0,1,0,…,0,0,0)
3) x_3 = (1,1,0,…,0,0,0)
4) x_4 = (0,0,0,…,0,0,1)
5) x_5 = (0,0,0,…,0,1,1).

First, let's consider x_1, x_2 and x_4. These three patterns present the same number of co-occurrences between them, that is $k-2$. Anyway x_4 has an unexpected (aberrant) structure with respect to the others (remind that the item marginal sums are ordered in decreasing order, $s_1 \ge s_2 \ge \ldots \ge s_k$). Then, although it has the same raw score (1) and the same number of co-occurrences, its similarity with patterns x_1 and x_2 should be lower than the similarity between those two (which are not aberrant). So we want $S(x_v, x_w)$ to be such that $S(x_1, x_2) \ge S(x_1, x_4)$.

Further, consider the couples x_1, x_3 and x_4, x_5: depending on the scores s_1, s_2, s_{k-1}, s_k we could desire $S(x_1, x_3) = S(x_4, x_5)$ or even $S(x_1, x_3) < S(x_4, x_5)$, giving more weight to similarity between "aberrant" patterns, which are, by definition, less frequent.

So, similarity index proposed will hopefully follow these properties, it will not be based only on the "rough" co-occurrence that should be weighted according to the item difficulty. In particular, our solution is to weight each outcome in a pattern with the corresponding item marginal sum.

From pattern x_v we create a vector that we call *conformity distribution* $f_v = (f_{v1},...,f_{vi},...,f_{vk})$, where:

$$f_{vi} = x_{vi} s_i + \bar{x}_{vi} (n - s_i),$$

According to the total scores s_i we can also define the pattern which follows the most logical structure (score 1 for the "easy" items and 0 for the "difficult" ones). Let x^* be any pattern with conformity distribution f^* such that:

$$f_i^* = \max\{ s_i , n - s_i \}$$

for $i = 1,2,...,k$ (notice that x^* could not belong to the dataset and, moreover, could not be univoquely identified - but this is not essential for our purposes).

First of all, we want to find an index to measure the "distance" between x_v and x^*. An effective way to measure distance between distributions should be based on the ratios f_{vi} / f_i^*. Intuitively, we could say that f_v is "close" to f^* if their ratio is (on average) close to 1. Suppose that we want to find which distribution, between f_v and f_w is

closer to f^*. Since $\sum_i f_{vi} \le \sum_i f_i^*$ and $\sum_i f_{wi} \le \sum_i f_i^*$, we can compare the ratio vectors f_v / f^* and f_w / f^* using weak majorization (form above) with respect to the measure defined by f^* (called μ^*). Notice that vectors $f_v / f^* = (f_{v1} / f_1^*, \ldots, f_{vk} / f_k^*)$ and $f_w / f^* = (f_{w1} / f_1^*, \ldots, f_{wk} / f_k^*)$ are μ^*-measurable functions.
Further, the reason why majorization has to be studied with respect to μ^* is that, since we divide f_v and f_w by f^*, the increasing re-ordering of these functions must consider the different "weights" given by μ^*.
So $\frac{f_v}{f^*} \prec^{w}_{(\mu^*)} \frac{f_w}{f^*}$ if and only if:

$$\int_0^x \left(\frac{f_v}{f^*}\right)_{\uparrow}^{(\mu^*)} dt \ge \int_0^x \left(\frac{f_w}{f^*}\right)_{\uparrow}^{(\mu^*)} dt, \quad \forall x \in [0, \sum_i f_i^*],$$

which means that f^* is "closer" to f_v than to f_w. Then, for the (μ)-majorization theorem we have:

$$\Phi(f_v, f^*) = \sum_i \varphi\left(\frac{f_{vi}}{f_i^*}\right) f_i^* \le \sum_i \varphi\left(\frac{f_{wi}}{f_i^*}\right) f_i^* = \Phi(f_w, f^*)$$

for any decreasing convex function φ.
This suggests to use, as *dissimilarity measure* between two conformity distributions f_v and f_w, the function:

$$d(f_v, f_w) = \left|\Phi(f_v, f^*) - \Phi(f_w, f^*)\right|.$$

From this result, we could define a similarity index $S(x_v, x_w)$ by the ratio between similarity (based on "rough" co-occurrences) and dissimilarity (based on distributions), that is:

$$S(x_v, x_w) = \frac{c_{vw}}{1 + [d(f_v, f_w)]^{\alpha}},$$

with $\alpha \ge 0$.
This defines a general class of similarity measures, depending on the choice of φ in $\Phi(f_v, f^*)$ and α, which can vary according on the weight we want to give to co-occurrence rather than dissimilarity. For simplicity, we can take φ = -ln since it is convex, decreasing and takes value 0 when $f_v = f^*$.
The following theorems show two fundamental properties of S that can be useful to understand its results in many particular situations.

Theorem 2
Let x_u, x_v, x_w, x_z be response patterns such that $c_{uv}=c_{wz}=k-1$. Suppose that $f_{vi} = f_i^* > f_{ui}$, for a fixed index i between 1 and k, and $f_{zj} = f_j^* > f_{wj}$, for a fixed index j between 1 and k. Then:

1. *$f_i^* > f_j^*$ if and only if $S(x_u, x_v) < S(x_w, x_z)$;*
2. *$f_i^* < f_j^*$ if and only if $S(x_u, x_v) > S(x_w, x_z)$;*
3. *$f_i^* = f_j^*$ if and only if $S(x_u, x_v) = S(x_w, x_z)$.*

Proof:
It's easy to see that

$$d(f_u, f_v) = \varphi\left(\frac{1 - f_i^*}{f_i^*}\right) f_i^*, \quad d(f_w, f_z) = \varphi\left(\frac{1 - f_j^*}{f_j^*}\right) f_j^* \quad .$$

$f_i^* > f_j^*$ if and only if $\frac{1 - f_i^*}{f_i^*} < \frac{1 - f_j^*}{f_j^*}$. Since φ is decreasing and $c_{uv}=c_{wz}$ we get $d(f_u, f_v) \ge d(f_u, f_w)$ and then $S(x_u, x_v) < S(x_w, x_z)$. Similarly we can prove 2) and 3).

Theorem 3
Let x_u, x_v, x_w be response patterns such that $c_{uv}=c_{uw}= k-2$. Suppose that $f_{ui} \ne f_{vi} = f_{wi}$, for a fixed index i between 1 and k, $f_{vj} \ne f_{uj} = f_{wj}$, for a fixed index j between 1 and k, and $f_{wl} \ne f_{ul} = f_{vl}$, for a fixed index l between 1 and k. Then:

1. *$s_i \ge s_j \ge s_l$ implies $S(x_u, x_v) \ge S(x_u, x_w)$;*
2. *$s_j \ge s_l \ge s_i$ implies $S(x_u, x_v) \ge S(x_u, x_w)$.*

Proof:
From the assumptions, we know that *u* and *v* present the same values in any cell except from cell *i* and cell *j*. We could say that the two patterns present two "non-co-occurrences". Similarly, also *u* and *w* present two non-co-occurences, in cell *i* and *l*. The proof will be divided analyzing four different stuations.
1) $s_i \geq s_j \geq s_l \geq n/2$.
Note that $s_i \geq s_j$ implies $1\text{-}s_i \leq 1\text{-}s_j$, so $\varphi\left(\frac{1-s_i/n}{s_i/n}\right)s_i/n \geq \varphi\left(\frac{1-s_j/n}{s_j/n}\right)s_j/n$ (since φ is decreasing). So we get:

$$d(f_u, f_v) = \left|\varphi\left(\frac{1-s_i/n}{s_i/n}\right)s_i/n - \varphi\left(\frac{1-s_j/n}{s_j/n}\right)s_j/n\right| = \varphi\left(\frac{1-s_i/n}{s_i/n}\right)s_i/n - \varphi\left(\frac{1-s_j/n}{s_j/n}\right)s_j/n .$$

Similarly, since $s_i \geq s_l$, we get:

$$d(f_u, f_w) = \varphi\left(\frac{1-s_i/n}{s_i/n}\right)s_i/n - \varphi\left(\frac{1-s_l/n}{s_l/n}\right)s_l/n .$$

The two dissimilarities differ only by the second addend. $s_j \geq s_l$ implies $1\text{-}s_j \leq 1\text{-}s_l$, so:

$$\varphi\left(\frac{1-s_j/n}{s_j/n}\right)s_j/n \geq \varphi\left(\frac{1-s_l/n}{s_l/n}\right)s_l/n$$

(since φ is decreasing), and then $d(f_u, f_v) \leq d(f_u, f_w)$. Finally, considering that $c_{uv}=c_{uw}$, we get $S(x_u, x_v) \geq S(x_u, x_w)$.
2) $s_i \geq s_j \geq n/2 \geq s_l$.
In this situation, $d(f_u, f_v)$ remains the same as before (point 1)), but for $d(f_u, f_w)$ we get

$$d(f_u, f_w) = \varphi\left(\frac{1-s_i/n}{s_i/n}\right)s_i/n + \varphi\left(\frac{s_l/n}{1-s_l/n}\right)(1-s_l/n)$$

(since $f_{ui}= f_i^*=s_i$, $f_{wi}=1\text{-}\ s_i < f_i^*$ and $f_{ul}= f_l^*=1\text{-}s_l$, $f_{wl}=s_l < f_l^*$). Then we surely have $d(f_u, f_v) \leq d(f_u, f_w)$ and so $S(x_u, x_v) \geq S(x_u, x_w)$.
3) $s_i \geq n/2 \geq s_j \geq s_l$.
In this situation, $d(f_u, f_w)$ is the same as before (point 2)) and similarly we get:

$$d(f_u, f_v) = \varphi\left(\frac{1-s_i/n}{s_i/n}\right)s_i/n + \varphi\left(\frac{s_j/n}{1-s_j/n}\right)(1-s_j/n)$$

The two dissimilarities differ only by the second addend. $s_j \geq s_l$ implies $1\text{-}s_j \leq 1\text{-}s_l$, so:

$$\varphi\left(\frac{s_j/n}{1-s_j/n}\right)(1-s_j/n) \leq \varphi\left(\frac{s_l/n}{1-s_l/n}\right)(1-s_l/n)$$

(since φ is decreasing), and then $d(f_u, f_v) \leq d(f_u, f_w)$ which implies $S(x_u, x_v) \geq S(x_u, x_w)$.
4) $n/2 \geq s_i \geq s_j \geq s_l$.
This situation is very similar to point 1), we get:

$$d(f_u, f_v) = \left|\varphi\left(\frac{s_i/n}{1-s_i/n}\right)(1-s_i/n) - \varphi\left(\frac{s_j/n}{1-s_j/n}\right)(1-s_j/n)\right| = \varphi\left(\frac{s_j/n}{1-s_j/n}\right)(1-s_j/n) - \varphi\left(\frac{s_i/n}{1-s_i/n}\right)(1-s_i/n) ,$$

$$d(f_u, f_w) = \varphi\left(\frac{1-s_l/n}{s_l/n}\right)s_l/n - \varphi\left(\frac{1-s_i/n}{s_i/n}\right)s_i/n .$$

Since we already know (from point 3)) that $\varphi\left(\frac{s_j/n}{1-s_j/n}\right)(1-s_j/n) \leq \varphi\left(\frac{s_l/n}{1-s_l/n}\right)(1-s_l/n)$, then we get $d(f_u, f_v) \leq d(f_u, f_w)$ which implies $S(x_u, x_v) \geq S(x_u, x_w)$.

As a particular consequence of these theorems, take simple patterns x_1, x_2, x_3 and x_4 considered previously.

First, since $s_1 \geq s_2 \geq \ldots \geq s_k$, from theorem 3) follows $S(x_1, x_2) \geq S(x_1, x_4)$ as we wished (take $x_1=x_u$, $x_2=x_v$, $x_4=x_w$; $i=1$, $j=2$, $l=k$). Further, regarding similarity between x_1, x_3 and x_4, x_5, from theorem 2) follows that, depending on the scores s_2 and s_{k-1} we may get different results. For example, take $s_2=n-s_{k-1}$ (which means $f_2^*= f_{k-1}^*$): then we get $f\, S(x_1, x_3)= S(x_4, x_5)$. Take now $s_2<n-s_{k-1}$: then we get $S(x_1, x_3)< S(x_4, x_5)$, otherwise, if $s_2>n-s_{k-1}$ (it may happen even if not very common in a dataset) we get $S(x_1, x_3)> S(x_4, x_5)$. Intuitively, the heavier the weight (given by f^*) in the "non-co-occurrent" data, the lower the similarity.
By the last intuitive explanation of S's property, we can also wish that the heavier the weights in the "co-occurrent" data, the higher the similarity. Here, the weights in the co-occurrent data set are given by f_v (or identically f_w). This suggests to simply consider the sum of the weighted co-occurrences between patterns x_v, x_w, so:

$$c^*_{vw} = \sum_C f_{vi} = \sum_C f_{wi},$$

where $C = \{i: f_{vi} = f_{wi}\}$. c^*_{vw} alone could be taken as a similarity measure but, since its value is influenced by the weights given by the scores, c^*_{vw} could get higher if the two patterns are close to x^*, and a lower if they are not (for example if they are aberrant). In the previous situation, we surely get $c^*_{13} \geq c^*_{45}$ only because $s_1 \geq s_k$. To avoid this situation, the second proposed similarity measure S' consists in a normalization of c^*_{vw}. In particular, we can divide c^*_{vw} by $m^*_{vw}= \max\{c^*_{vv}, c^*_{ww}\}$. This value decreases as the patterns x_v and x_w tend to be aberrant. The normalization leads to the following formula:

$$S'(x_v, x_w) = \frac{c^*_{vw}}{m^*_{vw}}.$$

4 Conclusion

Both indexes proposed, S and S', seem to meet the requirements which should be necessary to a similarity measure between response patterns. They are not only based on co-occurrence data but also on information provided by the item marginal sum, giving a different weight to each item.
The study of similarities in response patterns could be useful in IRT, in particular in comparison with Rasch measurement. With the RM, latent variable, such as individual ability, can be measured with maximum likelihood estimation. In particular, it is known that these estimates are strictly related to marginal sums r_v [2]. Otherwise said, within the Rasch measurement framework classification of individuals is only based on their raw scores [3]. Then, as a limit case, a couple of individuals that give opposite response to each item could, for example, get the same raw score r_v: nevertheless, it is hard to think that their abilities is completely equal and matched.
In this paper we presented a deterministic approach (as opposed to the probabilistic approach of Rasch) to the problem of measurement. Similarity measures could provide a deeper analysis of individuals and their abilities, not only related to the scores but also to the corresponding item difficulties. This study provided a new perspective in latent variable analysis, where also information from response patterns could play a key role and can be useful to data interpretation.

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Income Inequality and Gross Domestic Private Product Growth in Different Groups of Countries

Hana Lipovská[1], Jakub Fischer[2]

Abstract.

The aim of this paper is to analyse the relationship between economic growth, government expenditure and income inequality. Most studies suggest that there is a weak or non-existent relationship between growth and income inequality. We use Gross domestic private product growth instead of GDP growth. The analysis was performed by employing an OLS model for 73 countries. The clustering was used to classify data set into 5 subsets. A positive relationship between the GDPP growth and income inequality was found. This relationship is strong for developing and transitional economies and weak for the developed countries including the G7.

Keywords: economic growth, Gini index, government consumption

JEL Classification: H5, I3, O4
AMS Classification: 91B62

1. Introduction

The relationship between economic growth and income inequality measured by the Gini index is a frequently discussed macroeconomic topic addressed by numerous economic policies. Empirical studies discovered either no association or a negative relationship between economic growth and inequality in society (e.g.: Alesina, Rodrick [1]). Barro [2] proves that while the income inequality diminishes economic growth in poor states, it improves the growth rate in developed states, with GDP per capita equal to $2000 being the benchmark for sorting poor and developed states. Kuznets [5] suggested a widening inequality in the early phases of economic growth (e.g. developing countries) and a narrowing inequality in the later phases of industrialization and urbanization. There is no doubt, that the relationship is highly influenced by the way the GDP growth and income inequality is measured. Higgs [4] (based on Kuznets´ findings) pointed out that government product might be excluded from the GDP. Based on those studies, we developed adjusted macroeconomic indicator – Gross Domestic Private Product Growth (GDPP growth). The aim of this paper is to show that income inequality might contribute to the more rapid economic growth measured as Gross Domestic Private Product growth instead of Gross Domestic Product growth. The rest of the paper is organized as follows: we introduce briefly the theory of economic growth, show the importance of using GDPP instead of GDP and explain why income inequality accelerates the GDPP growth, while income equality contributes to the stagnation. We introduce the data as well as the statistical and econometrical methods we used. We then present results of our analysis. Firstly the relations between GDP growth rate and the Gini index and between GDPP growth rate and the Gini index are discussed. Secondly the impact of the Gini index, GDPP per capita and Gross Domestic Savings on GDPP growth is examined for 5 different groups of countries. Finally we analyze how the income inequality measured by the Gini index is influenced by GDPP growth, Gross Domestic Savings and Research and Development expenditures.

2. GDP and GDPP growth, Gini index

GDP and GDPP growth

Economic growth as the improving of economic production over time is rooted in several sources such as human capital, expected longevity, lower birthrate, stable formal institutions (e.g. legal or political), modest inflation and lower governmental consumption to GDP. There are three standard methods for estimating GDP – the output method, the income method and the expenditure method. For the aim of our contribution, the expenditure

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method is appropriate because it allowed us to eliminate government expenditures from the GDP to calculate GDPP. GDP growth is than calculated as the percentage change in GDP

$$GDP = C + I + NX + G, \qquad (1)$$

C being households consumption, I investments, NX net exports and G part of government expenditures. The government expenditures are the key part of this equity: government expenditures grew from the end of the 19th century to the 80´s in the 20th century. Panel data published by the World Bank suggests lower government expenditures, because the variable General government final consumption does not include the defense expenditures. The average government consumption for all economies in the dataset was 26%, although in developed and transitional countries it reached 32% and in African and Asian states it was just 17%. Government final consumption according to the World Bank includes cash payment for operating activities of the government in providing goods and services, compensation of employees, interest, subsidies, grants, social benefits or rents [9], but they do not include transfer payments. Barro pointed out, that government expenditures are one of the factors causing diminished economic growth [2]. For finding a relation between economic growth and income inequality, there are several reasons why government expenditures should not be part of the growth. According to Higgs [4] governments expenditures improve intermediate, rather than final production. Furthermore, it is difficult to calculate the market price of the goods and services purchased by government and finally the government expenditures are not influenced by the economic motives, but by political forces. This can be exemplified by the boom in government spending pulled by the need for supporting economic growth. Such spending is financed by tax-payers or by debt. In both cases, this artificial economic growth does not respond to the growth potential of such economies. We can illustrate it by a simple example: in the first year $GDP_1 = 3\,800$ and $G_1 = 760$, so $GDPP_1 = 3040$. Next year GDP_2 grew to 3857, so the GDP growth was 1.5%. However, this growth was influenced by the higher $G_2 = 877$, which means that $GDPP_2 = 2980$ and the GDPP growth was actually -1.9 %. In this case the GDPP growth explains the growth that is not influenced by governmental expenditures. Despite the questionable nature of such a claim, we believe, that GDP growth does not show the real potential growth of the national economy. We suggest using GDPP growth as the change of GDP adjusted by the government expenditure for calculating the relation between the income inequality and the tendency of the economies.

Gini index

The income inequality is often approximated by the Gini index measuring the spread between perfect equality and the real income (or wealth) distribution. Gini index is calculated as the ratio between the line of equality that forms a 45 degree angle with the x-axis and the Lorenz curve $L(x)$. The Gini index can be expressed as

$$GC = 1 - 2\int_0^1 L(x)dx, \qquad (2)$$

reaching the highest values around 65 (e.g.: Seychelles) and the lowest around 23 – 24 (Scandinavian economies).

This study and our findings could be improved with new proxy variable used instead of the Gini index. As Kuznets noticed, the income inequality should not be measured as the state in one moment, because of a migration among income groups. For this paper, however, the standard Gini index was used as we have not sufficient dataset available.

3. Data and models

A frequent problem when working with macroeconomic data is that only a sparse matrix is available, which means having to begin the analysis with as large a dataset as possible to get appropriately large dataset for the final modeling. We have used the World Bank data that is valuable for its high quality. Initially we used a data set of 214 cases; nevertheless only 73 cases could be used for our model. In spite of the fact that this article analyses cross-sectional data, for the primary data set data for the first decade of the 21st century (2001 – 2010) was used. As the benchmark year 2006 was chosen – this year was far away from the dot.com boom and following crises from the beginning of the millennium, and it was not heavily influenced by 2007 bubble and recent recession. That is the reason why data on GDP, GDP per capita, economic growth and inflation are referred to 2006. For the developed economies annual data is available, but this is not the case for developing or

underdeveloped countries, which provides this type of data sporadically. For all other variables excluding those above and GDPP growth the arithmetical average of available data from the period 2001 – 2010 was calculated.

After diminishing GDP by the government expenditures, the growth in GDPP was calculated. For each of the 73 states the average GDPP growth was calculated as the geometrical mean:

$$GDPP_{growth_i} = \sqrt[T]{\prod_{t=1}^{T} \frac{(1-0{,}01G_t) \times GDP_t - (1-0{,}01G_{t-1}) \times GDP_{t-1}}{(1-0{,}01G_t) \times GDP_{t-1}}},$$

$$i = 1, \dots 73, \min T = 1, \max T = 9. \qquad (3)$$

The Gini index was available for just several years of the discussed period moreover the World Bank data does not contain a Gini index for most of the member states of the Organization for Economic Co-operation and Development. If the World dataBank dataset was used, our analysis would miss data for e.g.: USA, Japan or Czech Republic. Aware of the risks of using in one analysis two types of data (and possibly two different methodologies), we have completed the data set with data from the OECD database. The arithmetical average of the Gini index enters our final model.

The final data set was created by excluding all economies, for which key macroeconomic data was not available. Furthermore, Bosnia and Herzegovina as well as Lesotho were excluded as outliers for a model using gross domestic savings to GDP. The final data set covers 75 % of world estimated GDP.

The OLS estimator was used to estimate a model explaining sources of income inequality and a model explaining sources of the GDPP growth. For macroeconomic data the risk of violation of homoscedasticity exists, therefore the White HCE estimator was used. The normality of residuals was not considered to be a problem, because the size of the data set allows us to assume asymptotical normality. World Bank as well as OECD data are widely valued for high quality. Nevertheless, there can be significant differences in methodology in space and time. The least developed countries as well as dictatorships are typical examples of this problem. The deliberate overestimation or underestimation of data can result in a significantly distorted model. On the other hand, excluding those economies from our model would lead to lack of spatial variety. For these reason, all the available data were finally used.

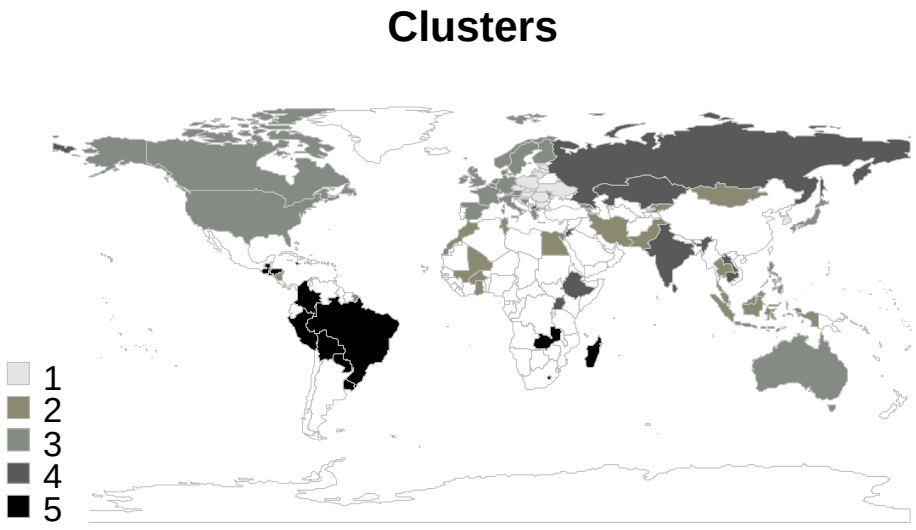

Figure 1 The spatial distribution of characteristic groups of states. (Source: author´s calculation and graphics)

Based on Kuznets [5] we expected differences between Gini index, GDPP growth and other variables among typical groups of states (according to the phase of their economies), so the cluster analysis was used. Five-means clustering was used on variables GDP per capita (PPP, 2006), GDPP growth and Gini index. The mean values for those variables and for all five clusters are depicted in Figure 2. The first cluster consists of transitional economies including the Czech Republic, as well as Israel and South Korea (as the only Asian state). The third cluster of Latin America states and four African states can be easily described. The remaining two clusters

cannot be clearly described in a geographical way, however all the states are similar in economic features: the second cluster includes several African and Asian states as well as Bosnia and Herzegovina, the forth cluster consists of African and Asian states, Russia and India (fig. 1). The forth cluster is the most important for our analysis, as it contains rapid GDPP growth and low Gini index (fig. 2). The impact of belonging to the clusters on the intercept is expressed by four dummies $D_i, i = 1, \dots 4$, the fifth cluster responds to the intercept of estimated models.

We used 22 variables included dummies. The first group of variables consists of variables connected to the gross domestic product: GDP and GDPP (measured in PPP), GDP per capita and GDPP per capita and GDP growth as well as GDPP growth. The second group of variables consists of $trade$ (export to GDP), income inequality proxy $Gini$ and gross domestic savings ($Savings$). Other variables describe government interventions, fiscal and monetary policy: $totaltax\_rate$ measures the percentage of aggregate company taxes, tax_rev measures economic freedom, governmental spending $expenses$, governmental spending on defense $military$, proxy of monetary policy $inflation$ and the unemployment rate U. The level of human capital was approximated by the total spending on research and development to GDP ($R\&D$), the demographical variables such as the size of population, population growth and rural population to the total population were used for the first models. However, for the OLS estimator just several of the variables above were statistically significant.

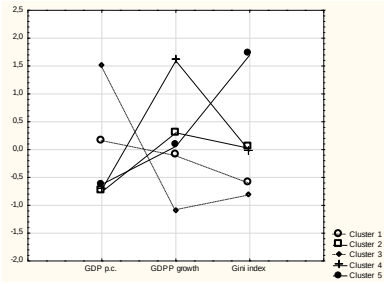

Figure 2 Graph of means (based on clustering analysis results)

4. Estimation results

No significant linear relationship between GDP growth and the Gini index was proved in our dataset. Part of the GDPP growth can be obviously explained by the income inequality measured by the Gini index however, this relation is still not very strong. Though, the relationship is positive: the higher values of Gini index are connected to the higher GDPP growth rate:

$$GDPP_{growth} = 3.06 + 0.078\, Gini. \qquad (4)$$

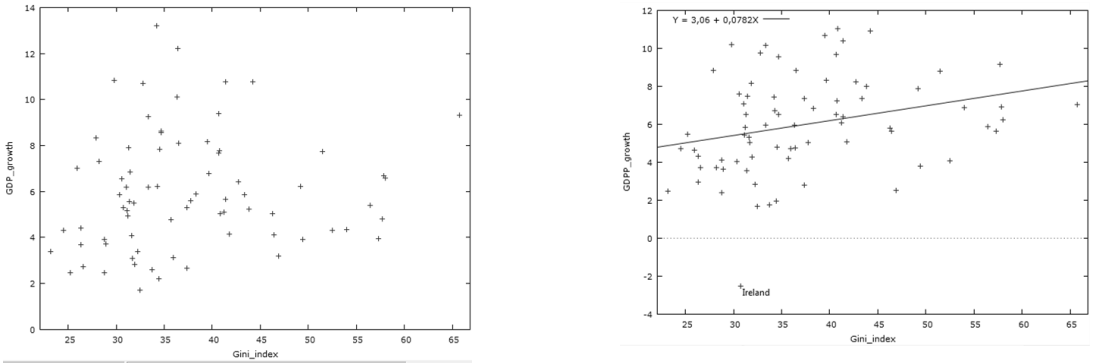

Figure 3 Relation between GDP growth and Gini index (left graph) and relation between GDPP growth and Gini index (right graph)

A correlation between GDP growth and Gini index $\mathrm{corr}(\mathrm{GDP}_{\mathrm{growth}}, \mathrm{Gini}) = 0.106$ is not significant at the p = 0.05 level, on the other hand correlation between GDPP growth and Gini index $\mathrm{corr}(\mathrm{GDP}_{\mathrm{growth}}, \mathrm{Gini}) = 0.282$ is significant on the p = 0.05 level.

	mean	minimum	maximum
Gini index	37.66	23.18	65.77
GDP per capita (PPP)	14813	711,16	53849
GDPP growth (%)	6.005	-2.53	11.04
GDP growth (%)	5.96	1.69	13.19

Table 1 Average, minimum and maximum values of key variables

Table 1 clearly shows the difference between GDP growth and GDPP growth. While there is no significant difference between the mean values, difference between minimal values is more than 4 percentage points and difference between maximum values is nearly 2 percentage points.

4. 1. GDPP growth model

Variable	Parameter estimate	SE	p-value	
$const.$	2,743	1,455	0,0638	*
$Gini\ index$	0,0624	0,0229	0,0084	***
$GDP\ p.c.$	$-7{,}181 \times 10^{-5}$	$2{,}978 \times 10^{-5}$	0,0187	**
$SAVINGS$	0,0479	0,0171	0,0066	***
D_1	1,139	0,475	0,0194	**
D_2	1,2397	0,615	0,0479	**
D_4	4,636	0,616	$1{,}88 \times 10^{-10}$	***
n	73			
R^2	0,71			

Table 2 Estimated OLS model of GDPP growth, with Gini index, GDP per capita, Gross domestic savings and dummies for characteristic groups of states as the explaining variables

The GDPP growth rate is explained by the income inequality, GDP per capita, Gross domestic savings and the clusters. As was expected, the higher Gini index and the Gross domestic savings effect the higher GDPP growth. The low base effect causes that the higher GDPP growth rate is typical for countries with lower GDP per capita. The GDPP growth in the South American and other states from the 5th cluster is *ceteris paribus* 2.74 %, transitive economies, included the Czech Republic, growth by 3.87 % and states from the 2nd cluster by 3.97 %. The highest growth (7.37 %) is in 4th cluster economies, including India and Russia. The average GDPP growth in those states was 10.15 % that is more than 4 percentage points higher in comparison with data set average. The GDPP growth in most developed states is significantly different from South America even if the explaining variables are taken in account: for the following average values $\overline{\mathrm{GINI}_2} = 37.858$, $\overline{\mathrm{GDPpc}_2} = 4433.5$, $\overline{\mathrm{SAVINGS}_2} = 16.206$ is the estimated GDPP growth in developed countries 5.53 %, for $\overline{\mathrm{GINI}_5} = 53.521$, $\overline{\mathrm{GDPpc}_5} = 6245.5$ and $\overline{\mathrm{SAVINGS}_5} = 13.127$ is the estimated GDPP growth in South America and several African states 6.22 %.

4. 2 GINI Index model

Variable	Parameter estimate	SE	p-value	
$const.$	39.309	3.458	3.46×10^{-17}	***
$SAVINGS$	-0.1733	0.089	0.0567	*
$\left(\frac{R\&D}{GDP}\right)$	-3.524	1.665	0.0381	**
$GDPP\ growth$	1.865	0.529	0.0008	***
D_1	-10.062	2.472	0.0001	***
D_2	-9.968	2.456	0.0001	***
D_4	-16.815	3.248	6.42×10^{-6}	***
n	73			
R^2	0.71			

Table 3 Estimated OLS model of income inequality measured by Gini index, with Gross domestic savings, Research and development expenditures, GDPP growth and dummies for characteristic groups of states as the explaining variables

The higher Gross domestic savings tend to lower the income inequality as do the Research and Development expenditures do. *R&D* expenditures are a proxy variable for the human capital and technology development: the high *R&D* rates are typical for Israel and Scandinavian countries. The transitional economies have a lower Gini index in comparison to South America and highly developed countries (the difference being 10 percentage points). The same effect is characteristic for the 2nd cluster economies. If the country is a member of the 4th group of countries (e.g.: India, Russia), the Gini index is the lowest. That is completely in accordance with our assumption, that in the fast developing countries the high GDPP growth rate and high income inequality are closely connected.

	1st cluster	2nd cluster	3th cluster	4. th cluster	5th cluster
$\overline{SAVINGS_\iota}$	21.299	16.206	23.791	15.978	13.127
$\overline{\frac{R\&D}{GDP_\iota}}$	1.2151	0.315	2.0925	0.349	0.216
$\overline{net\_growth_\iota}$	5.6854	6.786	3.168	10.119	6.152
$\overline{\mathrm{GINI_i}}$	31.9	36.8	33.78	37.3	47.7

Table 4 Estimation of Gini index for average values of explaining variables in different clusters

5. Conclusion

In our contribution, we have found a positive relation between income inequality measured by the Gini index and economic growth measured by the Gross domestic private product growth. Income inequality impact on the growth and vice versa is most significant among the different groups of states, which are similar in terms of Gross domestic product per capita, income inequality and growth rate. The countries of Latin America are those with the most pronounced income inequality differences but also with the most rapid growth. On the other hand, developed countries have modest inequality as well as modest growth. Those findings are completely in accordance with Kuznets´s [5] hypothesis.

As we have proven, economic growth (as the change in the status quo) might exist just in those societies, where the *space for growth* exists. Government interventions in form of government expenditures and other ways of narrowing the income gap between different groups of citizens may distort the natural potential for economic growth.

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Periodic Timetable of Routes with Closed Scheduling Vehicles and Time Coordination in One Point Optimization Model

Filip Lorenz[1], Dušan Teichmann[2], Michal Dorda[3]

Abstract. The aim of this paper is the time coordination of regular bus transport service connecting the city of Ostrava with the Leoš Janáček Airport in Mošnov. The subject of the proposal will be based on a mathematical model that would enable to build up a timetable. The introduction of the paper is dedicated to the importance of the problem, possible use of different types of optimization methods and analysis of available sources of input data. The main part of the paper is focused on the design of an original model for the design of timetable, which allows the direct application of the results of work in practice. Computational experiments are performed in optimization software Xpress – IVE in conditions of upcoming summer season 2013. In the conclusion, the attention is given to evaluation of achieved results.

Keywords: time coordination of connections, linear programming, optimization

JEL Classification: C 61
AMS Classification: 90C05

1 Motivation for solving the task and current state of knowledge

The motivation to solve the problem is to increase the fluency of the transportation process of passenger transport to and from the airport Ostrava by special bus route. Traffic on the route is subject to a special mode - it is required to provide a regular headway between lines in both directions which is equal to 60 min. All lines of the route are served by 2 vehicles. The problem lies in the fact that the current timetable has no dependence on the times of departures and arrivals of individual flights from / at the airport. Passengers who use bus lines are often forced to unnecessarily wait before their departure or after their arrival. Therefore, the bus route is not too attractive to passengers and is little used. The aim is to propose a mathematical model to create a timetable for the bus route connecting the airport without above mentioned delays of passengers. It is expected that properly designed timetable may increase interest in public transport to the airport.

Our problem belongs to the group of co called coordination tasks. The basic problem of time coordination of lines was formulated by authors of the book [1] for the first time. On the basis of their publication the staff of Institute of Transport at VŠB-TU Ostrava has begun with research of time coordination. In the past, there were solved some works relating the problem of time coordination in isolated nodes of transport network [2]. There were also executed some experiments in which public transport was coordinated in several nodes concurrently [3]. From other works, which were oriented on solving this and similar topics, we can mention, for example, coordination tasks for route sections [4] - [7]. Works, which solves coordination in transport nodes, use mathematical models which were assembled by prof. RNDr. Jaroslav Janáček, CSc. from the University of Žilina in the past. The model from 2007 (this model has not been published) was also the inspiration for the solution to our problem. However, the original model has to be modified because the original model did not work with regular headway between the lines and with so called closed operating mode on the route. By closed operating mode on the route we mean that all the lines of the route are served by the same set of vehicles. The consequence of it is that each time shift of a line causes the same time shift of all other lines of the route. Original model worked only with passenger transfers in one direction; our model must be able to work with transfers in both directions.

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2 Formulation of the model

Let be a transfer node (the airport Ostrava) given, for which we would like to coordinate arrivals and departures of buses and arrival and departure times of charter flights. Let us consider that following sets are given: the set *I* of bus arrival times at the airport, the set *J* of flight departure times, the set *K* of time slots for departures of buses, and the set *L* of flight arrival times. For each arriving bus line $i \epsilon I$ it is known the earliest possible time t_{1i} of its arrival at the airport and for each departing bus line $k \epsilon K$ the earliest possible time t_{3k} of its departure out of the airport. It is also known the time interval *N* within which it is possible to move with the arrivals and departures of the buses on the route. For each time point, at which a flight departure $j \epsilon J$ occurs, it is known its time position t_{2j} and the number of flight departures B_j for the given period. For each time point of flight arrival $l \epsilon L$ it is also known its time position t_{4l} and the number of flight arrivals A_l for the given period. Further, there are known minimal periods T_{dep} in which it is desirable that the bus lines should arrive before flight departure (these data we consider to be constant for all flight departures) and minimum time T_{arr} that must elapse after the flight arrival to enable easy passengers transfer. It is required that the bus lines should arrive and depart with the regular headway and the time period between the time of bus arrival at the airport and its departure back to Ostrava has to be equal to 20 minutes. The maximal allowable time shift *x* of individual bus lines can be 59 minutes (it is given by defined regular headway between two adjacent lines in one direction).

Now let us introduce variables of the model. The variable *x* is used to model the time shift of the whole system of the bus lines. For each flight departure time $j \epsilon J$ the variable h_j is defined.

This variable will model the time loss of travellers given by the interval between the moment of the bus line arrival before the flight departure $j \epsilon J$ and the beginning of the appropriate time T_{dep} necessary for check-in. For each flight arrival time $l \epsilon L$ the variable q_l is established. This variable will model the time loss which is given by the interval between the time T_{arr} of check-in termination after flight arrival $l \epsilon L$ and the time of the bus line departure. For each pair - the arriving bus line $i \epsilon I$ and the flight departure at the time position $j \epsilon J$ - we introduce a bivalent variable z_{ij}. This variable models the existence of transfer links between the bus line $i \epsilon I$ and the flight departure at the time position $j \epsilon J$. In the case that the passenger transfer between the bus line and the flight will be realized than the variable z_{ij} takes the value equal to 1, in the case that there is no passenger transfer than the variable takes the value equal to 0. Analogously, for each pair - the departing bus line $k \epsilon K$ and the flight arrival at the time position $l \epsilon L$ - a bivalent variable m_{kl} is also established.

This variable models the existence of transfer links between the bus line $k \epsilon K$ and the flight arriving at the time position $l \epsilon L$. If the passengers transfer from the arriving flight to the bus line than the variable m_{kl} is equal to 1, if there is no transfer than the variable m_{kl} takes the value 0.

The mathematical model will be as follows:

$$\min f(x, z, m, h, q) = \sum_{j \epsilon J} B_j \, h_j + \sum_{l \epsilon L} A_l q_l \tag{1}$$

subject to:

$$t_{2j} - (t_{1i} + x) - T_{dep} \geq T(z_{ij} - 1) \qquad for\ i \epsilon I, j \epsilon J \tag{2}$$

$$t_{2j} - (t_{1i} + x) - T_{dep} \leq h_j + T(1 - z_{ij}) \qquad for\ i \epsilon I, j \epsilon J \tag{3}$$

$$(t_{3k} + x) - t_{4l} - T_{arr} \geq T(m_{kl} - 1) \qquad for\ k \epsilon K, l \epsilon L \tag{4}$$

$$(t_{3k} + x) - t_{4l} - T_{arr} \leq q_l + T(1 - m_{kl}) \qquad for\ k \epsilon K, l \epsilon L \tag{5}$$

$$\sum_{i \epsilon I} z_{ij} = 1 \qquad for\ j \epsilon J \tag{6}$$

$$\sum_{k \epsilon K} m_{kl} = 1 \qquad for\ l \epsilon L \tag{7}$$

$$x \leq N \tag{8}$$

$z_{ij} \in \{0,1\}$ $for\ i \in I, j \in J$ (9)

$m_{kl} \in \{0,1\}$ $for\ k \in K, l \in L$ (10)

$x \geq 0$ (11)

$h_j \geq 0$ $for\ j \in J$ (12)

$q_l \geq 0$ $for\ l \in L$ (13)

Function (1) represents the optimization criterion which is the total time loss of passengers who use the airport shuttle service. The group of constraints (2) ensures that if it is not possible to transfer due to the time reasons, transfer link will not be created. The group of constraints (3) ensures linking between the constraints and the first term of the objective function. Group of constraints (4) has the same function as the group of constraints (2) for the relation flight arrival - bus departure. The group of constraints (5) has the same function as the group of constraints (3); however, the link is ensured to the second term of the objective function. The group of constraints (6) ensures that each flight departure is assigned to exactly one arriving bus.

Constraints (7) ensure that each flight arrival is assigned to exactly one departing bus. Constraint (8) ensures that the time shift of the arriving and departing bus links takes the value which will not exceed the permitted limit. Constraints (9), (10), (11), (12) and (13) are obligatory and define domains of definition for individual variables.

3 Computational experiments

In this section we will present the results of executed computational experiments. Computational experiments are carried out to verify the correctness of the proposed mathematical model and its applicability to real-life situations. Bus arrival times at their earliest possible time positions belong to first group of the input data that are necessary for experiments. The route is served by 13 lines in each direction per day. The first bus line arrives at the airport Ostrava at 8:25 (this time is the earliest possible arrival time), followed by other lines arriving with the 60 minutes headway. The last bus line arrives at 20:25. The first bus line departs from the airport at 8:45, followed by other departing also with 60 minutes headway, the earliest possible departure of the last bus line from the airport is at 20:45. Other input data relating to charter flights are summarized in Table 1 (the flight arrivals) and Table 2 (the flight departures).

Time position of flight arrivals and departures are expressed in minutes that elapse from the selected point in time 0 (7:00).

Table 1

Arrival number l	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Time position t_{4l}	135	150	170	180	190	230	235	240	255	270	315	320	345	420	455
Count A_l	18	1	1	15	3	22	9	22	11	68	11	11	16	17	14
Arrival number l	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Time position t_{4l}	465	495	505	515	530	570	595	600	630	650	660	670	705	760	770
Count A_l	14	11	29	5	13	15	10	16	11	24	11	15	21	11	11

Table 2

Departure number j	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Time position t_{2j}	195	230	250	275	280	310	320	325	330	355	380	385	480	510
Count B_j	18	15	3	9	22	10	39	15	11	11	11	16	17	14
Departure number j	15	16	17	18	19	20	21	22	23	24	25	26	27	28
Time position t_{2j}	515	555	565	575	600	660	680	700	710	720	740	750	820	850
Count B_j	15	32	9	5	15	10	11	11	49	34	12	9	11	12

Optimization experiments were carried out using the optimization software Xpress-IVE [8]. The measure of the solution quality is the difference between the value of the objective function for the solution obtained by solving the mathematical model and the value of the same criterion calculated for the current state. The values of the optimization criterion for the current state and for the state obtained by the model are shown in Table 3.

Table 3

The total time loss for the current state [min]	23 065
The total time loss gained as solution of the proposed mathematical model [min]	21 635
Difference [min]	1 430

From the comparison of results it is clear that the total time loss of passengers at the Ostrava airport was decreased by 1 430 minutes.

The result of the optimization calculation is also the time shift x of the whole system of bus lines; it takes the value 25 minutes. This value is added to the earliest possible time positions of all lines. After shifting the first bus line arrives at the airport in Ostrava at 8:50 and other lines will arrive with the regular headway equal to 60 minutes. The last line arrives at the airport in Ostrava at 20:50. The first bus line departs from the airport at 9:10 and others depart with the regular headway equal to 60 minutes. The last line departs from the airport in Ostrava at 21:10. Economic savings of the proposal based on the mathematical modelling result from the savings of time for passengers who use the bus service.

4 Conclusion

The article deals with the time coordination problem. The goal is to coordinate the lines of the bus route providing transportation to the airport with departures and arrivals of charter airlines. The article contains a linear mathematical model which has the optimization criterion based on the total time loss of passengers who use the bus line. The proposed mathematical model was tested in the conditions of the Ostrava Airport. Using the mathematical model it was achieved reduction equal to 1 430 minutes in time losses of passengers using the bus shuttle service. The mathematical model can be also applicable for other routes connecting smaller regional airports, for example, in Brno, Kosice, etc.

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Numerical investigation of 3-D nonlinear financial model

Ladislav Lukáš [1]

Abstract. The paper concerns with numerical investigation and some generalization of 3-D nonlinear financial model, which have been reported in the literature recently. The model consists of three simultaneous nonlinear ordinary differential equations consisting of three state variables: the interest rate, the investment demand, and the price index, in particular. We introduce time dependent constitutive coefficient representing rate of investment instead of time invariant one. Further, we admit more complex construction of right hand side terms including time delayed response of state variables. Numerical investigation of model with different forms of investment rate function is reported in detail. We analyze also an influence of different initial functions upon generated trajectories. All computations for solving initial value problems are performed by sw Mathematica.

Keywords: financial model, nonlinear financial model, initial value function, initial value problem, numerical solution, state variables.

JEL Classification: C63, G19
AMS Classification: 37N40, 91G80

1 Introduction

No doubt, dynamic models in finance and their numerical solutions have attracted scientists and financial engineers since computers appeared. At present, we may classify two main streams of research within those challenging topics. The first one is based upon theory of stochastic differential equations, and we refer to [5] just for a brief touch. The second one concerns, it is mainly focused on theory of deterministic nonlinear dynamic systems, which can produce chaotic behavior, strange attractors, etc., and we refer to [1] and [7] for further reading. In 2001, an interesting nonlinear dynamic financial model was published in [3] and [4]. The model was further investigated and developed for delayed feedback in [2], as well. We also refer to [6] and [8] for some general aspects of formulation and handling delays in dynamic financial systems.

The paper concerns with numerical analysis of two effects – i) introduction of time dependent constitutive coefficient representing rate of investment instead of time invariant one, ii) construction of right hand side terms including time delayed response of state variables thus enabling an analysis of different initial functions influence upon generated trajectories. The model contains three state variables denoted x, y, and z, which represent interest rate, investment demand and price exponent, in particular. The first derivatives $\mathrm{d}x/\mathrm{d}t$, $\mathrm{d}y/\mathrm{d}t$, $\mathrm{d}z/\mathrm{d}t$ with respect to time represent the changing rates of the state variables as usual, which govern evolution of the system. Following [3] and [4], a core of the model consists of three phenomenological equations giving components and their influences upon these rates.

The changing rate of interest rate x, i.e. $\mathrm{d}x/\mathrm{d}t$, is composed from two factors coupled in linear form (1), a) the surplus between investment and saving, and b) the structural adjustment induced by goods prices.

$$\mathrm{d}x/\mathrm{d}t = f_1(y - s)\, x + f_2\, z \tag{1}$$

where s is a parameter representing an amount of saving being constant in a certain period of time by assumption, and f_i , i=1,2 are two constitutive constants.

The changing rate of investment demand y, i.e. $\mathrm{d}y/\mathrm{d}t$, is given by equation (2) and is composed from three factors including quadratic influence of x, which maintains proper nonlinearity of the model.

$$\mathrm{d}y/\mathrm{d}t = f_3(e - \alpha y - \beta x^2) \tag{2}$$

where e stands for benefit rate of investment being assumed constant in a certain period of time, too, and f_3, α, β are constitutive constants, where α expresses opposite proportional influence of y upon its rate, and β moderates an influence of x^2 upon that rate, thus expressing a higher order investment propensity caused by interest rate thereon.

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Finally, the changing rate of price exponent z, i.e. $\mathrm{d}z/\mathrm{d}t$, is given by equation (3), and it takes a linear form again. It contains two constitutive constants f_i , i=4,5, so that f_4 passes an influence of market contradiction between supply and demand being expressed by variable z, and f_5 is to express the inflation rate influence.

$$\mathrm{d}z/\mathrm{d}t = -f_4\, z - f_5\, x \tag{3}$$

2 Simplified model and model with delayed feedback

Summarizing the model, we may inspect three state variables x, y, and z, and nine independent constitutive parameters f_i , i=1,.., 5 and s, e, α, β to be adjusted. An additional role of f_i is to maintain correct dimensioning. However following [3], we use a simplified model given by (4) with three non-negative constitutive parameters a, b, and c, only, which gauge saving amount, investment cost, and elasticity of demands of commercials transformed.

$$\begin{aligned} \mathrm{d}x/\mathrm{d}t &= z + (y-a)x, \quad x_0 = x(0) \\ \mathrm{d}y/\mathrm{d}t &= 1 - by - x^2, \quad y_0 = y(0) \\ \mathrm{d}z/\mathrm{d}t &= -x - cz, \quad z_0 = z(0) \end{aligned} \tag{4}$$

The model (4) can be simply extended to the model with delayed feedback, which is given by equations (5).

$$\begin{aligned} \mathrm{d}x/\mathrm{d}t &= z + (y-a)x + k_1(x - x(t-\tau_1)),\ t \geq 0, \quad x(t) = u_0(t),\ t \in [-\tau_1, 0] \\ \mathrm{d}y/\mathrm{d}t &= 1 - by - x^2 + k_2(y - y(t-\tau_2)),\ t \geq 0, \quad y(t) = v_0(t),\ t \in [-\tau_2, 0] \\ \mathrm{d}z/\mathrm{d}t &= -x - cz + k_3(z - z(t-\tau_3))\ t \geq 0, \quad z(t) = w_0(t),\ t \in [-\tau_3, 0] \end{aligned} \tag{5}$$

It contains three feedback intensities $k_j \in \mathrm{R}$, in general, delay times $\tau_j > 0$, j=1, 2, 3, and instead of initial values x_0, y_0, z_0 as in (4), it contains three initial histories $u_0(t)$, $v_0(t)$, $w_0(t)$. For $\tau_j = 0$ the model (5) turns to model (4).

3 Numerical results – case studies

All calculations were performed by Mathematica notebooks we developed for numerical investigation of both models (4) and (5) using extremely powerful command `NDSolve`. For the paper, we select just two topics to present – *i*) numerical investigation of model (4) with function $b(t)$ just replacing the constant b, *ii*) analysis of influence of different initial functions on state-space trajectories in model (5).

Figure 1 show trajectories of model (4) with different $t_{\max}$, $t_{\max}$ = 20 on the left, and 100 on the right, where we can identify also five loops run. Later, we compare the left image with other ones generated with model (5) with different initial history functions.

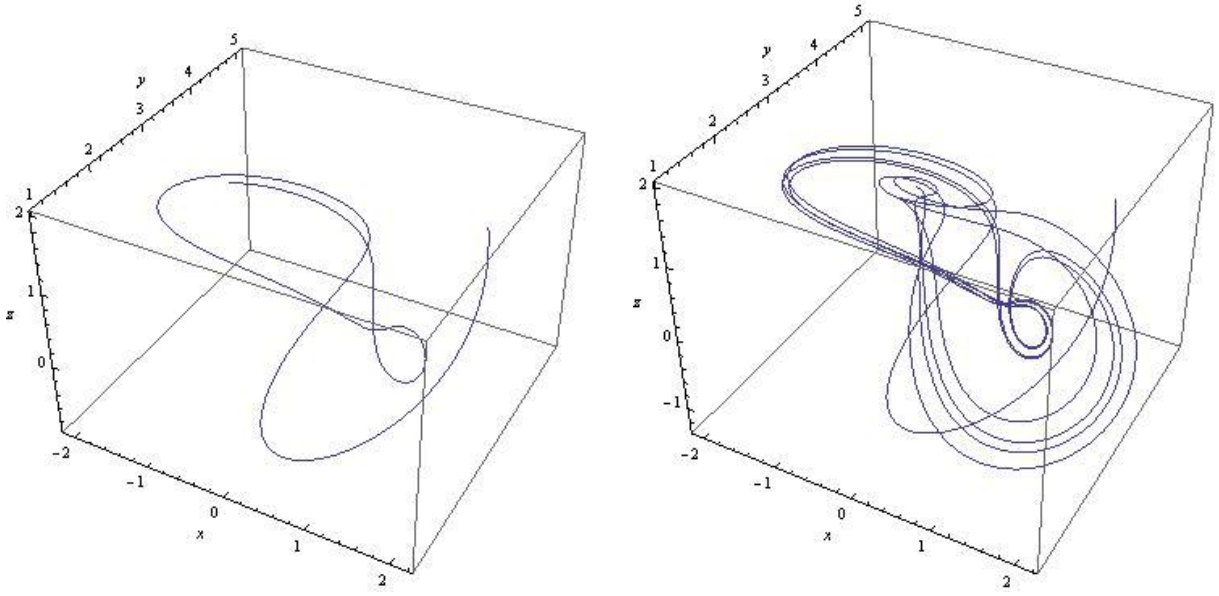
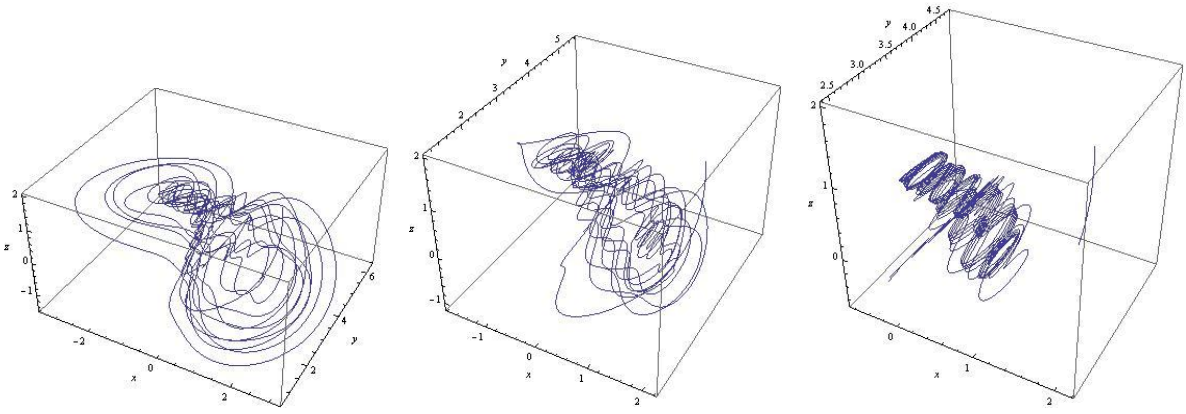


Figure 1 Model (4), a=3, b=0.1, c=1, x_0=2, y_0=3, z_0=2



β_1=0.1, β_2 =5, β_3=3, β_0=0 β_1=0.25, β_2 =10, β_3=3, β_0=0.25 β_1=0.28, β_2 =15, β_3=0, β_0=0.45

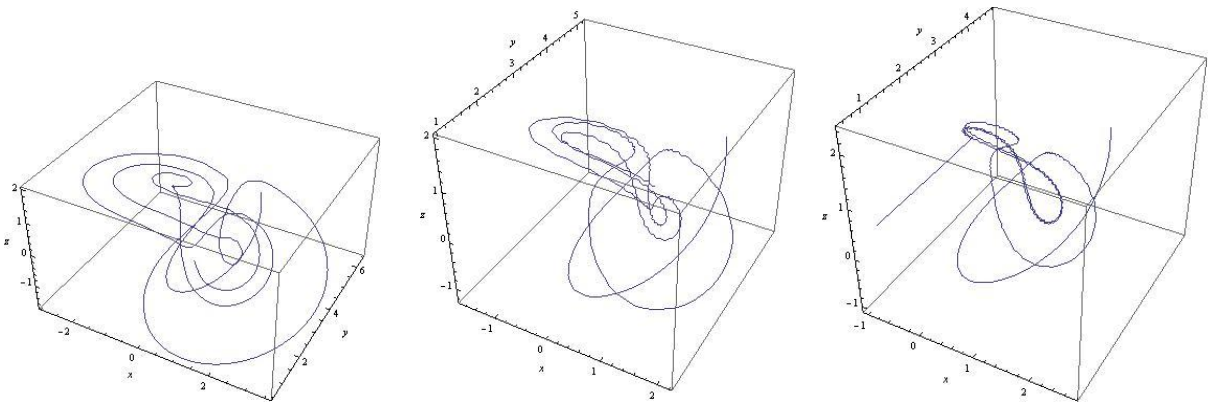
Figure 2 Model (4) with $b_1(t)$, a=3, c=1, x_0=2, y_0=3, z_0=2, t_{max} = 100

Figure 2 show trajectories of modified model (4), when parameter b is replaced by function $b_1(t)$ given by (6), which demonstrates the case that this parameter is not constant within time any more but undergoes scaled periodic forced changes elevated by β_0. The influence of increasing frequency and different elevation is evident.

$$b_1(t) = \beta_1 (\sin(\beta_2 t + \beta_3) + \beta_0) \tag{6}$$

Next, Figure 3 show similar numerical experiments but with shorter time t_{max} and another type of scaled periodic changes given by function (7), which takes non-negative values only.

$$b_2(t) = \beta_1 |\sin(\beta_2 t + \beta_3)| \tag{7}$$



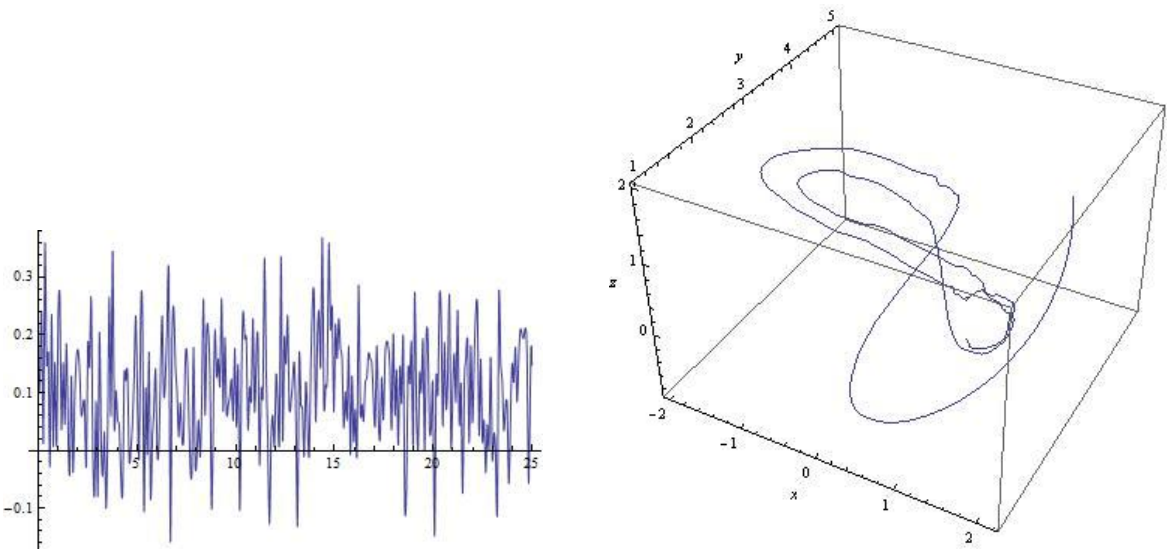
β_1=0.1, β_2 =5, β_3=3, β_0=0 β_1=0.25, β_2 =10, β_3=3, β_0=0.25 β_1=0.28, β_2 =15, β_3=0, β_0=0.45

Figure 3 Model (4) with $b_2(t)$, a=3, c=1, x_0=2, y_0=3, z_0=2, t_{max} = 50

We see different behavior of trajectories, which seems to be nothing else but due to non-negativity of changing values of the parameter b, which is replaced by the function $b_2(t)$, now. Inspecting the right most image of the Figure 3 thoroughly, we may discover rather fine vibrating modes being wound along the middle part of that trajectory, which is quite interesting at all. However, it is also interesting that `NDSolve` has reported exceeding of 10000 steps in numerical integration procedure allowed by default during calculation the given model. Increasing the value of scaling parameter β_1 gets `NDSolve` into serious numerical stability problems, at least when using default options only.

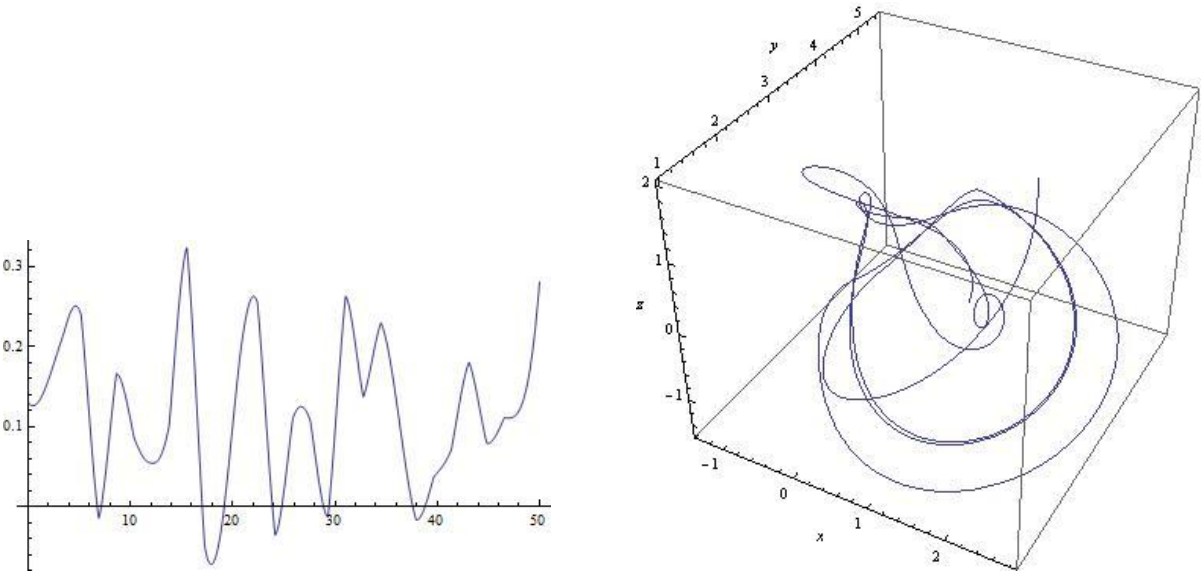
Finally, within this first investigation topic, we have selected still another type of replacement of the parameter b. Instead of deterministic periodic function, we used numerical simulated pseudorandom process built with cubic spline interpolation. Sure, we tried to submit a stochastic process into `NDSolve` directly, but it failed. Hence, we generate a set of n random values B_n sampled from normal distribution with given mean value μ and variance σ^2, first. Then we distribute them uniformly along the given time period [0, t_{max}], and using the Mathematica command `Interpolation` we create a cubic spline interpolant of the generated data. Such function is already acceptable by `NDSolve`. The Figures 4 and 5 show results of numerical experiments with two generated functions of such kind, $b_3(t)$ and $b_4(t)$, respectively. Comparing both results, we might conclude that

much noisy-looked function $b_3(t)$ caused just a low-sized perturbation of the calculated trajectory, whereas $b_4(t)$ produced smooth one.



$b_3(t)$, $t \in [0, t_{max}=25]$, $B_n \in N[\mu=0.1, \sigma^2=0.1]$, $n=300$

Figure 4 Model (4) with $b_3(t)$, $a=3$, $c=1$, $x_0=2$, $y_0=3$, $z_0=2$, $t_{max} = 25$



$b_4(t)$, $t \in [0, t_{max}=50]$, $B_n \in N[\mu=0.1, \sigma^2=0.1]$, $n=30$

Figure 5 Model (4) with $b_4(t)$, $a=3$, $c=1$, $x_0=2$, $y_0=3$, $z_0=2$, $t_{max} = 50$

The second topic we have investigated numerically concerns a short insight upon an influence of initial histories on launching part of trajectories generated by model (5), for $t \in [0,t_p]$, $t_p>0$. Our results are depicted on Figures 6, 7 and 8, respectively. In order to simplify the problem and to enable a comparison with already generated trajectory by model (4), as well, we assume initial histories $u_0(t)$, $v_0(t)$, $w_0(t)$ to take the following form (8).

$$u_0(t) = x_0\eta(t), \quad v_0(t) = y_0\eta(t), \quad w_0(t) = z_0\eta(t), \quad t \in [-\tau,0], \quad \tau_i = \tau, \; i=1,2,3, \; \tau>0, \quad \eta(0)=1 \tag{8}$$

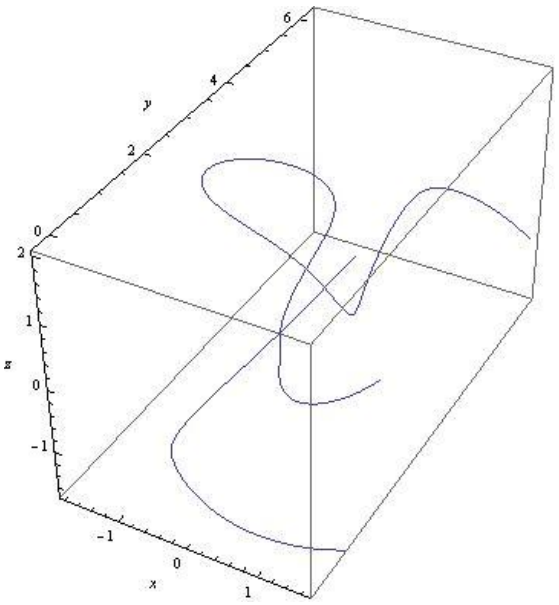
In that case, we select the function $\eta(t)$ only. All computations were performed by `NDSolve` again, which underlines its tremendous versatility. We make choice of five relatively simple classes of functions, which are listed below in formulas (9).

$$\begin{gathered} \eta_1(t)=\exp(\gamma_1 t), \quad \eta_2(t)=\cos(\gamma_2 t), \quad \eta_3(t)=\exp(\gamma_{3,1} t)\cos(\gamma_{3,2} t), \\ \eta_4(t)=1+\gamma_4 t, \quad \eta_5(t)=1+\gamma_{5,1} t + \gamma_{5,2} t^2, \quad t \in [\tau,0], \quad \gamma_i, \gamma_{j,1}, \gamma_{j,2} \in R, \; i=1,2,4, \; j=3,5 \end{gathered} \tag{9}$$

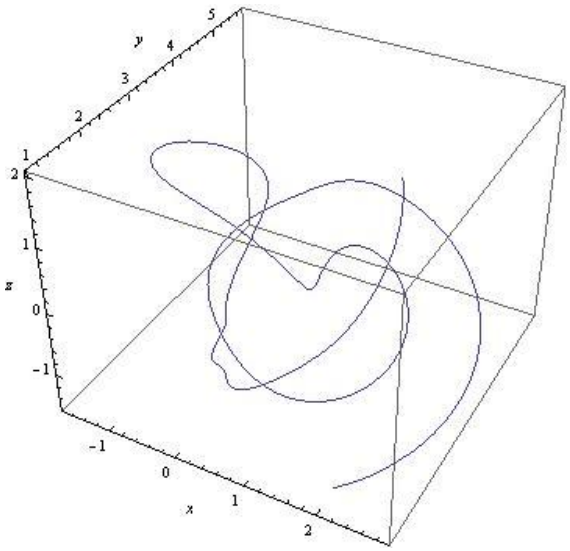
Hence, the initial histories of model (5) are given simply within `NDSolve` by following expressions (10).

$$x[t/;t\leq 0]==x0\eta[t],\ y[t/;t\leq 0]==y0\eta[t],\ z[t/;t\leq 0]==z0\eta[t] \tag{9}$$

where `x0, y0,` and `z0` stand for initial values x_0, y_0, z_0,and function $\eta[t]$ takes one of $\eta_k(t)$, k=1,.., 5. For all computations we set $t_p = 10\tau$, thus defining the period $[0,t_p]$ to be ten times longer than $[-\tau,0]$, the period of initial histories given, with τ =2. The feedback intensities are k_1=0.1, k_2=0.2, and k_3=0.1. Initial values x_0, y_0, and z_0 are set traditionally as follows x_0=2, y_0=3, z_0=2. In a similar way, we take values of constitutive parameters a=3, b=0.1, and c=1, which maintains the model (5) to be compatible with (4) one.

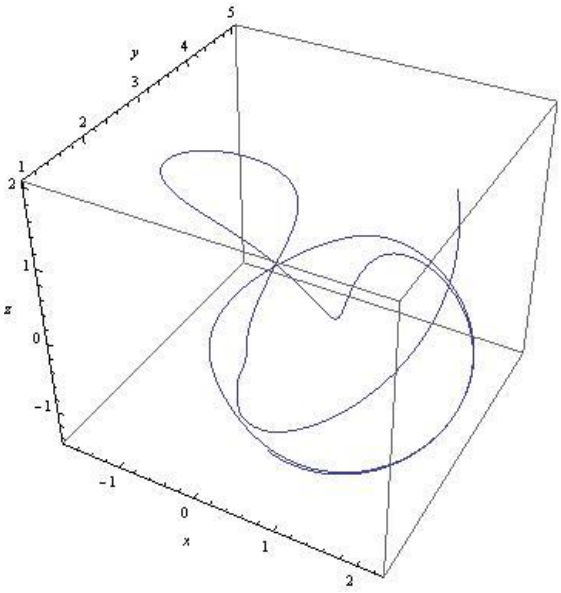


$\eta(t) = \eta_1(t)$, γ_1=5

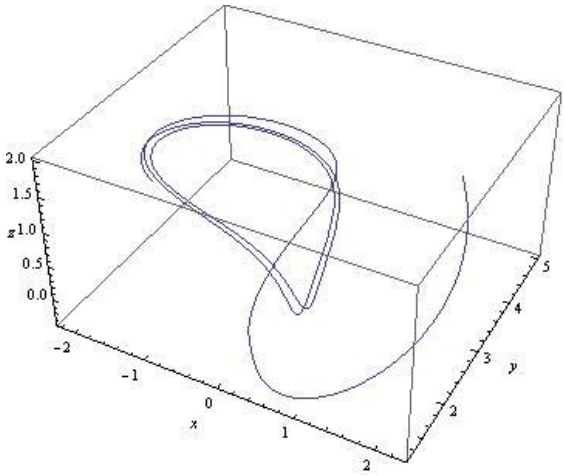


$\eta(t) = \eta_2(t)$, γ_2=10

Figure 6 Model (5)



$\eta(t) = \eta_3(t)$, $\gamma_{3,1}$=2.5, $\gamma_{3,2}$=15

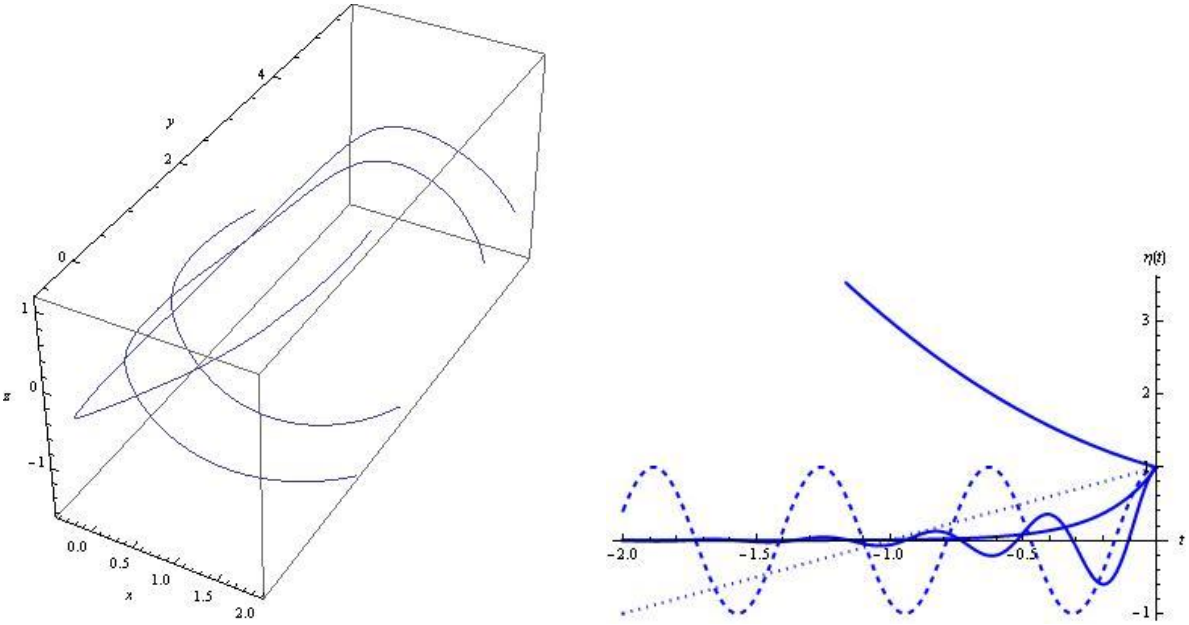


$\eta(t) = \eta_4(t)$, γ_4=1

Figure 7 Model (5)

Inspecting these results, we may just confirm expected dependence of starting parts of trajectories upon initial histories. Any of five trajectory images of model (5) can be directly compared with the trajectory generated by model (4), already presented on the left side of Figure 1, with similar constitutive and initial conditions to see the differences. Finally, we have to point out that presented numerical investigations form just a small fraction of numerical experiments we have already conducted with this dynamic nonlinear financial model in its basic simplified form, as well as with proposed generalizations. The full parametric space of the basic model (4) is R^3xR^3

for three constitutive parameters and three initial conditions. The complexity of model (5) increases dramatically, because of added feedbacks and initial history functions instead of just three numbers x_0, y_0, z_0.



$\eta(t) = \eta_5(t)$, $\gamma_{5,1} = -1$, $\gamma_{5,2} = 1$ functions $\eta_k(t)$, k=1,.., 5

Figure 8 Model (5)

4 Conclusions

We hope the large possibilities of numerical solution of nonlinear dynamic financial system with chaotic behavior using Mathematica command NDSolve were sketched. Proposed and discussed replacement of constitutive parameter b to be time dependent taking form of simple periodic function, pseudorandom process, etc., instead of constant one within the basic 3-D model (4) can be simply extended to other two parameters a and c, too.

Further research on the topic seems to be very challenging and will be focused both on complex program of numerical investigation of large parametric space and advanced theoretical development including analysis of bifurcation, as well as empirical verification of the models.

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Fuzzy Methods in Land Use Modeling for Archaeology

Tomáš Machálek[1], Richard Cimler[2],
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Abstract. The general objective of our research is to develop the complex agent-based simulation of ancient Celtic society population, food production and economics. To achieve this we proposed the agent-based agricultural model of land suitability for miscellaneous farming systems. The behaviour of agents is controlled by a fuzzy rule-based system. The system encodes farmers' agricultural knowledge as regards the decision where to apply specific farming method. Corresponding linguistic variables describe (1) the distance of individual land patches from the household, (2) the slope of the land, (3) the yield of individual land patches and (4) total harvest as a percentage of inhabitants' annual nutritional requirement. To prevent high-frequency spatial changes in land suitability caused by time-dependent stochastic nature of the crop yields, our model performs result smoothing using spatial signal filtering. The model is tested on the GIS data.

Keywords: Agent-based model, fuzzy sets, land-use model, social simulation.

JEL Classification: J10
AMS Classification: 68U20

1 Introduction

Due to the fragmented nature of data available we have only limited picture of how societies may have functioned in past. It is obvious that in order to investigate more complex issues sophisticated methods and tools are needed. Recent studies show that on top of a comprehensive collection and analyses of archaeological and environmental data, the building of explanatory models is a valid way of approaching the complexity of past societies (e.g. Kohler - van der Leeuw (eds.) 2007).

Models are heuristic tools, which not only can help to understand human-landscape interactions and processes of change, but also they are powerful for suggesting directions for future research (Wainwright 2008), by challenging traditional ideas and exploring new issues (Wright 2007). At the same time models do not attempt to become exact reflections of the past reality, they just aim to provide more precise research questions, select more appropriate attributes and factors to be further examined on an accurate spatial and temporal scale. This framework can be used to investigate socio-ecological interactions over a broad range of social, spatial, and temporal scales, allowing for a wide range of past archaeological issues to be concurrently addressed. With growing potential of computer power over last decades, agent-based models (ABM) provide an invaluable way how to explore and test various theoretical hypotheses especially in cases when detailed data is missing from the archaeological record. Ancient Celtic society is such a case.

The general objective of our research is to develop the complex agent-based simulation of ancient Celtic society population, food production and economics. The relationship between population growth and the development of the society depends on the availability of basic resources from a given unit of land and the ability of the population to exploit them. Relevant concepts are:

- *settlement ecosystem* (the system of land use based on the relationship between the landscape and human decision how to exploit it,
- *population pressure* (the point in population growth when it starts to press upon the resource base,
- *carrying capacity* (the limit that when is crossed the population must either diminish or innovate their exploitation strategies.

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Models of ancient agricultural strategies require defining resource levels of the ecosystem (i.e. productivity of the land), productivity potential of the population exploiting these resources (i.e. labour input, technology, and task management). Then we can test whether and under what conditions can certain resource become limiting and what outcomes can be driven from that. The increasing population trend is reflected in models of food production (i.e. area of fields and ability to cultivate them). The intention of our model is to search for the limit of the carrying capacity of the hinterland for the gradually increasing population. By this we may be able to explain the reason for population decline in case it was caused by reaching the limits of production. The objectives to ascertain are:

- size of the community and proportion of people engaged in agricultural work,
- organization of the working process,
- demands concerning land and labour,
- carrying capacity threshold,
- scale of sustainable agricultural production,
- stability of production (number of stress situations).

We are dealing with the subsistence strategies and the economic background of late Iron Age oppida, the fortified agglomerations occupied in the last two centuries BC. Oppida settlements represent complex systems (societies) with multiple functions. It is believed that they appeared as a part of economically advanced environment accompanied also by the distinct intensification of settlement pattern. This was reflected also by the rapidly growing population at the oppida which presumably doubled during the first two-three generations. However, the oppida were mostly built in landscapes considered to have been marginal in regard to what appeared to be the common settlement strategy. This phenomenon was probably influenced by a combination of factors (economic, security, exploitation ...) which are yet to be discovered. Due to the specific location and also due to the widespread evidence of late La Tène open lowland settlements (which were believed to supply the oppida with the necessary foodstuff) the agricultural potential of the oppida has usually been challenged. Also the probably quite rapid depopulation and collapse of these agglomerations during the second half of 1st century BC was, among all, attributed to the supply crisis (e.g. Salač 2000, 2009). However, the archaeological record reveals clear signs of at least some engagement in the agricultural production – by the evidence of crops cultivation, livestock breeding or traditional agricultural household units. Our model can provide substantial information about the limits of rural economy at a particular location and they may help to determine possible stress situations.

2 Model Description

The spatial structure of the fields cultivated is related to the scale of intensity of the labour input. This scale depends especially on the suitability of soils and the distance from the site. In order to model relevant farmers' decision processes we can create fuzzy expert system helping to provide initial hypotheses in terms of crop field layouts. While typical process of human-driven research in the field of archaeological research is the most often performed using a GIS tools with multi-criteria decision analysis (Eastman 2006), our model provides an alternative approach with possible extensions in terms of household/people interactions (Olševičová et al. 2012, Machálek et al. 2012).

NetLogo, our implementation tool, did not contain any fuzzy-related functionality but it allowed us building a plug-in based on jFuzzyLogic library (Cingolani 2012) which extended the core functionality in the way we needed.

Taking into account existing land suitability evaluation we have split the problem into two levels:

- First level represents evaluation of a land based only on relief's invariant properties (such as distance, slope, etc.). The problem how individual land characteristics together influence discussed suitability has already been addressed especially in human-landscape occupational rules but also in the field of agricultural practices (Jarosław – Hildebrandt-Radke 2009, Reshmidevi et al. 2009).
- Second level introduces dynamic factor (e.g. the harvest from the previous season). Evaluation process from the first level can be understood here as an initial step (time "zero") where inhabitants have no previous experience and also have not yet influenced the environment in any way. But in the next season, their knowledge becomes broader because they can evaluate real results of their previous assumptions on what part of the land is more or less suitable to be farmed with specific (or even any) intensity.

Attempting to encode the knowledge of Iron Age farmers in terms of land suitability and accepting their limited analytical capabilities we see a fuzzy rule-based system containing a set of *if-then* rules as a natural tool to express their decision processes (i.e. subjective, approximate, using terms like "near", "far", "weak", "strong", "fair"; cf. Ross 2010).

Fuzzy rule-based systems build on top of theory of fuzzy sets and fuzzy logic introduced by L. Zadeh in 1965. With classical sets we assign each object either "is-member" or "is-not-member" property, in case of fuzzy sets a membership degree function fits into the interval [0; 1] (see e.g. Babuška 2001, Ross 2010). This leads to different (but still consistent with our understanding of Boolean logic) concept of set operations and related logical operations. Fuzzy rule-based systems provide a way how to encode domain specific knowledge and control behaviour of a system or entity in conformity with this knowledge. The rules have following general form: *IF antecedent proposition THEN consequent proposition*.

Fuzzy propositions are statements like "*x* is *big*", where "*big*" is a linguistic label, defined by a fuzzy set on the universe of discourse of variable (Babuška 2001). Linguistic labels are also referred to as fuzzy constants, fuzzy terms or fuzzy notions.

Linguistic variable is a quintuple (Klir – Yuan 1995):

$$L = (x, A, X, g, m) \tag{1}$$

where x is the base variable (it also represents the name of the linguistic variable), $A = \{A_1; A_2; \ldots, A_n\}$ is the set of linguistic terms, X is the domain (universe of discourse) of x, g is a syntactic rule for generating linguistic terms and m is a semantic rule that assigns to each linguistic term its meaning (a fuzzy set in X).

The model simulates farmers' decision making process about suitability of individual land patches for crop (or animal) husbandry and about correction of predefined percentage of intensive farming according to the difference between actual harvest and annual nutritional requirements of the community. In addition, total harvest can be modified by long term trend function to simulate progressively decreasing or increasing carrying capacity of the area.

A single type of agent is defined – *a household* which represents one or more families living in a settlement (house or a group of houses) with some arable land around. We assume that the hinterland is based on exploring accessible area with centre in the settlement.

Proposed fuzzy inference requires four input linguistic variables:

1. distance of individual land patches from the household,
2. slope of the land,
3. yield of individual land patches,
4. total harvest (as a percentage of inhabitants' annual nutritional requirement).

Concrete form of membership functions related to these variables is the result of discussion with domain expert and also of empirical testing (see Figure 1).

A key parameter influencing both total and patch-level yields is a type of soil. The model works with 5 soil categories: unfarmed soil of the settlement, alluvial soil and three additional qualitative categories for arable land.

Stochastic nature of crop yields required selecting of proper random distribution. While there exist objections against normally distributed crop yields (e.g. Ramirez et al. 2001), modelling yields by normal distribution still cannot be refused in general (e.g. Upadhyay – Smith 2005). Lack of detailed evidence forced us to use normal distribution. The mean of the distribution separates aforesaid yield ranges into equal halves and standard deviation is defined so that the maximum and minimum values are at ±3σ. The estimated distributions N (μ; σ^2) were calculated as (2),(3):

$$\mu = \frac{(Y_{\min} + Y_{\max})}{2} \tag{2}$$

$$\sigma = \frac{(Y_{\max} - Y_{\min})}{6} \tag{3}$$

As an output of the fuzzy inference system, two linguistic variables have been defined:

- suitability – this variable quantifies suitability of a single land patch in terms of its usability for growing crops. Patches with suitability near value of 100 can be understood as very suitable (near the household, flat and with good yield potential). Patches near 0 are considered to be inappropriate (far lying, sloping, low yield potential).

- intensity – although the model operates with user-defined parameter *intensive-farming-percent* which specifies percentage of household's arable land cultivated by intensive farming system, it also provides auto-correction of this value according to the difference between required and actual harvest. The real proportion of intensively cultivated land is calculated as a product of *intensive-farming-percent* and *intensity*. Variable *intensity* is expected to be approximately 1 if the total harvest is about equal to the requested value (parameter *required-annual-yield*).

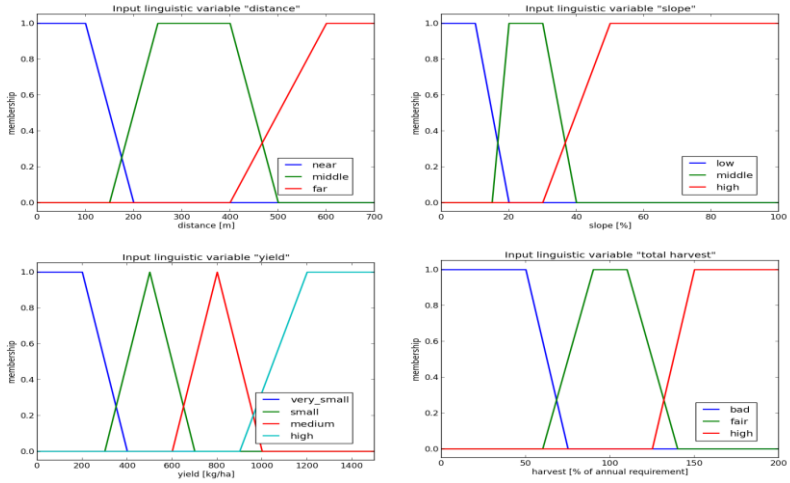

Figure 1 – input variables of the fuzzy inference system

To calculate yield y_p of a specific patch we have defined following function:

$$y_p = \frac{25 \cdot h \cdot y}{10000} \cdot T \cdot r$$

Where $\frac{25}{10000}$ recalculates per-unit yield to model's patch size, h is a yield per hectare, y is normal random variable with properties so that $h \cdot y$ has properties of the distribution defined in (2) and (3). T is a coefficient expressing an influence of a farming type (it has value of 1 for intensive farming and 0.3 for extensive farming) and r is an optional coefficient to apply long-term trends (i.e. climate change). For the fuzzy inference system, 13 rules have been defined as follows:

```
RULE 1: IF yield IS very_small THEN suitability IS low;
RULE 2: IF slope IS high THEN suitability IS low;
RULE 3: IF distance IS near AND (slope IS low OR slope IS middle) AND (yield IS high OR yield IS
medium) THEN suitability IS high;
RULE 4: IF distance IS near AND (slope IS high) AND (yield IS high) THEN suitability IS middle;
RULE 5: IF distance IS middle AND slope IS low AND (yield IS NOT very_small) THEN suitability IS high;
RULE 6: IF distance IS middle AND slope IS middle AND yield IS high THEN suitability IS high;
RULE 7: IF distance IS far AND (slope IS low) AND (yield IS medium OR yield IS high) THEN suitability
IS middle;
RULE 8: IF distance IS far AND (slope IS middle) AND (yield IS NOT very_small) THEN suitability IS low;
RULE 9: IF distance IS near AND slope IS low AND yield IS NOT very_small THEN suitability IS middle;
RULE 10: IF distance IS middle AND slope IS low AND (yield IS small OR yield IS very_small) THEN
suitability IS low;
RULE 11: IF harvest IS fair THEN intensity IS normal;
RULE 12: IF harvest IS bad THEN intensity IS high;
RULE 13: IF harvest IS high THEN intensity IS low;
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Operator AND was defined using *min* function, operator OR was defined using *max* function, for rule activation, *min* function was used. For rule accumulation, bounded sum method (i. e. $\min(1, \mu_A + \mu_B)$) was used. Fuzzy implication in based on Mamdani's inferencing scheme.

3 Results

Figure 2 presents an example of land evaluation. Land area in the model is represented by a grid of discrete patches of the same size. Stochastic properties of the model cause that even several neighbouring patches may sometimes differ significantly in terms of calculated suitability. Such result is hard to interpret directly because we cannot expect that real farmer was mixing crop husbandry practices (including the "no use" one) every few meters. To resolve this problem we have proposed to post-process resulting "suitability map" using linear filtering:

$$g(i, j) = \sum_{k,l} f(i+k, j+l)h(k,l) \qquad (4)$$

Here *f* represents an input signal, *h* is called linear convolution kernel. As a kernel we have been using bilinear kernel:

$$h(k,l) = \frac{1}{16}\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix} \tag{5}$$

Suitability map can be understood as a spatial signal where proper linear convolution kernel serves as a low-pass filter which attenuates higher frequencies from the signal (Szeliski 2010).

While treating the suitability as a continuous variable is convenient in terms of described calculations, it is more complicated to apply such variable directly in practice (i.e. what should farmer do exactly if the suitability is *x*?). Because of that we defined a mapping from continuous suitability variable to an ordinal set of suggested farming types (see Figure 2: dark grey area around the center corresponds to the land of the best suitability for intensive farming, light grey area around the center corresponds to the best suitability for extensive farming or pastures, black area corresponds to the evaluated land of lower suitability).

Figure 3 shows a comparison between discussed fuzzy model and an alternative multi-criteria based solution (which considered only distance and soil type). Given required target yield we can see that the fuzzy model is capable to produce results with comparable accuracy with only exception of low intensity farming which is caused by the fact that the model evolves from defined starting property size and is unable to reach required area in model's running time limit (it evaluates too many patches as unsuitable to be able to reach required yield).

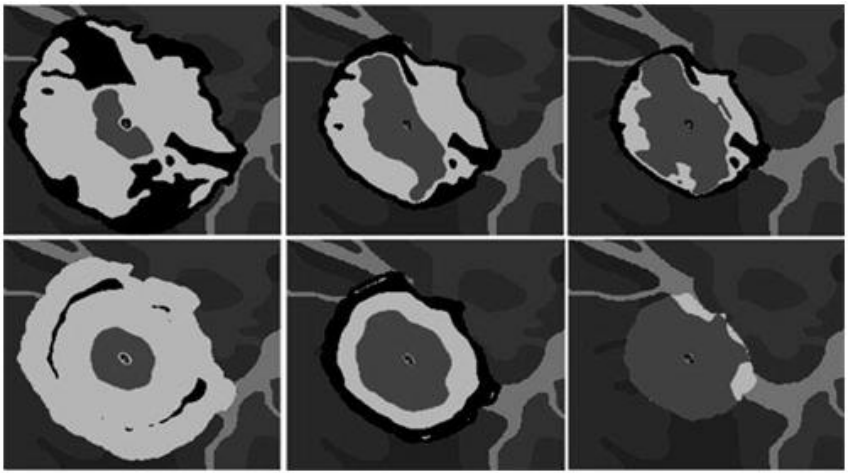

Figure 2 – Fuzzy approach with linear filtering (1st row) and alternative (2nd row) models; parameters: required annual yield = 9000kg, required ratios of intensive farming = 10% (left), 50% (center), 90% (right)

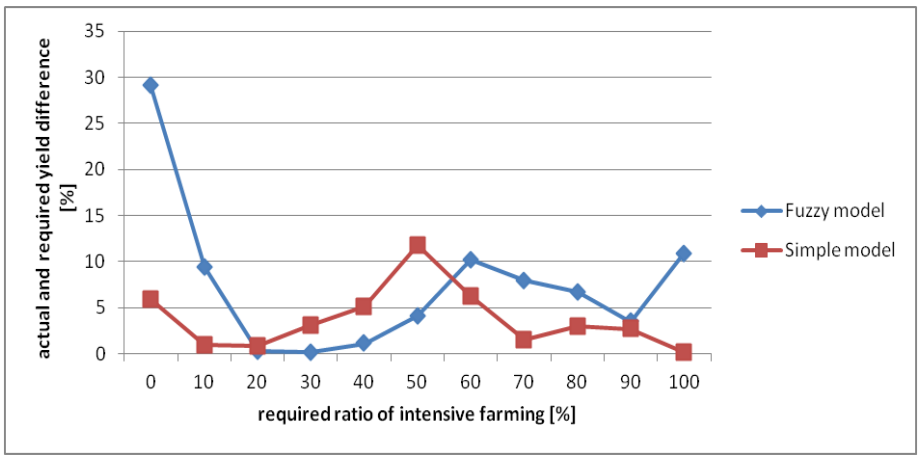

Figure 3 –Fuzzy (including linear filtering) and alternative model – target yield accuracy

4 Conclusion

This contribution has attempted to discuss modeling of sustainability of subsistence production under different agricultural strategies likely employed by the late Iron Age population of the oppida agglomerations. By presentation of the model of adaptability potential of land-use strategies we intended to demonstrate the ability to move from a static data set (archaeological and environmental records) to dynamic modelling that incorporates feedback mechanisms, system integration, and nonlinear responses to a wide range of input data.

Future studies can build upon the model. The applied approach can be adapted in other regions, and other economic strategies can be explored. Our next objectives are to investigate further the population structure of the oppidum by incorporating immigration and different types of households and related attributes of the individuals. The agricultural model is planned to be completed by more detailed weather data, more detailed analyses of the animal production and related labour input resulting into the ratio of producers and non-producers in the society. Also, elaboration of the decision process of Iron Age farmers promises an encouraging way where to direct our further research. At the moment our models end with the limits of the given agricultural practice when reaching certain population density. By including the social variables representing farmers´ independent decisions to change from one economic strategy into another (or to adopt new one) in order to cope with worsening conditions of the sustainable agricultural practice our model can approach the past social complexity studies on a new level.

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Shortest Path Problem In Huge Graphs Making Use of Reachness of Vertices

Tomáš Majer[1]

Abstract. Gutman in [4] presents an algorithm to find the shortest path in directed graph making use the reachness of vertices. In this paper we present how to compute the reachness in parallel on cluster with about 500 CPU cores.

Keywords: graph, digraph, shortest path

JEL classification: C61
AMS classification: 68R10,05C85

1 Introduction

The part of graph theory concerning shortest path problem seemed to be closed. Many types of shortest path algorithms were published, implemented and succesfully applied in practice. European street network contains approximately 20 million vertices and 50 million arcs. These numbers are rising since the street models contain more and more details – e. g. not only streets but also street lines, not only street crossings but also important points of street crossings, etc.

Most of the fast shortest path algorithms use some precomputed characteristics of the network modeled by directed graph. This values can be computed on servers with multiple fast CPUs and huge amount of operational memory. This data is than used as input data on devices with low computational power i.e. smartphones and GPS navigators.

1.1 REACH algorithm

Gutman introduced this algorithm in paper [4]. Algorithm is a modification of Dijkstra algorithm [1]. This algorithm makes use of the reachness (importance) of every vertex and of the lower bound of distance between vertices. The main idea of this algorithm is that vertices with low reachness can be omitted during processing what makes the computation faster.

Definition 1. Let $G = (V, A, c)$ be a arc weighted directed graph with non negative arc weight. Let $\mu(u, v)$ be the shortest $u - v$ path from vertex $u \in V$ to vertex $v \in V$. Let the shortest path contain a vertex $w \in V$, we say $w \in \mu(u, v)$.
The reachness of the vertex w on the shortest path $\mu(u, v)$ is defined as:

$$R(w, \mu(u, v)) = \min\{\mathrm{d}(u, w), \mathrm{d}(w, v)\} \tag{1}$$

where $\mathrm{d}(i, j)$ is distance from vertex i to vertex j – the length of the shortest i–j path.

The reachness of the vertex w in digraph G is:

$$R(w, G) = R(w) = \max_{\mu}\{R(w, \mu) | \mu \text{ is shortest path, } w \in \mu\}. \tag{2}$$

Modified Dijkstra's algorithm processes only vertices $w \in V$ which meat the condition

$$R(w) \geq t(u, w) \text{ or } R(w) \geq \hat{\mathrm{d}}(w, v) \tag{3}$$

where $t(u, w)$ is the length of the best known shortest $u - w$ path (label t of vertex w in Dijkstra's algorithm) and $\hat{\mathrm{d}}(w, v)$ is lower bound of the distance from vertex w to vertex v. This lower bound can be obtained i.e. by Haversine method from vertices GPS values.

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Author writes in his paper, that this modified algorithm should be ten times faster than original Dijkstra's algorithm. However, computation of reachness $R(v)$ of every vertex $v \in V$ in directed graph G is very difficult.

2 Computation of reachness

An algorithm to compute reachness of every vertex in directed graph $G = (V, A, c)$ can be formulated as follow:

Step 1. *Initialization.*
$\forall i \in V$: $R(i) = 0$

Step 2. For every shortest $u - v$ path $\mu(u, v)$ and every vertex $x \in \mu(u, v)$ on this path let

$$R(x) := \max\{R(x), R(x, \mu(u, v))\} = \max\{R(x), \min\{\mathrm{d}(u, x), \mathrm{d}(x, v)\}\}.$$

We need to find the shortest path between all pairs of vertices to compute reachness of every vertex. We can use Dijkstra's *point-to-all* algorithm applied for every starting vertex $u \in V$ to find the shortest path $\mu(u, v)$ to every finishing vertex $v \in V$. The reachness $R(x, \mu(u, v))$ of the vertex $x \in \mu(u, v)$ can be computed using labels $t(v)$ and $t(x)$

$$R(x, \mu(u, v)) = \min\{\mathrm{d}(u, x), \mathrm{d}(x, v)\} = \min\{t(x), t(v) - t(x)\}. \tag{4}$$

Let us define $R_u(x)$ as the reachness of vertex $x \in V$ on all shortest paths starting at vertex $u \in V$ as follows:

$$R_u(x) = \max_{v \in V} \begin{cases} R(x, \mu(u, v)) \text{ if } x \in \mu(u, v) \\ 0 \text{ if } x \notin \mu(u, v) \end{cases}. \tag{5}$$

Then we can compute $R_u(x)$ for different vertices $u \in V$ in different threads of parallel process. After computing $R_u(x)$ for all $u \in V$ the last thing we must make is to compute reachness $R(x)$ from reachness $R_u(x)$ as follows:

$$R(x) = \max_{u \in V} R_u(x). \tag{6}$$

Let μ be the shortest $v_i - v_j$ path

$$\mu(v_i, v_j) = (v_i, (v_i, v_{i+1}), v_{i+1}, \ldots, v_k, \ldots, v_{j-1}, (v_{j-1}, v_j), v_j), \tag{7}$$

let λ be the $v_i - v_{j+1}$ shortest path which is an extension of the shortest path μ by vertex v_{j+1}:

$$\lambda(v_i, v_{j+1}) = (v_i, (v_i, v_{i+1}), v_{i+1}, \ldots, v_k, \ldots, v_{j-1}, (v_{j-1}, v_j), v_j, (v_j, v_{j+1}), v_{j+1}). \tag{8}$$

Then it holds for every vertex $v_k \in \mu$

$$R(v_k, \lambda) = \min\{\mathrm{d}(v_i, v_k), \mathrm{d}(v_k, v_{j+1})\} \geq R(v_k, \mu) = \min\{\mathrm{d}(v_i, v_k), \mathrm{d}(v_k, v_j)\}. \tag{9}$$

So it suffices to process only non-extendable shortest paths i.e. it suffices to process only leaf vertices in shortest path tree acquired by Dijkstra's *point-to-all* algorithm. This optimization was used in the first version of our algorithm for computing reachness of vertices.

Let two shortest paths $\nu(u, v)$ a $\xi(u, w)$ acquired by Dijkstra's algorithm contain vertex $x \in V$. Then the parts from vertex u to vertex x of both this paths are the same. Let $d = \mathrm{d}(u, x)$, then the reachness of vertex x on path ν is

$$R(x, \nu) = \min\{\mathrm{d}(u, x), \mathrm{d}(x, v)\} = \min\{d, \mathrm{d}(u, v) - d\} \tag{10}$$

and the reachness of vertex x on path ξ is

$$R(x, \xi) = \min\{\mathrm{d}(u, x), \mathrm{d}(x, w)\} = \min\{d, \mathrm{d}(u, w) - d\}. \tag{11}$$

Let the length of ν is no greater than the length of ξ, i.e. $\mathrm{d}(u,v) \leq \mathrm{d}(u,w)$. Then $R(x,\nu) \leq R(x,\xi)$. If we process shortest paths acquired by Dijkstra's *point-to-all* algorithm from longest to shortest, we never need to update label $R_u(x)$. This can be simply done by storing vertices in LIFO stack in the same order in which Dijkstra's *point-to-all* attaches definite labels to them.

The computational complexity of this enhanced version of algorithm for transportation network is only $O(n^2 \cdot \log n)$. (We obtain all shortest path from vertex u making use of Label set implementation of Dijkstra's algorithm in time $O(n \cdot \log n + m)$. Since $m < K \cdot n < K \cdot n \cdot \log n$ in digraphs modeling transportation networks we obtain all shortest path from vertex u in time $O(n \cdot \log n)$. Then we compute labels $R_u(x)$ for all vertices x in time $O(n)$. All of them must be repeated n times for every vertex u, what implies the complexity $O(n^2 \cdot \log n)$.)

3 Experiments

We have implemented algorithm to compute reachness of vertices in directed graph making use of `OpenMPI` implementation of *Message Passing Interface*. We have computed reachness of Slovakia road network in parallel process at HPC cluster of University of Žilina with total of 46 nodes of 2 x 6 cores Intel(R) Xeon(R) CPU L5640 @2.27GHz and 48 GB RAM.

The computation time of the first version of the algorithm on 40 nodes of HPC was little less than 10 minutes. The computation time of the second enhanced version on 10 nodes only was 3 minutes and 20 seconds.

The vertices in road network of Slovakia with reachness more than 200 km are shown on Figure 1.

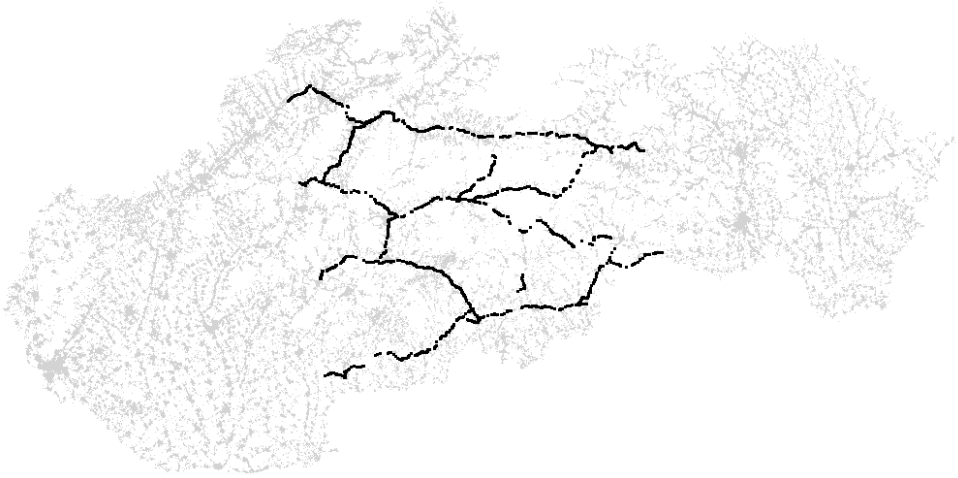


Figure 1: Vertices in road network of Slovakia with reachness more than 200 km

3.1 Comparison of Dijkstra's and Reach algorithm

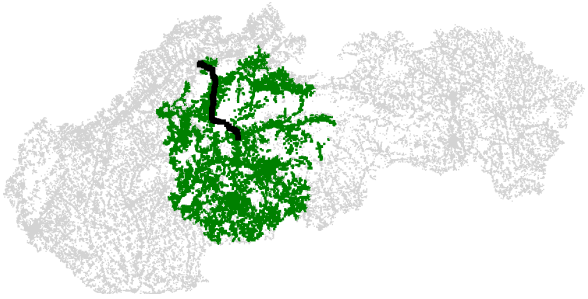
We implemented Dijkstra's and Reach algorithm in `C language` making use of POSIX thread `libpthread` library in `OS Debian Linux` for `AMD 64`. Average processing time to find shortest path between randomly chosen vertices in road network of Slovakia on desktop PC with 2x2 Intel(R) i3 @2.8 GHz processors and 4 GB of main memory are shown in Table 1. It is very interesting that parallel bidirectional implementation of Reach algorithm is slower than one direction only implementation.

Processed vertices in a shortest path between Banská Bystrica and Žilina in the road network of Slovakia by both algorithms (Dijkstra's and Reach) are shown on Figure 2 and Figure 3.

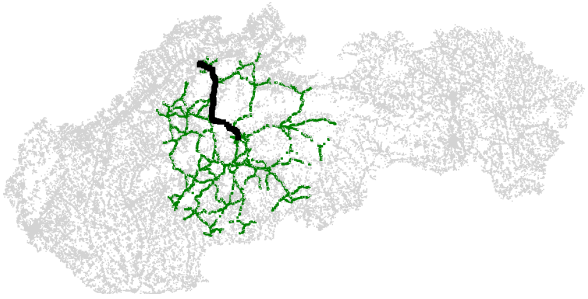
Algorithm	One direction only				Bidirectional parallel			
	n	l	t_1	t_2	n	l	t_1	t_2
Dijkstra's	109 840	194,881	**29,3**	29,3	84 991	194,881	**17,3**	28,6
Reach	11 222	194,881	**6,3**	6,3	9 810	194,881	**7,4**	8,7

n – number of processed vertices, l – average length of the shortest path [km],
t_1 – monolithic time [ms], t_2 – CPU time [ms]

Table 1: Processing time to find shortest path in road network of Slovakia



Processed vertices by Dijkstra's algorithm



Processed vertices by Reach algorithm

Figure 2: Processed vertices by one direction only algorithm

4 Conclusion

In this paper we presented an enhanced algorithm for compute of reachness of vertices of directed graph and its implementation in parallel environment using *Message Passing Interface*. We computed reachness of vertices of road map of Slovakia and computed values are used to make comparison of quickness of Dijkstra's and Reach algorithm for finding shortest paths.

Very fast algorithm A^* was presented by authors Hart, Nilsson and Raphael in paper [5]. Modification of A^*, ALT algorithm was then proposed by Goldberg and Harrelson in paper [2]. ALT algorithm can be easy combined together with Reach algorithm and this REAL algorithm can be found in Goldberg presentation [3].

Acknowledgements

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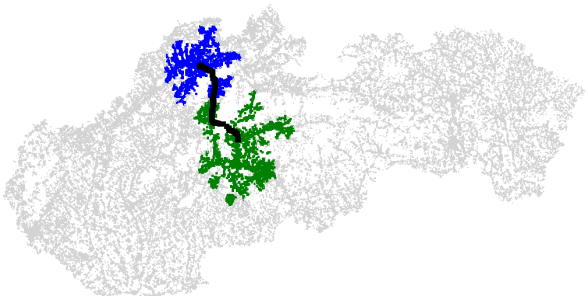
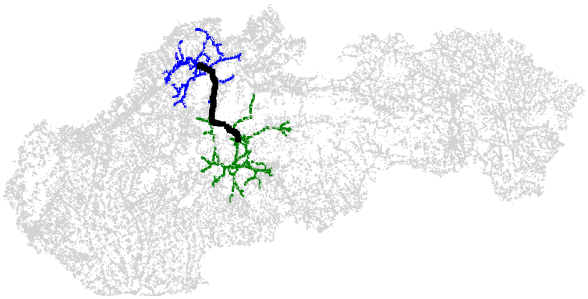


Processed vertices by Dijkstra's algorithm



Processed vertices by Reach algorithm

Figure 3: Processed vertices by bidirectional algorithm

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Verification of the Okun's Law Regarding Selected Newly Industrialized Countries

Ingrid Majerová[1], Elena Mielcová[2]

Abstract. The widely-used relationship between gross domestic product (GDP) growth and unemployment rate change time series – so-called Okun's Law – was originally revealed by an economist Arthur Okun in early 1960s. Okun observed the negative correlation between the two variables. The relation shows persistency in results over the long-term period. Now, students of economics learn that for 1 % excess of the natural unemployment rate, 2-3 % gross domestic product gap occurs. However, these numbers are based on empirical analysis from the data of the US economy.

The main aim of this article is to ascertain validity of the Okun's Law regarding the selected Newly Industrialized Countries - Hong Kong, Singapore, Korea, Philippines, Indonesia, Malaysia and Thailand in 1990 - 2010. The analysis is based on two versions of Okun's law – the difference and the gap version of the Okun's law. Estimations based on the chosen data sets give different results comparing to those from the US time series.

Keywords: newly industrialized countries, Okun´s law, unemployment rate, gross domestic product growth.

JEL Classification: C02
AMS Classification: 91B84, 62M10

1 Introduction

The aim of any economic policy is economic growth, connected with optimal levels of unemployment, low inflation and balanced current account of balance of payments. Arthur Melvin Okun dealt with the relationship of the first two macroeconomic variables - output and unemployment - in 1962 and his measurements entered the economic history as Okun's law (sometimes also called Okun´s rule of thumb, because it is primarily an empirical observation rather than a result derived from theory). In his work Okun [19] concluded that between the growth of gross domestic product and unemployment, there is an inverse relationship: if the product is growing, the unemployment is falling and vice versa. Depending on the type of product (potential or real), he found out - using the three measurements - that if unemployment grows to about one percentage point, the product will fall by 2.8 to 3.3 percentage points.

These estimations are based on empirical analysis from data of the US economy and are slightly changing through the time, they record a decline in a magnitude - based on data from more recent days show predicted 2% decrease in output for every 1% increase in unemployment (Silvapulle et al. [21] or Abel and Bernanke [1]). One implication of Okun's law is that an increase in labor productivity or an increase in the size of the labor force can mean that real net output grows without net unemployment rates falling, so called phenomenon of "jobless growth".

In general, there are three main ways to interpret the Okun's law – the gap version, the difference version and the dynamic version. The gap version of the Okun's law was originally derived from a production function. This model is difficult to estimate because of unmeasured values of optimal level of output and unemployment, so their time series predictions are used by quadratic trend, the Hodrick-Prescott filter and the Baxter-King filter as detrending methods (Mielcová [17]). That is the reason why the difference version of the Okun's law was introduced – it relates changes in unemployment rate to changes in output. The third way – the dynamic version of Okun's law relates the current change of unemployment rate to the current real output growth, past real output growth and past changes in unemployment rate.

The main aim of this article is to ascertain validity of the Okun's Law regarding the Newly Industrialized Economies (NIE´s), using data of the countries such as Hong Kong, Singapore, Korea, Philippines, Indonesia,

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Malaysia and Thailand in 1982 - 2010. The analysis is based on two versions of Okun's law – the difference and the gap version of the Okun's law. Estimations based on the chosen data sets give different results comparing to those from the US time series. It might be surprising why the NIE´s are subjected to analysis of effect of economic growth on unemployment. There are two reasons - the first is that the key to success of newly industrialized economies was the mobilization of production factors, which also include labour. The second reason is that the unemployment rate was always at very low levels in these economies (Hanusch [8]) and the rising growth could show on validity of Okun´s law.

Newly industrialized economies (sometimes called advanced developing countries), NIE´s, are developing economies with high dynamic growth, the parameters of which are approaching economically advanced economies. They are so-called Asian Tigers, or the first generation of NIE´s – such as South Korea, Taiwan, Hong Kong and Singapore and NIE´s of the second generation, which began to develop twenty years later (in 1990´s) and which exhibit slower growth than the economies of the first generation. The examples of NIE countries of the second generation can be found in Latin American for example Argentina, Brazil, Mexico and Chile and in Asia for example Philippines, Thailand, Malaysia and Indonesia (Majerová and Nezval, [16]). These economies are characterized by economic growth, rapid industrialization, gradual attenuation of economic importance of agriculture (related to urbanization), export orientation and the free market, by large inflows of foreign investments and greater freedom and respect for human rights than in other developing countries. As Schmigelow [20] stated their economy is characterized by active government involvement, a reluctant mercantilism, infant industry protection, currency undervaluation and dynamic structure of capitalism.

The article is divided into four parts - the first part deals with the theoretical approach to Okun's law and the characteristics of the newly industrialized countries, the second part is devoted to literature review of research articles and publications that deal with Okun's law and its application in specific economies. The third section describes the estimation results and discussion, followed by conclusion and a list of references.

2 Literature Review

Okun with his work, in which he explained the relations between GDP and unemployment, started a far-reaching discussion, continuing to the present, about the validity of his claims. The research on Okun's law and its verification could be divided, regarding the focus of the article, into two main groups. The first group deals with the verification of the law´s validity in developed economies and the second group is engaged in checking the validity of Okun's law in less developed parts of the world.

Using correct regression, the verification of Okun´s law was provided by Barreto and Howland [5] concerning Japan and they founded that wrong regression could generate much confusion and severe overestimation of the relevant parameters. Sögner and Stiassny [23] studied the 15 OECD countries, for ten of them (Switzerland, Germany, Denmark, Finland, France, Great Britain, Japan, the Netherlands, Norway and Sweden) they obtained changes in Okun´s law and for the rest (Austria, Belgium, Canada, Italy and the US) inferred the stable Okun relationship. Harris and Silverstone [9] tested symmetric relationship between changes in unemployment and real output in OECD countries such as Australia, Canada, Germany, Japan, New Zealand, the United Kingdom and the USA using asymmetric model. They found out that it is possible to establish co-integration and to show that for most countries a short-run adjustment to disequilibrium differs according to whether up-turns or down-turns in the business cycle are considered. International Monetary Found [12] published a study about Okun´s law of twenty advanced countries in WEO, in which it was founded that responsiveness of unemployment to output has increased over past twenty years in many countries. In 2010 Beaton [16] made a time variation of Okun´s law for Canada and the USA and founded that this law exhibits structural instability in both countries. Kitov [15] in his work concluded that the high level of unemployment in developed countries cannot be reduced without fast economic growth well above 2% per year. Hutengs and Stadtmann [10] made a comparison between Germany and the U.S. and the result was that the law is valid for the USA, but not for Germany. Ball et al. [4] confirmed the results about declining growth caused a decreasing level of employment by having tested the U.S. economy since 1948 and twenty advanced economies since 1980 in a short run.

Unlike the verification Okun's law in the advanced economies, the number of publications that deal with the verification in developing countries is smaller. Research conducted by Geidenhyus and Marinkov [7] in South Africa during the years 1970 to 2005. The results of their work indicate that a statistically significant relationship exists between cyclical output and cyclical unemployment and also indicate the presence of an Okun's law relationship in South Africa over the period 1970-2005 with more evidence in favour of asymmetries during recessions. Malaysian economy and testing of relationship between output and unemployment was the subject of interest of Noor et al. [18]. Their result was the verification of Okun´s law in this economy. Islas-Camargo and Cortez [13] estimated the correlation between the transitory components of unemployment and output and found

out that 1 percent decrease in that kind of unemployment corresponded to 1.8 % increase in transitory real GDP. Two papers are dealing with verification of Okun's law in Pakistan: the first one, written by Ahmad et al. [2] demonstrated, using annual time series data on growth rate of GDP and unemployment in 1974-2009, that Okun's law in this economy is not valid. On the other hand, the second one for the period 1981-2005 showed that in a short run, one percent of economic growth caused 3.1 percent decrease in the unemployment rate (Javeid, [14]). Similarly, Okun's law was also tested in two studies in Nigeria. Sodipe and Oluwatobi [22] came to conclusion that a long run inverse relationship between unemployment and GDP exists, thus Okun´s law exists in the Nigerian economy. Arewa and Nwakanma [3] found out that consistent decrease in the gap between natural rate of unemployment and current rate of unemployment lead to a correspondent bud not necessarily proportionate decrease between a potential and real GDP – thus Okun´s law is invalid in Nigeria.

A paper by Hanusch [8], which deals with the East Asian region, namely a newly industrialized countries (with a modification - Indonesia was replaced by China), is useful for our research of Okun´s law. The author asks whether the growth in NIE´s is jobless. Hanusch analyzes only one-sectoral unemployment, namely agricultural one, because he is convinced that agricultural unemployment reveals a stronger relationship between the growth and employment. The result of it was that the countercyclical relationship between the growth and jobs exists regarding agriculture, which is a reverse Okun´s law.

3 Estimation Results and Discussion

As it has already been mentioned, seven newly industrialized economies of Asia have been selected for the analysis of Okun's Law, as the first generation (excluding Taiwan) and the second generation. Data are drawn from the statistics of the International Monetary Fund [11]. Data were applied to the two versions of this law – difference version and gap version.

3.1 Difference version of Okun's law

The difference version of Okun's law relates changes in the unemployment rate to real output growth:

$$\Delta u_t = a + b\frac{\Delta Y_t}{Y_t} + \varepsilon_t \tag{1}$$

where changes in the unemployment rate Δu_t are derived from the observed unemployment rate u_t such that $\Delta u_t = u_t - u_{t-1}$; $\frac{\Delta Y_t}{Y_t}$ is the real output growth, where Y_t is the observed real output, and differences are derived such that $\Delta Y_t = Y_t - Y_{t-1}$. Disturbances are denoted by ε_t. Parameter b – the Okun's coefficient – is expected to be negative.

Results of OLS procedure are given in Table 1. Results of the difference of Okun's law model for the seven chosen countries confirm the persistence of the negative relationship between the unemployment rate change and the output growth rate – all calculated b coefficients are negative, three of them statistically significant at 0.01 level (in the case of Hong Kong, Singapore, and Korea), two of them statistically significant at 0.1 level (Philippines and Thailand), and Malaysia results are significant at 0.05 level.

Country	a	b	t-ratio	p-value	R^2	-a/b
Hong Kong	0.535	-8.390***	-3.676	0.0010	0.317	0.064
Singapore	0.621	-7.469***	-4.817	$5.45.10^{-5}$	0.472	0.083
Korea	0.381	-5.518***	-7.535	$2.63.10^{-8}$	0.662	0.309
Philippines	0.309	-4.205*	-1.749	0.091	0.095	0.073
Indonesia	0.186	-0.703	-1.145	0.270	0.080	0.265
Malaysia	0.230	-2.663**	-2.742	0.012	0.255	0.086
Thailand	0.215	-3.305*	-1.971	0.059	0.126	0.065

Table 1 OLS estimation of the difference version of Okun's law model. Source: Own calculations.

The ratio *–a/b* gives the output growth rate under the condition of stable unemployment rate. This coefficient varies from 0.64 (Hong Kong) to 0.265 (Indonesia) in chosen countries. Moreover, the relationship expresses, that each increase of GDP growth by 1 causes the decrease in unemployment by the rate from 0.703 % (Indonesia) to 8.39 % (Hong Kong).

3.2 Gap version of Okun's law

The gap version of Okun's law relates the level of unemployment rate to output gap:

$$u_t = u_t^* + b'(y_t - y_t^*) + \varepsilon_t \tag{2}$$

where the output gap is defined as the difference between the level of observed real output y_t and potential output y_t^* (both terms in natural logarithms). Constant term, denoted as u_t^*, determines the level of natural unemployment rate – the unemployment rate related with full employment level. As in the previous model, the regression coefficient b' is expected to be negative. Estimations of the output gap were based on the use of the Hodrik-Prescott filter ($\lambda = 100$). Results are given in Table 2.

Country	u^*	b'	t-ratio	p-value	R^2
Hong Kong	3.750	-9.208**	-2.431	0.021	0.165
Singapore	3.434	-7.992***	-7.422	$4.4.10^{-8}$	0.663
Korea	3.562	-5.821***	-5.022	$2.19.10^{-5}$	0.457
Philippines	8.137	-7.883**	-2.431	0.021	0.165
Indonesia	6.368	-0.800	-0.228	0.821	0.003
Malaysia	4.041	-5.283**	-2.576	0.0163	0.210
Thailand	2.053	-5.495***	-3.728	0.001	0.324

Table 2 OLS estimation of the gap version of Okun's law model for unemployment rate. Source: Own calculations.

The natural unemployment rate u^* varies from 2.053 (Thailand) to 8.137 (Singapore). Coefficient b' is negative and almost all estimated coefficients of b' are statistically significant.

In general, the estimation of Okun's relationship does not cover time variation in coefficients; however, there is a possibility to reveal time variance of natural unemployment rate using so-called rolling regressions. A rolling regression estimates a particular relationship over many different moving sample periods; if the relationship is stable over time, then the estimated coefficients should be similar over time. Estimated values of natural unemployment rate using a 15-year interval rolling regressions are given in Figure 1 and 2. The two countries of the second generation - the Philippines and Indonesia - were excluded from the analysis due to incomplete time series. Monitoring the natural rate of unemployment was split between the selected NIE's of the first and second generation, as their development of macroeconomic variable tend to be different - while the countries of the first generation responded to the Asian financial crisis in 1998, the second group of economies did not reflect it concerning the unemployment.

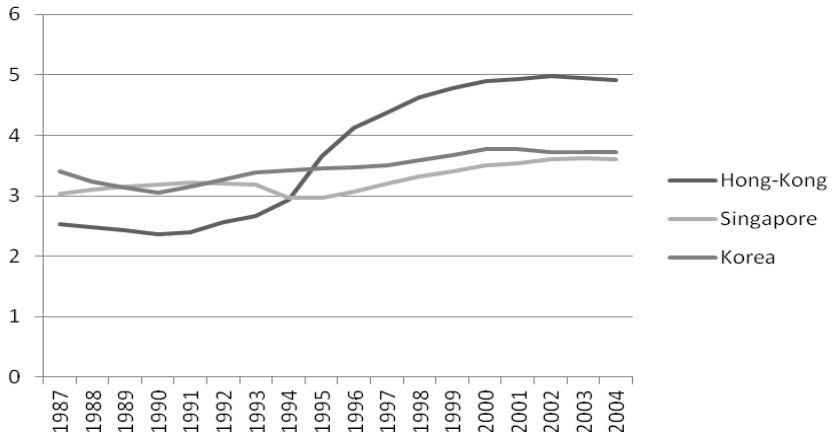

Figure 1 Level of natural unemployment rate in Hong Kong, Singapore, and Korea in 1987-2004 period. Source: Own calculations.

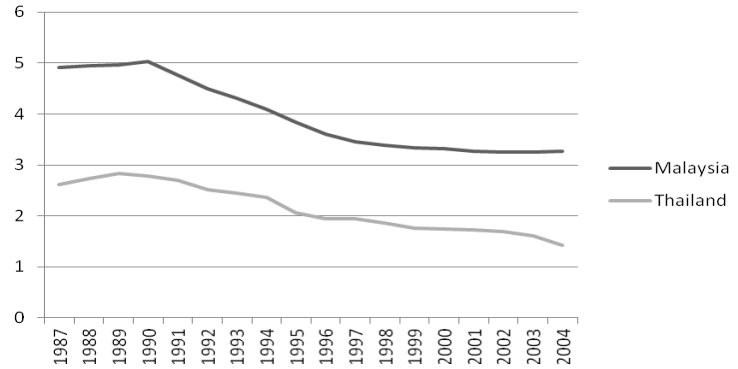

Figure 2 Level of natural unemployment rate in Malaysia, and Thailand in 1987-2004 period. Source: Own calculations.

Another possibility of the gap Okun's law model formulation can be of the form:

$$y_t - y_t^* = \alpha + \beta(u_t - u_t^*) + \varepsilon_t \tag{3}$$

where $(y_t - y_t^*)$, as in the previous model, denotes an output gap, while $(u_t - u_t^*)$ is an unemployment rate gap. Coefficient β is expected to be negative. The main problem with the gap Okun's model is the fact, that there are no observable values of y_t^*, and u_t^*. The usual way to handle this problem is to generate trend series for both output and unemployment gap time series. This analysis uses Hodrick-Prescott filter as a de-trending technique. Because of missing data we could use the Hodrik-Prescott filter only on the data from Hong Kong, Korea and Philippines; results are given in Table 3. The β coefficients are negative and statistically significant at 0.01% level. The results indicate, that for 1 % excess of the natural unemployment rate, a 0.057 %, 0.114% and 0.053 % GDP gap can be predicted for Hong Kong, Korea, and Philippines, respectively.

Country	α	β	t-ratio	p-value	R^2
Hong Kong	0.000	-0.057***	-4.478	0.0001	0.401
Korea	0.000	-0.114***	-6.148	$9.22.10^{-7}$	0.558
Philippines	0.000	-0.053***	-3.511	0.0014	0.291

Table 3 OLS estimation of the gap version of Okun's law model for GDP gap variable. Source: Own calculations.

4 Conclusion

Okun´s law shows, that if unemployment will grow to about one percentage point, the product will fall by 2.8 to 3.3 percentage points, or regarding difference version, if GDP increases about 1 percentage, unemployment decreases about 0.3 percentage. The main aim of this paper was to verify the Okun's law of NIE's, which are considered to be developing economies, but which are close to advanced countries with their economic levels and in terms of economic growth, these countries even surpass them.

This verification was done on the basis of two versions of Okun's law – difference version and gap version. The results of the difference version confirm the persistence of negative relationship between the unemployment rate change and the output growth rate. The increase of GDP by approximately 1 percent causes the decrease in unemployment by the rate from 0.703 to 8.39 percent - Okun's law has been verified, although with higher causality. In the gap version we founded out that with 1 percent excess of the natural unemployment rate a 0.053-0.114 percent GDP gap can be predicted. In this case the Okun's law has been confirmed, but relative negative correlation between both variables is not so high and fluctuations in unemployment in these economies do not affect GDP too much.

We have also monitored the development of natural unemployment for each group of NIE's separately and the development had a different trend. In economies such as Hong Kong, Singapore and Korea there can be found an increase in the value of natural unemployment (especially in Hong Kong), that was caused by the deterioration of economic situation as the result of the Asian financial crisis. This crisis was approved mostly in NIE's of the first generation, which had highly profitable financial markets attracting a large number of risk

capital by then. It is interesting that the NIE's of the second generation developed conversely - although these economies are also affected by the financial crisis, its impact on unemployment was reversed. Based on the rolling regression, we found out that the highest natural rate of unemployment was in both economies in 1986 and then they already did not reach such high levels. Since 2004 (after the end of financial crisis in Argentina) natural unemployment has reached a stable level in all economies and was not affected by the financial crisis in 2008.

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Application of probability weighted method of moments to estimation of financial return distribution tail

Marta Małecka[1], Dorota Pekasiewicz [2]

Abstract. Fitting the tail of the random variable with an unknown distribution plays a pivotal role in finance since it gives grounds for estimation of high quantiles and in consequence offers risk measures. The parametric estimation of fat tails is based on the convergence to the generalized Pareto distribution (GPD). The paper explored the probability weighted method of moments (PWMM) applied to estimation of the GPD parameters.
The PWMM algorithm requires specification of the CDF estimate of the so-called excess variable and depends on the choice of the order of the probability weighted moments. Through the simulation study, the paper investigated statistical properties of the GPD parameter estimates with reference to the PWMM algorithm specification. The simulation experiment was designed with the use of fat tailed distributions with parameters assessed on the basis of the empirical daily data for DJIA index. The results showed that, in comparison to the commonly used CDF formula, the choice of the alternative empirical distribution function improved the statistical properties of PWMM estimates. As a complementary analysis, the PWMM tail estimate of DJIA log returns distribution was presented.

Keywords: PWMM, generalized Pareto distribution, tail estimate, distribution function estimate, EVT.

JEL Classification: C02, C19, C53
AMS Classification: 62F12, 62E20

1 Introduction

In finance the estimates of the loss distribution tails are used for risk valuation. Using the extreme value theory, risk is assessed on the basis the extreme quantile of the loss distribution or the expected overshoot of a specified threshold. The estimation of the fat tails of the loss distribution is based on the empirical distribution of the auxiliary variable which is defined as the excess over the threshold and converges asymptotically to the generalized Pareto distribution (GPD).

The paper explored the probability weighted method of moments (PWMM) applied to estimation of the parameters of the generalized Pareto distribution. The main advantege of PWMM over commonly used maximum likelihood method is that its application is not restricted to large samples. This is an important practical asspect since scarcity of observations is inherent to extreme events [2, 8]. The PWMM requires specification of the CDF estimate of the excess variable and depends on the choice of the order of the probability weighted moments used for parameters estimation. The paper investigated statistical properties of the GPD parameter estimates with reference to the CDF estimate and weighted moment order. The bias and variance of the estimates were evaluated in the simulation study. The simulation experiment was designed with the use of fat tailed distributions: t-Student, Pareto, log gamma and Burr with parameters assessed on the empirical daily data for DJIA index. As a complementary analysis, the PWMM tail estimate of DJIA log returns distribution was presented.

The paper is organized as follows. The next section sets the notation and introduces the probability weighted method of moments. Section 3 outlines the estimation procedure of the generalized Pareto distribution parameters and the tail of the underlying loss variable with the use of the PWMM. That section is followed by a comparative study of PWMM estimates of GPD parameters according to the chosen empirical distribution function and order of moments. At the end of section 4 we turn to empirical study for the daily DJIA data. The final section summarizes and concludes the article.

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2 Probability weighted method of moments

Probability weighted method of moments is the procedure of estimation of the distribution parameters $\theta_1,...,\theta_s$ of the random variable X with CDF F that utilizes the idea of probability weighted moments of the distribution and their estimates. The probability weighted moment of the X distribution is defined as (if the relevant expectation exists):

$$M_{1,j,0} = E\left(xF^j(x)\right), \tag{1}$$

$$M_{1,0,k} = E\left(x\left(1-F(x)\right)^k\right), \tag{2}$$

for $j,k = 0,1,...$ [7].

Suppose that we have a sequence of i.i.d. observations $X_1, X_2, ..., X_n$ from an unknown distribution F. $M_{1,j,0}$ and $M_{1,0,k}$ estimates are sample probability weighted moments given by:

$$m_{1,j,0} = \frac{1}{n}\sum_{i=1}^{n} X_{(i)}^{(n)}\left[F_n\left(x_{(i)}^{(n)}\right)\right]^j, \tag{3}$$

$$m_{1,0,k} = \frac{1}{n}\sum_{i=1}^{n} X_{(i)}^{(n)}\left[1-F_n\left(x_{(i)}^{(n)}\right)\right]^k, \tag{4}$$

where $X_{(i)}^{(n)}$ are position statistics and F_n is the empirical CDF.

Solving the system of s equations of the form:

$$\begin{cases} m_{1,j,0} = M_{1,j,0}, \\ m_{1,0,k} = M_{1,0,k}, \end{cases} \tag{5}$$

where $j = 0, 1, ..., s_1$, $k = 1, ..., s_2$ and $s_1 + s_2 = s$, gives the estimates of $\theta_1,...,\theta_s$.

Statistical properties of the estimates depend on the form of empirical distribution function. We considered three types of CDF estimates [4]:

$$F_n(x) = \frac{1}{n-1}\sum_{i=1}^{n}\left(I_{(-\infty,x\rangle}\left(x_{(i)}^{(n)}\right)-1\right), \tag{6}$$

$$F_n(x) = \sum_{i=1}^{n} w_{n,i} I_{(-\infty,x\rangle}\left(x_{(i)}^{(n)}\right), \tag{7}$$

$$\text{where} \quad w_{n,i} = \begin{cases} \frac{1}{2}\left[1-\frac{n-2}{\sqrt{n(n-1)}}\right] & \text{for } i = 1, n, \\ \frac{1}{\sqrt{n(n-1)}} & \text{for } i = 2, 3, ..., n-1, \end{cases} \tag{8}$$

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{(-\infty,x\rangle}\left(x_{(i)}^{(n)}\right). \tag{9}$$

3 Estimation of the generalized Pareto distribution parameters and the distribution tail

Fitting the tail of the variable X distribution with the CDF F gives grounds for estimation of high quantiles and is therefore used for risk valuation. Of particular importance are distributions with fat tails which demonstrate financial data features. In such cases the estimate of the distribution function $F(x)$ for $x > u$ is given by [6]:

$$\hat{F}(x) = 1 - \frac{n_u}{n}\left(1 + \hat{\xi}\frac{x-u}{\hat{\beta}}\right)^{-\frac{1}{\hat{\xi}}}, \tag{10}$$

where u is the fixed threshold, n_u is the number of observations larger than u and $\hat{\xi}, \hat{\beta}$ are the estimates of the ξ, β parameters of the generalized Pareto distribution with the distribution function:

$$F_{\beta,\xi}(x) = \begin{cases} 1 - \left(1 + \xi\frac{x}{\beta}\right)^{\frac{-1}{\xi}} & \text{for} \quad \xi > 0,\ x \geq 0. \end{cases} \tag{11}$$

The GPD parameters are estimated from the sample $Y_1, ..., Y_{n_u}$, where $Y_i = X_i - u$ and $Y_i > 0$ for $i = 1, ..., n_u$. If $\xi < \frac{1}{2}$, parameters ξ and β of the GPD may be estimated by the method of moments [3]. Then $\hat{\xi} = \frac{1}{2} - \frac{\bar{Y}^2}{2S^2}$ and $\hat{\beta} = \frac{\bar{Y}\left(\bar{Y}^2/S^2 + 1\right)}{2}$, where $\bar{Y}$, S^2 are subsequently the mean and the variance of n_u-element sample $Y_1, ..., Y_{n_u}$.

If $0.5 < \xi < 1$ only the expectation of the GPD exists and in that case the estimates of the parameters are given by:

$$\hat{\xi} = 2 - \frac{\bar{Y}}{\bar{Y} - 2\alpha}, \tag{12}$$

$$\hat{\beta} = \frac{2\alpha\bar{Y}}{\bar{Y} - 2\alpha}, \tag{13}$$

where α is the estimate of the moment $M_{1,0,1}$. Depending on the chosen form of the empirical distribution function we have subsequently:

$$\alpha = \frac{1}{n_u}\sum_{i=1}^{k}\frac{n_u - i}{n_u - 1}Y_{(i)}^{(k)}, \tag{14}$$

$$\alpha = \frac{1}{n_u}\left(\frac{1}{2} + \frac{n_u - 2}{2\sqrt{n_u(n_u - 1)}}\right)Y_{(1)}^{(n_u)} + \frac{1}{n_u}\sum_{i=2}^{n_u - 1}\left(\frac{1}{2} + \frac{n_u - 2i}{2\sqrt{n_u(n_u - 1)}}\right)Y_{(i)}^{(n_u)}, \tag{15}$$

$$\alpha = \frac{1}{n_u}\sum_{i=1}^{k}\frac{n_u - i}{n_u}Y_{(i)}^{(k)}, \tag{16}$$

where $Y_{(i)}^{(k)}$ is the i-th position statistic.

For $\xi \geq 1$ the GPD has only the probability weighted moments. The estimates of its parameters are based on moments $M_{1,0,k}$ and $M_{1,0,k+1}$ (if they exist) and it holds that:

$$\hat{\xi} = \frac{(k+1)^2\alpha_1 - (k+2)^2\alpha_2}{(k+1)\alpha_1 - (k+2)\alpha_2}, \tag{17}$$

$$\hat{\beta} = \frac{(k+2)(k+1)\alpha_1\alpha_2}{(k+1)\alpha_1 - (k+2)\alpha_2}, \tag{18}$$

where $\alpha_1 = \frac{1}{n_u}\sum_{i=1}^{n_u}\frac{(n_u - i)(n_u - i - 1)...(n_u - i - k + 1)}{(n_u - 1)(n_u - 2)...(n_u - k)}Y_{(i)}^{(n_u)}$, $\alpha_2 = \frac{1}{n_u}\sum_{i=1}^{n_u}\frac{(n_u - i)(n_u - i - 1)...(n_u - i - k)}{(n_u - 1)(n_u - 2)...(n_u - k - 1)}Y_{(i)}^{(n_u)}$.

4 Statistical properties of estimates of GPD parameters for chosen return distributions

Statistical properties of estimates of GPD parameters ξ, β were examined in a simulation study. We considered four fat tailed distributions: t-Student, Pareto, log gamma and Burr. The distribution parameters assumed for the purpose of the simulation experiment were assessed on the basis of the historical distribution of the daily DJIA return so that they reflect features of real financial series [5]. The threshold u was fixed at the level of the quantile of order 0.95 and the sample size was set to 1000 which guaranteed the number of observations over u large enough for the use of the asymptotic theorems.

Distribution	Moments	$\hat{\xi}-\xi$	$D^2\left(\hat{\xi}\right)$	$MSE\left(\hat{\xi}\right)$
t-Student $\xi=0.2$	$M_{1,0,0}$, $M_{1,0,1}$	-0.099	0.041	0.051
	$M_{1,0,1}$, $M_{1,0,2}$	-0.118	0.116	0.130
	$M_{1,0,2}$, $M_{1,0,3}$	-0.134	0.292	0.310
Pareto $\xi=0.2$	$M_{1,0,0}$, $M_{1,0,1}$	-0.027	0.029	0.030
	$M_{1,0,1}$, $M_{1,0,2}$	-0.020	0.093	0.094
	$M_{1,0,2}$, $M_{1,0,3}$	-0.023	0.257	0.257
log gamma $\xi=0.2$	$M_{1,0,0}$, $M_{1,0,1}$	-0.023	0.030	0.031
	$M_{1,0,1}$, $M_{1,0,2}$	-0.016	0.093	0.093
	$M_{1,0,2}$, $M_{1,0,3}$	-0.021	0.254	0.254
Burr $\xi=0.2$	$M_{1,0,0}$, $M_{1,0,1}$	0.130	0.043	0.060
	$M_{1,0,1}$, $M_{1,0,2}$	0.188	0.113	0.149
	$M_{1,0,2}$, $M_{1,0,3}$	0.208	0.253	0.297

Table 1 Statistical properties of parameter $\xi=0.2$ estimates for different weighted moments

Distribution	Moments	$\hat{\xi}-\xi$	$D^2\left(\hat{\xi}\right)$	$MSE\left(\hat{\xi}\right)$
t-Student $\xi=0.8$	$M_{1,0,0}$, $M_{1,0,1}$	-0.180	0.062	0.095
	$M_{1,0,1}$, $M_{1,0,2}$	-0.059	0.076	0.080
	$M_{1,0,2}$, $M_{1,0,3}$	-0.054	0.180	0.183
Pareto $\xi=0.8$	$M_{1,0,0}$, $M_{1,0,1}$	-0.176	0.061	0.093
	$M_{1,0,1}$, $M_{1,0,2}$	-0.051	0.075	0.078
	$M_{1,0,2}$, $M_{1,0,3}$	-0.041	0.181	0.182
log gamma $\xi=0.8$	$M_{1,0,0}$, $M_{1,0,1}$	-0.127	0.044	0.060
	$M_{1,0,1}$, $M_{1,0,2}$	0.041	0.075	0.077
	$M_{1,0,2}$, $M_{1,0,3}$	0.061	0.175	0.179
Burr $\xi=0.8$	$M_{1,0,0}$, $M_{1,0,1}$	-0.136	0.044	0.063
	$M_{1,0,1}$, $M_{1,0,2}$	0.043	0.070	0.072
	$M_{1,0,2}$, $M_{1,0,3}$	0.075	0.172	0.178

Table 2 Statistical properties of parameter $\xi=0.8$ estimates for different weighted moments

The historical data analysis showed that the real values of ξ are below 1. It was therefore possible to use, for GPD parameter estimation, the expected value and further probability weighted moments of orders higher than 0. We used moments of orders 0 to 3 to show how the moment choice affects the estimate properties. The results, obtained over 20000 replications, are presented in Tables 1 and 2.

The simulation results showed that, while the estimate bias did not always exhibit a regular behavior, the variance influence resulted in the mean square error (MSE) decrease for higher weighted moments. Hence the results support the choice of the expectation and the weighted moment of order 1 for ξ estimation, according to formulas (12) and (13).

In the next step we evaluated, over 20000 repetitions[3], the bias and the variance of estimates $\hat{\xi}_1, \hat{\xi}_2, \hat{\xi}_3$, for empirical distribution functions (6), (7) and (8). The results are presented in Table 3. For $\xi < 1$, in comparison to the commonly used empirical CDF formula (6), the choice of formulas given by (7) or (8) improved both the bias and the variance of parameter ξ estimates. The estimate bias for the commonly used formula $\hat{\xi}_1$ was mainly negative and reached values up to 0.18, which showed that this formula produced substantially lower estimates than the real parameter value. For the examined fat tailed distributions the best statistical properties were obtained for estimate $\hat{\xi}_2$, which used the formula (7) for the estimate of the empirical distribution function.

Distribution	Bias			Variance		
	$\hat{\xi}_1 - \xi$	$\hat{\xi}_2 - \xi$	$\hat{\xi}_3 - \xi$	$D^2\left(\hat{\xi}_1\right)$	$D^2\left(\hat{\xi}_2\right)$	$D^2\left(\hat{\xi}_3\right)$
t-Student(5)	-0.099	-0.049	-0.064	0.041	0.029	0.032
t-Student(1.25)	-0.180	-0.160	-0.169	0.062	0.052	0.056
Pareto(1, 5)	-0.027	0.018	0.004	0.029	0.025	0.025
Pareto(1, 1.25)	-0.176	-0.156	-0.165	0.061	0.052	0.055
Log gamma(2, 0.2,0)	-0.023	0.022	0.008	0.030	0.026	0.026
Log gamma(2, 0.8,0)	-0.127	-0.109	-0.117	0.044	0.037	0.039
Burr(0.5,10)	0.130	0.166	0.153	0.043	0.051	0.047
Burr(0.5,2.5)	-0.136	-0.118	-0.126	0.044	0.037	0.040

Table 3 Bias and variance of parameter $\xi = 0.2$ and $\xi = 0.8$ estimates for different CDF estimates

In finance, the estimates of the generalized Pareto distribution are used for fitting tails of fat-tailed return distributions. The functional form of the tail gives basics for calculating high quantiles of the distribution and in consequence offers risk measures. We used estimate in the form $\hat{\xi}_2$, characterized by the lowest bias and variance, to calculate the tail estimate from the sample of 1000 observations of DJIA dating back to March 2009 (Fig. 1). The estimated distribution function takes the form $\hat{F}_2(x) = 1 - \frac{1}{20}\left(1 + 0.104\frac{x - 0.025}{0.008}\right)^{-\frac{1}{0.104}}$ compared to $\hat{F}_1(x) = 1 - \frac{1}{20}\left(1 + 0.052\frac{x - 0.025}{0.008}\right)^{-\frac{1}{0.052}}$ for the estimate $\hat{\xi}_1$ and $\hat{F}_3(x) = 1 - \frac{1}{20}\left(1 + 0.088\frac{x - 0.025}{0.008}\right)^{-\frac{1}{0.088}}$ for $\hat{\xi}_3$.

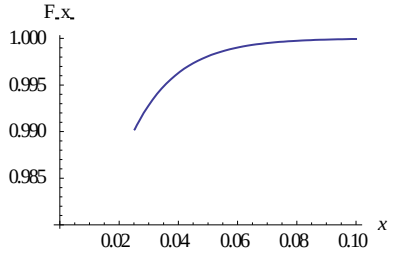

Figure 1 Tail estimate for formula $\hat{\xi}_2$, sample of 1000 daily observations of DJIA, March 2009-March 2013

In empirical analysis the value of parameter ξ, referred to as a tail index, is often of particular interest, since it gives information about the weight of the distribution tail. In Figure 2 we presented the estimated values of this parameter for DJIA index, with the use of the three considered CDF estimates. The results were obtained from

[3] In most cases 10000 is a sufficient number of repetitions in simulations [1], however in our study we noticed differences in the second decimal point when diminishing the number of repetitions from 20000 to 10000.

the rolling estimation with the window length of 1000 observations and the step of 20 observations. The dates included in the picture indicate the end point of the estimation window: starting sample ranged from March 2005 to March 2009 and the final sample range was from March 2009 to March 2013. The graphical presentation confirmed that parameter values had tendency to lie below the level of 1. The estimates $\hat{\xi}_1$ obtained with the use of the empirical distribution function in the form (6) lied at some distance form estimates $\hat{\xi}_2$ and $\hat{\xi}_3$, characterized by lower bias and variance, and were systematically below them.

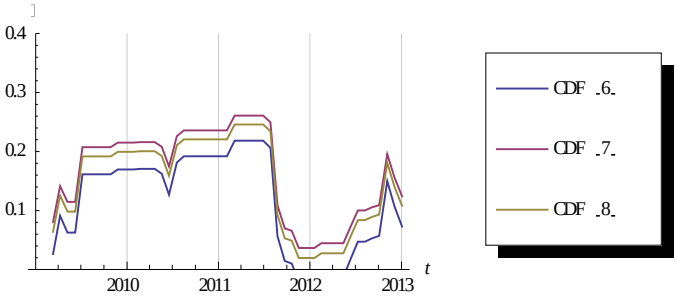

Figure 2 Parameter ξ estimates from the rolling estimation based on daily DJIA observations

5 Conclusion

The paper explored the probability weighted method of moments applied to estimation of the parameters of the generalized Pareto distribution. Statistical properties of GPD parameters ξ, β estimates were examined in a simulation study. We considered four fat tailed distributions: t-Student, Pareto, log gamma and Burr.

Since the PWMM method depends on the choice of the order of the probability weighted moments used for parameters estimation, the first step of the presented study concentrated on estimate statistical properties with relation to the chosen moment. It was shown that there is a general decreasing tendency in mean square error with increasing the moment order. This result supported the choice of the expectation and the weighted moment of order 1 for ξ estimation.

Following the moment choice analysis we turned to examining statistical properties of generalized Pareto distribution parameter estimates with reference to the CDF estimate required for parameter estimation by PWMM. The simulation study results indicated that for $\xi < 1$ it was possible to improve estimate statistical properties by the choice of the empirical distribution function given by formulas different that the commonly used CDF estimate.

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Probability Models for Wage Distributions

Lubos Marek[1], Michal Vrabec[2]

Abstract. In our article we try to contribute to the discussion of the possibility to predict wage distribution trends in the Czech Republic. Classical models use probability distributions such as log-normal, Pareto, etc., but their results are not very good. We suggest using a mixture of normal probability distributions (normal mixture) in our models. We focus mainly on the possibility of constructing a mixture of normal distributions based on parameter estimation. We estimate these parameters (mainly mean and variance) on the basis of their evolution in time. For estimation we use time series analysis, specifically trend analysis.
We work with data collected in the last 16 years. The data are divided in groups with respect to gender, age and region. For all groups we are able to determine percentile measures and other characteristics.
When we estimate the future shape of the distribution, we can calculate the expected frequency in each income interval. Then we are able to estimate how wages affect taxes, social security, health insurance, etc. We are also able to use the model for calculation of confidence intervals and for estimation of other important characteristics

Keywords: probability, model, wage distribution, normal mixture.

JEL Classification: C44
AMS Classification: 90C15

1 Introduction - Time Evolution of Wages in the Czech Republic

In this article we analyze the time evolution of wages in the Czech Republic in the years from 1995 to 2012. Our data source is the Trexima Company. The annual data is reported in quarterly units; our study observes the average wages in the second quarter of each year. The scope of the data set on which the analyses were carried out was gradually increased from more than 300,000 observations in 1995 to approximately two million in 2013. This data is structured in a very detailed way. The salary values are divided into intervals whose widths are 500 CZK. Such a detailed structure enables us to achieve quite accurate results. We have at our disposal basic characteristics of wages in the entire period under consideration: arithmetic mean, standard deviation, median, upper and lower quartiles and 10% and 90% quantiles. In particular, the arithmetic mean and standard deviation are very important for us – without those we would not have been able to estimate the density of probability distribution for wages. The said characteristics were calculated for the entire Czech Republic and also in categories by gender, age (with three groups: up to 30 years; 30 to 50 years; and above 50 years of age), and regions. The overall number of characteristic categories hence was $2 \times 3 \times 14 = 84$.

The analyses were aimed at formulating a model for probability distribution of wages (estimating the shape of the probability density). We are then able to work with this model and use it for subsequent calculations – whether predicting the future shape of the distribution, determining the confidence intervals, predicting characteristics of wages, etc. Most calculations were carried out with the aid of SAS software using literature [11], [13].

2 Materials

2.1 Empirical distribution of wages

Data is structured in intervals 500 CZK wide – see an example in Table 1. Such a detailed structure enables us to achieve quite accurate results.

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lower bound		upper bound	absolute frequency
15,000	-	15,500	41,541
15,500	-	16,000	42,942
16,000	-	16,500	44,960

Table 1 Wage intervals

Let us view the frequency distributions for average wages in each year (up to an amount of 60,000 CZK) for the entire Czech Republic. Leftmost is the year 1995, rightmost is 2012. It is obvious that the distribution's position, variability, skewness, and kurtosis significantly change between years. In the early years the shape of the distribution is relatively smooth, while in the later years the empirical density's curve is much less smooth – this is, above all, a consequence of the growing variability.

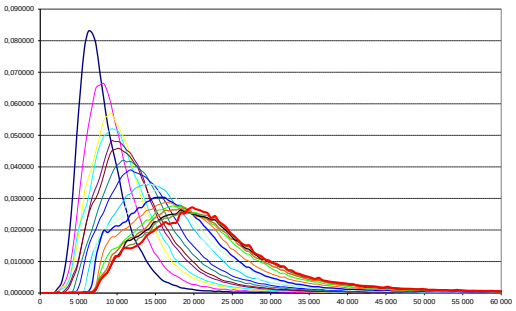

Figure 1 Empirical distribution of wages

In the following two Figures we can see the evolution of average wages and standard deviations. In both instances the characteristics' evolution in time is linear, and the trends can be modeled by lines. The index of determination is close to 1 in both instances, meaning that a better trend curve would hardly be achievable.

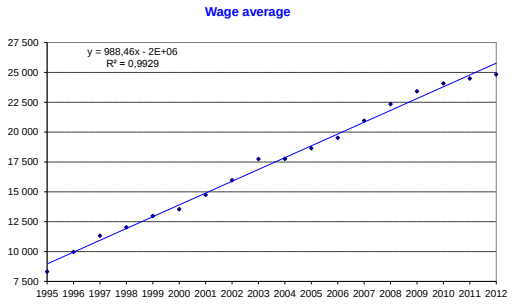

Figure 2 Average of wages

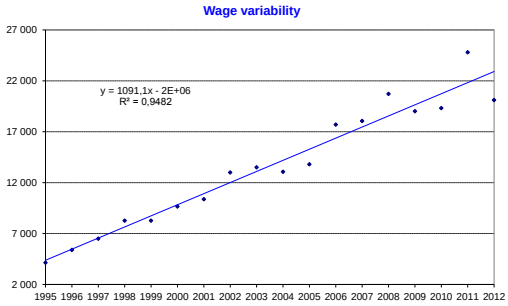

Figure 3 Standard deviation of wages

2.2 Theoretical distribution of wages

A logarithmic-normal distribution is often used for modeling wages– cf. [4], [5], [6]. This distribution can have two or four parameters and in the early years its fit to the data's empirical distribution was very good – cf. [1], [2], [3]. Other distributions could also be used, of course, such as Log-Logistic, Burr, or Frechet. However, with the growing variability in later years the results achieved with the aid of classical distributions are less and less accurate.

In our opinion, the situation can be resolved using the so-called mixtures of normal or log-normal distributions [12]. The main idea is based on the fact that the average wage over the entire Czech Republic is a composition of average wages for different categories with different weights. The probability distribution in each category is of the same type, but their parameters – and consequently the density shapes – are different. In articles [7], [8] we treated constructions based on the normal distributions. The resulting models were getting worse in more recent years. Improved results were achieved when we used mixtures of the logarithmic-normal distributions.

Let us, for example, consider average wages of men and women in 2012 and depict the shapes of the empiric densities.

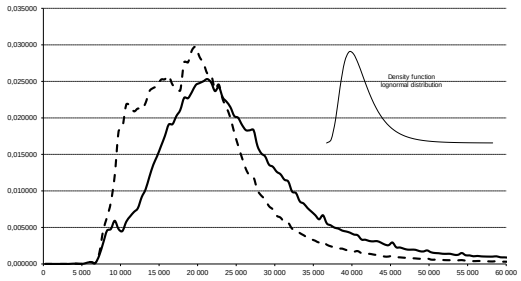

Figure 4 Distribution of relative frequency – men vs. women

At first look we can see that the distribution is of the same type (say, logarithmic-normal), but with different parameters – see, e.g., [4]. The situation is clearer in the following Figure with four empiric densities for four selected regions (14 regions are at our disposal, we chose only four for illustration), again in 2012 only.

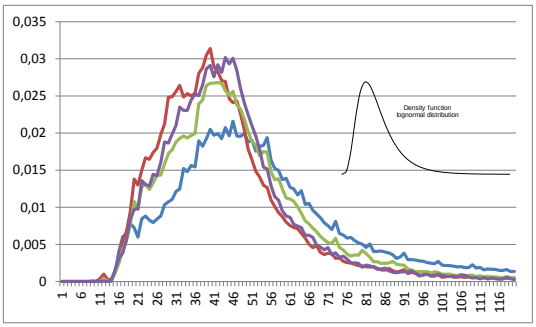

Figure 5 Distribution of relative frequency – four selected regions

In the Figures we can see that the density for the overall average wage can be viewed as a mixture of the partial densities (partial distributions). It is therefore suggestive to calculate this mixture to create a probability model for the resulting density. Since there are 84 categories, the resulting density can be a mixture of up to 84 partial densities.

3 Methods - Log-normal mixture

We take logarithmic-normal distributions with two parameters, μ and σ^2, as the probability model for the density of wages. The shape of this density is

$$\begin{aligned} f(x) &= \frac{1}{\sigma x\sqrt{2\pi}} e^{\frac{-(\ln x-\mu)^2}{2\sigma^2}}, & x>0, \\ &= 0, & x\le 0. \end{aligned} \tag{1}$$

We need to estimate the mean value and variance for each category. This poses no problem since we have at our disposal the arithmetic mean (as an estimate of the mean value) and the empirical standard deviation (as an estimate of the standard deviation). Let us make use of the relationships between the distribution's parameters and the mean value and variation

$$E(X) = \exp\left(\mu + \frac{\sigma^2}{2}\right) \tag{2}$$

$$D(X) = \exp(2\mu+\sigma^2)\left[\exp(\sigma^2)-1\right] \tag{3}$$

Substituting for $E(X)$ and $D(X)$ we then easily estimate the parameters. The same procedure is repeated for each category to get the parameter estimates for each partial density. Let us denote density (1) by $f_i(x_i,\mu_i,\sigma_i)$ to emphasize different parameter values for each category.

The probability density for a general model of a normal mixture can be written as follows

$$f(x,n,p,m,s) = \sum_{i=1}^{n} p_i \cdot f_i(x_i,\mu_i,\sigma_i) \tag{4}$$

where $f(x,n,p,m,s)$ is a probability density of a mixture of log-normal distributions (log-normal mix), x is the argument, n is the number of components in the mixture, and p is the vector of weights, for which it holds

$$0 < p_i < 1,\ \forall i,\ \sum_{i=1}^{n} p_i = 1, \tag{5}$$

m and *s* are vectors of mean values and standard deviations of individual components in this mixture, and $f_i(x_i,\mu_i,\sigma_i),\ i=1,...,n$ are individual log-normal densities in the mixture.

4 Results

4.1 Final model

When formulating the model, we first estimated the distribution of wages in 2012 on the basis of the data from years 2000 to 2011. We compared the results obtained with the actual wages of 2012 to evaluate the fit of the model to the data. The fit was very good and the model was thus shown to be fully adequate. Then we used the data from years 2000 to 2012 and predicted 2013. A prediction for 2013 requires estimates of the mean value and standard deviation. For these estimates, we will make use of time series analysis, namely, trend analysis. Both the mean value and standard deviation have linear trends; hence, getting a good-quality prediction for one year ahead is not difficult. This stage of our analysis successfully combines the classical probability approach with that of time series analysis. Without the trend analysis we would not have been able to estimate the future values of the parameters, needed for the future shape of the probability density of the wage distribution. We use general probability theory – see [9], [10].

A comparison between the model and the 2012 data came first. The resulting mixture of log-normal densities was calculated three times, as mixtures of:

- two densities (according to gender);
- three densities (corresponding to age groups: up to 30; between 30 and 50 ; and above 50 years of age);
- 14 densities (corresponding to 14 regions in the Czech Republic).

We also experimented with finer divisions, but without significantly improving the results. A comparison of the results obtained is shown in the following Figure.

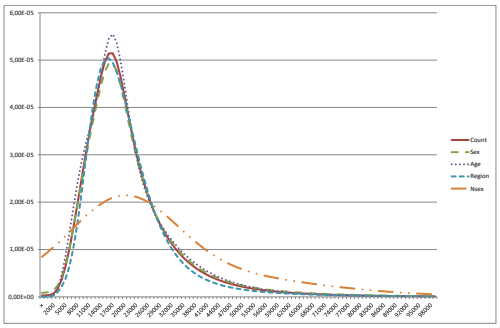

Figure 6 Distribution of relative frequency – four selected regions

Count shows empiric relative frequencies (data); curves **Sex**, **Age**, **Region** are the resulting mixtures of densities. We can see that all three resulting mixtures have rather similar shapes. The fourth mixture (**NSex**), composed of normal-distribution densities of two categories divided by gender, provides an utterly unsuitable model.

The results obtained are too numerous (several dozen pages of computer output), so we only show one selected model for illustration. A mixture composed of three partial log-normal densities is used (corresponding to the three age groups). The following Table shows a goodness-of-fit test between the theoretical estimates of the parameters and their empirical counterparts (for 2012) and three age groups (for the three partial probability densities).

The parameters of the three partial densities were estimated from the data as shown in the following Table.

parameters	under 30	30 to 50	over 50
μ	9.8441336	9.9038454	9.8552531
σ	0.1602285	0.5377981	0.5164149

Table 2 Parameters of partial densities

These parameters are used in the calculation of the resulting mixture of densities. The cited code of SAS software shows not only the parameter values, but also the weight values used in the calculation.

```
f(x)=
0.155827142295696*pdf('lognormal',x,9.84413360026477,0.160228477025705)+
0.543699942098089*pdf('lognormal',x,9.90384544335508,0.537798090570426)+
0.300472915606215*pdf('lognormal',x,9.85525310618347,0.516414923348181);
```

The resulting mixture is shown in Figure 6. We further tested goodness of fit between the empirical and theoretical distributions. A null hypothesis says that there is no significant difference between the empirical and theoretical frequencies. As the Table indicates, the null hypothesis cannot be rejected on the test level $\alpha = 0.05$. The correlation between the theoretical and empirical frequencies nearly equals one, which is another indication of a good fit between these frequencies.

Age	0,00001		t-Ratio	4,716771
Count	0,00001		DF	100
Mean Difference	4.78e–7		Prob > |t|	<.0001*
Std. Error	.01e–7		Prob > t	<.0001*
Upper 95%	6.8e–7		Prob < t	1.0000
Lower 95%	2.77e–7			
N	101			
Correlation	0.99792			

Table 3 Goodness-of-fit test between the empirical and theoretical frequencies

We further applied the Wilcoxon Signed Rank test to verify the fit between the model and reality This test also confirmed the fit between the theoretical and empirical frequencies.

Wilcoxon Signed Rank	Age-Count
Test Statistic S	1,376.50
Prob>|S|	<.0001*
Prob>S	<.0001*
Prob<S	1,0000

Table 4 Wilcoxon Signed Rank test

In other words, the evaluation of the test provided good results from all viewpoints. Nothing prevents us from predicting the density for 2013. The resulting curve for the mixture of log-normal densities is shown in the following Figure. This model cannot be compared with reality, of course. On the basis of the results achieved in 2012 we have very good reason to expect that this model corresponds to the future values as well as to the most recent years' data.

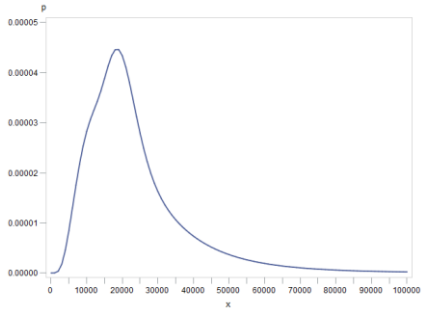

Figure 7 Density of log-normal mix for age

4.2 Using of reached results

The main aim of this study was finding a suitable model of wages. Is much better to work with a given model than with 2 million observations. Knowledge of the theoretical model allows us to perform all probability calculations such as

- construction of confidence intervals,
- estimation of the relative and absolute frequency of employees in wage intervals,
- estimation of the future value of average wages,
- estimation of the future trend of variability,
- estimation of the future values of percentiles measures etc.

Achieved estimates can also be consequently used in estimating tax revenues, in social politics (construction of minimum wage) and in many other areas of state administration. The authors will continue this study and will publish practical applications of their research in other articles.

5 Conclusions

The analyses were aimed at formulating a model for probability distribution of wages (estimating the shape of the probability density) and estimating its future shape. While older data complied with models based on single density curves, such an approach has been less successful in the more recent years. The reason for that phenomenon is given by quickly changing parameters of the wage distributions, in particular, their growing variability. Hence we suggested a new approach based on mixtures of logarithmic-normal distribution densities. In order to estimate future shape of the probability density, we had to estimate the parameters for the partial densities in our mixture. Trend analysis was utilized to this end. The results achieved for 2012 were compared with reality; the model's very good quality was thus proven. Moreover, the model underwent a series of verification tests. After that, we arrived at our estimate for 2013. The wage distribution is relatively stable in time and its time evolution is – regardless of the economic crisis – quite predictable. This fact enables us to predict future evolution further ahead than one year. Knowing the future shape of the wage distribution enables us to carry out many analyses and calculations useful in many areas of economics.

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Monte Carlo Simulation Methods as an Estimation Tool for Capital Requirements

Petra Matušková

Abstract. Risk management and solvency play a key role in the financial institutions and their functions. To ensure the solvency of financial institutions, and therefore the ability to meet their obligations at any time, the institutions must hold a certain amount of capital for risk coverage. Capital requirements are regulated by legislative framework and the main method for their determination is the Value at Risk. The Monte Carlo simulation is flexible and valuable tool for estimating Value at Risk. The basic procedure for Monte Carlo simulation is based on generating a large number of scenarios, and therefore other procedures, which reduce the number of scenarios and thus improving estimation, are applied. Monte Carlo simulation, Antithetic Sampling Monte Carlo and Latin Hypercube Sampling Monte Carlo with dependence will be used to estimate the probability distribution. Based on the results, the capital requirements will be estimated according to the rules of relevant legislation. The aim of this paper is comparison different methods of Monte Carlo simulation to estimate capital requirements for currency risk in insurance or banking sector.

Keywords: Capital requirements, Value at Risk, Monte Carlo simulation
JEL Classification: G17, G32
AMS Classification: 65C05, 60J65,

1 Introduction

With the development of economy and European market it comes to an integration and harmonization in the area of financial markets, and at the same time the emphasis is put on credibility, transparency and stability of financial institutions, which at the time of political and economic instability and not ending economic crises is getting bigger. In order to ensure protection of clients and all other aforementioned aspects, it is necessary to determine certain rules for entrepreneurship in this area, including designation of supervisory institutions, which monitor and control compliance of determined rules. Significant tool of regulation in insurance and banking sectors is monitoring of their solvency, thus insurance company's or a bank's ability to secure permanent fulfillment of liabilities by their own sources.

In banking and insurance sector the way of regulation and monitoring are gradually regulated at international level approximately from the 70's of last century. With the development of both markets and inception of new instruments, the regulatory framework was considered insufficient and therefore it is continuously replaced by new regulatory regimes. In banking it is the legislative amendment Basel II (currently its refinement within legislative framework Basel III[1] is being prepared). In insurance, it is the system Solvency II[2], which replaces current Solvency I regime (the directive should be fully implemented in the year 2004, see EIOPA (2013).

Both regimes are based on the system of three pillars, which are concentrating on capital requirements for individual risks, area of risk management, controlling, rights and responsibilities of regulatory bodies and obligations pertaining to information release and institution's transparency.

Legislatively respected method for determination of capital requirements is method Value at Risk (VaR), which is widely used in the area of finance both for risk management and for regulatory purposes.No standard method of calculation for VaR exists. One of the possibilities how to determine VaR is the utilization of Monte Carlo simulation (MC). The method is based on utilizing large amount of portfolio's value development simulations. With the emphasis on improving the estimation's effectiveness, thus decreasing the estimation's error and decreasing the amount of generated scenarios, various methods of this simulation technique were introduced, in literature referred to as reduction methods, see Boyle et all [2].

[1] More detailed information about this problematic can be found at website of Bank for International Settlements
[2] More detailed information pertaining to problematic Solvency II can be found at website of European Insurance and Occupational Pensions Authority

Objective of this paper is application and comparison of Monte Carlo's various methods simulation for the estimation of capital requirements for currency risk in insurance and bank sector. The following methods will be applied: Monte Carlo method, Antithetic Sampling Monte Carlo and Latin Hypercube Sampling with dependence Monte Carlo. Capital requirements are expressed based on Solvency II and Basel II as Value at Risk at significance levels 85 %, 99 % and 99.5 %. The comparison is executed based on various portfolios and various time periods for which the capital requirements are enumerated.

2 Value at Risk

In practice, developed and widely used method for measuring and managing risks is the value index of risk, also called Value at Risk. This method can be used to calculate capital requirements, financial risks management, integration of risks into one value etc. VaR represents the risk value, which with given probability α will not be exceeded during certain time period N, see Hull [4]. Mathematically VaR can be expressed as one sided quantile of distribution of profits and losses for certain time of holding, and it is determined based on certain historical period. It is a function which consists of two parameters: time horizon (N) and the significance level (α %). A formal equation follows:

$$\Pr\left(\Delta \widetilde{\prod} \leq -VaR\right) = \alpha. \tag{2}$$

VaR is, from theoretical point of view, relatively simple and straight forward conception; however, practical determination can be a significant statistical problem. No standard calculation method exists for the determination of VaR. The differences among individual methods lie especially in the methods of simulating changes of risk factors and in methods transforming risk factors to change portfolio's value. Alexander [1] shows, that there are three basic methods used to calculate VaR in practice: **Variance and covariance method**, which is used for estimation of potential portfolio's losses volatility and correlation, which are acquired from historical data, **Historical simulation**, where the potential future loss is estimated based on losses which happened in the past, **Monte Carlo simulation**, which works with large number of simulations of portfolio's value development and which will be further described in detail.

In case of bank subject, the capital requirement (CR) of market risk is outlined as VaR at 99% significance level for 10 days' time horizon supposing liquidation positions are held. The banks have to determine this requirement daily[3]. In case of insurance subject Solvency Capital Requirement (SCR), which corresponds to VaR, the 0,5% significance level is defined. SCR is calculated once a year.[4] Due to long-term investment horizon Minimum Capital Requirement (MCR) is further defined. It represents such a capital value under which the insurers will be subjected to unacceptable risk level. To determine MCR, the level of risk in primary capital of insurance is used at the significance level 85 % and at the time horizon of one year. At the same time, MCR must comply with other requirements based on directives Solvency II.[5]

3 Monte Carlo simulation

Monte Carlo simulation (MC) is a flexible tool modeling stochastic processes and is used to determine the value of non-linear instruments or can be used where mathematical methods fail, see Alexander [1]. Method is deriving from the law of big numbers; where the large numbers of randomly generated risk factors with selected characteristics come close to theoretical assumption, see Tichý [7]. When executing Monte Carlo simulation the following procedure can be used; based on selected probability distribution (e.g. Gaussian, Poisson, Student's distribution etc.), a vector of random numbers is generated, see below. In case that the portfolio does not contain more files, it is necessary to estimate also correlation structure by e.g. Cholesky algorithm:

$$\vec{z}^T = \vec{\varepsilon}^T \cdot P\,, \tag{3.1}$$

where $\vec{\varepsilon}^T$ represents vector of independent, random variables from distribution N(0;1), P represents upper triangular matrix derived from covariance matrix C, so called Cholesky's matrix and $\vec{z}^T$ is transposed vector $\vec{\varepsilon}^T$.

[3] Directive of European Parliament and Council 2006/49/ES dated June 14th, 2006 about capital adequacy of investment companies and credit institutions (amended).
[4] Directive of European Parliament and Council 2009/138/ES dated November 25th, 2009 about the approach to insurance and securing activity and their performance (Solvency II).
[5] Directive of European Parliament and Council 2009/138/ES dated November 25th, 2009 about the approach to securing activity and its performance (Solvency II).

Subsequently development of yield (x) of assets with selected model specifying behavior of individual portfolio instruments is simulated e.g. Brownian motion, Levy's model etc. Specifically for Brownian motion, which will be used in the paper, the development of yields can be defined as:

$$x^i = \mu \cdot \Delta t + \sigma \cdot \tilde{z}^i \cdot \sqrt{\Delta t}\ , \tag{3.2}$$

where μ is average yield , σ is standard deviation, $\tilde{z}$ is a random number from normalized normal distribution N(0;1), see (3.1), Δt is increase of time, *i* expresses *i*-th asset.

3.1 Methods of Monte Carlo simulation

As it was mentioned above, MC is based on generating large numbers of random scenarios, whose selected characteristics will come close to theoretical assumption. Estimation's error than corresponds to standard deviation of result. In 1997, Boyle et. all [2], introduced techniques, which are trying to lower estimation error (dispersion of result) and by this increasing simulation's Monte Carlo effectiveness. At the same time, it comes to reducing the number of generated scenarios and decreasing the time requirement of Monte Carlo simulation. Among these procedures of MC belong: Antithetic Sampling Monte Carlo, Stratified Sampling Monte Carlo, Control variants Monte Carlo, Moment matching Monte Carlo and others.

When applying **Monte Carlo simulation (PMC)** random elements are generated so that they correspond to characteristics of selected probability distribution. This technique is relatively quick, but the estimation will be sufficiently accurate only for large number of random scenarios. Large numbers of scenarios lead to higher time demand of simulation, see Tichý [7]. From this reason technique which enables to improve effectiveness of SMC, can be applied; e.g. Glasserman [3] or Jäckel [5] are dealing with this.

Antithetic Sampling Monte Carlo (ASMC) is for its simplicity and comprehensibility used very often in finance. The method is based on negative correlation among vectors of random elements, meaning $\rho(X, \overline{X}) = -1$, see Tichý [7]. Supposing *X* random elements $\tilde{z}$ from normalized normal distribution then by multiplying vector with coefficient -1 we will get vector $\overline{X}$ of random elements $-\tilde{z}$. By unification of both vectors we can achieve double the amount of random elements, which better fulfill characteristics of selected distribution. It is a method, which leads to the decrease of the time consumption and achievement of zero mean, meaning the symmetry of probability distribution. The method's limitations lie in the fact that it can be used only when generating random elements from symmetric probability distributions.

Stratified Sampling Monte Carlo (SSMC) is another very effective method, which in comparison to ASMC is determined by more complex procedure when it comes to division of random elements sample to smaller parts in a way that probability of random elements' occurrence $\tilde{z}$ from selected probability distribution will be the same for all partial parts. The method is useful when generating numbers with almost required characteristics for low number of scenarios and can be used also in other than just symmetrical distribution. SSMC can be divided to path-integral stratification method for given distribution or non-integral method where the inverse transformation is used. Latin hypercube sampling or Bridge sampled represent more sophisticated procedures of SSMC, see Jäckel [5], Glasserman [3] or Tichý [7].

Latin hypercube sampling (LHS) represents a procedure which efficiently extends stratified sampling to random vectors whose components are independent. This method is similar to stratified selection; its only difference is that for each variable the sequence of sub-intervals are permuted separately. This method enables generating two or more mutually independent groups of random numbers and can be used for generating processes, which consist of random numbers from the distributions with various characteristics. It is generated *i* independent samples ($k^1 \ldots k^d$) and is generated *m* independent permutations ($P^1 \ldots P^d$) of {1…n} drawn from the distribution that makes all permutations equally probable. LHS is given by:

$$L = \frac{P_i^j - 1}{n} + \frac{K_i^j}{n} \quad j = 1,\ldots d, \quad i = 1,..n \tag{3.3}$$

This method can be with dependence **(LHSD)**. Essentially, LHSD extends LHS to random vectors with dependent components. Proceed as above practice, but correlation matrix must be equal correlation matrix of sort generated component.

4 The application of Monte Carlo simulation

4.1 Problem's definition

In the case of internationally oriented market subjects the role of monetary risks plays an important role; this derives from unexpected changes of foreign currency exchange rates and as a result it changes the market price and also the subject market's position. Since insurers invest especially on European and American markets, the daily closing prices of three stock indexes, which are denominated in three different currencies, were used as input data: Down Jones Industrial Average (DJI) denominated in USD, Deutscher Aktien IndeX (DAX) denominated in EUR and FTSE 100 (FTSE) denominated in GBP. Prices of individual shares were inquired on daily basis in the period of January 1st, 2003 to January 1st, 2013 (D.).[6] For the same period exchange rates of foreign currencies to CZK were also inquired. The exchange rates announced by Czech National Bank[7] were considered. In case that some stock markets did not trade, the missing data were substituted by the value from previous business day. We have available time line of 2591 daily logarithmic yields of stock markets and exchange rates.

Further, we suppose three portfolios, which are: portfolio with minimal risk (A), market portfolio (M) and balance portfolio (B). Portfolios were determined based on conditions of Markowitz model and their composition is shown in Tab.4.1.

	DJI (%)	DAX (%)	FTSE (%)	$E(R_p)$ (%)	$\sigma(R_p)$ (%)	S	K
A	25.6835	31.4803	42.8363	0.0066	1.0372	0.1212	9.6736
M	28.9679	32.5953	38.4367	0.0069	1.0387	0.1349	9.7632
B	33.3333	33.3333	33.3333	0.0072	1.0450	0.1450	9.8462

Table 4.1 Portfolio's composition

From the aforementioned table it is evident that the portfolios do not have normal distribution. The values of describing characteristics for portfolios reached very similar values. The data are more or less symmetrically distributed around mean value; there is identified low positive skewness (S) and higher kurtosis (K) compared to normal distribution.

VaR will be calculated by methods PMC, ASMC and LHSD, at the significance levels and time horizon, which correspond to SCR, MCR a CR, meaning at 0,5% and 15% significance level for a yearlong time horizon and at 1% significance level for 10 days' time horizon. We suppose that yields have multi-dimensional normal distribution, behavior of individual portfolio's instrument follows Brown's process and 100 000 random scenarios are generated. Subsequently, used methods will be compared. It is assumed only multi-dimensional normal distribution for comparison individual MC methods, although yields have not normal distribution, because it is first step for more detailed analysis. Calculations were performed in Wolfram Mathematica 8.0.

Then, the data of initial period will be divided into four parts based on the development of economic situation on market, and capital requirements will be compared based on these periods. First period (I.) is historic series for period from 1. 1. 2003 to 31. 12. 2006 (before economic crises), next part (II.) is period from 1. 1. 2007 to 29. 8. 2008 (beginning of economic crises), third period (III.) includes data from 1. 9. 2008 to 31. 12. 2010 (economic crises) and last part (IV.) is period from 1. 1. 2011 to 1. 1. 2013 (retreat of economic crises).

Capital requirements will be determined also for data modified by the 10-days' (V.) and 250-days (VI.) moving average.

4.2 Problem's solution

By PMC, ASMC and LHSD are estimated probability distributions for each portfolio and determined portfolio´s composition, which is shown in Table 4.2.

	PMC				ASMC				LHSD			
	$E(R_p)^*$	$\sigma(R_p)^*$	S	K	$E(R_p)^*$	$\sigma(R_p)^*$	S	K	$E(R_p)^*$	$\sigma(R_p)^*$	S	K
A	0.0100	1.0391	0.0064	3.0105	0.0066	1.0314	0.0000	3.0015	0.0100	1.0391	0.0064	3.0105
M	0.0108	1.0402	0.0073	3.0126	0.0069	1.0329	0.0000	2.9997	0.0106	1.0402	0.0073	3.0126
B	0.0111	1.0463	0.0084	3.0140	0.0072	1.0393	-0.0000	2.9971	0.0111	1.0463	0.0084	3.0140

* In %

Table 4.2 Portfolio´s composition for MC methods

[6] Data available at: <http://finance.yahoo.com/>

[7] Data available at :< http://www.cnb.cz/cs/index.html>

From the above table it can be seen that individual estimations are very close to normal distribution, which is characterized by zero mean and standard deviation of one. Skewness should be at value of zero and kurtosis value of three. PMC and LHSD methods provide the same results. ASMC method provides the best estimation of the normal distribution.

The most time-consuming is LHSD method, because random values not only generated, but also it is made permutation and than acquired data it must be sorted. The least time-consuming is ASMC method, it is generated only half number of scenario.

Based on selected data and aforementioned procedures the capital requirements for currency risk were calculated for various portfolios and various time periods. The results are shown in Tab. 4.3.

PMC	**Portfolio with minimum risk**			**Market portfolio**			**Balanced portfolio**		
	SCR	MCR	CR	SCR	MCR	CR	SCR	MCR	CR
D.	40.0102	14.5373	7.5371	39.7779	14.3846	7.5092	39.9782	14.3316	7.5536
I.	28.0400	6.2752	6.2578	28.1567	6.3928	6.2737	28.3825	6.6196	6.2871
II.	53.6113	31.0737	7.4115	53.0959	30.5689	7.3777	52.9817	30.1903	7.3958
III.	53.2816	19.2511	10.2122	52.7963	18.9312	10.1774	52.8923	18.7631	10.2412
IV.	32.2983	10.0971	6.4570	32.1558	9.9443	6.4722	32.4343	9.7842	6.5387
V.	11.8757	3.5252	2.4553	11.7711	3.4085	2.4487	11.6239	3.3363	2.4361
VI.	1.4240	-0.4686	0.5034	1.3088	-0.5668	0.5062	1.2013	-0.6501	0.4834
ASMC	**Portfolio with minimum risk**			**Market portfolio**			**Balanced portfolio**		
	SCR	MCR	CR	SCR	MCR	CR	SCR	MCR	CR
D.	40.4120	15.2211	7.5183	40.3284	15.1518	7.5260	40.4482	15.2142	7.5583
I.	28.6919	7.0395	6.2584	28.9823	7.2141	6.2747	29.2202	7.4887	6.3043
II.	54.0166	31.6601	7.3979	53.2849	31.2305	7.3789	53.2849	30.9164	7.4143
III.	53.8025	20.1666	10.0462	53.6644	19.9652	10.0105	53.4567	19.8751	10.0833
IV.	32.6298	10.6927	6.4392	32.5848	10.5981	6.4883	32.8656	10.5358	6.5579
V.	12.1322	3.8437	2.4550	12.0252	3.7629	2.4508	11.9655	3.6986	2.4419
VI.	1.4993	-0.3823	0.5057	1.3848	-0.4817	0.4967	1.2718	-0.5648	0.4859
LHSD	**Portfolio with minimum risk**			**Market portfolio**			**Balanced portfolio**		
	SCR	MCR	CR	SCR	MCR	CR	SCR	MCR	CR
D.	40.0102	14.5373	7.5371	39.7779	14.3846	7.5092	39.9782	14.3316	7.5536
I.	28.0400	6.2752	6.2578	28.3825	6.2737	6.2578	28.3825	6.6196	6.2871
II.	40.0102	14.5373	7.5371	39.7779	14.3846	7.5092	39.9782	14.3316	7.5536
III.	53.2816	19.2511	10.0588	52.7963	18.9312	10.0233	52.8923	18.7631	10.0443
IV.	32.2983	10.0971	6.4570	32.1558	9.9443	6.4722	32.4343	9.7842	6.5387
V.	11.8757	3.5052	2.4553	11.7711	34.0845	2.4487	11.6239	3.4085	2.4361
VI.	1.4240	-0.4686	0.5034	1.3088	-0.6501	0.4936	1.2013	-0.6501	0.4834

Table 4.3 Capital requirements for given portfolios and selected periods in %

It is very important to realize that with growing level of significance also grows the value of VaR and by this also capital requirements value. The same stands for time horizon, for which VaR is calculated, with growing time horizon the requirements for held capital are growing. As a result of this, SCR will always reach the highest values because it is calculated at the highest levels of significance and for a year, on the contrary CR will reach always the lowest value especially because of the time horizon for which the VaR is determined.

From the capital requirements' comparison based on individual portfolio it is evident that capital requirements reach similar values and are different in tenths in all researched periods. This is given especially by portfolio's structure, which is very similar.

More distinctive differences are found within the comparisons for individual periods. Especially in periods I.- IV., which show different stages of economic development, it can be seen, that capital requirements take under consideration risk profile of given subject and if the bank or insurance company faces higher risk, it must hold more capital for undergoing risks and vice versa.

In methods PMC a ASMC in the period of stable economic situation I. the SCR is at the 28 % level, MCR at 6-7 % and CR at 6 % for all portfolios. In the period of economic crises inception II. the requirements for held capital got higher about 25 p.b., with SCR and MCR, with CR about 1 p.b. In III. period similar results are achieved as in previous period with SCR, however in MCR the values lowered, which is caused by the data selection. CR in this period grew to approximately 10 %, which correspond to theoretical specification.

Estimation of capital requirements with LHSD method better correspond with theory. Whereas, in the period of stable economic situation I. the SCR is at the 28 % level, MCR at 6 % and CR at 6 % for all portfolios. In the period of economic crises beginning II. the requirements for held capital got higher about 12 p.b. with SCR, about 8 p.b. with MCR, and with CR about 1 p.b. In III. period SCR grew again to about 53 %, MCR to about 19 % and CR to about 10 %. In last period the values lowered in SCR to about 32 %, in MCR to about 10 % and in CR to about 6 %.

Very significant differences were found in the results for initial historical period D and initial period which was modified by moving average. They were supposed to ensure that capital requirements are really calculated for one period of one year, or more precisely for 10 days. These values should better describe currency risk of given subject and by this better estimate requirements for held capital. However, results for this period do not confirm this statement and the subject's risk is either significantly underestimated or it reaches negative values, which is not realistic for all methods.

5 Conclusion

Financial institutions must hold a certain amount of capital for risk coverage. Especially risk management and solvency play a key role in the fulfillment of financial institutions 'functions. Main method for estimation capital requirements is Value at Risk. Value at Risk can be quite well determined by the Monte Carlo simulation. In this method many scenarios are generated, which leads to higher time consumption of the estimation. In order to reduce generated scenarios, and lower time consumption and by this effectiveness of estimation, various methods of Monte Carlo simulation can be applied.

From results it follows that individual methods provides similar estimation and the best estimation of normal distribution provides ASMC method, this method is also the least time-consuming. Estimation of capital requirements are similar for given portfolios, whose structures are not different too. For given periods (I.-IV.) it was found that with growing level of risk also capital requirements grow.

Achieved results are comparable and applicable, but this problem can be further extended not only by another alternatives of Monte Carlo simulation methods, but also by the possibility to select different probability distribution, for example Student distribution, which captures risk of company better. Next possibility is to use Conditional Value at Risk, which captures losses exceeding Value at Risk.

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Multiple criteria decision making with ordinal preferences

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Abstract. Multiple criteria decision making (MCDM) is an important part of everyday's life as well as it is successfully applied in many areas of human action such as management, marketing, social science, politics, etc. When preferences of a decision maker with regard to given criteria are in the form of rankings (orderings) of n alternatives from the 1st to the n^{th} place, then it is MCDM with ordinal preferences. Although such preferences are common in many situations (rankings of enterprises, products, sportsmen, political parties, favourite meals, etc.), the theory focused mainly on a group decision making with the one overall criterion (a part of the social choice theory), and ordinal information about multiple criteria was usually converted into cardinal information by Borda-Kendall's method of marks or a similar method. So far, there is no theory of MCDM with strictly ordinal preferences. Hence, the aim of the article is to propose a method for the solution of the MCDM with ordinal preferences based on pair-wise comparisons of all alternatives with regard to each and every criterion and a dominance relation enabling the selection of the best alternative. The proposed approach is natural, computationally simple and can be useful in many areas of human action. The use of the method is illustrated by an example of a selection of the most suitable school.

Keywords: multi-criteria decision making, pair-wise comparisons, ordinal preferences, ranking.

JEL Classification: C61, D89
AMS Classification: 62F07

1 Introduction

In multi-criteria decision making (MCDM) with ordinal information a set of n objects is ranked from the 1st to the n^{th} position (from the best to the worst) with regard to a set of k criteria, which are also ranked from the 1st to the k^{th} position according to their importance, and no cardinal information (especially about criteria weights) is available.

Usually, in such setting ordinal information (about criteria weights) is turned into cardinal one. One easy way is to use Borda-Kendall's method of marks (counts), see [1] or [5]. In this approach each alternative is assigned the number of points (marks) corresponding to its rank under given criterion, and an alternative with the lowest total sum (or average) of marks is the best. However, this and similar methods treat positions in rankings as weights, but this is ad-hoc assumption as real weights are not known. Cook and Kress (see [3-4]) proposed more sophisticated methods to generate cardinal weights of criteria out of ordinal information. They proposed to use weights of positions in a ranking proportional to the factor $1/n$, where n is a position in a ranking, and to use a data envelopment analysis (DEA) approach, in which criteria weights are searched by DEA method so that a utility function for all alternatives (called *composite ranking*) attains its maximum. Also, several methods for group decision making with ordinal information were proposed, see e.g. [2] or [6-8], but none of these methods is suitable for multiple criteria.

At present there is no strictly ordinal MCDM method known to author though ordinal information might be sufficient to compare alternatives in many cases. This is somewhat surprising as ordinal way of comparison of objects is more general and also more natural than cardinal approach. We rank objects in accord with our preferences every time when we are selecting a car, a computer, a drink or a dinner in a restaurant, a holiday destination, etc. We rank objects by criteria such as beauty, style, design, safety, taste, etc., which cannot be naturally converted into some numerical, e.g. Saaty's 1-9 fundamental, scale.

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Hence, the aim of the article is to propose a simple method for MCDM with strictly ordinal information about criteria and alternatives. The use of the method is illustrated by examples, and its results are compared to results obtained by Borda-Kendall's method of marks and Cook-Kress method.

2 Formulation of the problem and the method for its solution

Setting of the problem *of multi-criteria decision making with ordinal information* considered in this article is as follows:

Let $C = \{C_1,...,C_k\}$ be the set of criteria and let $AL = \{A, B, C, ..\}$ be the set of *n* alternatives. Let the ordering of all criteria according to their importance be as follows: $C_1 \succ C_2 ... \succ C_k$, so C_1 is the most important criterion and C_k is the least important criterion. Let all alternatives be ranked from the best to the worst by all criteria C_i (such rankings are nothing else than permutations of all alternatives). The goal of the problem is to rank the alternatives from the best to the worst with regard to all criteria.

Definition 1. *For each pair of alternatives A and B there is a binary index-vector* $I_{(A,B)}$ *such that* $I_{(A,B)} = (i_1,...,i_k)$ *, where* $i_j = 1$ *if an alternative A is ranked better than B with regard to a criterion j, otherwise* $i_j = 0$.

Example 1. Consider four criteria (from the most important C_1 to the least important C_4) and an alternative *A* is ranked better than *B* by criteria C_2 and C_4. Then $I_{(A,B)} = (0,1,0,1)$. As a consequence of Definition 1 there exists also an index-vector $I_{(B,A)}$ such that $I_{(B,A)} = (1,...,1) - I_{(A,B)}$, so: $I_{(B,A)} = (1,0,1,0)$. Both $I_{(A,B)}$ and $I_{(B,A)}$ are binary and are inverse to each other. If both alternatives are ranked in the same position (see section 5) by a given criterion, then relation $I_{(B,A)} = (1,...,1) - I_{(A,B)}$ is still applicable.

Index-vectors provide immediate information about alternatives' pair-wise comparisons by each and every criterion, and they can be used for determining overall dominance between two alternatives. If an alternative *A* dominates some other alternative *B*, it means that *A* is better (ranked higher) than *B* overall (by all criteria).

Definition 2. Let $H_{(A,B)} = (h_1,...,h_k)$ *be a cumulative index-vector such that* $H_{(A,B)} = \left(i_1, \sum_{j=1}^{2} i_j, ..., \sum_{j=1}^{k} i_j \right)$. $H_{(B,A)}$ *is defined analogously from the index-vector* $I_{(B,A)}$ *, and let* $H_{(B,A)} = (h_1^*,...,h_k^*)$.

Definition 3a *(a pair-wise dominance relation). An alternative A dominates an alternative B (* $A \succ B$ *) iff :*

$$h_i \geq h_i^* \quad \forall i \in \{1,...,k\}, \tag{1}$$

and at least one inequality in (1) is strict (and vice versa).

Example 2. From Example 1 we have $H_{(A,B)} = (0,1,1,2)$ and $H_{(B,A)} = (1,1,2,2)$. According to Definition 3a) an alternative *B* dominates an alternative *A*, because: $h_i^* \geq h_i \quad \forall i \in \{1,...,4\}$, and from these four inequalities two inequalities (for *i* = 1 and *i* = 3) are strict.

A pair-wise dominance relation from Definition 3a) can be equivalently reformulated without the use of mathematical notation either:

Definition 3b *(equivalent formulation of a dominance relation). An alternative A dominates an alternative B (* $A \succ B$ *) iff for each criterion by which B is ranked better than A, there is a unique and more important criterion, by which A is ranked better than B.*

Consequence of Definitions 3a)-b). If an alternative *A* dominates an alternative *B* then *A* is ranked better than *B* by the same or higher number of criteria than vice versa. And also, an alternative *A* is ranked better than *B* by the most important criterion. On contrary if *A* is ranked better than *B* by the most important criterion, then *A* dominates *B* or both alternatives are incomparable.

According to Definition 3b) if one alternative is ranked better by more important criteria (and also by majority of criteria or at least by the same number of criteria) then it should be ranked better overall too. In Example 1 an alternative *A* is ranked better than *B* by criteria C_2 and C_4, so an alternative *B* is ranked better than

A by criteria C_1 and C_3. No matter what are the (unknown) weights of all criteria, *B* should be ranked better than *A* overall, because it is ranked better by more important criteria (C_1 is more important than C_2, and C_3 is more important than C_4) than *A*.

The dominance relation from Definition 3 induces only quasi-order on a set of all alternatives. Hence, there might be pairs of alternatives which are incomparable, as in the following Example 3.

Example 3. Consider four criteria (from the most important C_1 to the least important C_4) and an alternative *A* is ranked better than *B* by criteria C_1 and C_4, so $I_{(A,B)} = (1,0,0,1)$ and $I_{(B,A)} = (0,1,1,0)$, $H_{(A,B)} = (1,1,1,2)$ and $H_{(B,A)} = (0,1,2,2)$. As can be seen from both cumulative index-vectors, there is no dominance between *A* and *B*, because $H_{(A,B)}$ has greater value on the first position, while $H_{(B,A)}$ has greater value on the third position. Hence, we cannot conclude, which alternative is ranked better overall.

In Example 3 for some set of weights an alternative *A* can be evaluated better in overall (typically when a criterion C_1 acquires large weight), while for some other set of weights an alternative *B* might be evaluated better (when C_2 and C_3 acquire almost the same weight as C_1, and C_4 acquires low weight).

3 The model for MCDM with ordinal preferences

The proposed model for MCDM with ordinal information proceeds in the following 5 steps:

1. All criteria are ranked from the most important to the least important.
2. All alternatives are ranked from the best to the worst with regard to all criteria.
3. Index-vectors and cumulative index-vectors for all pairs of alternatives are established.
4. All pairs of alternatives are compared by a pair-wise dominance relation (1).
5. All alternatives are ranked into one final ranking or more final rankings in a case of ties among alternatives. The final overall ranking of all alternatives is every ranking consistent with all alternative pair-wise comparisons by Definition 3.

As for a solution of the MCDM problem with ordinal information by the presented model in general, there are three possible cases:

- *Case 1*. The final overall ranking of all alternatives constitutes a complete order on a set of alternatives. Then the best alternative is unique and constitutes the best solution to a given problem.
- *Case 2*. The final overall ranking of all alternatives constitutes only a quasi-order (some alternatives cannot be compared and are regarded as 'ties') on a set of all alternatives, but the best alternative is unique and constitutes the best solution to a given problem. The fact that some other (worse) ranked alternatives are incomparable is usually unimportant.
- *Case 3*. The final overall ranking of all alternatives constitutes only a quasi-order on a set of all alternatives, and the best alternative is not unique. In such a case the best solution to a given problem cannot be found by the described method. All best alternatives can be considered equally good. A decision maker might try to repeat the method with only these best alternatives, or he/she can use some other method of MCDM, this time with cardinal information (such as AHP/ANP), which might be more suitable for a given problem.

4 A numerical example

Consider situation where parents have to choose the most suitable school for their child from six available schools (abbreviated by letters A, B, C, D, E and F). They have four criteria:

- *Location* (C_1): parents must consider the distance to a given school and its accessibility by foot, bike or a bus.
- *School educational programme* (C_2): in the Czech Republic each elementary and secondary school has its unique school educational programme (called 'školní vzdělávací program' in Czech), usually with emphasis on foreign languages, science, mathematics or sport.
- *Quality of education* (C_3): available school might differ significantly in quality due to different quality of teachers, management, equipment (such as computers, Internet, smartboards, specialized classrooms for science or arts lessons, etc).

- *Offer of after-school group hobbies and sport activities* (C_4): some schools offer variety of activities ranging from sports to arts.

All the criteria above are ordinal in nature, as there is no sense in using cardinal criteria for the evaluation of e.g. quality of education; hence the problem is suitable for the presented MCDM with ordinal preferences. The goal is to find the overall ranking of all schools (Case 1) or at least (in the Case 2) the unique best school.

For the solution of the problem the method described in previous Section is used:

Step 1: All criteria are ranked from the most important (1.) to the least important (4.):

1. Location , 2. School educational programme, 3. Quality of education, 4. Offer of after-school group hobbies and sport activities.

In other words, the relation of importance is as follows: $C_1 \succ C_2 \succ C_3 \succ C_4$.

Step 2: All alternatives as ranked from the best to the worst with regard to all criteria, see Table 1.

Criterion/position	**1.**	**2.**	**3.**	**4.**	**5.**	**6.**
$\boldsymbol{C_1}$	A	D	C	E	B	F
$\boldsymbol{C_2}$	C	A	B	F	E	D
$\boldsymbol{C_3}$	E	B	A	D	C	F
$\boldsymbol{C_4}$	D	A	C	B	F	E

Table 1 Rankings of schools with regard to all criteria.

Step 3: Index-vectors and cumulative index-vectors for all pairs of alternatives are established:

- Pair *A*-*B*: $I_{(A,B)} = (1,1,0,1)$, $I_{(B,A)} = (0,0,1,0)$, $H_{(A,B)} = (1,2,2,3)$, $H_{(B,A)} = (0,0,1,1)$.
- Pair *A*-*C*: $I_{(A,C)} = (1,0,1,1)$, $I_{(C,A)} = (0,1,0,0)$, $H_{(A,C)} = (1,1,2,3)$, $H_{(C,A)} = (0,1,1,1)$.
- Pair *A*-*D*: $I_{(A,D)} = (1,1,1,0)$, $I_{(D,A)} = (0,0,0,1)$, $H_{(A,D)} = (1,2,3,3)$, $H_{(D,A)} = (0,0,0,1)$.
- Pair*A*-*E*: $I_{(A,E)} = (1,1,0,1)$, $I_{(E,A)} = (0,0,1,0)$, $H_{(A,E)} = (1,2,2,3)$, $H_{(E,A)} = (0,0,1,1)$.
- Pair*A*-*F*: $I_{(A,F)} = (1,1,1,1)$, $I_{(F,A)} = (0,0,0,0)$, $H_{(A,F)} = (1,2,3,4)$, $H_{(F,A)} = (0,0,0,0)$.
- Pair *B*-*C*: $I_{(B,C)} = (0,0,1,0)$, $I_{(C,B)} = (1,1,0,1)$, $H_{(B,C)} = (0,0,1,1)$ $H_{(C,B)} = (1,2,2,3)$.
- Pair *B*-*D*: $I_{(B,D)} = (0,1,1,0)$, $I_{(D,B)} = (1,0,0,1)$, $H_{(B,D)} = (0,1,2,2)$ $H_{(D,B)} = (1,1,1,2)$.
- ...

Step 4: All pairs of alternatives are compared by a pair-wise dominance relation (1):

- Pair *A*-*B*: $A \succ B$, because each coordinate of $H_{(A,B)} = (1,2,2,3)$ is equal or higher than corresponding coordinate of $H_{(B,A)} = (0,0,1,1)$.
- Similarly (after all 15 pairs were compared), the following 13 dominance relations were obtained:

$$A \succ B,\ A \succ C,\ A \succ D,\ A \succ E,\ A \succ F,\ C \succ B,$$

$$C \succ E,\ C \succ F,\ D \succ C,\ D \succ F,\ E \succ B,\ E \succ F,\ B \succ F.$$

Step 5: The final overall ranking of all alternatives have to be consistent with these 13 relations. Two remaining pairs were incomparable, namely: *B*-*D* and *D*-*E*. The (only) final overall ranking is:

$$(A, D, C, E, B, F)$$

Hence, the best (most suitable) school is ***A***.

This result was compared to the result of Borda-Kendall's method of marks, see Table 2. By this method we obtain the final ranking:

$$(A, C, D, B, E, F).$$

The difference between both methods rests only in the ranking of pairs *C-D* and *B-E*.

Overall ranking	**1.**	**2.**	**3.**	**4.**	**5.**	**6.**
Alternative	A	C	D	B	E	F
Sum of points	8	12	13	14	16	21

Table 2 Rankings of schools with regard to Borda-Kendall's method of marks.

Now suppose (in accord with [3]) that the weights of criteria are in the form of $w_j = c / j$, where j is the ranking of the j^{th} criterion and c a constant such that $\sum_{i=1}^{k} w_i = 1$, and a value of an alternative ranked at i^{th} position in a ranking is $f_i = d / i$, where d is a constant such that $\sum_{i=1}^{n} f_i = 1$. Then we obtain: c = 12/25 and d = 49/20, weights of criteria are provided in Table 3 and weights of positions in Table 4. Finally, each alternative is assigned its overall value (score) for attained positions in all rankings by simple multiplication of weights of positions and weights of criteria, and alternatives are ranked from the best to the worst, see Table 5. The result differs from the proposed method in the ranking of the pair *C-D*.

Criterion	C_1	C_2	C_3	C_4
Weight	0.48	0.24	0.16	0.12

Table 3 Weights of criteria.

Position	**1.**	**2.**	**3.**	**4.**	**5.**	**6.**
Weight	0.408	0.204	0.136	0.102	0.082	0.068

Table 4 Weights of positions in a ranking.

Overall Position	**1.**	**2.**	**3.**	**4.**	**5.**	**6.**
Alternative	A	C	D	E	B	F
Score	0.291	0.193	0.180	0.142	0.117	0.078

Table 5 Overall ranking of all alternatives with their scores.

5 Extensions

The proposed method can be extended to include ties between alternatives or uncertainty in a ranking of two or more alternatives. In the latter case an expert might express an intensity of preference between alternatives. Also, the method can be adopted for group decision making if criteria are substituted by decision maker's preferences.

- *Ties*: When a decision maker is not sure, whether an alternative *A* is better than an alternative *B* or vice versa, than he/she can assign both alternatives 0.5 points. In such a case index-vectors are not binary any more, but the method (the model) does not change. From index-vectors cumulative index-vectors are derived, and the dominance relation (1) is used in the same way as before to acquire the final ranking of alternatives.

- *Uncertainty*: The approach above can be further generalized to include uncertainty or intensity of preference. If $p_{j,k} \in [0,1]$ is the intensity of preference of an alternative *J* over an alternative *K* from a

set of alternatives A, $p_{k,j} \in [0,1]$ vice versa, and $p_{k,j} + p_{j,k} = 1$, then the relation $p : A \times A \to [0,1]$ is called the fuzzy preference relation. Index-vectors now include values from the interval $[0,1]$, and the method can be used as before again, see [6-7].

- *Group decision making*: if criteria are replaced by decision makers, than we obtain a group decision making problem, where the final ranking of alternatives expresses a consensus among decision makers.

Example 4. Consider five criteria (from the most important C_1 to the least important C_5) and two alternatives: A and B. An alternative A is ranked better than B by criteria C_1 and C_2, the same as B by a criterion C_3, and worse than B by remaining criteria. Then:

$$I_{(A,B)} = (1,1,0.5,0,0)\,,\; I_{(B,A)} = (0,0,0.5,1,1)$$
$$H_{(A,B)} = (1,2,2.5,2.5,2.5)\,,\; H_{(B,A)} = (0,0,0.5,1.5,2.5)\,.$$

According to the dominance relation (1) an alternative A dominates B.

6 Conclusions

The aim of the article was to demonstrate new method for multi-criteria decision making with (only) ordinal preferences about criteria and alternatives. The proposed method is computationally simple and intuitive, so it can be used in many areas of multi-criteria decision making where the problem is rather ordinal than cardinal in nature. Further research may focus for example on problems with incomplete rankings or problems with dependent criteria, as in this paper it was assumed that all citeria were independent.

To facilitate decision making in problems with the larger number of alternatives, the software support tool in the form of Microsoft Excel Add-in would follow soon.

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On Decision Making under Risk with Continuous States of Nature

Jiří Mazurek[1]

Abstract: Many real-world decision making situations, especially in business, are associated with uncertainty regarding future state of the World. Traditionally, in such situation different (discrete or fuzzy) *scenarios* – future states of nature – are considered. This domain of decision making is denoted as decision making under risk. However, limitation to some set of discrete states of nature is somewhat unnatural as future reality might not choose one of considered states, but some other state or a state in between. The aim of this paper is to propose the more natural approach with continuum states of nature, where all possible future states expressed by their probability density function from some reasonable interval are taken into consideration. Alternatives are assigned an expected (mean) value, which enables finding the best alternative. This approach is illustrated by a numerical example where future states of nature are distinguished by different economic growth, and also it is compared with the corresponding decision making under risk with discrete and fuzzy states of nature.

Keywords: continuous states of nature, decision making under risk, discrete states of nature, expected mean value, fuzzy states of nature.

JEL: D81

AMS: 90B50,91B06,91B16,91B30.

1. Introduction

Many real-world decision making situation in economics, politics, environmental protection, etc. are associated with uncertainty regarding future state of the World. When the probabilities of future states of the World (also called 'scenarios') are known, then the problem is referred as *decision making under risk*. When the probabilities of scenarios are not known, then the problem is referred as *decision making under uncertainty*. To deal with the both situations the *expected utility function approach* was proposed already by von Neumann and Morgenstern in [8]: the best alternative is the alternative with the highest expected value of a utility function over all scenarios. Later, this approach was enriched by a notion of a *risk aversion*, see [2] or [6], and today decision making under risk or uncertainty has numerous applications in many areas of human action, see for example [1] or [5]. Works of Klir and Folger [7] or Raiffa [9] can be considered a modern primer on decision making under uncertainty in general.

Now, let's consider economic situation at the beginning of the year 2013 from a point of view of a small enterprise. Will the world experience another recession or not? Should they hire new employees and expand, or should they rather turn into cost-saving regime?

Traditionally, in such situation different (and discrete) *scenarios* – future states of nature – are considered. Imagine that one possible scenario would be small economic growth (say 1% of GDP), the second stagnation (0% of GDP), and the third small decline (–1% of GDP). Each scenario can be assigned its probability (e.g. by some financial expert). The best alternative for firm's behavior can be evaluated from the utility function for all alternatives under all scenarios, with the best alternative achieving the highest value. This approach is called the *decision making under risk with discrete states of nature*.

However, one doesn't have to limit his thoughts to just three scenarios (cases), which are often somewhat artificial (why we use exactly 1%, and not, let's say, 0.87%?). More natural approach would be to evaluate all scenarios between some reasonable limits, say between –1% to +1% of GDP growth in the next year. This approach can be called the *decision making under risk with continuous states of nature*.

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The aim of this article is to formulate the decision making under risk with continuum states of nature, illustrate its use on a numerical example, and to compare it with decision under risk with discrete and fuzzy states of nature.

The paper is organized as follows: Section 2 decision making under risk with discrete states of nature is briefly described, in Section 3 a continuum of states of nature is introduced and decision making with fuzzy states of nature is described in Section 4. A numerical example with all three approaches is provided in Section 5. Conclusions close the article.

2. Decision making under risk with discrete states of nature

Let S_1 to S_n be future states of nature with probabilities p_1 to p_n, such that $p_i \geq 0$ and $\sum_{i=1}^{n} p_i = 1$; let A_1 to A_k be alternatives and v_{ij} pay-offs for *i*-th alternative under *j*-th state of nature. All this information can be described in a *decision matrix* (see Table 1). Furthermore, it is supposed that higher pay-offs are considered 'better'.

	S_1	S_2	...	S_n
A_1	v_{11}	...	...	v_{1n}
A_2	...	...	...	...
...	...	...	...	...
A_k	v_{k1}	...	...	v_{kn}
	p_1	p_2	...	p_n

Table 1 A general form of a decision matrix in a decision making under risk.

A utility function of an alternative *j* under all states of nature (all scenarios) is given as [10]:

$$E_j = \sum_{i=1}^{n} p_i \cdot v_{ij} \tag{1}$$

Hence, each alternative *j* is assigned the expected value E_j, and the best alternative is that with the maximum expected value:

$$\max_k E_k = \sum_{i=1}^{n} p_i \cdot v_{ij} \tag{2}$$

A numerical example of this approach is provided in Section 4.

3. Decision making under risk with continuous states of nature

Let $S(x)$ be a continuum of states of nature, $x \in [a,b]$, let $\upsilon_i(x)$ be a pay-off function for the *i*-th alternative and let $p(x)$ be a probability density function of states $S(x)$, satisfying the following conditions:

$$p(x) \geq 0 \text{ for } x \in [a,b] \tag{3a}$$

$$\int_a^b p(x)dx = 1 \tag{3b}$$

Then, by analogy with the utilitily function (1), each alternative *j* is assigned the number (expected value of a utility function) over all scenarios:

$$E_j = \int_a^b p(x)v_j(x)dx \tag{4}$$

Again, the best alternative attains the maximum expected value (4).

The venue of this approach is that we don't have to break our considerations into small number of (somewhat artificially and deliberately chosen) cases, as we are considering all possibilities (future scenarios) bounded by some limits *a* and *b*. Secondary, we can focus on arbitrary subinterval of interest between limits *a* and *b* (if there is a good reason to narrow the limits), which is not possible with discrete scenarios.

The weakness of this approach rests in the fact that we have to describe $p(x)$ and $v_1(x)$ as functions, which might be difficult in practice especially for the probability density function due to constraints (3a) and (3b). However, complicated functions can be approximated by simpler ones, such as linear, quadratic, exponential or logarithmic functions.

4. Decision making under risk with fuzzy states of nature

Apart from discrete and continuous states of nature also fuzzy states of nature might be considered, see [3] or [4]. In such setting S_i are different future states of nature $\left(i \in \{1,...,M\}\right)$, A_j alternatives $\left(j \in \{1,...,K\}\right)$ and F_k different fuzzy states of nature $\left(k \in \{1,...,N\}\right)$. Furthermore, υ_{ij} is a pay-off of an alternative A_i under the fuzzy state of nature F_j, $\mu_{F_k}(S_i)$ is a membership value of a state S_i within a fuzzy state of nature F_k and $p(S_i)$ is a probability of a state of nature S_i.

Membership values $\mu_{F_k}(S_i)$ must satisfy so called orthogonality condition (5), by relation (6) probabilities of fuzzy states of nature are computed, and expected mean values (EMVs) of alternatives' pay-offs are obtained by relation (7), so alternatives can be sorted from the best to the worst.

$$\sum_{i=1}^{N} \mu_{F_i}(S_k) = 1 \tag{5}$$

$$p(F_i) = \sum_{k=1}^{M} \mu_{F_i}(S_k) \cdot p(S_k) \tag{6}$$

$$EMV_j = \sum_{i=1}^{K} v_{ji} \cdot p(F_i) \tag{7}$$

5. A numerical example

In this section an illustrative example on both approaches is provided. We turn back to the example in Introduction: Let there be scenarios with the different economic growth (from 0% to 4% GDP) and three alternative strategies of some enterprise with the corresponding pay-offs. The goal is to find the best strategy under given assumptions.

At first we consider the discrete states of nature: Let S_1 to S_5 be (distinct) future states of nature with probabilities p_1 to p_5, such that $p_i \geq 0$ and $\sum_{i=1}^{5} p_i = 1$; let A_1 to A_3 be alternative strategies and let v_{ij} be pay-offs for *i*-th alternative under *j*-th state of nature, see Table 2.

	S_1 (0%)	S_2 (1%)	S_3 (2%)	S_4 (3%)	S_5 (4%)
A_1	4	5	6	7	8
A_2	1	2	4	8	16
A_3	3	5	7	9	11
p_i	0	0.25	0.5	0.25	0

Table 2 A decision matrix of an illustrative example.

Using relation (1) we obtain the expected values of a utility function for all three alternative strategies:

$$E_1 = \sum_{i=1}^{5} p_i \cdot v_{ij} = 0 \cdot 4 + 0.25 \cdot 5 + 0.5 \cdot 6 + 0.25 \cdot 7 + 0 \cdot 8 = 6\,,$$

$E_2 = 4.5$,

$E_3 = 7$

Hence, the best strategy is A_3.

When continuous states of nature are considered, $p(x)$ and $v_i(x)$ for each alternative i must be expressed as functions. In this case we might construct these functions from the discrete case in Table 2, as row values of Table 2 can be regarded as an output of linear or exponential functions. The function $p(x)$ for all scenarios is shown in Figure 1.

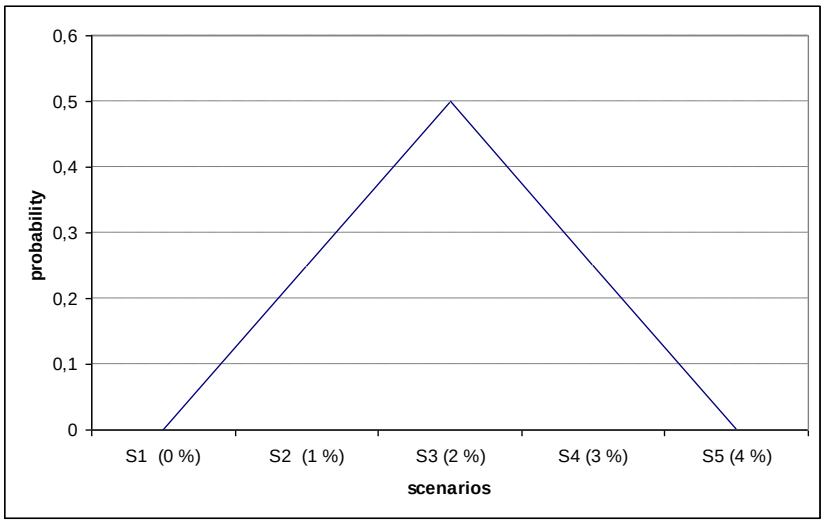

Figure 1 The probability density function of a continuum of states of nature.

Let $p(x) = 0.25x$ for $x \in [0,2]$ and $p(x) = 1 - 0.25x$ for $x \in [2,4]$, (it is easy to verify that constraints (3a) and (3b) for the probability density function are satisfied),

$v_1(x) = x + 4$,

$v_2(x) = 2^{x-1}$,

$v_3(x) = 2x + 3$,

$a = 0, b = 4$.

Using relation (4) we obtain the expected values of a utility function for all three alternative strategies:

$$E_1 = \int_0^4 p(x)v_1(x)dx = \int_0^2 0.25x \cdot (x+4)\,dx + \int_2^4 (1 - 0.25x) \cdot (x+4)\,dx = 6,$$

$E_2 = 4.683,$

$E_3 = 7.$

The highest expected value is attained by the strategy A_3. This result is the same as for the discrete case (because pay-off functions were symmetric and linear, but for other pay-off functions the result would differ in general). Nevertheless, the continuous approach is more general in nature, as $p(x)$ and $v_i(x)$ are functions, not only given numbers, which opens more space for modelling scenario probabilities as well as pay-offs of alternatives. Also, this approach can be easily extended to group decision making under risk with the use of suitable aggregation operators. A risk aversion could be included into a continuous model as well.

In terms of decision making under risk with fuzzy states of nature Example 4 can be reformulated in the following way: Let S_1 to S_5 be future states of nature, let A_1 to A_3 be alternative strategies, and let υ_{ij} be a pay-off of an alternative A_i under the fuzzy state of nature F_j. Now, let F_1 to F_3 be three fuzzy states of nature corresponding to *low*, *medium* and *high* economic growth. A decision matrix with alternatives' pay-offs υ_{ij} under different fuzzy states of nature is shown in Table 3 (the values correspond to the pay-offs in Table 2). Table 4 provides membership values of all considered states of nature (given by some expert) and in Table 5 probabilities of fuzzy states of nature are shown.

With the use of relations (5)-(7) expected mean values (EMVs) of alternatives' pay-offs were obtained and are shown in Table 3, the last column. For example expected mean value of the alternative A_1 is obtained from (7) as follows:

$$EMV_1 = \sum_{i=1}^{3} v_{1i} \cdot p\left(F_i\right) = 4.5 \cdot 0.2 + 6 \cdot 0.7 + 7.5 \cdot 0.1 = 5.85 .$$

According to these EMVs, the best alternative is A_3 again.

Alternatives	F_1	F_2	F_3	EMV
A_1	4.5	6	7.5	5.85
A_2	1.5	4	12	4
A_3	4	7	10	6.7

Table 3 Alternatives' pay-offs under different fuzzy states of nature.

States of nature	F_1	F_2	F_3
S_1	1	0	0
S_2	0.8	0.2	0
S_3	0	1	0
S_4	0	0.6	0.4
S_5	0	0	1

Table 4 Membership values of states of nature.

$p(F_1)$	0.2
$p(F_2)$	0.7
$p(F_3)$	0.1

Table 5 Probabilities of fuzzy states of nature

6. Conclusions

The aim of this article was to demonstrate the use of a decision making under risk with continuum states of nature, which can be considered more natural and also more general approach than a decision making under risk with discrete states of nature. In this approach probability density function of states is modeled by a suitable (linear, quadratic, exponential, etc.) function. This approach was compared to the decision making under risk with discrete and fuzzy states of nature respectively. While the use of discrete states of nature is appealing for its simplicity, it is also unnatural and somewhat ad-hoc approach. Application of fuzzy states of nature seems to be more realistic, but it also requires more input data in the form of probabilities of different crisp states of nature into fuzzy states of nature, which might be difficult to assess in real-world problems.

Future research might focus on a comparison of similarity between decision making under risk with discrete and continuous states respectively, or on empirical analysis of suitability of the latter approach in different areas of (economic) decision making.

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On the analytic hierarchy process with conflicting criteria

Jiří Mazurek[1], Zuzana Kiszová[2]

Abstract: In the Analytic Hierarchy Process (AHP) developed by T. L. Saaty elements from one hierarchical level (such as alternatives) are pair-wise compared with regard to each element from immediately higher level of hierarchy (e.g. by criteria) on 1-9 scale. By this procedure weights of criteria are established by the eigenvalue method and the best alternative is found with regard to a given goal. However, in real-world multiple criteria decision making (MCDM) problems some criterion might be in conflict with other criteria, but this problem is completely neglected in the standard AHP. The aim of the article is to propose a new method for the evaluation of criteria weights based both on their pair-wise comparisons and the degree of a conflict/agreement among criteria. This degree for each pair of alternatives is expressed by a value from [-1,1] interval, where positive values denote an agreement, while negative values denote a conflict, and all the degrees form a conflict matrix. Information from the conflict matrix is converted into criteria (sub)weights, and the final weights of criteria are given as an aggregation of weights from pair-wise comparisons and from the conflict matrix. The use of the method is illustrated by an example.

Keywords: Analytic hierarchy process (AHP), conflicting criteria, criteria weights, multiple criteria decision making

JEL Classification: C02, C44
AMS Classification: 90B50, 91B06, 15A16, 15A18, 15B51

1 Introduction

The objective of multiple criteria decision making (MCDM) is to select the best alternative from a set of feasible alternatives with regard to a given set of criteria. One of the most popular tools for MCDM is the analytic hierarchy process (AHP) of T. L. Saaty, see [8-10]. AHP approach was successfully used in many areas of economics, see e.g. [1-3], [6-7] or [9-11].

However, sometimes criteria might be in a conflict. By a conflict we mean a negative dependence between criteria, as criteria are seldom independent. For example applicant for a job is supposed to be young, dynamic and flexible on one hand, also experienced on the other hand. But experience is usually gained by time. Also, when selecting the most suitable location for a new house, people often prefer a quiet place, but they want to live nearby a town's centre so they can use its infrastructure. But these requirements are clearly contradicting, as it is difficult to find a calm place near cities' centres.

The problem of dependence among criteria is studied intensively for example in the analytic network process (ANP) framework (see e.g. [10-11]) or in the context of fuzzy cognitive maps ([4]), but the problem of conflicting criteria was rather neglected so far.

Hence, the aim of the article is to demonstrate how conflicting criteria can be handled within AHP framework. The proposed method uses Saaty's eigenvalue method for the evaluation of criteria weights w, but is also uses a fuzzy cognitive map approach (see [4]), enabling to express dependence and feedback among concepts with different intensity given by a real number from [–1,1] or [0,1] interval. In the proposed method criteria are assigned also weights m corresponding to the degree of their concord or conflict with other criteria. Concordant criteria attain generally larger weights, while conflicting criteria attain smaller weights (some criteria might achieve zero weights, if they are strongly contradicting other criteria), and final criteria weights v are obtained by aggregation of partial weights w and m. The method is described in detail in section 3, while section 2 provides brief description of AHP, in section 4 the use of the method is illustrated by an example, and Conclusions close the article.

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2 The analytic hierarchy process (AHP)

The main principal of AHP is to organize elements as goal, criteria (sub-criteria) or alternatives in a hierarchical structure. Pair-wise comparison matrix is the main tool of AHP. It is irreducible nonnegative $n \times n$ matrix which entries are results of pair-wise comparisons. The objects $x_1, x_2, \ldots, x_n$ are compared in pairs to each other to express preference intensity on the scale presented by the closed interval: $s_{ij} \in [1/9, 1/8, 1/7, \ldots, 1, \ldots, 7, 8, 9]$. Pair-wise comparison matrix $S = \{s_{ij}\}$ satisfies the *multiplicative reciprocity* property:

$$s_{ij} \cdot s_{ji} = 1, \text{ for all } i, j = 1, 2, \ldots, n. \tag{1}$$

If object *i* is more preferred than object *j*, the result s_{ij} satisfies the inequality $1 < s_{ij} \leq 9$. If object *i* is less preferred than object *j*, the entry s_{ij} satisfies the condition $1/9 \leq s_{ij} < 1$. If these two objects are equally preferred, then $s_{ij} = 1$. The equality $s_{ii} = 1$ for all *i* is obvious. Entries of pair-wise comparison matrix express estimation of weight ratio of two compared objects. The real weights are calculated using the characteristic equation:

$$Sw = \lambda_{\max} w, \tag{2}$$

where $\lambda_{\max}$ is the maximal eigenvalue and w is corresponding eigenvector, which elements express weights of compared objects.

When weights of criteria and weights of alternatives with respect to given criterion are gained, it is possible to calculate overall weights of alternatives with regard to the goal. If weight of *i*-th criterion is w_i and weight of *j*-th alternative with respect to criterion f_i is $v_j(f_i)$, the overall weight E_j of *j*-th alternative is

$$E_j = \sum_{i=1}^{m} w_i \cdot v_j(f_i), \tag{3}$$

where $j = 1, 2, \ldots, n$. On the basis of overall weights it is possible to rank alternatives from the best to the worst. Of course the best alternative gains the highest weight and vice versa.

3 AHP with conflicting criteria

In AHP criteria weights are derived from pair-wise comparisons of criteria's relative importance. However, this method does not take into account that some criteria might be in conflict with criteria. To incorporate this feature into AHP, we propose a two stage method described in this section.

In our approach criteria weights *w* are established by the standard AHP procedure with pair-wise comparisons of criteria, and weights of criteria are estimated by the eigenvalue method (the same comparison is applied for alternatives).

In the second stage another criteria weights *m* are established from criteria' conflicts expressed by a value from $[-1, 1]$ interval (which is then converted into $[0, 1]$ interval). Negative values indicate conflict (or inverse proportionality) between criteria, while positive values indicate a concord (direct proportionality). The final weights of criteria *v* are obtained by aggregation (averaging) of both partial weights *w* and *m*.

Written in steps, the method proceeds as follows:

1. Criteria C_i are pair-wised compared with regard to a goal on Saaty's fundamental scale from 1 to 9, and the pair-wise comparison matrix S (s_{ij}) is formed.
2. Criteria's weights w_i are estimated by the eigenvalue method.
3. Pair-wise conflict/concord value a_{ij} among criteria is estimated by an expert (or from the data if there are any), $a_{ij} \in [-1, 1]$, $a_{ii} = 1$, and conflict matrix $A = \{a_{ij}\}$ is formed.

Elements of the matrix *A* can be regarded as influence or dependence between criteria, and because this dependence can be also indirect (in the form of chains or loops), matrix *A* is raised to powers, and overall influence is given by limiting powers to infinity in the same way as in ANP, where supermatrix is formed and the limiting supermatrix is found with columns corresponding to weights. However, matrix *A* is not suitable for

the limiting process as it contains negative numbers in general (it cannot be turned into column stochastic matrix as desired), so in the following step a linear transformation from $[-1,1]$ interval to $[0,1]$ interval takes place:

4. Elements $a_{ij} \in [-1,1]$ of the matrix *A* are linearly transformed into elements $e_{ij} \in [0,1]$ of a matrix E:

$$e_{ij} = \frac{a_{ij} + 1}{2} \tag{4}$$

5. Matrix E is converted into matrix *E** with all column sums equal to 1, which is called column stochastic. Column stochasticity of the matrix *E** is necessary for the convergence in the Step 6.
6. The matrix *E** is raised to powers and the limiting matrix *M* is found:

$$\lim_{n\to\infty} (E^*)^n = M \tag{5}$$

The convergence of a limit (5) is ensured by column stochasticity of a matrix *E**. If there is no single limit, but a limit cycle consisting of *k* matrices E_k*, then the limiting supermatrix *M* is given by Cesáro summation [5]:

$$\lim_{n\to\infty} \frac{1}{k} \sum_{i=1}^{k} (E_k^*)^n = M \tag{5'}$$

7. Criteria weights *m* corresponding to their concord/conflict are found as column vectors of *M*.
8. The criteria weights *w* and *m* are aggregated into one final weights *v* by the following general formula:

$$v = f(w,m) \tag{6}$$

where $f(w,m)$ is an aggregation operator. Here, we use the arithmetic mean:

$$v_i = \frac{1}{2} \sum_{i=1}^{n} (w_i + m_i) \tag{7}$$

9. In the final step alternatives are pair-wise compared and the best alternative according to the relation (3) with regard to a goal is found in the same way as in AHP.

Now we turn to some special cases. It is easy to demonstrate that the proposed method has some 'reasonable' properties:

- If all criteria are independent, that is $a_{ij} = 0, i \neq j$, then the weights *m* of all criteria are equal, hence, the result is the same as for standard AHP.
- If all criteria are in absolute concord, $a_{ij} = 1; \forall i, j$, then the weights *m* of all criteria are also equal, and the result is the same as for standard AHP.
- If one criterion is in absolute conflict with all other criteria, while other criteria are in absolute concord, then the weight of the conflicting criterion *m* = 0.

4 The numerical example

In this section illustration of the method and its comparison with the standard AHP is provided in the example from human resources management: a selection of the most suitable person for a given job.

4.1 Statement of the problem

Let A, B, C, D and E be five applicants for a managerial position. The best applicant from the group has to be selected, and the selection is based on four criteria: education, experience, flexibility and salary expectations.

Education (ED) is a maximization criterion with three categories: university education in the branch = 7 points, university education in other branch = 4 points, other education = 1 point. Experience (EX) is a maximization criterion, expressed as time spent in managerial position (in years). Flexibility (F) represents willingness to work overtime (in hours per month) and it is a maximization criterion. Salary expectation (SE) is a minimization criterion expressed in EURs per year excluding bonus. The data are shown in Table 1.

	A	B	C	D	E
ED	7	4	7	7	4
EX	15	20	13	20	18
F	215	230	26	245	220
SE	30.000	35.000	28.000	33.000	32.000

Table 1 The data about applicants

4.2 Standard AHP solution

In our case the goal of decision-making is to choose the optimal applicant according to four given criteria. The pair-wise comparisons of criteria are based on expert experiences of the human resources manager. The pair-wise comparison matrix is presented in Table 2.

	ED	EX	F	SE
ED	1	1/3	1/2	1/2
EX	3	1	2	2
F	2	1/2	1	1
SE	2	1/2	1	1

Table 2 Pair-wise comparison matrix of criteria

Weights of criteria are calculated by relation (2). By relation (3) overall weights of alternatives (i.e. applicants) are achieved and their ranking is established (Table 3).

criterion	weight
ED	0.122
EX	0.424
F	0.227
SE	0.227

a)

alternative	weight	ranking
A	0.203	4
B	0.212	2
C	0.151	5
D	0.231	1
E	0.203	3

b)

Table 3 a) Weights of criteria, b) Weights (priorities) and rankings of alternatives

It is evident that the optimal applicant in our case is D. On the contrary, the least appropriate applicant is C.

4.3 AHP with conflicting criteria solution

Step 1: see Table 2.

Step 2: Weights of criteria in the order (ED, EX, F, SE) derived from their relative importance are the same as in standard AHP: $w = (0.122, 0.424, 0.227, 0.227)$.

Step 3: Concord/conflict among criteria (given by an expert or based on previous experience with admissions) is shown in Table 4a). Table 4a) is a matrix A with elements a_{ij} (of course, the values are subjective, as this is just illustrative example).

Step 4: Elements $a_{ij} \in [-1,1]$ of the matrix A are linearly transformed by relation (4) into elements $e_{ij} \in [0,1]$ of a matrix E, see Table 4b).

Step 5: Matrix E is converted into matrix E^* with all column sums equal to 1, see Table 5a).

Step 6: The matrix E^* is raised to powers and the limiting matrix M is found, see Table 5b).

Step 7: Criteria weights m corresponding to their concord/conflict are found as column vectors of M. Hence, we obtain, in the order (ED, EX, F, SE), the following weights: $m = (0.279, 0.248, 0.212, 0.261)$. The criterion ED attains the highest weight, because it is in concord with other criteria, while the criterion F receives the lowest weight, because it is the most conflicting criterion.

Step 8: The criteria weights w and m are aggregated into one final weights v by averaging (7): $v = (0.200, 0.336, 0.220, 0.244)$.

Step 9: Weights (priorities) and rankings of all alternatives with regard to the goal are shown in Table 6 (pair-wise comparisons of alternatives with regard to the goal were performed in the same way as in the previous section).

A comparison with standard AHP, see Table 3b), shows that there are only minor changes in alternatives' priorities, the best alternative is D again, while alternatives A, B and E changed their rankings.

	ED	EX	F	SE
ED	1	0.4	0.3	0.6
EX	0.4	1	-0.3	0.5
F	0.3	-0.3	1	-0.2
SE	0.6	0.5	-0.2	1

a)

	ED	EX	F	SE
ED	1	0.7	0.65	0.8
EX	0.7	1	0.35	0.75
F	0.65	0.35	1	0.4
SE	0.8	0.75	0.4	1

b)

Table 4 a) Concord/conflict among criteria – matrix *A*, b) matrix *E* .

	ED	EX	F	SE
ED	0.32	0.25	0.27	0.27
EX	0.22	0.36	0.15	0.25
F	0.21	0.13	0.42	0.14
SE	0.25	0.27	0.17	0.34

a)

	ED	EX	F	SE
ED	0.279	0.279	0.279	0.279
EX	0.248	0.248	0.248	0.248
F	0.212	0.212	0.212	0.212
SE	0.261	0.261	0.261	0.261

b)

Table 5 a) Matrix *E**, b) The limiting matrix *M*.

alternative	weight	ranking
A	0.209	2
B	0.204	3
C	0.160	5
D	0.230	1
E	0.198	4

Table 6 Final weights (priorities) and rankings of all alternatives

5 Conclusions

In this article a new method for AHP with conflicting criteria is proposed. In this method criteria in concord with other criteria attain the higher overall weights than criteria which are in conflict with others. The main benefit of this approach is that it incorporates potential conflicts among criteria which are often present in real-world decision making situations, but this problem is rather neglected in the standard AHP. Furthermore, the method is natural and computationally simple, and its use was demonstrated by the example from human resources management, namely in the selection of the most appropriate applicant for a given position. Future research might focus on examination of similarity between the proposed method and the standard AHP.

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Mathematical Modeling and Optimization of Viral Marketing

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Abstract: The aim of the article is to introduce a new mathematical model of viral marketing and its software implementation VIRAMARK. The model consists of a number of messages sent by a messenger to a set of target groups, a number of target groups, retransmission coefficients among groups, time period, weights of target groups and a (floor) utility function, which enables to express an impact (a revenue) of viral marketing under different configurations of model's inputs. Within model's framework several different viral marketing optimization problems are formulated and solved with the use of VIRAMARK, a free MS Excel add-in. Presented optimization model of viral marketing can aid and support a successful advertising campaign, and also it enables simulation of marketing's outcomes under different conditions. The use of the model is illustrated by examples.

Keywords: mathematical modeling, model, optimization, viral marketing.

JEL classification: 15A24, 40C05, 65K05, 90C05, 90C35

AMS classification: C61, M31, M37

1 Introduction

Viral marketing is relatively new and powerful direction in advertising enabled by internet and smartphones, which utilizes entertaining video clips, images, text messages, which are shared by users of different social networks such as Facebook, Twitter, LinkedIn, etc. Because spreading of such advertisements is similar to spreading of a virus, this approach is called 'viral'. Recently, even large international enterprises such as Volvo, Burger King, Old Spice, Anheuser-Busch or 20th Century Fox with their blockbuster movie Prometheus take advantage of this advertising tool.

However, there is no mathematical model of viral marketing known to authors so far, though there is a vast literature on the use of networks (mainly in the context of learning and decision making), such as neural networks, fuzzy decision maps, Petri nets, etc., see e.g. [1], [2] or [7].

Hence, the aim of the paper is to introduce a simple model of viral marketing, which consists of the following main components:

- a messenger (usually an enterprise),
- messages (video and/or music clips, images, etc.),
- a network of target groups (users of social networks, who communicate with each other),
- a utility function expressing viral marketing revenue.

In the model described in detail thereinafter a messenger sends a message (or messages) to one or more target groups. During some time period messages are repeatedly retransmitted among target groups, and an 'impact' on each target group is expressed as a number of messages received by a given group. It is assumed that the larger is the amount of messages (an impact) a target group receives, the higher is the expected revenue expressed by a value of the corresponding utility function (repeating advertisements have larger influence or effect on consumers than just one advertisement, and that's why we see ads for various washing powders on TV every day).

Model's framework enables modeling of different situations, as well as it allows to formulate and solve various optimization problems, e.g. including minimization of number of messages necessary to achieve a given value of the utility function, as it is assumed that creation and sending of a message (such as video clip) is associated with some (relatively high) costs.

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To facilitate solution of some optimization problems, a software tool called VIRAMARK was developed and can be downloaded from http://www.opf.slu.cz/kmme/VIRAMARK.

The paper is organized as follows: in section 2 notation and variables of the model are provided, in section 3 several optimization problems are formulated, and in section 4 an illustrative example of the use of the model is provided.

2. The model properties, notation and variables

In this section notation and variables of the model are provided with a brief explanation:

- n...a number of target groups,
- $g = (g_1, ..., g_n)$ ...the vector of target groups g_i,
- τ...time period (one day, week, etc.),
- $r_{ij} \in [0,1]$ ... the rate of messages retransmitted by a group i to a group j by a period of time τ,
- R... $n \times n$ retransmission matrix with elements $r_{ij} \in [0,1]$,
- $v_0 = (v_{01}, ..., v_{0n})$ ...the initial (row) vector, v_{0i} is a number of messages sent to a target group i by an messenger at time $\tau = 0$,
- $v_j = (v_{j1}, ..., v_{jn})$ ... the state (row) vector, v_{ji} is a number of messages retransmitted from all other target groups at time $\tau = j$,
- w...the vector of message worth (weights) associated with a vector of target groups g, $w_j \geq 0$, $\sum_{j=1}^{n} w_j = 1$,
- $U(r, w, v_{0,} \tau)$ ...the utility function of viral marketing revenue,
- $U^*(r, w, v_{0,} \tau)$ ...a floor utility function of viral marketing revenue.

The state vectors are computed from preceding vectors with a recurrent formula (1):

$$v_k = v_{k-1} \cdot R \tag{1}$$

The utility function $U(r, w, v_{0,} \tau)$ is defined as follows:

$$U(R, w, v_0, \tau) = \sum_{j=1}^{n} \left(\sum_{i=0}^{\tau} v_i \right) \cdot w_j \tag{2}$$

The floor utility function $U^*(r, w, v_{0,} \tau)$ is defined as follows:

$$U^*(R, w, v_0, \tau) = \sum_{j=1}^{n} \left\lfloor \sum_{i=0}^{\tau} v_i \right\rfloor \cdot w_j \tag{3}$$

- Model's *input* variables include: n, g, τ, r_{ij}, w

- Variables v_0, k, and U/U^* can be both *input* or *output* variables depending on the formulation of a problem, see section 3.

- The model iteratively computes state vectors v_k from a preceding state vector by a formula (1). After all iterations are done, the value of a utility function is established by relations (2) or (3).

- The floor utility function U^* is introduced because one may ask what does for example $v_k = (0.88, 1.65, 0.57)$ mean: how many messages are received, when the values of v_k are not integers? Non-integer values can be understood as probabilities (value 1.65 means that 1 message was surely received and the second message was received with 65% probability by the second target group). But if one takes into account only messages that were surely received, then he/she can use the floor utility function.

- The utility function $U(r, w, v_{0,} \tau)$ is assumed to be directly proportional to the viral marketing revenue, so the higher is $U(r, w, v_{0,} \tau) / U^*(r, w, v_{0,} \tau)$, the higher is the revenue. By the value of $U(r, w, v_{0,} \tau) / U^*(r, w, v_{0,} \tau)$ all feasible initial states (v_0 vectors) can be compared and ordered from the best to the worst.

- Utility function (2) is linear in v, as $U(v_{01} + v_{02}) = U(v_{01}) + U(v_{02})$ and $U(kv_0) = kU(v_0)$, so the problem belongs to (multiobjective) linear modeling, and techniques such as simplex method can be used for the solution, see e.g. [3], [4] or [5].

- Square retransmission matrix R is neither stochastic nor symmetric generally. But if it is stochastic, then state vectors are Markov chains. If some $r_{ij} = 0$, then it can be interpreted as there is no communication from a group i to a group j, $r_{ii} = 0$ Also, it is assumed that elements r_{ij} are constant in time.

To show how the model works consider the following example.

Example 1. Consider three target groups (n = 3), five messages (k = 5), initial vector $v_0 = (2,1,2)$ $\tau = 1$, $w = (0.5, 0.3, 0.2)$ and the matrix $R = \begin{pmatrix} 0 & 0.1 & 0.4 \\ 0.2 & 0 & 0.2 \\ 0.3 & 0.1 & 0 \end{pmatrix}$. Find the utility value (2).

Solution:

According to (1) $v_1 = v_0 \cdot R = (0.8, 0.4, 1)$.

From (2) we obtain the utility function: $U(R, w, v_0, \tau) = \sum_{j=1}^{3} \left(\sum_{i=0}^{1} v_i \right) \cdot w_j = (2.8, 1.4, 3) \cdot (0.5, 0.3, 0.2) = 2.42$.

One can easily verify that if all 5 messages were send to the group 1, so $v_0 = (5,0,0)$, then U would be higher (U = 3.05), so it makes good sense to find the initial vector maximizing the utility function under given conditions.

3. Formulation of selected optimization problems

In the previous section 2 it was demonstrated that the utility function U depends on the initial vector v_0 *ceteris paribus*. In this section this and other viral marketing optimization problems are formulated:

Problem no. 1*: Find the initial vector v_0 so that the (floor) utility function is maximal for a given number of k messages (for a given cost) and given time τ.*

Maximize: $U(R, w, v_0) / U^*(R, w, v_0, \tau)$ (4)

s.t.

$$|v_0| = k,\ k \in N,$$
$$r_{ij} \in [0,1],$$
$$\tau \in N,$$
$$\sum_{j=1}^{n} w_j = 1,\ w_j \geq 0$$

On the other hand, when a messenger wants to spread his advertisement among a given number of recipients (for example 10 million Internet users in 100 groups), he may ask how many messages have to be sent to a given social network (in other words how many messages have to be produced at a given cost). This results in formulation of the second problem of viral marketing:

Problem no. 2*: Find the minimal initial vector v_0 so that the (floor) utility function attains (exceeds) a given value K at given time τ.*

Minimize: v_0 (5)

s.t. $U \geq K$ / $U^* \geq K$,

$$r_{ij} \in [0,1],$$
$$\tau \in N,$$
$$\sum_{j=1}^{n} w_j = 1, w_j \geq 0$$

A minimal initial vector is a vector with a minimal length given by l_1 metric.

Problem no. 3*: Find the minimal initial vector v_0 so that all target groups are impacted at least by L messages at given time τ.*

$$\text{Minimize: } v_0 \tag{6}$$
$$\text{s.t.} \quad v_{\tau i} \geq L \text{, for } i = 1,2,\ldots,n$$
$$r_{ij} \in [0,1],$$
$$\tau \in N,$$
$$\sum_{j=1}^{n} w_j = 1, w_j \geq 0$$

***Problem no. 4**: Find the minimal initial vector v_0 so that all target groups are impacted at least by L messages at given time τ, while the (floor) utility function attains (exceeds) a given value K at given time τ.*

$$\text{Minimize: } v_0 \tag{7}$$
$$\text{s.t.} \quad v_{\tau i} \geq L \text{, for } i = 1,2,\ldots,n$$
$$U \geq K \,/\, U^* \geq K,$$
$$r_{ij} \in [0,1],$$
$$\tau \in N,$$
$$\sum_{j=1}^{n} w_j = 1, w_j \geq 0$$

In problem 3 it is required to deliver at least a given number L of messages to all groups, so all groups are impacted by a viral marketing to some (desired) degree. As viral marketing might be multiobjective, in problem 4 both conditions from Problems 2 and 3 are combined into one problem.

Of course, also other interesting problems can be formulated, which might be important for a messenger, but are omitted here due to the limitation of space.

4. Illustrative example

In this Section solving of Problem no. 1 is illustrated by the use of Microsoft Excel Add-in called VIRAMARK. VIRAMARK works with all current versions of Microsoft Excel from version 97. It consists of four individual files:

- VIRAMARK.xla – main module with user interface, it is written in VBA (Visual Basic for Applications),
- VIRAMARK.dll – it contains special functions used by the application, it is written in C#,
- VIRAMARK.xll – it contains library for linking C# modules with Excel called Excel-DNA (http://exceldna.codeplex.com),
- VIRAMARK.dna – configuration file for Excel-DNA module.

All four files must be placed in the same folder and macros must be permitted before running the module (see Excel documentation for details). VIRAMARK itself can be executed by double clicking on the file VIRAMARK.xla. After executing the add-in there will appear a new menu item “VIRAMARK” in the Add-ins ribbon (in older Excel versions the menu item “VIRAMARK” will appear in the top level menu). A new problem can be generated by clicking on “New problem” item in the main VIRAMARK menu, see figure 1.

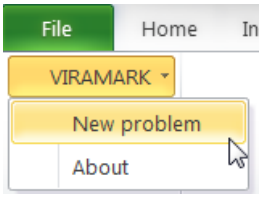

Figure 1 New problem menu

Then there will be shown a form with main problem characteristics, see figure 2.

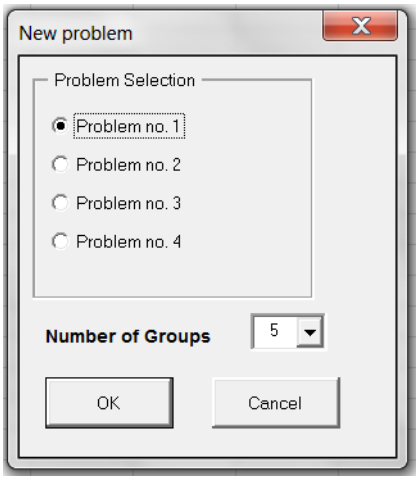

Figure 2 New problem characteristics

In the top panel we select the problem we want to solve based on the problems definition in the section 3 and the second parameter determines the number of groups which should be affected by our messages. When user submits his options a new Excel sheet with forms is created, where user can put the values of the retransmission matrix R and other parameters specific to the selected problem. Generated Excel sheet for the Example 2 defined below is shown on figure 4.

Example 2. Consider 5 target groups with the network given by the matrix R (see Figure 3), $\tau = 3$, $w = (0.1, 0.3, 0.2, 0.1, 0.3)$ and k = 8. The goal is to maximize $U(R, w, v_0, \tau)$, see (4). Hence, a messenger wants to maximize the utility function for 8 messages under given circumstances.

$$R = \begin{pmatrix} 0 & 0.4 & 0.2 & 0.1 & 0 \\ 0.4 & 0 & 0.3 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0 & 0.4 & 0.3 \\ 0.2 & 0.2 & 0.2 & 0 & 0.1 \\ 0.2 & 0.2 & 0.1 & 0.2 & 0 \end{pmatrix}$$

Figure 3 Retransmission matrix

	A	B	C	D	E	F
1	**VIRAMARK - Viral Marketing Solver**					
2						
3	**Number of time periods:**			**3**		
4	**Number of messages:**			**8**		
5						
6	**Message weights:**					
7	**0.1**	**0.3**	**0.2**	**0.1**	**0.3**	
8						
9	**Retransmission matrix:**					
10	**0**	**0.4**	**0.2**	**0.1**	**0**	
11	**0.4**	**0**	**0.3**	**0.1**	**0.4**	
12	**0.5**	**0.2**	**0**	**0.4**	**0.3**	
13	**0.2**	**0.2**	**0.2**	**0**	**0.1**	
14	**0.2**	**0.2**	**0.1**	**0.2**	**0**	
15						
16	**Minimal initial vector:**					
17						
18						
19	**Utility function:**				Calculate	
20						

Figure 4 Input values for the problem

Once all input values are entered we need to click on "Calculate" button below the form, which will calculate values of minimal initial vector based on the selected problem, see figure 5.

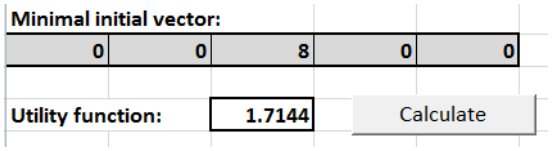

Figure 5 Solution of the problem

As we can see we should send all 8 messages to group no. 3 to get maximal value of the utility function. If we want to modify any of the input values we can easily do it and after clicking on "Calculate" button we get instantly new solution of the problem.

5. Conclusions

The aim of the paper was to present a new mathematical model of viral marketing and to show its use on an example. To facilitate computations the MS Excel add-in VIRAMARK was developed, which is free and can be downloaded from http://www.opf.slu.cz/kmme/VIRAMARK. Though the presented model is simplified, we believe that it can be useful in practice and that it might contribute to a successful advertising campaign, nevertheless, readers are kindly welcomed to modify or improve the model.

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Non-Performing Loans in the Czech Republic: Lag Structure Identification of the Macroeconomic Determinants

Aleš Melecký[1], Monika Šulganová[2]

Abstract. In recent years, mainly as a result of the financial crisis, the interest in the macroprudential policy and its role as a part of reformed regulatory framework has been rising. Problems in the financial (banking) system may occur if systemic risks, especially credit risk, are not properly managed. In this situation, the financial instability/crisis has a large negative impact on economic activity. Non-performing loans (NPLs) are one of the commonly used indicators of credit risk which, at the system level, show cyclical behavior. The aim of this paper is to identify the structure of time lags of the most important macroeconomic determinants of non-performing loans in the Czech Republic using a linear regression model. The main determinants of NPLs used in the model are its lagged value, real economic growth, inflation, exchange rate and lending rates. Estimation results suggest that non-performing loans are medium persistent and that the lag length of its main determinants ranges from two to seven quarters.

Keywords: Non-performing loans, determinants of non-performing loans, time lag structure, Czech Republic, linear regression model.

JEL Classification: E32, E58, G28
AMS Classification: 62M10

1 Introduction

The last financial crisis, as well as the former ones, highlighted the need of wider regulation and supervision framework of financial system. Therefore, this paper is framed by macroprudential policy framework and the idea of researching relations between the financial stability indicators and macroeconomic variables is the main function of the macroprudential analysis [1].

Financial instability might occur if the systemic risks, especially credit risk, are not properly managed [10]. Kaufmann and Scott [9] defined systemic risk as the probability that the whole financial (banking) system will collapse, therewith the majority or even all parts of the financial system are interconnected.

Credit risk materialization manifests itself in a rise of credit losses, which negatively affect investment activity through the limited credit supply. Also the consumption of economic agents might decrease because of weaker stream of financial revenues and higher debt burden. As a result, economic activity is constrained on an aggregate level. In the empirical literature, non-performing loans are broadly used as an indicator of credit risk materialization [see eg. 3, 11, 4, 8].

According to IMF Financial Soundness Indicators the loan is classified as non – performing if (i) payments of principal and interest are past due by three months (90 days) or more; (ii) interest payments equal to three months (90 days) interest or more have been capitalized (re-invested into the principal amount), refinanced or rolled over (i.e. payment has been delayed by arrangement) [2].

On the systemic level we can assume cyclical behavior of the NPLs [7], which is influenced by macroeconomic, microeconomic, institutional and other determinants. Thus, monitoring of credit losses and their macroeconomic determinants is an important role of macroprudential supervision.

The aim of this paper is to identify the structure of time lags of the most important macroeconomic determinants of non-performing loans in the Czech Republic using a linear regression model. The main determinants of NPLs used in the model are its lagged value, real economic growth, inflation, exchange rate and lending rates.

The rest of the paper is organized as follows. The second part briefly describes the data we used for the empirical analysis, model and the estimation method. The third part of this paper is dedicated to discussion of the estimation results and the last part concludes the analysis.

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2 Macroeconomic determinants of non-performing loans

The determinants of the non-performing loans play the key role for their (NPLs') analysis. The stylized theory distinguishes between the macroeconomic and banking determinants. The latter are specific for particular institution, i. e. they capture an idiosyncratic risk. Macroeconomic determinants, which influence an amount of NPLs, resulted from the macroeconomic imbalances. Thus these risk factors are systemic and affect every institution in the financial sector.

Empirical findings confirmed the negative relation between the non–performing loans and economic growth [see eg. 3, 11, 4]. In economic boom agents demand more products and services due to their better financial position which increases credit supply. In contrast, in a phase of economic bust, agents start to have difficulties to repay their debts on time and thus amount of NPLs rises. This relation was confirmed also in the Czech Republic by Babouček and Jančar [1] and Jakubík [7].

An increase of interest rates (in the case of loans with floating rate) leads to the higher cost in servicing debt (the interest rates payments go up). Due to the higher debt burden of economic agents the amount of non–performing loans rises. This positive correlation between NPLs and interests rates is also well-established in the empirical literature [see eg. 11, 4], for the case of the Czech Republic see eg. [7] or [5].

The impact of inflation on the volume of non–performing loans is not clear. Empirical results are not uniformed due to different economic conditions in individual countries or different research methods. Inflation can improve borrowers' ability to meet obligations by eroding the real value of repayment [12, 6]. On the other hand, growing inflation worsens the decision making process of economic agents and thus might weaken an economic activity in the future and result in the higher credit losses [5]. For the Czech Republic the latter impact was confirmed by Babouček and Jančar [1].

The nominal exchange rate affects the value of loans denominated in foreign currencies (unless they are secured). Depreciation of domestic currency (i. e. rise in the nominal exchange rate) increases the debt burden of economic agents with subsequent growth of credit losses. Depreciation of the domestic currency is also reflected in an improvement of international competitiveness, which is crucial factor for export-oriented industries. Firms with better financial condition may thus increase repayment capacity, what would indicate a decrease in volume of NPLs [3]. Therefore the relationship between the nominal exchange rate and NPLs is not clear, and the resulting impact of depreciation of domestic currency depends on whether the positive effect of improvement of competitiveness and international trade outweighs the negative impact on asset quality due to unsecured positions of debtors in foreign currencies. Babouček and Jančar [1] and Jakubík [7] researched the non-performing loans against the real effective exchange rate of the Czech Crown. Both authors confirmed the positive effect of real depreciation on the amount of NPLs (i. e. the positive influence of depreciation on international competitiveness of the country).

3 Data and methodology

3.1 Data

For the empirical analysis performed in this paper, we used quarterly data for the Czech Republic ranging from 2002Q1 to 2012Q3. Basic characteristic of the data as well as their sources are listed in the following table.

Variable	Characteristic	Source
NPLs ratio	Calculated as share of non-performing loans to total volume of nominal loans	ARAD, CNB
Lending rate - companies	Nominal lending rates for companies	ARAD, CNB
Lending rate - households	Nominal lending rates for households	ARAD, CNB
Nominal exchange rate	Nominal exchange rate of CZK to EUR	ARAD, CNB
CPI	Consumer price index, 2005=100	ARAD, CNB
Real GDP	Real gross domestic product, seasonally adjusted, in 2005 prices	ARAD, CNB

Table 1 Data overview

Because of higher volatility of the variables in the Czech Republic we calculated from nominal exchange rates, CPI and real GDP (presented in table 1) annual growth rates as suggested by Festić and Bekő [5] or Buncic

and Melecký [3]. To avoid potential stationarity problems we therefore used the first order differences of all variables. After this modification and based on stationarity and unit root testing we can conclude that all data series are stationary.

3.2 Model and estimation method

We used dynamic linear regression model of the Czech economy for the empirical analysis of the effects of macroeconomic determinants on NPLs ratio (NPLR) as well as identification of their lags effects. This model explains changes in NPLs ratio by lagged values of the NPLs ratio and the main macroeconomic variables. The selection of the macroeconomic variables is motivated by the economic theory described above as well as findings of other authors it this research area. We focused only on the most important macroeconomic determinants, which impact was confirmed in the various empirical papers (see table 3 in appendix). Based on the estimation results, the relation between NPLs ratio and main macroeconomic determinants can be described as follows:

$$NPLR_t = \alpha + \beta_{NPLR} NPLR_{t-1} + \beta_{REG} REG_{t-4} + \beta_{INFL} INFL_{t-3} + \beta_{ER} ER_{t-2} + \beta_{LRH} LRH_{t-7} + \beta_{LRC} LRC_{t-5} + u_t \quad (1)$$

Where α is a constant and parameters β_{xx} denote values of coefficients of respective explanatory variables. So NPLR denotes change in NPLs ratio, REG change in annual real GDP growth, INFL is change in annual inflation calculated from CPI index, ER is change in CZK\EUR exchange rate, LRH and LRC stands for changes in companies and households lending rates respectively, and u_t is *i.i.d.* normally distributed shock with mean zero and variance σ.

For estimation of the parameters of the selected model we used ordinary least squares (OLS) method with empirically motivated lags. Lag length selection was based upon values of the information criterions (Akaike, Schwartz, HQ) and statistical significance of the model as well as *t*-statistics of the estimated coefficients.

4 Discussion of the estimation results

Main results from the OLS estimation of the selected model are summarized in the Table 2. Now we will discuss our main findings. According to our estimation, there is a statistically significant persistence in the change of NPLs ratio with the estimated coefficient of 0.4 and statistical significance at 1% level. This means that non-performing loans show some persistence and do not perform a leap changes.

In line with economic theory changes in the NPLs ratio reacts negatively to changes of the real economic growth with estimated coefficient of -0.1 and identified lag of 4 periods. This effect is caused by decrease in income of the economic agents and reduced ability to pay off their debts. Estimated coefficient for real economic growth is statistically significant at 1% level.

Our estimates suggest a positive relationship between changes in inflation and NPLR, which is statistically significant at 5% significance level; however, the estimated coefficient is quite small. This result is in line with studies of Babouček and Jančar [1] and Buncic and Melecký [3].We identified, that it takes 3 quarters until changes in inflation cause change in the NPLs ratio.

The effect of appreciation/depreciation of the Czech Crown with respect to Euro is close to zero as suggested by the model. This may be caused by two contradictory effects. Firstly, depreciation of the Czech Crown vis-a-vis Euro increases the debt burden of economic agents with unsecured positions in foreign currencies, which can affect the growth of credit losses. On the contrary, depreciation improves the terms of trade and increases competitiveness of the Czech economy. In our case it seems that the first effect slightly prevails. According to our estimates, changes in the exchange rates influence the changes in NPLs ratio with 2 periods lag and respective estimated coefficient is statistically significant at 1% level.

There is also statistically significant relationship between changes in NPLs ratio and lending rates. However the effect of lending rates is different for households and companies, where the estimated coefficients are 1.13 and 0.34 respectively. Positive values of the estimated interest rate coefficients, suggesting that the rise in interest rate increases the debt burden and decreases the ability to pay off the debts, support findings of Jakubík [7] and Festić and Bekő [5]. Changes in household and companies lending rates affect NPLs ratio with quite long lag, when identified lags suggested by the model are 5 and 7 quarters respectively. Estimated coefficient shows a bit higher significance in case of households (at 1% level) than companies (at 5% level).

Parameter	Estimated coefficient	Std. Error
α	0.0676*	0.0345
β_{NPLR}	0.4058***	0.1123
β_{REG}	-0.1006***	0.0268
β_{INFL}	0.0727**	0.0280
β_{ER}	0.0248***	0.0081
β_{LRH}	1.1263***	0.2940
β_{LRC}	0.3357**	0.1274
Adj. R-squared	0.653598	
Prob(F-statistic)	0.000002	
Akaike info criterion	-0.488537	
Schwarz criterion	-0.177468	
Hannan-Quinn criter.	-0.381156	

Note: Stars symbols indicate statistical significance of the estimated coefficients, so that ***, **,* means coefficient is significant at 1%, 5% and 10% respectively.
Source: Author's calculations

Table 2 Estimation results

Overall the model is statistically significant and shows satisfactory data fit (with adjusted $R^2 = 0.65$)[3] with respect to the limited length of available data, as well as the results of other empirical works, see e. g. [3] or [4]. Also values of Akaike, Schwartz and Hannan-Quinn information criterions seem to be satisfactory. The model's residuals where tested for normality, autocorrelation and heteroscedasticity. The main tests results are presented in Table 3 in appendix.

5 Conclusions

Macroeconomic conditions are significant determinants of development of the NPLs ratio in the Czech Republic as confirmed by our empirical results, which were obtained from linear regression model with carefully selected time lags. We identified time lags of the main macroeconomic determinants (real economic growth, inflation, changes in exchange rate and lending rates) in the range of 2 to 7 quarters, with the longest lag in the households' lending rate.

Even though, the NPLs ratio shows some persistence, the main effect comes from changes in the lending rates for companies and especially for households. It is thus important, for the respective authorities, to carefully watch development of the lending rates. Also deterioration of the economic performance of the Czech Republic as well as rising inflation negatively influences development of the NPLs ratio. Although effect of the exchange rate is statistically significant, its estimated value is very small. This result might be caused by the fact that changes in exchange rate affect NPLs ratio by two contrary effects (specifically, they change debt burden of economic agents with foreign currencies exposures and competitiveness of the Czech economy). More detailed analysis of the role of the exchange rate will be needed to distinguish those effects from each other.

In the future work we want to focus on situation in specific sectors of the Czech economy, where we assume different impact of corresponding variables as well as lag structure. This assumption is also supported by results of our paper where lending rates for households and companies affect NPLs ratio with different lags.

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[3] We also estimated VAR model containing the same variables as the OLS model. The results show that careful theoretically based lag length selection in OLS model is more efficient compared to the VAR model (with adjusted R^2=0.57).

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Appendix

Author (year)	REG	IR↑	u ↑	π ↑	NER↑	RER↑	EX↑	I↑	FDI↑
Babouček a Jančar (2005)	↓	n. s.	↑	↑	-	↓	-	-	-
Festić a Bekő (2008)	-	↑	↑	-	-	↓	↓	-	↓
Festić et al. (2011)	-	-	-	-	-	-	↓	↓	↑
Gerlach (2005)	↓	↑	-	↓	-	-	-	-	-
Jakubík (2005)	↓	↑	-	↓	-	-	-	-	-
Jiménez a Saurina (2006)	↓	↑	-	-	-	-	-	-	-
Louzis et al. (2012)	↓	↑	↑	-	-	-	-	-	-
Melecký a Buncic (2012)	↓	↑	-	↑	n. s.	-	-	-	-
Shu (2002)	↓	↑	n. s.	↓	↑	-	-	-	-

Source: self-elaboration.

Note: ↑= increase in volume of non-performing loans; ↓= decrease in volume of non-performing loans; n. s. = non-significant effect; REG = real economic growth; IR = interest rate; u = unemployment rate; π = inflation rate; NER = nominal exchange rate; RER = real exchange rate; EX = export; I = investment; FDI = foreign direct investment.

Table 3 Impact of macroeconomic determinants on volume of NPLs

Breusch-Godfrey Serial Correlation LM Test	
F-statistic	Prob. F(8,20)
0.55	0.80
White (Heteroskedasticity) Test	
F-statistic	Prob. F(27,7)
0.71	0.76
Normality Test	
Jarque-Bera statistic	Prob.
2.21	0.33

Source: Author's calculations

Table 4 Autocorrelation, heteroskedasticity and normality tests

Comparing of EU15 and EU12 countries efficiency by application of DEA approach

Lukáš Melecký[1]

Abstract. The paper deals with evaluation of efficiency level in EU countries by Data Envelopment Analysis (DEA) application. The aim of the paper is to measure and assess relative rate of technical efficiency, that can be consider as a source of competitiveness potential, in EU15 countries and EU12 countries in reference years 2000, 2008 and 2011. Use of DEA approach has been chosen because DEA approach is based on ratio between input and output indicators and measures thus efficiency with EU countries are able to transform their inputs into outputs. Data base for efficiency analysis consists of selected indicators which are part of Country Competitiveness Index (CCI) measurement. Theoretical part of the paper is focused on the background of competitiveness in the context of efficiency and introduction to DEA approach. Empirical part is concentrated on evaluation of EU15 and EU12 countries efficiency rates calculated using the Charnes-Cooper-Rhodes output oriented model (CCR-O) with constant returns to scale (CRS) and Andersen-Petersen`s output oriented model (APM-O) of super efficiency. EU countries are compared, in terms of calculated DEA models, within each group of EU countries. Based on super efficiency analysis results, the comparison of efficiency scores between these two groups of EU countries is included.

Keywords: Andersen-Petersen`s model, country competitiveness index, CCR CRS model, DEA approach, efficiency, super efficiency, EU15, EU12.

JEL Classification: C67, C82, O11, O33, Y10
AMS Classification: 62H99, 90C05, 93B15, 93D25

1 Introduction

The process of achieving an increasing trend of performance and a higher level of competitiveness is significantly difficult by the heterogeneity of different areas mainly in the European Union (EU) where are many heterogeneous units presented by EU Member States and their regions. Although the EU is one of the most developed parts of the world with high living standards, there exist significant and huge economic, social and territorial disparities having a negative impact on the balanced development across Member States and their regions, and thus weaken EU's performance and competitiveness in a global context. The European integration process is thus guided by striving for two different objectives: *to foster economic competitiveness* and *to reduce differences* [9]. From the long-term perspectives, competitiveness requires paying attention not only to economic but also to social and environmental factors, in recent years especially to territorial characteristics of areas.

To *define competitiveness* is difficult because of the *lack of mainstream view* for understanding this term. Competitiveness can be understood in different ways and levels despite widespread acceptance of its importance. The concept of competitiveness is distinguished at three basic levels - *microeconomic, macroeconomic and regional*. Anyway, there are some differences between these approaches; see e.g. [8], [10] or [11]. Although there is no uniform definition and understanding, competitiveness remains one of the *fundamental criteria of economic performance.*

In relation to competitiveness objective, *performance* and *efficiency* are *complementary objectives*, which determine the long-term development of countries. Competitiveness in the level of performance is a major obstacle to the balanced and harmonious development of the territory. Analysis of efficiency brings the important information about the key problematic issues in countries on the one side and its development and competitive potential on the other side. Efficiency is a central issue in analyses of economic growth, the effects of fiscal policies, the pricing of capital assets, the level of investments, the technology changes and production technology, and other economic topics and indicators. In general sense, the *efficiency can be achieved under the conditions of maximizing the results of an action in relation to the resources used, and it is calculated by comparing the effects obtained in their efforts* [7]. In a competitive economy, therefore, the issue of efficiency can be resolved by comparing these economic issues.

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2 Measurement of efficiency by DEA method

Techniques to measure efficiency are improved and investigations of efficiency become more frequent. Nevertheless, the measurement of efficiency of countries and regions, resp. their factors, *remains a conceptual challenge*, because there are *difficulties in efficiency measuring*. Measurement of efficiency is highly sensitive to the data sets being used. Good quality data are needed because the techniques available to measure efficiency are sensitive to outliers and may be influenced by exogenous factors. Data used for international comparisons require a minimum level of homogeneity. The analysis of efficiency is about the relationships between inputs, outputs and outcomes. In 1957, Farrell already investigated the question how to measure efficiency and highlighted its relevance for economic policy makers. "*It is important to know how far a given industry can be expected to increase its output by simply increasing its efficiency, without absorbing further resources*" [5]. Farrell confined his numerical examples and discussion to single output situations, although he was able to formulate a multiple output case. Twenty years after Farrell's model in 1978, Charnes et al. [3], responding to the need for satisfactory procedures to assess the relative efficiencies of multi-input multi-output production units, introduced a powerful methodology in the form of Charnes-Cooper-Rhodes (CCR) model assuming constant returns to scale (CRS), which has subsequently been titled as *Data envelopment analysis* (DEA) [4]. Nevertheless, the measurement of efficiency and effectiveness of countries and regions, respectively their factors, remains a conceptual challenge.

The original idea behind DEA was to provide a methodology whereby, within a set of comparable *decision making units* (DMUs), those exhibiting best practice could be identified, and would form an *efficient frontier* Furthermore, the methodology enables one to measure the level of efficiency of non-frontier units, and to identify benchmarks against which such inefficient units can be compared [4]. The efficiency analysis, based on application of DEA approach is nowadays used also for evaluating national development quality and potential (with respect to the national factors endowment). DEA method becomes a suitable tool for ranking competitive (uncompetitive) position of countries based on efficiency within the group of evaluated countries.

The initial *hypothesis of paper* is based on the comparison of efficiency within EU15 and EU12 countries. We assume that group of EU15 countries are countries best (better) at converting inputs into outputs (mainly *Scandinavian countries* and *Western European countries*) and therefore having greater performance and productive potential than group of EU12 countries (especially *South Eastern European countries* and *Baltic countries).*

2.1 Theoretical background of DEA method

Data envelopment analysis is a tool for measuring the *relative efficiency* and comparison of decision making units (DMU). The DMUs are usually characterized by several inputs that are spent for production of several outputs. The aim of DEA method is to examine DMU if they are effective or not effective by the size and quantity of consumed resources by the produced outputs. DEA can successfully separate DMUs into categories which called efficient and inefficient DMUs [3]. Efficient DMUs have equivalent efficiency score. However, they don't have necessarily the same performance. DMU is efficient if the observed data correspond to testing whether the DMU is on the virtual production possibility frontier. All other DMU are inefficient, and DEA identifies a set of corresponding efficient units that can be utilized as benchmarks for improvement of inefficient units. If we want to evaluate the degree of efficiency, it is appropriate to use the formulation of DEA model. Consider a set of n DMUs, with each DMU_j, (j = 1,. . .,n) using m inputs x_{ij} (i = 1,. . .,m) and generating s outputs y_{rj} (r = 1,. . ., s). If the prices or multipliers u_r , v_i associated with outputs r and inputs i, respectively, are known, then borrowing from conventional benefit/cost theory, one could express the efficiency e_j of DMU_j as the ratio of weighted outputs to weighted inputs, as follows [4]:

$$Efficiency\ of\ \ DMU = \frac{\text{weighted sum of outputs}}{\text{weighted sum of inputs}} = \frac{\sum_r u_r y_{rj}}{\sum_i v_i x_{ij}}. \tag{1}$$

The basic principle of the DEA in evaluation of efficiency of the DMU_j, (j = 1,. . .,n) consists in looking for a *virtual unit* with inputs and outputs defined as the weighted sum of inputs and outputs of the other units in the decision set Xλ a Yλ, where $\lambda=(\lambda_1, \lambda_2, \ldots,\lambda_n)$, $\lambda>0$ is the vector of weight of the DMUs. The virtual unit should be better (or at least not worse) than the analysed unit DMU_j. The problem of looking for virtual unit can generally be formulated as standard linear programming problem; see e.g. [3], [4], [6]. For calculations of national efficiency of EU15 and EU12 countries in following efficiency analysis, the basic *Charnes-Cooper-Rhodes output oriented model* (CCR-O) assuming *constant returns to scale* (CRS) have been used. This model was chosen because of the aim of maximization of output presented by CCI output indicators in given CCI input indicators. Selected CCR-O model also assumes constant return to scale, i.e. we simply consider that such doubling of inputs leads to a doubling of outputs in all EU countries.

CCR model evaluates the efficiency of DMUs for any number of inputs and outputs. The coefficient of efficiency is the ratio between the weighted sum of outputs and the weighted sum of inputs. Each DMU selects input and output weights that maximize their efficiency score. The coefficient of efficiency takes values in the interval <0, 1>. In DEA models aimed at outputs the efficiency coefficient of efficient DMU equals 1, but the efficiency coefficient of inefficient DMU is greater than 1. *Dual version of CCR-O model* (2) assuming CRS is formulated as follows [6]:

$$\max g = \theta_q + \varepsilon(e^T s^+ + e^T s^-), \tag{2}$$

subject to

$$X\lambda + s^- = x_q,$$
$$Y\lambda - s^+ = \theta_q y_q,$$
$$\lambda, s^+, s^- \geq 0,$$

where g is the coefficient of efficiency of DMU_q; θ_q is radial variable indicates required rate of increase of output; ε is infinitesimal constant (usually 10^{-6} or 10^{-8}) ; e^T is vector; in the case of CRS $e^T = (1, 1, \ldots, 1)$; s^+, and s^- are vectors of slack variables expressing the difference between virtual inputs/outputs and appropriate inputs/outputs of the observed DMU_q; λ represent vector of weights assigned to individual units; x_q means vector of input of DMU_q; y_q means vector of output of DMU_q; X is input matrix; Y is output matrix.

Depending on chosen model, but also on the relationship between number of DMUs and number of inputs and outputs, we can assume that number of efficient units can be relatively large. Due to the possibility of efficient units' classification, it is used *Andersen-Petersen's model* (APM) of super efficiency [1]. Model (3) is case of output oriented dual version of Andersen-Petersen`s model (APM-O) assuming CRS formulated as follows:

$$\max g = \theta_k + \varepsilon(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+), \tag{3}$$

subject to

$$\sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j x_{ij} + s_i^- = x_{ik},$$
$$\sum_{\substack{j=1 \\ j\neq k}}^{n} \lambda_j y_{rj} - s_r^+ = \theta_k y_{rk},$$
$$\lambda_j, s_r^+, s_i^- \geq 0,$$
$$j = 1, 2, \ldots, n; r = 1, 2, \ldots, s; i = 1, 2, \ldots, m.$$

where x_{ij} and y_{rj} are i-th inputs and r-th outputs of $DMUj$; θ_k is efficiency index (intensity factor) of observed DMU_k; λ_j is dual weight which show DMU_j significance in definition of input-output mix of hypothetical composite unit, DMU_k directly comparing with. The rate of efficiency of inefficient units ($\theta_k > 1$) is identical to model (2); for units identified as efficient in model (2), provides APM-O (4) the rate of super efficiency lower than 1, i.e. $\theta_k \leq 1$.

2.2 Fundamental basis of efficiency analysis

The efficiency analysis starts from building database of indicators that are part of a common approach of World Economic Forum (WEF) and EU in the form of *Country Competitiveness Index* (CCI). The aim of this approach is to develop a rigorous method to benchmark national competitiveness and to identify the key factors which drive the low competitiveness performance of some countries [2]. The reference to CCI is the well-established *Global Competitiveness Index* (GCI) by WEF. Complete set of all CCI indicators consists of *81original indicators* classified in 11 pillars. All CCI indicators are further classified in two main groups – inputs and outputs according to its nature, respectively into those which describe driving forces of competitiveness, and those which are direct or indirect outcomes of a competitive society and economy; for more details see [2]. The indicators selected for the CCI framework are all of quantitative type (hard data) with different range of values and different units. Complete set of all CCI indicators are not shown in the paper because of the limited scope of the paper.

For efficiency analysis, set of 61 available variables for 27 EU countries has been compiled for three years (2000, 2008 and 2011). In order to *ensure comparability* between all EU countries and *methodological requirements of efficiency analysis* using selected DEA model, all variables have been pre-processed by exploratory *factor analysis* that reduced the number of variables and detected structure in the relationships between vari-

ables. Firstly, it was necessary to standardized set of all 61 available CCI indicators for all EU countries thus to unify their standards. Like used method of standardization, *Z-score* transformation has been used. Secondly, a *correlation analysis* was performed in each referred year separately for 36 standardized variables of inputs and 25 standardized variables of outputs. We have obtained three pairs of *correlation matrixes* between input and output variables. Based on used data standardization method, *Pearson's correlation coefficient* was chosen as a measure of correlation. Appropriate values of correlation coefficient have been found in cases where correlation degree of variables did not fall below 0.3. Like would not fall below 0.3, correlation coefficients should appropriate variables or vice versa exceed 0.9. Elaboration of correlation analysis has been processed by the *IBM SPSS Statistics 20*. On the basis of defined procedures, new data set of CCI indicators has consisted of *12 indicators – 7 input* and *5 output indicators*, thus 29 variables for inputs and 20 variables for outputs were excluded. New reduced data set of indicators determinate from extracted factors is shown in Table 1. Extracted factors explained approximately 82 % of average variability of indicators; for more details see [9].

Dimension	Pillar	Indicator of input
Inputs	**Macroeconomic Stability**	Gross Fixed Capital Formation, Labour Productivity per Person Employed
	Primary, Secondary and Tertiary Education; Training and Lifelong Learning	Total Public Expenditure at Tertiary Level of Education, Participants in Early Education, Participation in Higher Education, Lifelong Learning
	Indicators for Technological Readiness	Level of Internet Access
Dimension	**Pillar**	**Indicator of output**
Outputs	**Labour Market Efficiency**	Public expenditure on Labour Market Policies
	Market Size	Gross Domestic Product, Disposable income
	Business Sophistication	Gross Value Added in sophisticated sectors
	Innovation	Total intramural R&D expenditure

Table 1 Indicators of inputs and outputs relevant for DEA analysis
Source: based on [11], own modifications, 2013

3 Results of efficiency analysis in EU15 and EU12 countries

Efficiency analysis using CCR output oriented model assuming CRS and output oriented AP model of super efficiency assuming CRS has been calculated by the *DEA Frontier* software. The following results of DEA approach are based on DEA models (2) and (3) with *7 indicators of input* and *5 indicators of output*. Models are calculated for group of *27 DMUs* representing all EU Member States in three different years (2000, 2008 and 2011).

In the first step, CCR-O CRS model of efficiency has been solved for all DMUs in the reference period. So, efficient and inefficient DMUs where determined in group of EU15 countries. Complete *results of efficiency analysis of EU15 countries* are presented in Table 2. CCR-O CRS model singled out that most of EU15 countries present efficient DMUs with efficiency coefficient equals 1. Only *Italy* (IT) and *Portugal* (PT) are cases of inefficient DMUs in all three years where efficiency coefficients are greater than 1. In the second step, Andersen-Petersen's model of super efficiency (APM-O CRS) has been solved for the group of efficient EU15 countries in three referred years. Based on calculations of Andersen-Petersen's model, more variable results of super efficiency of EU15 countries can be determined and ranked in alphabetical order as it shown in Table 2. The best results of super efficiency analysis have been shown in the case of *Luxembourg* (LU) where coefficient of super efficiency is the smallest one in all three years. To the group of EU15 countries with the best results in coefficient of super efficiency in referred years belong *France* (FR) and *Finland* (FI) in 2000, *Belgium* (BE) and *Greece* in 2008, *Ireland* (IE) and *Belgium* (BE) in 2011. In Table 2, the most efficient countries are highlighted by bold and dark grey colour. The group of EU15 countries with the worst results in coefficient of super efficiency in referred years is presented by *Spain* (ES) and *Irelan*d (IE) in 2000, *Austria* (AT) in 2008 and 2011. In Table 2, the most inefficient EU15 countries are highlighted by italics and light grey colour. Some of EU15 countries have reached improving results in efficiency rate through the time period, e.g. *Belgium* (BE), *Greece* (EL) and mainly *Spain* (ES) or *Ireland* (IE) that improved its position in 2011 more than doubled compared to year 2000. Other EU15 countries have reached declining results of super efficiency rate in referred years, e.g. *Finland* (FI), *Sweden* (SE) and *France* (FR) where was the biggest worsening in super efficiency results between the years 2000 and 2011. Balanced results in coefficient of super efficiency we can observe in referred years in the case of *Denmark* (DK), *Netherlands* (NL) and *United Kingdom* (UK).

EU15 Countries	CCR-O CRS 2000		Rank*	CCR-O CRS 2008		Rank*	CCR-O CRS 2011		Rank*
	Efficiency	*Super efficiency*		*Efficiency*	*Super efficiency*		*Efficiency*	*Super efficiency*	
AT	1.000	0.904	11	*1.008*	*1.008*	*13*	*1.000*	*0.920*	*13*
BE	1.000	0.610	5	**1.000**	**0.460**	**2**	**1.000**	**0.580**	**3**
DE	1.000	0.762	9	1.000	0.796	8	1.000	0.807	9
DK	1.000	0.724	8	1.000	0.788	6	1.000	0.819	12
EL	1.000	0.711	7	**1.000**	**0.685**	**3**	1.000	0.600	4
ES	*1.161*	*1.161*	*14*	1.000	0.984	12	1.000	0.737	6
FI	**1.000**	**0.513**	**3**	1.000	0.781	5	1.000	0.795	7
FR	**1.000**	**0.328**	**2**	1.000	0.836	10	1.000	0.817	11
IE	*1.160*	*1.160*	*13*	1.000	0.903	11	**1.000**	**0.520**	**2**
IT	*1.200*	*1.200*	*15*	*1.075*	*1.075*	*15*	*1.026*	*1.026*	*14*
LU	**1.000**	**0.244**	**1**	**1.000**	**0.229**	**1**	**1.000**	**0.156**	**1**
NL	1.000	0.526	4	1.000	0.736	4	1.000	0.661	5
PT	1.107	1.107	12	*1.034*	*1.034*	*14*	*1.087*	*1.087*	*15*
SE	1.000	0.689	6	1.000	0.794	7	1.000	0.816	10
UK	1.000	0.789	10	1.000	0.804	9	1.000	0.801	8

Note: * Rank of EU15 countries is based on super efficiency results (in alphabetical order)

Table 2 Results of CCR-O CRS model for EU15 countries

Source: Own calculation and elaboration, 2013

Complete *results of efficiency analysis in EU12 countries* are presented in Table 3. CCR-O CRS model singled out much more variable results according to the values of efficiency coefficient for the group of EU12 countries. There is no country in the group of EU12 countries as efficient DMU with efficiency coefficient equals 1in all three referred years but most of EU12 countries presented cases of inefficient DMUs in all three years where efficiency coefficients are greater than 1. Andersen-Petersen's model of super efficiency has been also calculated for the group of efficient EU12 countries in all three years. Based on this calculation, results of super efficiency of EU12 countries has been determined and ranked in alphabetical order as it shown in Table 3. The best results of super efficiency analysis have been shown in the case of *Bulgaria* (BG) and *Slovenia* (SI) where coefficients of super efficiency are the smallest ones in most years. To the group of EU12 countries with the best results in coefficient of super efficiency in referred years accept countries mentioned above, belong also *Romania* (RO) and *Slovakia* (SK) in 2000, *Cyprus* (CY) in 2008 and *Estonia* (EE) in 2011. In Table 3, the most efficient EU12 countries are highlighted by bold and dark grey colour. The group of EU12 countries with the worst results in coefficient of super efficiency in referred years is presented by *Estonia* (EE), *Latvia* (LV) and *Malta* (MT) in 2000, *Lithuania* (LT), *Latvia* (LV) and *Slovakia* in 2008, *Lithuania* (LT), *Poland* (PL) and again *Latvia* (LV) in 2011. In Table 3, the most inefficient EU12 countries are highlighted by italics and light grey colour. To the group of EU12 countries that have reached improving results in efficiency rate through the referred years belong *Cyprus* (CY), *Hungary* (HU) and mainly *Estonia* (EE) that enormously improved its position in 2011 in comparison with two previous years. The group of EU12 countries that have reached declining results of super efficiency rate in referred years is presented by *Poland* (PL), *Slovakia* (SK) and mainly *Romania* (RO) where was the biggest worsening in super efficiency results in the years 2011 compared to 2000. Balanced trend in results in coefficient of super efficiency can be seen in the case of *Bulgaria* (BG), *Czech Republic* (CZ) and Latvia (LV).

EU12 Countries	CCR-O CRS 2000		Rank*	CCR-O CRS 2008		Rank*	CCR-O CRS 2011		Rank*
	Efficiency	*Super efficiency*		*Efficiency*	*Super efficiency*		*Efficiency*	*Super efficiency*	
BG	**1.000**	**0.909**	**2**	**1.103**	**1.103**	**2**	**1.000**	**0.936**	**3**
CY	1.192	1.192	6	**1.000**	**0.971**	**1**	1.000	0.942	4
CZ	1.172	1.172	5	1.235	1.235	4	1.049	1.049	6
EE	*2.214*	*2.214*	*12*	1.497	1.497	9	**1.000**	**0.920**	**2**
HU	1.531	1.531	9	1.244	1.244	5	1.000	0.987	5
LT	1.455	1.455	8	*1.827*	*1.827*	*12*	*1.740*	*1.740*	*12*
LV	*2.126*	*2.126*	*11*	*1.746*	*1.746*	*11*	*1.516*	*1.516*	*10*
MT	*1.813*	*1.813*	*10*	1.347	1.347	7	1.451	1.451	8
PL	1.361	1.361	7	1.474	1.474	8	*1.568*	*1.568*	*11*
RO	**1.000**	**0.860**	**1**	1.278	1.278	6	1.177	1.177	7
SI	1.153	1.153	4	**1.148**	**1.148**	**3**	**1.000**	**0.787**	**1**
SK	**1.000**	**0.965**	**3**	*1.716*	*1.716*	*10*	1.493	1.493	9

Note: * Rank of EU12 countries is based on super efficiency results (in alphabetical order)

Table 3 Results of CCR-O CRS model for EU12 countries

Source: Own calculation and elaboration, 2013

The results generated by used DEA models and displayed in Tables 2 and 3 can provides a comparison of the both groups of EU countries among themselves. Looking at the calculate results of efficiency by CCR-O CRS

model in face of initial hypothesis, we can generally *confirmed predefined hypothesis*. The number of efficient DMUs in the group of EU15 countries is much higher than number of efficient DMUs in the group of EU12 countries which is represented by 3 efficient countries (BG, RO, SI) in the year 2000, only one efficient country (CY) in 2008 and again by 3 efficient countries (BG, EE, SI) in the year 2011. Also most of EU12 countries reached higher value of coefficient of super efficiency in comparison with EU15 countries in each year.

4 Conclusion

Competitiveness and efficiency are complementary objectives which determine the long-term development of countries. Evaluation of efficiency in the context of competitive potential can be performed only if it is used existing concept of these terms or selected mainstream. The concept of Country competitiveness index is based on indicators of inputs and outputs and therefore used DEA approach seems to appropriate for efficiency analysis in EU countries. Our analysis has detected the significant differences in efficiency and super efficiency scores between EU15 and EU12 countries and also inside these groups. Analysing the results of super efficiency analysis in detail way, we find out some surprising facts. Super efficiency score of *Scandinavian countries* (Sweden, Denmark) is worse than has been assumed in comparison with other advanced EU15 countries. Typical representatives of *advanced Western European countries* (Germany or Austria) achieved worse rates of super efficiency coefficient than *Greece* in all years. The efficiency analysis brings interesting results also in the group of EU12 countries, especially in the case of *Bulgaria* and *Romania* as EU Member States that joined the EU in 2007. In these two cases we can find out some anomalies in values of coefficients of efficiency. DEA method evaluates, in fact, the volume of inputs for given outputs, which in case of Bulgaria and Romania seems to be effective, although these countries generally belong to the least developed within EU12 countries and also EU27 countries. This fact could be a prerequisite for further research on evaluation of national efficiency by specification of performed DEA model.

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Concept of spatial power indices with applications on real voting data from the Lower House of the Czech Parliament

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Abstract. Decision-making process of parliamentary voting has long attracted attention of political scientists, as well as economists and mathematicians. In general, taking into account game-theoretical approach, any parliamentary voting can be described as a cooperative game with transferable utility function. Moreover, in real world, agents of the game – usually political parties – act not strictly as predicted in theory. As all real systems are full of an uncertainty, also parliamentary voting can be described up to some degree of freedom. The concept of Shapley value, first introduced by L.S.Shapley in 1953, was the first attempt to evaluate players of these game types. Since then, many adjustments to the basic theory were done in order to improve real data results. One of such a transformation, the Owen and Shapley spatial index, took into account both the effect of agenda and the distribution of power. Adjustments of the index were done by Barr, Pasarelli and Benatiand, Marzetti, who tested the theory on the decision-making process in the European Union. To incorporate the coalition forming influence, Bilal, Albuquerque and Hosli proposed to consider additional weights to possible coalitions into power indices. This article applies the concept of additional weights to calculate power in a real voting, namely the data from the Lower House of the Czech Parliament with the emphasis on the State Budget voting issues.

Keywords: Shapley-Shubik power index, power distribution, Czech Parliament.

JEL Classification: C71
AMS Classification: 91B12

1 Introduction

Simple cooperative games, as special types of N-player cooperative games with transferable utility functions, are widely used to description of coalition formation in real voting bodies. The studies of cooperative games with transferable utility function always cover idea of game values. The concept of Shapley value, first introduced by L.S.Shapley in 1953 [6], was the first attempt to evaluate players of such cooperative games. The transformation of Shapley value to the context of simple cooperative games was done by Shapley and Shubik in 1954 [8]. The so-called Shapley-Shubik power index serves as a-priori evaluation of coalitions in voting bodies. The Shapley-Shubik power index is based only on information of decision-making rule.

The extension of this concept, done by Shapley and Owen (1989) [7], is based on the idea that the power of an agent depends not only on the voting rule of decision making, but also on the position of agents in political space; roughly speaking, some voters are more likely to vote similarly than others. Thus, the index is dependent on the consistency of coalitions positions based on the group preferences. Adjustments of the Shapley-Owen index were done by Barr and Pasarelli (2009) [2] and Benatiand Marzetti (2011) [3], who tested the theory on the decision-making process in the European Union. Godfrey (2005) [5] proposed algorithm for computation of Shapley-Owen index in two dimensions, and applied results to various voting bodies, e.g. US Senate committees, IMF board voting, or UN Security Council voting. Aleskerov and Otchour (2007) [1] applied the same approach to the power distribution among political parties in the State Duma of the Russian Federation. Bilal, Albuquerque and Hosli (2001) [4] combined probabilistic definition of power indices with model of unidimensional and multidimensional policy spaces, using exponential relationship.

The main aim of this article is to perform similar analysis on the data from the Czech parliament, namely state budget voting data from the 2006-2010 electoral period. Parliamentary discussions of state budget has to be finished before the end of the preceding year; these discussions usually cover tens of amendment votes to a state budget proposal, many of them related to one specific issue. Hence, the number of votes related to state budget issues is different throughout years, namely 414 votes in 2006, 61 votes in 2007, 130 votes in 2008, and 44 votes

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in 2009. The outcome of every vote for every member can be "no", "yes", "present, abstain", "absent". Every bill to be passed needs at least as many "yes" votes as quota. Quota is based on the sum of all present legislators, that means outcome "present, abstain" serves as "no" outcome; in this analysis this outcome is reclassified as "no" outcome. During the studied period, there were five political parties in the Lower House of the Czech Parliament; three of them created governmental coalition: Civic Democratic Party (ODS), Christian and Democratic Union – Czechoslovak People's Party (KDU-ČSL), and Green Party (SZ), while other two political parties – Czech Social Democratic Party (ČSSD) and Communist Party of Bohemia and Moravia (KSČM) – stayed in opposition. Basic information on the Czech Parliamentary system as well as the set of all historical votes can be found at the official web site of the Lower House of the Czech Parliament URL: www.psp.cz [10].

2 Power indices and probability of coalition creation

This analysis is based on the Shapley–Shubik power index [8], which was created to a-priori evaluation of the power division among bodies in committee system. Its main advantage is in the possible application under many different circumstances, for example in simple and weighted voting games as well as in multi-cameral systems. The derivation of the member's Shapley-Shubik power index is based on the number of cases when the member is in ordering pivotal.

The game in characteristic function form v for which the characteristic function can reach only values 1 or 0, $v(S) \in \{0,1\}$, for all coalitions S is called a simple game. If *S* is the winning coalition in a simple game, than $v(S) = 1$, otherwise $v(S) = 0$. Let *N* be the set of all players with cardinality *n*. Let Π denotes a set of all *n!* permutations of *N*. For every permutation $\boldsymbol{\pi} \in \Pi$ of the form $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)$ there exists a unique *k* such that $\{\pi_1, \pi_2, \ldots, \pi_k\}$ wins and $\{\pi_1, \pi_2, \ldots, \pi_{k-1}\}$ does not win. Player π_k is the pivot of permutation $\boldsymbol{\pi}$. Denote Π_p the set of all permutations with pivot *p*: $\Pi_p = \{\boldsymbol{\pi} \in \Pi;\ p \text{ pivot of } \boldsymbol{\pi}\}$. Then the Shapley-Shubik power index can be expressed as (Shapley and Shubik, 1954, [8]):

$$\varphi_p(v) = \frac{|\Pi_p|}{|\Pi|} = \frac{|\Pi_p|}{n!} \tag{1}$$

This approach is based on original Shapley idea (Shapley, 1953, [6]):

$$\varphi_i(v) = \sum_{i \in T \subset N} \frac{(t-1)!(n-t)!}{n!} . \left(v(T) - v(T-i)\right) \tag{2}$$

where *t* denotes the position of the pivotal player *i* in the ordered coalition *T*. The first term of the equation

$$Q(T,i) = \frac{(t-1)!(n-t)!}{n!} \tag{3}$$

is, in general, an expression of probability of occurrence of coalition T. However, in the real world, some coalitions are formed more likely than others. To incorporate the coalition forming influence, Bilal, Albuquerque and Hosli (2001) [4] proposed to consider additional weights of possible coalitions into the quantity *Q(T,i)*:

$$\hat{Q}(T,i) = \frac{Q(T,i).W(T)}{\sum_{\substack{S \subset N \\ i \text{ pivotal}}} Q(S,i).W(S)} \tag{4}$$

where *S* runs through all coalitions in which the player *i* is pivotal. In one-dimensional case, the weight of coalition of two players depends on the mutual distance $d_{i,j}$ of respective players *i, j*:

$$W(i,j) = e^{-d_{i,j}} \tag{5}$$

In the case of coalition of more players, the weight of coalition is determined by a distance of the two most distant players of the coalition.

$$W(C) = W(p_{\min}, p_{\max}) \tag{6}$$

In general, each vote stands for a preference relation of a respective player on specific issue. Taking into account m votes, we have preferences in *m* dimensions. The overall preference of each player is represented by a point in $\mathbf{R}^m$. Mutual distances of players in $\mathbf{R}^m$ play important role in their voting power.

3 Power calculations – Czech Parliament 2006-2009 state budget voting

The distribution of seats at the beginning of the 2006-2010 electoral period in the Lower House of the Czech Parliament did not allow simple government setting – both left-wing (ČSSD, KSČM) and right-wing political parties (ODS, KDU-ČSL, and SZ) gained 100 votes, not enough to pass any vote. The a-priori power distribution at the beginning of the period measured by Shapley-Shubik power index together with number of seats of political parties in the Lower House of the Czech Parliament is given in Table 1.

Political Party	**ODS**	**CSSD**	**KSCM**	**KDU-CSL**	**SZ**
Seats	81	74	26	13	6
Power Index	0.3667	0.2833	0.2833	0.0333	0.0333

Table 1 Shapley-Shubik power index of political parties in the Lower House of the Czech parliament at the beginning of the 2006-2010 electoral period. Source: Own calculations.

Power calculations in this article take into account seats distribution as it was at the beginning of the period. For every parliamentary party and every relevant vote, the party decision result was calculated. For each pair of political parties, the distance in the form of $d(\mathbf{p},\mathbf{q})$ was set:

$$d(\mathbf{p},\mathbf{q}) = \sqrt{\sum_{i=1}^{n}(p_i - q_i)} \tag{7}$$

In the relation (7), summation is taken through all votes to number of votes in respective state budget voting *n*, variables **p** and **q** are vectors of votes of compared political parties. The voting game is a simple game, thus the **p** and **q** vectors are composed of *n* values from the set $\{0,1\}$. Calculated Euclidian distances between political parties during respective state budget voting are given in Tables 2-5.

	ČSSD	ODS	KSČM	KDU-ČSL	SZ
ČSSD	-	7.616	13.266	8.124	12.923
ODS	7.616	-	13.856	8.944	12.767
KSČM	13.266	13.856	-	13.638	14.248
KDU-ČSL	8.124	8.944	13.638	-	12.845
SZ	12.923	12.767	14.248	12.845	-

Table 2 Distances of political parties in 2007 state budget voting. Source: Own calculations.

	ČSSD	ODS	KSČM	KDU-ČSL	SZ
ČSSD	-	6.633	4.796	6.557	6.083
ODS	6.633	-	6.856	1	2.646
KSČM	4.796	6.856	-	6.782	6.928
KDU-ČSL	6.557	1	6.782	-	2.449
SZ	6.083	2.646	6.928	2.449	-

Table 3 Distances of political parties in 2008 state budget voting. Source: Own calculations.

The interesting point is that, in calculated distances, the scale of results varies in different years. These differences are consequences of different number of votes. Even though we could expect that calculated numbers should be very similar, higher amount of votes increases possibility for any pair of political party to differ. That is the reason why, for example, mutual distances in 2007 state budget voting (represented by 414 votes) varied from 7 to 14, while in 2008 state budget voting (represented by 61 votes) varied approximately from 1 to 7. Similarly, mutual distances in 2009 state budget voting (represented by 130 votes) varied approximately from 1 to 11, while in 2008 state budget voting (represented by 44 votes) varied approximately from 2 to 6.

	ČSSD	ODS	KSČM	KDU-ČSL	SZ
ČSSD	-	10.488	3.162	10.536	10.535
ODS	10.488	-	10.770	1.000	1.000
KSČM	3.162	10.770	-	10.724	10.817
KDU-ČSL	10.536	1.000	10.724	-	1.414
SZ	10.535	1.000	10.817	1.414	-

Table 4 Distances of political parties in 2009 state budget voting. Source: Own calculations.

	ČSSD	ODS	KSČM	KDU-ČSL	SZ
ČSSD	-	4.243	4.123	3.742	4.000
ODS	4.243	-	5.196	4.000	2.828
KSČM	4.123	5.196	-	5.000	5.385
KDU-ČSL	3.742	4.000	5.000	-	3.742
SZ	4.000	2.828	5.385	3.742	-

Table 5 Distances of political parties in 2010 state budget voting. Source: Own calculations.

Another interesting point is to compare mutual distances of governmental coalition political parties – ODS, KDU-ČSL, and SZ. In order to compare their relative distances through four years, their mutual positions are adjusted such that the distance between two main political rivals, ODS and ČSSD, is set to be 1 in respective years. Adjusted differences of coalitional parties are given in Table 6. Graphical comparison of the respective distances is given in Figure 1.

Adjusted distances	ODS and KDU-ČSL	ODS and SZ	KDU-ČSL and SZ
2007 State Budget	1.18	1.68	1.69
2008 State Budget	0.15	0.39	0.37
2009 State Budget	0.10	0.10	0.13
2010 State Budget	0.94	0.67	0.88

Table 6 Adjusted distances of political during state budget voting. Source: Own calculations.

Calculated distances reflect situation in the Lower House of the Czech Parliament. The first, 2007 state budget voting was held in December 2006, while the government was approved after months of discussions in January 2007. Detailed description of the situation after 2006 Parliamentary Elections can be found in Škochová (2008) [9].

After the government was set, the governmental coalition composed of ODS, KDU-ČSL, and SZ worked together. However, differences in political programs and disagreement between governmental political parties ended up in a downfall of the government; on March 24, 2009, the Lower House of the Czech Parliament approved no-confidence of the government. The caretaker government was set and functioned till elections in 2010 Detailed description of the no-confidence voting in the Czech Parliament can be found in [11].

The political situation is visible at mutual distances between governmental political parties. The interesting point in 2007 state budget voting is the fact, that two political parties, usually staying in opposition – ODS and CSSD, are more close to each other than any pair of later governmental political parties – ODS, KDU-CSL, and SZ (Table 2). In 2008 and 2009 state budget voting all governmental political parties stayed relatively closer, while the distance increased in 2010 state budget voting (Table 6, Figure 1).

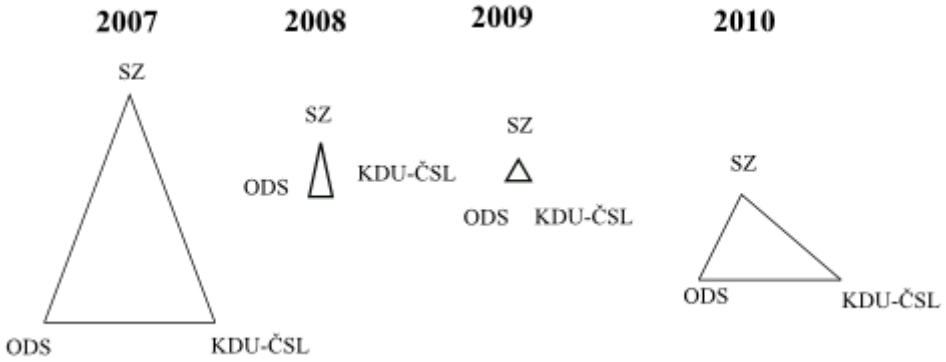

Figure 1 Adjusted distances of political during state budget voting. Source: Own calculations.

The situation is also reflected on calculated power of political parties in state budget voting 2007-2010 (Table 7). In 2006 (that means 2007 state budget voting), the cooperation of two political parties with the highest amount of seats (ODS and CSSD) almost ruled out other three political parties – their power decreased to zero, while power of the influential political parties was distributed approximately 1:1. In the 2008 and 2009 state budget voting, the influence of CSSD decreased. The voting discipline of governmental parties in 2009 state budget voting caused increase in power of all three governmental parties. However, the power distribution in the last studied 2010 state budget voting reveals increase influence of oppositional political parties – ODS and KDU-CSL.

	ODS	CSSD	KSCM	KDU-CSL	SZ
2007 State Budget	0.498	0.497	0.004	0.001	0.000
2008 State Budget	0.367	0.308	0.264	0.033	0.028
2009 State Budget	0.340	0.019	0.019	0.287	0.335
2010 State Budget	0.413	0.361	0.180	0.027	0.019

Table 7 Estimated power of political parties in 2007-2010 state budget voting.

4 Conclusion

This article compares the calculated a-priori Shapley-Shubik power index with decisional power of political parties calculated using state budget voting data from the 2006-2010 electoral period. The real decisional power was calculated in accord with spatial power distribution as presented in [4]. The results of calculations of mutual distances as well as results of power distribution show differences through years. The situation during 2006-2010 electoral period in the Chamber of Deputies of the Czech Parliament was evolving, votes of legislators reflected given situation. The obtained results show, that governmental political party – ODS, KDU-CSL, and SZ – had highest influence (measured as voting power) during 2009 state budget voting, even though the biggest parliamentary party – ODS – had higher real power index in 2007 state budget voting. After the change of the government in March 2009, the consequent state budget voting had shown increase in power of political parties staying originally in opposition – CSSD and KSCM.

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Cyclical unemployment behavior and labor markets institutions

Štěpán Mikula[1]

Abstract. This paper deals with relationship of labor-market institutions and cyclical unemployment described by unemployment gap. Results obtained from unbalanced panel regression for OECD countries in period 1990–2012 brings an evidence that there indeed is a relationship between institutional settings of labor markets and the amplitude of cyclical unemployment. Where the effects of institutional setting vary in dependence on the phase of the business-cycle. If the output is below the trend the more protective institutions mostly tend to reduce the amplitude. In the opposite case – i.e. if the output is higher than trend – the effect of institutions is mixed.

Keywords: cyclical unemployment, institutions, OECD countries

JEL classification: E24,E02
AMS classification: 91B40

1 Introduction

The institutions understood – in accordance with North [6] – as *"the rules of the game"* are subject of many theoretical as well as empirical works. These works are primarily focused on the influence of institutions on the economic performance or economic growth. However the output is not the only one characteristic of economy dependent on the institutional setting. Relationship of the labor market performance and institutions also deserves closer attention. The institutions determine rigidity or flexibility of the labor markets and some authors treat *"adverse institutions"* and *"labor market rigidities"* as equivalent terms (for example see [3]). Such role of the labor market institutions emphasize the need of their research – the understanding of their role should provide valuable information for macroeconomic modeling.

This paper is not of course the first one that investigate the role of labor market institutions. However the most attention was so far paid to the explanation of the variability of unemployment and the shifts of its level over time (see for example [2, 3, 1]). This paper uses similar methods and data as previously mentioned papers however it asks a different question. The goal is not to explain some property of the unemployment trend but to explain cyclical component amplitude and also investigate its possible changes in the current global recession commonly called the economic crisis. Unemployment development in economic crisis is already described and explained (even considering the role of institutions) by Furceri & Mourougane [5]. However their research is focused specifically on the structural unemployment. The goal of this paper is more general – it deals with the cyclical component regardless its composition.

The research is conducted on the homogeneous sample of OECD countries for the period 1990–2012. This time period contains long and quite stable period of growth followed by the economic crisis. It allows to assess the role of institutions in both situations – crisis (as especially deep recession) and in common business-cycle. Therefore models described in following section are estimated three times – for the whole period and for the subsamples of crisis and the rest (i.e. pre-crisis sample). The boundary between these two subsamples is set to the year of 2008 – which is the first year of crisis in the USA.

Results obtained from unbalanced panel regression brings an evidence that there indeed is a relationship between institutional settings of labor markets and the amplitude of cyclical unemployment. Where the effects of the institutional setting vary in dependence on the phase of the business-cycle. If the output is below the trend the more protective institutions mostly tend to reduce the amplitude. In the opposite case – i.e. if the output is higher than trend – the effect of institutions is mixed.

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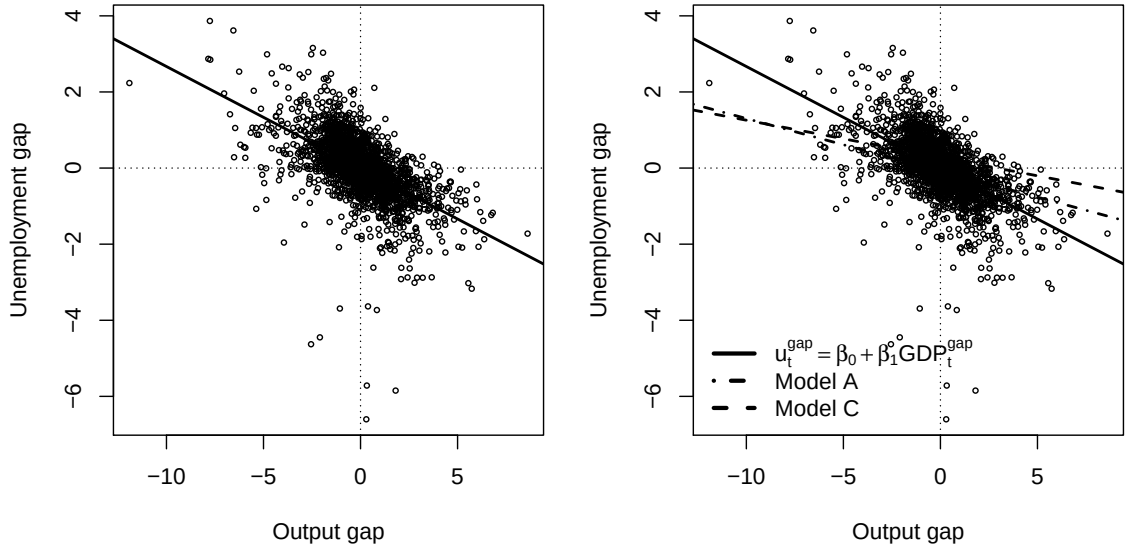
2 Model and data

This paper seeks an explanation of the cyclical unemployment behavior. The cyclical component is represented by the unemployment gap (u^{gap}) – i.e. by the percentage deviation of observed unemployment from the trend (which can be understood as the natural rate of unemployment). An unemployment gap is obtained from quarterly rate of seasonally adjusted harmonized unemployment provided by OECD [7]. The trend a cyclical component is identified using Hodrick-Prescott filter with $\lambda = 1600$. The descriptive statistics for all samples are provided (as well as for all other variables) in the Table 2.

The first choice in explanatory variables is clear. All above mentioned papers use output as the predictor of unemployment. This paper is no different. The first explanatory variable is output gap (GDP^{gap}) obtained from quarterly seasonally adjusted data on real GDP per capita provided by OECD [7]. The gap is yet again computed using Hodrick-Prescott filter in the same setting. There is a clear relationship between output and unemployment gap – as it is depicted in the Figure 1a. This figure illustrates well-known fact that development of the unemployment rate is contracyclical. The dependence of u^{gap} on the output gap is described by Model A for the time t and the country i:

$$u_{t,i}^{\text{gap}} = \beta_0 + \beta_1 \text{GDP}_{t,i}^{\text{gap}} + \beta_2 \text{GDP}_{t-1,i}^{\text{gap}} + \varepsilon_{t,i} \tag{1}$$

The lag order of GDP^{gap} is chosen on the basis of maximizing of coefficient of determination.



(a) Relationship of the current output gap and the current unemployment gap

(b) Relationship of the current output gap and the current unemployment gap controlled for laged output gap and institutional variables

Figure 1: Correlation of the output gap and the unemployment gap (both in % and at the same time)

The Model A is used as a base model for the testing of the influence of institutions. Model in specification B is obtained by adding the institutional variables from Botero et al. [4]:

$$u_{t,i}^{\text{gap}} = \beta_0 + \beta_1 \text{GDP}_{t,i}^{\text{gap}} + \beta_2 \text{GDP}_{t-1,i}^{\text{gap}} + \beta_3 \text{labor7a}_i + \beta_4 \text{industrial4a}_i + \beta_5 \text{socseca}_i + \beta_5 \text{cra}_i + \varepsilon_{t,i} \tag{2}$$

I use following institutional variables:

- labor7a: measures the protection provided by employment laws. It is based on assessment of costs of alternative contracts, increasing hours worked, firing workers and dismissal procedures.
- industrial4a: measures the protection of collective relations laws.
- socseca: measures social benefits. It takes into account unemployment benefits as well as old age benefits, disablity and sickness benefits.
- cra: measures the extent of protection of vulnerable groups against discrimination.

Higher values of all indexes indicates stronger protection. Data from Botero et al. [4] are unfortunately available only for one year. (Dataset reflects situation around year 2000.) However the institutions are stable in time and also alternative indicators available on yearly basis are in the long run stable or even constant. Therefore using values from Botero et al. [4] for the whole period of interest probably does not distort the results too much.

The Model B is further adjusted to specification C by multiplying of institutional variables by dummies *dp* and *dn*:

$$\begin{aligned} u_{t,i}^{\text{gap}} = \beta_0 + \beta_1 \text{GDP}_{t,i}^{\text{gap}} + \beta_2 \text{GDP}_{t-1,i}^{\text{gap}} + \beta_3 (dp_{t,i} \times \text{labor7a}_i) + \beta_4 (dp_{t,i} \times \text{industrial4a}_i) + \\ + \beta_5 (dp_{t,i} \times \text{socseca}_i) + \beta_6 (dp_{t,i} \times \text{cra}_i) + \beta_7 (dn_{t,i} \times \text{labor7a}_i) + \beta_8 (dn_{t,i} \times \text{industrial4a}_i) + \\ + \beta_9 (dn_{t,i} \times \text{socseca}_i) + \beta_1 0 (dn_{t,i} \times \text{cra}_i) + \varepsilon_{t,i} \end{aligned} \tag{3}$$

$$dp_{t,i} = \begin{cases} 1 & \text{if } \text{GDP}_{t,i}^{\text{gap}} > 0 \\ 0 & \text{if } \text{GDP}_{t,i}^{\text{gap}} \leq 0 \end{cases} \tag{4}$$

$$dn_{t,i} = \begin{cases} 1 & \text{if } \text{GDP}_{t,i}^{\text{gap}} \leq 0 \\ 0 & \text{if } \text{GDP}_{t,i}^{\text{gap}} > 0 \end{cases} \tag{5}$$

Employment of *dp* and *dn* allows to study the relationship in the different phases of the business-cycle.

The alternative measures of institutional setting of labor markets come from OECD. These variables *ept* and *epr* are available on yearly basis[1] and are employed in model specifications D and E analogous to B and C:

$$u_{t,i}^{\text{gap}} = \beta_0 + \beta_1 \text{GDP}_{t,i}^{\text{gap}} + \beta_2 \text{GDP}_{t-1,i}^{\text{gap}} + \beta_3 \text{epr}_{t,i} + \beta_4 \text{ept}_{t,i} + \varepsilon_{t,i} \tag{6}$$

$$\begin{aligned} u_{t,i}^{\text{gap}} = \beta_0 + \beta_1 \text{GDP}_{t,i}^{\text{gap}} + \beta_2 \text{GDP}_{t-1,i}^{\text{gap}} + \beta_3 (dp_{t,i} \times \text{epr}_{t,i}) + \beta_4 (dp_{t,i} \times \text{ept}_{t,i}) + \\ + \beta_5 (dn_{t,i} \times \text{epr}_{t,i}) + \beta_6 (dn_{t,i} \times \text{ept}_{t,i}) + \varepsilon_{t,i} \end{aligned} \tag{7}$$

Employed OECD's indicators are following:

- ept (EPT_v1 in OECD's database): measures dismissal procedures of employees on regular contracts.
- epr (EPR_v1 in OECD's database): measures strictness of regulation on temporary contracts.

OECD's database contains more indicators however the chosen ones are available for the longest period (1985–2008). Unfortunately the length of the time-series does not allow analysis of crisis effects. Both indicators are scaled on scale from 0 to 6 where higher values indicate higher restrictions.

3 Results

All random effects[2] models are estimated using feasible generalized least squares (FGLS) on unbalanced panel of OECD countries. The estimated coefficients are presented in the Table 1.

All estimated coefficient for output gap are statistically significant and with exception of crisis sample in Model C have expected impact on unemployment gap. The exception – even if it is statistically significant – has a minimal impact. Its absolute value is by far the lowest from all estimated coefficients for output gaps.

Coefficients for institutional variables are generally statistically significant. It generally supports hypothesis of existence of relationship between institutions and cyclical unemployment amplitude. Of course there are some exceptions. Index socseca is significant only in two cases (of six) on sample including all observations (Model C) and sample from crisis (Model B). Such results suggest that relationship between socseca and u^{gap} might not be robust.

A special attention should be paid to Model C and D, which allow to distinguish the influence of institutions in different phases of business-cycle. In Model C the more protective institutions in recession

[1] It is assumed that the institutional setting is constant for the whole year – i.e. for four subsequent quarters. This assumption comes from common praxis to execute major legislature changes to the beginning of the year.

[2] Variables constant in time rule out the possibility of using fixed effects models.

Specification	Model A			Model B		Model C			Model D	Model E
Observations	all	pre-crisis	crisis	all	pre-crisis	crisis	all	pre-crisis	crisis	all	all
Intercept	-0.076^{***}	-0.008^{***}	-0.055^{***}	-0.023	-0.04	0.24^{***}	0.283^{***}	-0.077^{*}	0.371^{***}	0.017	-0.019
	(0.004)	(0.002)	(0.003)	(0.037)	(0.034)	(0.074)	(0.026)	(0.043)	(0.1)	(0.013)	(0.015)
Output gap_t	-0.138^{***}	-0.146^{***}	-0.087^{***}	-0.166^{***}	-0.154^{***}	-0.047^{***}	-0.097^{***}	-0.143^{***}	0.01^{***}	-0.158^{***}	-0.181^{***}
	(0.002)	(0.001)	(0.004)	(0.003)	(0.003)	(0.002)	(0.003)	(0.004)	(0.001)	(0.004)	(0.005)
Output gap_{t-1}	-0.133^{***}	-0.13^{***}	-0.205^{***}	-0.16^{***}	-0.161^{***}	-0.255^{***}	-0.187^{***}	-0.147^{***}	-0.246^{***}	-0.203^{***}	-0.179^{***}
	(0.002)	(0.001)	(0.003)	(0.003)	(0.003)	(0.001)	(0.003)	(0.004)	(0.001)	(0.004)	(0.004)
labor7a				0.155^{***}	0.042^{***}	0.004					
				(0.041)	(0.015)	(0.058)					
industrial4a				-0.222^{***}	0.125^{***}	-0.115^{**}					
				(0.056)	(0.026)	(0.047)					
socseca				0.054	0.042	-0.189^{**}					
				(0.062)	(0.056)	(0.074)					
cra				0.024	-0.07^{***}	-0.154^{***}					
				(0.016)	(0.024)	(0.039)					
dp × labor7a							-0.145^{***}	-0.288^{***}	0.184^{**}		
							(0.048)	(0.023)	(0.079)		
dp × industrial4a							0.298^{***}	0.364^{***}	0.504^{***}		
							(0.053)	(0.032)	(0.069)		
dp × socseca							-0.21^{***}	0.173^{***}	-0.501^{***}		
							(0.047)	(0.067)	(0.094)		
dp × cra							-0.427^{***}	-0.121^{***}	-0.867^{***}		
							(0.032)	(0.031)	(0.016)		
dn × labor7a							-0.037	0.395^{***}	-0.293^{***}		
							(0.046)	(0.024)	(0.079)		
dn × industrial4a							-0.038	-0.085^{***}	-0.605^{***}		
							(0.05)	(0.031)	(0.069)		
dn × socseca							-0.341^{***}	-0.009	0.015		
							(0.046)	(0.066)	(0.094)		
dn × cra							0.127^{***}	-0.063^{**}	0.263^{***}		
							(0.031)	(0.03)	(0.016)		
epr										-0.004	
										(0.006)	
ept										0.013^{***}	
										(0.003)	
dp × epr											0.015^{**}
											(0.007)
dp × ept											0.03^{***}
											(0.004)
dn × epr											-0.019^{**}
											(0.008)
dn × ept											0.045^{***}
											(0.003)
Multiple R^2	0.386	0.416	0.384	0.408	0.437	0.399	0.420	0.442	0.430	0.477	0.472
n (countries)	32	32	32	30	30	30	30	30	30	28	28
N (observations)	2356	1723	633	2245	1652	593	2245	1652	593	1584	1584

Table 1: Panel regression results (random effects) for unemployment gap

Stars denote significance level of estimated coefficients: * stands for 10 % level, ** for 5 % level and *** for 1 % level. Parentheses contains standard errors.

	Min.	Median	Mean	Max.	S.E.
Sample: all					
Unemployment gap	−6.601	0.000	0.000	3.865	0.798
Output gap	−11.911	−0.068	0.000	8.609	1.773
labor7a	0.161	0.520	0.519	0.809	0.193
industrial4a	0.188	0.469	0.463	0.667	0.139
socseca	0.478	0.723	0.715	0.873	0.087
cra	0.468	0.734	0.695	0.933	0.118
epr	0.170	2.250	2.157	4.830	0.821
ept	0.250	1.630	2.028	5.380	1.491
Sample: pre-crisis					
Unemployment gap	−4.450	−0.007	−0.003	3.865	0.694
Output gap	−9.433	−0.048	0.099	8.609	1.600
epr	0.170	2.250	2.155	4.830	0.824
ept	0.250	1.630	2.038	5.380	1.503
Sample: crisis					
Unemployment gap	−6.601	0.030	0.009	3.145	1.058
Output gap	−11.911	−0.137	−0.287	6.813	2.177

Table 2: Descriptive statistics of variables used in regression

Values for variables from Botero et al. [4] are the same for all three samples.

tend to lower the amplitude of u^{gap}. It does not entirely hold for the index cra. Higher protection against discrimination tend to increase the amplitude especially in the time of crisis. Yet another exception is labor7a, which seems to increase the amplitude in the pre-crisis period. However the difference between coefficient of labor7a in pre-crisis period and in crisis is not statistically significant ($t = 207.678^{***}$, $df = 629.877$). ()Equality of estimated coefficients is tested by Welch's two sample t-test.)

The effects in the phase of expansion are mixed. Higher values of industrial4a tend to decrease amplitude on the other hand the cra yet again increases the amplitude. Effects of remaining indicators differ in pre-crisis and crisis period.

The Welch's t-test can be also used to investigate whether there are differences between effects in recession and expansion. For the purpose of this test the coefficients estimated for $dp = 1$ are multiplied by -1 in order to scale the effects. (Negative values for $dp = 1$ increase the amplitude and for $dn = 1$ decrease the amplitude.) The values for pre-crisis period significantly differs in all cases except cra – which seems to be quite universal cause of the higher amplitudes. In the case of the crisis sample the test do not reject the null hypothesis in all cases except effect of socseca – which is for $dn = 1$ statistically insignificant. Equality of coefficients in crisis sample might be caused by lower sample and especially by the fact, that there quite little observations with $dp = 1$, which can of course distort the results. If the coefficient from Model C estimated on full sample are put into the test the null hypothesis is rejected in the case of cra and labor7a. The results for cra seems to be generally robust – cra tends to increase the amplitude with different effect in recession and expansion.

The data on institutions provided by OECD (Model E) allows just a limited analysis. More strict institutions tend to decrease the amplitude with exception of ept for $dp = 1$. The effect of ept also significantly differs with the phases of business-cycle.

4 Conclusion

This paper brings an evidence that there indeed is a relationship between institutional settings of labor markets and the amplitude of cyclical unemployment. Where the effects of institutional setting vary in dependence on the phase of the business-cycle. If the output is below the trend the more protective institutions mostly tend to reduce the amplitude – which might be the goal of economic policy. The

opposite case – i.e. if the output is higher than trend – the effect of institutions are mixed.

My results also indicates, that in some cases the effect of institutions differ during the crisis and pre-crisis period. The cause of this finding is unclear. Difference between coefficients may in fact reveal some kind of nonlinearity. Depth of the downswing may cause that that output gap exceed some possible structural break.

Presented results rise serious questions about the nature of current crisis – especially whether is this crisis different from other recessions. With regard to used data the results cannot be understood as some proof of uniqueness of this crisis. Results obtained for the crisis should be compared with effects estimated on the another historical cases of the deep recessions. The another way how to improve the results is using method of seemingly unrelated regressions (SUR) which allows to assess country-specific relationship between institutions and unemployment gap.

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Robustness of (0–1) Monge fuzzy matrices

Monika Molnárová[1]

Abstract. Robustness of (0–1) Monge matrices over max-min algebra (fuzzy matrices) is studied. The max-min algebra is an extremal algebra with operations maximum and minimum. Equivalent conditions for (0–1) Monge fuzzy matrices to be robust were proved. Polynomial algorithm for checking the necessary and sufficient conditions for robustness of (0–1) Monge fuzzy matrices is introduced.

Keywords: (max, min) algebra, robustness, Monge matrix

JEL classification: C02
AMS classification: 08A72, 90B35, 90C47

1 Introduction

To model applications of fuzzy discrete dynamic systems (DDS) are usually operations of addition and multiplication replaced by operations of maximum and minimum. The resulting extremal algebra is the so–called $(\max, \min)$–algebra and the matrices representing DDS are the so–called fuzzy matrices. The Monge matrices and their applications in many different areas were studied in [1], [2]. Properties of fuzzy matrices were described in [3]. The problem of robust matrices in fuzzy algebra was presented [6]. Sufficient and necessary conditions for checking robustness of interval fuzzy matrices with corresponding polynomial algorithms were introduced in [5].

2 Preliminaries

The fuzzy algebra $\mathcal{B}$ is a triple $(B, \oplus, \otimes)$, where $(B, \leq)$ is a bounded linearly ordered set with binary operations *maximum* and *minimum*, denoted by $\oplus$, $\otimes$. The least element in B will be denoted by O, the greatest one by I. We consider in this paper $B = \{O, I\}$. By $\mathbb{N}$ we denote the set of all natural numbers. The greatest common divisor of a set $S \subseteq \mathbb{N}$ is denoted by $\gcd S$, the least common multiple of the set S is denoted by $\operatorname{lcm} S$. For a given natural $n \in \mathbb{N}$, we use the notation N for the set of all smaller or equal positive natural numbers, i.e., $N = \{1, 2, \ldots, n\}$.

For any $m, n \in \mathbb{N}$, $B(m, n)$ denotes the set of all matrices of type $m \times n$ and $B(n)$ the set of all n-dimensional column vectors over $\mathcal{B}$. The matrix operations over $\mathcal{B}$ are defined formally in the same manner (with respect to $\oplus$, $\otimes$) as matrix operations over any field. The rth power of a matrix $A \in B(n, n)$ is denoted by A^r, with elements a_{ij}^r. For $A, C \in B(n, n)$ we write $A \leq C$ if $a_{ij} \leq c_{ij}$ holds for all $i, j \in N$.

A *digraph* is a pair $G = (V, E)$, where V, the so–called vertex set, is a finite set, and E, the so–called edge set, is a subset of $V \times V$. A digraph $G' = (V', E')$ is a subdigraph of the digraph G (for brevity $G' \subseteq G$), if $V' \subseteq V$ and $E' \subseteq E$. A path in the digraph $G = (V, E)$ is a sequence of vertices $p = (i_1, \ldots, i_{k+1})$ such that $(i_j, i_{j+1}) \in E$ for $j = 1, \ldots, k$. The number k is the length of the path p and is denoted by $\ell(p)$. If $i_1 = i_{k+1}$, then p is called a cycle. For a given matrix $A \in B(n, n)$ the symbol $G(A) = (N, E)$ stands for the complete, edge–weighted digraph associated with A, i.e., the vertex set of $G(A)$ is N, and the capacity of any edge $(i, j) \in E$ is a_{ij}. In addition, for given $h \in B$, the *threshold digraph* $G(A, h)$ is the digraph $G = (N, E')$ with the vertex set N and the edge set $E' = \{(i, j);\ i, j \in N,\ a_{ij} \geq h\}$. The following lemma describes the relation between matrices and corresponding threshold digraphs.

Lemma 1. [5] Let $A, C \in B(n, n)$. Let $h, h_1, h_2 \in B$.

(i) If $A \leq C$ then $G(A, h) \subseteq G(C, h)$,
(ii) if $h_1 < h_2$ then $G(A, h_2) \subseteq G(A, h_1)$.

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By a *strongly connected component* of a digraph $G(A,h) = (N,E)$ we mean a subdigraph $\mathcal{K} = (N_\mathcal{K}, E_\mathcal{K})$ generated by a non-empty subset $N_\mathcal{K} \subseteq N$ such that any two distinct vertices $i,j \in N_\mathcal{K}$ are contained in a common cycle, $E_\mathcal{K} = E \cap (N_\mathcal{K} \times N_\mathcal{K})$ and $N_\mathcal{K}$ is the maximal subset with this property. A strongly connected component $\mathcal{K}$ of a digraph is called non-trivial, if there is a cycle of positive length in $\mathcal{K}$. For any non-trivial strongly connected component $\mathcal{K}$ is the *period* of $\mathcal{K}$ defined as $\operatorname{per}\mathcal{K} = \gcd\{\,\ell(c);\ c \text{ is a cycle in } \mathcal{K},\ \ell(c) > 0\,\}$. If $\mathcal{K}$ is trivial, then $\operatorname{per}\mathcal{K} = 1$. By $\mathrm{SCC}^\star(G)$ we denote the set of all non-trivial strongly connected components of G.

Let $A \in B(n,n)$ and $x \in B(n)$. The sequence $O(A,x) = \{x^{(0)}, x^{(1)}, x^{(2)}, \dots, x^{(n)}, \dots\}$ is the orbit of $x = x^{(0)}$ generated by A, where $x^{(r)} = A^r \otimes x^{(0)}$ for each $r \in \mathbb{N}$.
For a given matrix $A \in B(n,n)$, the number $\lambda \in B$ and the n–tuple $x \in B(n)$ are the so–called *eigenvalue* of A and *eigenvector* of A, respectively, if they are the solution of the *eigenproblem* for matrix A, i.e. they satisfy the equation $A \otimes x = \lambda \otimes x$. The corresponding *eigenspace* $V(A,\lambda)$ is defined as the set of all eigenvectors of A with associated eigenvalue λ, i.e., $V(A,\lambda) = \{x \in B(n);\ A \otimes x = \lambda \otimes x\}$.
Let $\lambda \in B$. A matrix $A \in B(n,n)$ is *ultimately λ–periodic* if there are natural numbers p and R such that the following holds: $A^{k+p} = \lambda \otimes A^k$ for all $k \geq R$. The smallest natural number p with above property is called the period of A, denoted by $\operatorname{per}(A,\lambda)$. In case $\lambda = I$ we denote $\operatorname{per}(A,I)$ by abbreviation $\operatorname{per} A$.

According to [3] we define $\mathrm{SCC}^\star(A) = \cup\{\mathrm{SCC}^\star(G(A,h));\ h \in \{a_{ij}; i,j \in N\}\}$.

Theorem 1. [3] *Let $A \in B(n,n)$. Then* $\operatorname{per} A = \operatorname{lcm}\{\operatorname{per}\mathcal{K};\ \mathcal{K} \in \mathrm{SCC}^\star(A)\}$.

Definition 1. Let $A = (a_{ij}) \in B(n,n)$, $\lambda \in B$. Let $T(A,\lambda) = \{x \in B(n);\ O(A,x) \cap V(A,\lambda) \neq \emptyset\}$. A is called *λ–robust* if $T(A,\lambda) = B(n)$. A λ–robust matrix with $\lambda = I$ is called *a robust matrix*.

The following result (adapted for $\lambda = I$) was proved in [6].

Lemma 2. [6] Let $A = (a_{ij}) \in B(n,n)$. Then A is robust if and only if $\operatorname{per} A = 1$.

3 Monge matrices and robustness

Definition 2. We say, that a matrix $A = (a_{ij}) \in B(m,n)$ is a convex Monge matrix (concave Monge matrix) if and only if

$$a_{ij} \otimes a_{kl} \leq a_{il} \otimes a_{kj} \qquad \text{for all} \quad i < k,\ j < l$$
$$\big(\, a_{ij} \otimes a_{kl} \geq a_{il} \otimes a_{kj} \qquad \text{for all} \quad i < k,\ j < l \,\big).$$

In this paper, we assume that the considered matrix is convex.

Theorem 2. *Let $A \in B(n,n)$ be a matrix with no zero rows and no zero columns. A is a Monge matrix if and only if $a_{ij} = 0$ then*

$$\textit{either} \qquad a_{kl} = 0 \qquad \textit{for all} \qquad k \leq i,\ l \leq j \tag{1}$$
$$\textit{or} \qquad a_{kl} = 0 \qquad \textit{for all} \qquad k \geq i,\ l \geq j. \tag{2}$$

Proof. Let $A \in B(n,n)$ be a Monge matrix with no zero rows and no zero columns. Then there exists an element in ith row equal to 1. There are two possibilities.
Case 1. Let $r, j \in N$ be arbitrary such that $r > j$ and the equality $a_{ir} = 1$ holds. Using the Monge property $a_{kj} = 0$ for all $k < i$. Since there is no zero column in A, there exists $p > i$ with $a_{pj} = 1$. We shall show that $a_{kl} = 0$ for all $k \leq i$, $l < j$. Let us assume an element $a_{st} = 1$ for $s \leq i$, $t < j$. We obtain $a_{st} \otimes a_{pj} > a_{sj} \otimes a_{pt}$ what is a contradiction with the Monge property.
Case 2. Let $a_{il} = 0$ for all $l > j$. Consequently there exists $r < j$ with $a_{ir} = 1$. Using the Monge property $a_{kj} = 0$ for all $k > i$. Since there is no zero column in A, there exists $p < i$ with $a_{pj} = 1$. We shall show that $a_{kl} = 0$ for all $k \geq i$, $l > j$. Let us assume an element $a_{st} = 1$ for $s \geq i$, $t > j$. We obtain $a_{st} \otimes a_{pj} > a_{sj} \otimes a_{pt}$ what is a contradiction with the Monge property.
For the converse implication let us denote the set of indices corresponding to zero elements in left upper corner of A by $L = \{(i,j) \in N \times N;\ a_{kl} = 0 \text{ for all } k \leq i,\ l \leq j\}$ and the set of indices corresponding to zero elements in right lower corner in A by $R = \{(i,j) \in N \times N;\ a_{kl} = 0 \text{ for all } k \geq i,\ l \geq j\}$. Since there is no zero row and no zero column in A, $L \cap R = \emptyset$. Moreover the set of indices $C = \{(i,j) \in N \times N;\ (i,j) \notin L \cup R\}$ corresponding to elements equal to 1 in A is non-empty. We shall prove that the matrix with above structure has the Monge property. Let $i,j,k,l \in N$ such that $i < k$ and $j < l$. There

are two possibilities.
Case 1. Let at least one of the couples (i,j) and (k,l) be from $L \cup R$. Then $a_{ij} \otimes a_{kl} = 0$ and the assertion follows.
Case 2. Suppose that both (i,j) and (k,l) are from the set C. Thus $a_{ij} \otimes a_{kl} = 1$. Let us assume that $a_{il} = 0$. Then $(i,l) \in L$ and consequently $a_{ij} = 0$, a contradiction or $(i,l) \in R$ and consequently $a_{kl} = 0$, a contradiction. By analogy we can show that $a_{kj} = 1$ as well. Hence $a_{il} \otimes a_{kj} = 1$ and the assertion follows. □

Remark 1. It is clear that a Monge matrix with no zero rows and no zero columns with minimal number of elements equal to 1 is of the form matching with $C = \{(1,n), (2,n-1), \ldots, (n,1)\}$. Let us denote the corresponding matrix by I_{ad}. The digraph $G(I_{ad}, 1)$ consists of $\lceil \frac{n}{2} \rceil$ strongly connected components $\{\mathcal{K}_1, \ldots, \mathcal{K}_m\}$, where $m \in \{\frac{n}{2}, \lceil \frac{n}{2} \rceil\}$. If n is even, then $N_{\mathcal{K}_1} = \{1,n\}$, $N_{\mathcal{K}_2} = \{2,n-1\}$, $\ldots$, $N_{\mathcal{K}_{\frac{n}{2}}} = \{\frac{n}{2}, \frac{n}{2}+1\}$ and the only cycle in each component is of the length 2. If n is odd, then $N_{\mathcal{K}_1} = \{1,n\}$, $N_{\mathcal{K}_2} = \{2,n-1\}$, $\ldots$, $N_{\mathcal{K}_{\lceil \frac{n}{2} \rceil - 1}} = \{\lceil \frac{n}{2} \rceil - 1, \lceil \frac{n}{2} \rceil + 1\}$, $N_{\mathcal{K}_{\lceil \frac{n}{2} \rceil}} = \{\lceil \frac{n}{2} \rceil\}$ and the only cycle in each component is of the length 2 except the last component with loop on the only node $\lceil \frac{n}{2} \rceil \in \mathcal{K}_{\lceil \frac{n}{2} \rceil}$. Note that the nodes i and $n-i+1$ lie in the same component.

Every matrix $A \in B(n,n)$ satisfying $A \geq I_{ad}$ is a matrix with no zero rows and no zero columns.

Definition 3. Let A, $B \in B(n,n)$. We define the minimum of matrices A and B, in notation $\min(A,B)$ as the matrix $D \in B(n,n)$ with $d_{ij} = a_{ij} \otimes b_{ij}$.

It is easy to see that the left upper corner of a Monge matrix with no zero rows and no zero columns can be created as the minimum of finite number (at most $\mid L \mid$) of matrices corresponding to set L. Let us denote the obtained matrix by A_L. By analogy we can derive the right lower corner corresponding to set R and denote the resulting matrix by A_R. Finally, $A = \min(A_L, A_R)$.

Example 1. For $A \in B(4,4)$ Monge matrix holds

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \min(A_L, A_R) = \min\left(\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right).$$

It is not enough to check two consecutive rows and columns to verify the Monge property of a matrix in max-min algebra as illustrated in next example.

Example 2. For the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

the consecutive columns satisfy the Monge property. Unfortunately $a_{11} \otimes a_{23} > a_{13} \otimes a_{21}$.

However, if we do not allow the matrix to have zero row or zero column, this case will not appear.

Theorem 3. *Let $A \in B(m,n)$ be a matrix with no zero rows and no zero columns. Then A is a Monge matrix if and only if all submatrices of A consisting of two consecutive rows and columns are Monge matrices.*

Proof. Let $i \in M$ and $j \in N$ be arbitrary. Let us consider two consecutive rows

$$a_{ij} \otimes a_{i+1j+1} \leq a_{ij+1} \otimes a_{i+1j} \quad (3)$$

$$a_{ij+1} \otimes a_{i+1j+2} \leq a_{ij+2} \otimes a_{i+1j+1} \quad (4)$$

We shall prove that $a_{ij} \otimes a_{i+1j+2} \leq a_{ij+2} \otimes a_{i+1j}$. Let us assume that $a_{ij} \otimes a_{i+1j+2} > a_{ij+2} \otimes a_{i+1j}$. Then $a_{ij} \otimes a_{i+1j+2} = 1$ implies $a_{ij} = a_{i+1j+2} = 1$ and $a_{ij+2} \otimes a_{i+1j} = 0$ implies $a_{ij+2} = 0$ or $a_{i+1j} = 0$. There are two possibilities.
Case 1. Let $a_{ij+2} = 0$. Since $a_{i+1j+2} = 1$, to satisfy (4) must $a_{ij+1} = 0$. Since $a_{ij} = 1$, to satisfy (3) must $a_{i+1j+1} = 0$. Thus the $(j+1)$st is a zero column, what is a contradiction.
Case 2. Let $a_{i+1j} = 0$. Since $a_{ij} = 1$, to satisfy (3) must $a_{i+1j+1} = 0$. Since $a_{i+1j+2} = 1$, to satisfy (4) must $a_{ij+1} = 0$. Thus the $(j+1)$st is a zero column, what is a contradiction.
We can prove by analogy that considering two executive columns would contradict with assumption no zero row in A. □

The above theorem improves the $O(n^3)$ ([1], [2]) computational complexity for checking the Monge property of a (0–1) fuzzy matrix with no zero columns and no zero rows.

Corollary 1. There is an algorithm $\mathcal{A}$ with computational complexity $O(n^2)$ to verify for a matrix $A \in B(n,n)$ whether A is a matrix with no zero rows and no zero columns and check the Monge property in positive case.

Theorem 4. *Let $A \in B(n,n)$ be a Monge matrix of type A_L with no zero rows and no zero columns. Then A is robust.*

Proof. According to Theorem 1 we shall consider the threshold digraph $G(A,1)$. Using Theorem 2 and the fact that there is no zero column and no zero row in A we obtain $a_{kn} = a_{nl} = 1$, for $k, l = 1, 2, \ldots, n$. Thus the threshold digraph $G(A,1)$ is strongly connected with loop on vertex n. Consequently per $A = 1$. Thus the matrix A is robust. □

Theorem 5. *Let $A \in B(n,n)$ be a Monge matrix of type A_R with no zero rows and no zero columns. Then A is robust.*

Proof. According to Theorem 1 we shall consider the threshold digraph $G(A,1)$. Using Theorem 2 and the fact that there is no zero column and no zero row in A we obtain $a_{k1} = a_{1l} = 1$, for $k, l = 1, 2, \ldots, n$. Thus the threshold digraph $G(A,1)$ is strongly connected with loop on vertex 1. Consequently per $A = 1$. Thus the matrix A is robust. □

It requires $O(n)$ operations to check whether $a_{kn} = a_{nl} = 1$, for $k, l = 1, 2, \ldots, n$ and according to Theorem 1 another $O(n^2)$ operations to check the Monge property. Thus the total computational complexity for checking the robustness of fuzzy matrix of type A_L is $O(n^2)$. The same holds for checking the robustness of a matrix of type A_R.

Lemma 3. [4] Let $c = (i_1, i_2, \ldots, i_k, i_1)$ be a cycle of length $k \geq 3$. Then there are arcs (i_j, i_{j+1}) and (i_l, i_{l+1}) in c such that

$$i_j < i_l \quad \text{and} \quad i_{j+1} < i_{l+1}.$$

Lemma 4. Let $A \in B(n,n)$ be a Monge matrix. Let $\mathcal{K} \in \mathrm{SCC}^\star(G(A,1))$. Let c be a cycle of odd length $\ell(c) \geq 3$ in $\mathcal{K}$. Then there is a node in c with loop.

Proof. Let $c = (i_1, i_2, \ldots, i_k, i_1)$ be a cycle of odd length $\ell(c) \geq 3$ in $\mathcal{K}$. By Lemma 3 there are arcs (i_j, i_{j+1}) and (i_l, i_{l+1}) in c such that $i_j < i_l$ and $i_{j+1} < i_{l+1}$. Using the Monge property the component $\mathcal{K}$ contains the arcs (i_j, i_{l+1}) and (i_l, i_{j+1}) as well. Thus the cycle c splits into cycles $c_1 = (i_1, i_2, \ldots, i_j, i_{l+1}, \ldots, i_1)$ and $c_2 = (i_{j+1}, \ldots, i_l, i_{j+1})$ with $\ell(c) = \ell(c_1) + \ell(c_2)$. Moreover, one of the cycles has odd length. There are two possibilities. If one of the equalities $i_j = i_{l+1}$ or $i_{j+1} = i_l$ holds exclusively, then the cycle c_1 or c_2 is a loop and the assertion follows. If not, then the cycle with odd length can split by the same procedure into two cycles again. This procedure can be repeated until the length of the cycle with odd length is equal to 3. Then there is a pair of nodes which are identical and the assertion follows. □

Theorem 6. *Let $A \in B(n,n)$ be a Monge matrix. Then* per $A = 1$ *if and only if each $\mathcal{K} \in \mathrm{SCC}^\star(G(A,1))$ contains a loop.*

Proof. Let $A \in B(n,n)$ be a Monge matrix. Let per $A = 1$. Then $\mathcal{K} \in \mathrm{SCC}^\star(G(A,1))$ contains a cycle c with odd length. There are two cases. The length $\ell(c) = 1$ and the assertion follows immediately or $\ell(c) > 1$ and the assertion follows by Lemma 4.
The converse implication is trivial. □

Theorem 7. *Let $A \in B(n,n)$ be a Monge matrix. Then A is robust if and only if each $\mathcal{K} \in \mathrm{SCC}^\star(G(A,1))$ contains a loop.*

Proof. The theorem is a consequence of Theorem 6 and Lemma 2. □

In view of Theorem 2 and Theorem 7 it is necessary to achieve that each node is in a strongly connected component with loop.

Lemma 5. Let $A = (a_{ij}) \in B(n,n)$ be a Monge matrix with $A \geq I_{ad}$. Let n be an even number and the nodes 1, n, $\frac{n}{2}$, $\frac{n}{2}+1$ be in the same non-trivial strongly connected component of $G(A,1)$. Then the digraph $G(A,1)$ is strongly connected.

Proof. By Remark 1 the nodes i and n on one side and the nodes $\frac{n}{2}$ and $\frac{n}{2}+1$ on other side lie on common cycles of length 2. Let us assume that these nodes lie in the same non-trivial strongly connected component $\mathcal{K}$ of $G(A,1)$. Let edges $(1, \frac{n}{2}+1)$ and $(\frac{n}{2}+1, 1)$ be there. Thus $a_{1,\frac{n}{2}+1} = 1$ and by Theorem 2

$$a_{ij} = 1 \quad \text{for } i,j \quad \text{satisfying} \quad 1 \leq i \leq \tfrac{n}{2} \quad \text{and} \quad \tfrac{n}{2}+1 \leq j \leq n-i+1 \tag{5}$$

and $a_{\frac{n}{2}+1,1} = 1$ and by Theorem 2

$$a_{ij} = 1 \quad \text{for } i,j \quad \text{satisfying} \quad \tfrac{n}{2}+1 \leq i \leq n \quad \text{and} \quad 1 \leq j \leq n-i+1. \tag{6}$$

We shall prove that for each pair of nodes $k, l \in G(A,1)$ there is a common cycle containing nodes k and l. There are four possibilities.
Case 1. Let $1 \leq k \leq \frac{n}{2}$ and $1 \leq l \leq \frac{n}{2}$. Without loss of generality we can assume that $k \leq l$ (otherwise the order of constructed paths would be opposite). By (5) there is an edge from k to $n-l+1$, if $n-l+1 \leq n-k+1$, what holds for $k \leq l$. Since l and $n-l+1$ lie on common cycle (see Remark 1) there is a path from k to l. For converse path from l to k we use again the fact that the nodes l and $n-l+1$ lie on common cycle and since $k \leq n-(n-l+1)+1 = l$ holds, by (6) there is an edge from $n-l+1$ to k.
Case 2. Let $1 \leq k \leq \frac{n}{2}$ and $\frac{n}{2}+1 \leq l \leq n$. We shall distinguish $\frac{n}{2}+1 \leq l \leq n-k+1$ and $l > n-k+1$. In first case there is an edge from k to l by (5) and an edge from l to k by (6). In second case applying (5) there is an edge from k to $\frac{n}{2}+1$. By assumption there is an edge from $\frac{n}{2}+1$ to 1 and by (5) an edge from 1 to l. For converse path from l to k we use (6) for an edge from l to 1. By assumption there is an edge from 1 to $\frac{n}{2}+1$. Applying (6) again $(n-(\frac{n}{2}+1)+1 = \frac{n}{2} \leq \frac{n}{2})$ we get an edge from $\frac{n}{2}+1$ to k.
Case 3. Let $\frac{n}{2}+1 \leq k \leq n$ and $1 \leq l \leq \frac{n}{2}$. This is a dual case to Case 2.
Case 4. Let $\frac{n}{2}+1 \leq k \leq n$ and $\frac{n}{2}+1 \leq l \leq n$. Without loss of generality we can assume that $k \geq l$ (otherwise the order of constructed paths would be opposite). Since k and $n-k+1$ lie on common cycle of length 2, there is an edge from k to $n-k+1$. Since $\frac{n}{2}+1 \leq l \leq n-(n-k+1)+1 = k$ holds, there is and edge from $n-k+1$ to l by (5). For converse path from l to k we can apply (6) for an edge from l to $n-k+1$ since $n-k+1 \leq n-l+1$ for $k \geq l$. We complete the path from l to k using the fact that $n-k+1$ and k lie on common cycle again.
We can prove the assertion of the lemma by analogy taking instead of edges $(1, \frac{n}{2}+1)$ and $(\frac{n}{2}+1, 1)$ another couple of edges to get the nodes 1, $\frac{n}{2}$, $\frac{n}{2}+1$, n into the same non-trivial strongly connected component. □

By slight modification of the above proof we can prove the following lemma in case that n is odd.

Lemma 6. Let $A = (a_{ij}) \in B(n,n)$ be a Monge matrix with $A \geq I_{ad}$. Let n be an odd number and the nodes 1, n, $\lceil \frac{n}{2} \rceil$ be in the same non-trivial strongly connected component of $G(A,1)$. Then the digraph $G(A,1)$ is strongly connected.

Theorem 8. *Let $A = (a_{ij}) \in B(n,n)$ be a Monge matrix with $A \geq I_{ad}$. Then A is robust if and only if $G(A,1)$ is strongly connected and contains a loop.*

Proof. Let $A \in B(n,n)$ be robust. By Theorem 7 each $\mathcal{K} \in \mathrm{SCC}^{\star}(G(A,1))$ contains a loop. Since $A \geq I_{ad}$ by Lemma 1 $G(I_{ad},1) \subseteq G(A,1)$. The digraph $G(I_{ad},1)$ consists of $\lceil \frac{n}{2} \rceil$ strongly connected components (see Remark 1). There are 3 possibilities.
Case 1. There is a loop on node 1. By Theorem 2 $a_{ij} = 1$, for all i and $1 \leq j \leq n-i+1$. Consequently $G(A,1)$ is strongly connected.
Case 2. There is a loop on node n. By Theorem 2 $a_{ij} = 1$, for all i and $n-i+1 \leq j \leq n$. Consequently $G(A,1)$ is strongly connected.
Case 3. There is a loop on node $\frac{n}{2}$, for n even, or loop on node $\lceil \frac{n}{2} \rceil$, for n odd, respectively and the nodes 1, $\frac{n}{2}$, $\frac{n}{2}+1$, n or 1, $\lceil \frac{n}{2} \rceil$, n, respectively, lie in the same strongly connected component. By Lemma 5 and Lemma 6 $G(A,1)$ is strongly connected.
The same procedure would follow, if there would be a loop on the node i for $1 < i < \lceil \frac{n}{2} \rceil$ (or dually on i for $\lceil \frac{n}{2} \rceil < i < n$). Consequently the nodes $1, 2, \ldots, i, n-i+1, n-i+2, \ldots n$ would lie in the same strongly connected component. Moreover by Theorem 2 the nodes i, $n-i+1$, $i+1$, $n-i$, …, $\frac{n}{2}$, $\frac{n}{2}+1$,

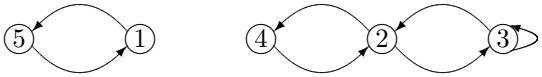

Figure 1: Threshold digraph in non–robust case

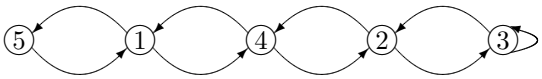

Figure 2: Threshold digraph in robust case

for n even, or i, $n-i+1$, $i+1$, $n-i$, $\ldots$, $\lceil \frac{n}{2} \rceil$, for n odd, respectively, lie in the same strongly connected component. Consequently $G(A,1)$ is strongly connected.
For the converse implication let us assume that $G(A,1)$ is strongly connected and contains a loop. Using Theorem 7 A is robust. □

Example 3. Let us consider the bellow Monge matrices A, $B \in B(5,5)$ with $A \geq I_{ad}$ and $B \geq I_{ad}$

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The digraph $G(A,1)$ is not strongly connected (see Figure 1). Hence the considered matrix is not robust. The matrix B is a slight modification of the matrix A and the answer is positive. The digraph $G(B,1)$ is strongly connected with a loop (see Figure 2). Thus the matrix B is robust.

Theorem 9. *There is an $O(n^2)$ algorithm $\mathcal{A}$, which decides for a given matrix $A = (a_{ij}) \in B(n,n)$ whether A is robust.*

Proof. The algorithm consists of four parts. First, we check the inequality $A \geq I_{ad}$ in $O(n)$ time. Second, we verify the Monge property according to Theorem 1 in $O(n^2)$ time. Third, we check the strong connectivity in $O(n^2)$ time. Fourth, we check the existence of a loop by $a_{ii} = 1$, for some $i \in N$ in $O(n)$ time. Thus the total computational complexity is $O(n^2)$. □

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Robustness of Interval Matrices in max-drast Algebra

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Abstract.

Max-drast fuzzy algebra uses instead of conventional operations for multiplication of vectors and matrices the operations maximum and one of the triangular norms, drast norm. Transition matrices in max-drast fuzzy algebra and their power sequences occur in describing complex fuzzy systems in which extreme demands are put on the reliability of the system. Robust matrix is a matrix for which the orbit reaches an eigenvector with any nontrivial starting vector. We give a necessary and sufficient condition for the robustness of a matrix in max-drast fuzzy algebra. However, in practice we deal often with inexact input data. This leads to a requirement to replace the scalar matrices with so-called interval matrices. We define the notions of the universal robustness and the possible robustness of interval max-drast matrices and give necessary and sufficient conditions for them.

Keywords: max-drast fuzzy algebra, robust matrix, interval matrix, possible robustness, universal robustness.

JEL classification: C02
AMS classification: 15A18; 15A80; 65G30

1 Introduction

Discrete time systems are often described with transition matrices with fuzzy values. In dependence on the character of the system, various triangular fuzzy t-norms are used. If the elements of a transition matrix correspond to the flow capacities, then the behavior of such system is described by powers of matrices in max-min algebra. Periodic behavior of the power sequence in max-min algebra was studied by many authors, see e.g. [4], [6], [7].

The drastic triangular norm is the basic example of a non-divisible t-norm on any partially ordered set, see [3]. Max-drast matrix concerns the transitions reliability in the system. Interpretation of a complex fuzzy systems in max-drast fuzzy algebra reflects the situation when extreme demands are put on the reliability of the system. The problem of robust matrices, whose eigenvectors are reached with any initial vector x, in max-plus algebra has been studied in [1], [2] and in max-min algebra in [13].

In this paper we derive a necessary and sufficient condition for a matrix in max-drast fuzzy algebra to be robust. Further, we introduce the concept of the robustness of max-drast fuzzy matrices with inexact data (interval matrices) and derive algorithms for verifying the corresponding properties of interval max-drast fuzzy matrices.

2 Preliminaries

By *max-drast fuzzy algebra* we understand a triplet $(\mathcal{I}, \oplus, \otimes_d)$, where $\mathcal{I}$ is the unit interval $\langle 0, 1\rangle$, $\oplus$ is the maximum and $\otimes_d$ is the binary drast operation (drastic triangular norm) defined as follows

$$\mathrm{drast}(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ 0 & \text{if } \max(x,y) < 1. \end{cases} \tag{1}$$

Proposition 1. *Let $a \in \mathcal{I}$ be arbitrary. Then $a \otimes_d 0 = 0$ and $a \otimes_d 1 = a$.*

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By $\mathbb{N}$ we denote the set of all natural numbers. The greatest common divisor of a set $S \subseteq \mathbb{N}$ is denoted by $\gcd S$ and the least common multiple by $\operatorname{lcm} S$. For a given natural number $n \in \mathbb{N}$, we use the notations $N = \{1, 2, \ldots, n\}$.

For any $n \in \mathbb{N}$, $\mathcal{I}(n,n)$ denotes the set of all square matrices of order n and $\mathcal{I}(n)$ denotes the set of all n-dimensional column vectors over $\mathcal{I}$. The matrix operations over $\mathcal{I}$ are defined formally in the same manner (with respect to $\oplus$, $\otimes_d$) as matrix operations over any field. The r-th power of a matrix A is denoted by A^r, with elements $(A^r)_{ij}$.

For $A = (a_{ij}) \in \mathcal{I}(n,n)$, $C = (c_{ij}) \in \mathcal{I}(n,n)$ we write $A \leq C$ if $a_{ij} \leq c_{ij}$ holds for all $i, j \in N$.

By digraph we understand a pair $\mathcal{G} = (V_\mathcal{G}, E_\mathcal{G})$, where $V_\mathcal{G}$ is a non-empty finite set, called the node set, and $E_\mathcal{G} \subseteq V_\mathcal{G} \times V_\mathcal{G}$, called the arc set. A digraph $\mathcal{G}' = (V_{\mathcal{G}'}, E_{\mathcal{G}'})$ is a subdigraph of digraph $\mathcal{G}$, if $V_{\mathcal{G}'} \subseteq V_\mathcal{G}$ and $E_{\mathcal{G}'} \subseteq E_\mathcal{G}$. Specially, $\mathcal{G}/V_{\mathcal{G}'}$ stands for the subdigraph induced by vertex set $V_{\mathcal{G}'}$.

A walk in a digraph $\mathcal{G}$ is the sequence of nodes and arcs $\mathcal{P} = (v_0, e_1, v_1, e_2, v_2, \ldots, v_{l-1}, e_l, v_l)$ such that $e_k = (v_{k-1}, v_k) \in E_\mathcal{G}$ for $k = 1, 2, \ldots, l$. A walk in $\mathcal{G}$ is called a trail if all its arcs are distinct. The number l is the length of the trail $\mathcal{P}$ and is denoted by $\ell(\mathcal{P})$. If $v_0 = v_l$, then $\mathcal{P}$ is called a cycle. A cycle is elementary if all nodes except the terminal node are distinct. A digraph is called strongly connected if any two distinct nodes of $\mathcal{G}$ are contained in a common cycle. By a strongly connected component of $\mathcal{G}$ we mean a maximal strongly connected subdigraph of $\mathcal{G}$. A strongly connected component $\mathcal{K} = (V_\mathcal{K}, E_\mathcal{K})$ is called non-trivial if there is a cycle of positive length in $\mathcal{K}$. For any non-trivial strongly connected component $\mathcal{K}$ is the *period* of $\mathcal{K}$ defined as

$$\operatorname{per} \mathcal{K} = \gcd \{ \ell(c);\ c \text{ is a cycle in } \mathcal{K},\ \ell(c) > 0 \}. \tag{2}$$

If $\mathcal{K}$ is trivial, then $\operatorname{per} \mathcal{K} = 1$. By $\mathrm{SCC}^\star\, \mathcal{G}$ we denote the set of all non-trivial strongly connected components of $\mathcal{G}$.

Lemma 1 (Myšková [11])**.** Let $\mathcal{K} \in \mathrm{SCC}^\star\, \mathcal{G}$, $\mathcal{K}' \in \mathrm{SCC}^\star\, \mathcal{G}'$ and $\mathcal{K} \subseteq \mathcal{K}'$. Then $\operatorname{per} \mathcal{K}' \mid \operatorname{per} \mathcal{K}$.

For a given matrix $A \in \mathcal{I}(n,n)$ the symbol $\mathcal{G}(A) = (V_{\mathcal{G}(A)}, E_{\mathcal{G}(A)})$ stands for the complete, edge-weighted digraph associated with A, i.e., the vertex set of $\mathcal{G}(A)$ is N, and the capacity of any edge $(i,j) \in E_{\mathcal{G}(A)}$ is a_{ij}. In addition, for a given $h \in \mathcal{I}$, the *threshold digraph* $\mathcal{G}(A,h)$ is the digraph with the vertex set $V_{\mathcal{G}(A,h)} = N$ and the edge set $E_{\mathcal{G}(A,h)} = \{(i,j);\ i,j \in N,\ a_{ij} \geq h\}$.

Lemma 2 (Molnárová–Myšková–Plavka [9])**.** Let $A, C \in \mathcal{I}(n,n)$. Let h, h_1, $h_2 \in \mathcal{I}$.

1. If $A \leq C$ then $\mathcal{G}(A,h) \subseteq \mathcal{G}(C,h)$,
2. if $h_1 < h_2$ then $\mathcal{G}(A,h_2) \subseteq \mathcal{G}(A,h_1)$.

3 Robust matrices

Let $A \in \mathcal{I}(n,n)$ and $x \in \mathcal{I}(n)$. The orbit $\mathcal{O}(A,x)$ of $x = x^{(0)}$ generated by A is the sequence

$$x^{(0)}, x^{(1)}, x^{(2)}, \ldots, x^{(n)}, \ldots,$$

where $x^{(r)} = A^r \otimes_d x^{(0)}$ for each $r \in \mathbb{N}$.

For a given matrix $A \in \mathcal{I}(n,n)$, the element $\lambda \in \mathcal{I}$ and the n-tuple $x \in \mathcal{I}(n)$ are the so-called *eigenvalue* of A and *eigenvector* of A, respectively, if

$$A \otimes_d x = \lambda \otimes_d x.$$

The *eigenspace* $V(A,\lambda)$ is defined as the set of all eigenvectors of A with associated eigenvalue λ, i.e.,

$$V(A,\lambda) = \{x \in \mathcal{I}(n);\ A \otimes_d x = \lambda \otimes_d x\}.$$

Lemma 3. Let $A \in \mathcal{I}(n,n)$ and $\lambda \in \mathcal{I}$ be arbitrary. Then $V(A,\lambda) \neq \emptyset$.

Proof. Let $A \in \mathcal{I}(n,n)$ and $\lambda \in \mathcal{I}$ be arbitrary. Define the vector $o = 0 \in \mathcal{I}(n)$. Then

$$[A \otimes_d o]_i = \bigoplus_{j \in N} a_{ij} \otimes_d o_j = \bigoplus_{j \in N} a_{ij} \otimes_d 0 = 0 = \lambda \otimes_d o_i$$

for each $i \in N$. Since the vector o is an eigenvector of A associated with λ, we have $V(A,\lambda) \neq \emptyset$. □

Let $\lambda \in \mathcal{I}$. A matrix $A \in \mathcal{I}(n,n)$ is *ultimately λ-periodic* if there are natural numbers p and R such that the following holds:

$$A^{k+p} = \lambda^p \otimes_d A^k \quad \text{for all} \quad k \geq R.$$

The smallest natural number p with the above property is called the period of A, denoted by $\mathrm{per}(A,\lambda)$.

In this paper we shall deal with the case $\lambda = 1$. In this case, we shall use the notations per A and $V(A)$ instead of $\mathrm{per}(A,1)$ and $V(A,1)$, respectively. Moreover, we shall call ultimately 1-periodic matrix in short *periodic*. Similarly as in the max-min algebra, by max-drast operations no new elements (except 0) are created. Consequently, the matrices in the power sequence of a matrix in max-drast algebra only contain the entries from A and 0. This implies that the matrix powers sequence in max-drast algebra is always ultimately periodic.

The following theorem gives the formula for computing the matrix period.

Theorem 2 (Gavalec–Němcová [8])**.** *Let $A \in \mathcal{I}(n,n)$. Then*

$$\mathrm{per}\ A = \mathrm{lcm}\{\mathrm{per}\,\mathcal{K};\ \mathcal{K} \in \mathrm{SCC}^\star\,\mathcal{G}(A,1)\}. \tag{3}$$

Remark 1. In the case that $\mathrm{SCC}^\star\,\mathcal{G}(A,1) = \emptyset$ there exists $r \in \mathbb{N}$ such that $A^k = 0$ for each $k \geq r$, so $\mathrm{per}\,A = 1$.

According to [8] computing the matrix period in max-drast algebra requires $O(n^3)$ arithmetic operations.

Let us denote

$$T(A) = \{x \in \mathcal{I}(n);\ \mathcal{O}(A,x) \cap V(A) \neq \emptyset\}.$$

Definition 1. Let $A = (a_{ij}) \in \mathcal{I}(n,n)$. A matrix A is called *robust* if $T(A) = \mathcal{I}(n)$.

Theorem 3 (Plavka–Szabó [13])**.** *Let $A = (a_{ij}) \in \mathcal{I}(n,n)$. Then A is robust if and only if* per $A = 1$.

Proof. If $\mathrm{per}\,A = 1$ then there exists $k \in \mathbb{N}$ such that $A^{k+1} = A^k$. Then for each $x \in \mathcal{I}(n)$ we have $A \otimes_d (A^k \otimes_d x) = A^k \otimes_d x$ which means that $A^k \otimes_d x \in V(A)$. Thus $T(A) = \mathcal{I}(n)$, i.e., the matrix A is robust.

For the converse implication suppose that a matrix A is robust. Define the vectors $a^{(j)}$, where $a^{(j)}$ is the vector with the entries equal to j-th column of A, for each $j \in N$. The robustness of A implies that for each $j \in N$ there exists $r_j \in \mathbb{N}$ such that $A \otimes_d (A^{r_j} \otimes_d a^{(j)}) = A^{r_j} \otimes_d a^{(j)}$. Let us set $r = \max\limits_{j \in N} r_j$. Since for any $x \in \mathcal{I}(n)$ if $x \in V(A)$ then $A \otimes x \in V(A)$, we obtain

$$A^{r+1} \otimes_d a^{(j)} = A^r \otimes_d a^{(j)} \tag{4}$$

for each $j \in N$. From the definition of $a^{(j)}$ it follows that the system of the equalities of the form (4) is equivalent to the equality $A^{(r+2)} = A^{(r+1)}$ which implies $A^{k+1} = A^k$ for each $k \geq r+1$. Thus $\mathrm{per}\,A = 1$. □

The robustness of max-plus matrices was studied in [1]. Plavka and Szabó [13] dealt with robust matrices in max-min algebra.

4 Robustness of interval matrices

In practice, the elements in a transition matrix may depend on outside conditions, so the values a_{ij} are from intervals of possible values. In this section we shall deal with matrices with interval elements. Similarly to [5], [9], [10], [12], [11], [14] we define an interval matrix $\boldsymbol{A}$.

Definition 2. Let $\underline{A}, \overline{A} \in \mathcal{I}(n,n)$, $\underline{A} \leq \overline{A}$. An interval matrix $\boldsymbol{A}$ with bounds $\underline{A}$ and $\overline{A}$ is defined as follows

$$\boldsymbol{A} = [\underline{A}, \overline{A}] = \left\{ A \in \mathcal{I}(n,n);\ \underline{A} \leq A \leq \overline{A} \right\}.$$

Investigating the robustness of an interval matrix $\boldsymbol{A}$, the following questions can arise: *Is A robust for some $A \in \boldsymbol{A}$ or for all $A \in \boldsymbol{A}$?*

Definition 3. An interval matrix $\boldsymbol{A}$ is called

- *possibly robust* if there exists a matrix $A \in \boldsymbol{A}$ such that A is robust,
- *universally robust* if each matrix $A \in \boldsymbol{A}$ is robust.

The possible and universal robustness of max-min interval matrices were studied in [9].

4.1 Possible robustness

Theorem 4. *An interval matrix $\boldsymbol{A}$ is possibly robust if and only if for each $\mathcal{K} \in \mathrm{SCC}^\star\, \mathcal{G}(\overline{A}, 1)$ such that $\operatorname{per} \mathcal{K} \neq 1$ the digraph $\mathcal{G}(\underline{A}, 1)/V_{\mathcal{K}}$ is acyclic.*

Proof. Suppose that there exists $\mathcal{K} \in \mathrm{SCC}^\star\, \mathcal{G}(\overline{A}, 1)$ such that $\operatorname{per} \mathcal{K} \neq 1$ and the digraph $\mathcal{G}(\underline{A}, 1)/V_{\mathcal{K}}$ contains a cycle c. Let $A \in \boldsymbol{A}$ be arbitrary. Since $\mathcal{G}(\underline{A}, 1) \subseteq \mathcal{G}(A, 1)$ there exists $\mathcal{K}' \in \mathrm{SCC}^\star\, \mathcal{G}(A, 1)$, $\mathcal{K}' \subseteq \mathcal{K}$ such that $c \in \mathcal{K}'$. By Lemma 1 we obtain $\operatorname{per} \mathcal{K}' \neq 1$ which implies that, according to Theorem 2, $\operatorname{per} A \neq 1$ for each $A \in \boldsymbol{A}$. In view of Theorem 3 there is no robust matrix in $\boldsymbol{A}$. Thus the interval matrix $\boldsymbol{A}$ is not possibly robust.

For the converse implication suppose that the digraph $\mathcal{G}(\underline{A}, 1)/V_{\mathcal{K}}$ is acyclic for each $\mathcal{K} \in \mathrm{SCC}^\star\, \mathcal{G}(\overline{A}, 1)$ such that $\operatorname{per} \mathcal{K} \neq 1$. Denote by $\mathcal{K}^1, \mathcal{K}^2, \ldots, \mathcal{K}^m$ the nontrivial strongly connected components of $\mathcal{G}(\overline{A}, 1)$ with period equal to 1 (if there exist).

Define the matrix $\tilde{A} = (\tilde{a}_{ij}) \in \boldsymbol{A}$ as follows:

$$\tilde{a}_{ij} = \begin{cases} \overline{a}_{ij} & \text{if } (i,j) \in \bigcup\limits_{s=1}^{m} E_{\mathcal{K}^s}, \\ \underline{a}_{ij} & \text{otherwise.} \end{cases} \tag{5}$$

We will prove that the matrix $\tilde{A}$ is robust. The acyclicality of the digraph $\mathcal{G}(\underline{A}, 1)/V_{\mathcal{K}}$ for each $\mathcal{K} \in \mathrm{SCC}^\star\, \mathcal{G}(\overline{A}, 1)$ such that $\operatorname{per} \mathcal{K} \neq 1$ implies that $\mathrm{SCC}^\star\, \mathcal{G}(\tilde{A}, 1) = \{\mathcal{K}^1, \mathcal{K}^2, \ldots, \mathcal{K}^m\}$. Since $\operatorname{per} \mathcal{K}^s = 1$ for each $s \in \{1, 2, \ldots, m\}$, by (3) we have $\operatorname{per} \tilde{A} = 1$. If $\mathrm{SCC}^\star\, \mathcal{G}(\tilde{A}, 1) = \emptyset$ then we get $\operatorname{per} \tilde{A} = 1$ according to Remark 1. Thus the matrix $\tilde{A}$ is robust and $\boldsymbol{A}$ is possibly robust. □

Example 1. Let $\boldsymbol{A}$ be of the form

$$\boldsymbol{A} = \begin{pmatrix} [0.5, 0.8] & [1, 1] & [0.7, 0.9] & [0.8, 1] & [0.5, 0.8] \\ [0.8, 1] & [0.5, 0.6] & [1, 1] & [0.5, 0.9] & [0.6, 0.8] \\ [1, 1] & [0.8, 0.9] & [0.2, 1] & [0.5, 0.7] & [0.4, 0.9] \\ [0.5, 0.5] & [0.6, 0.7] & [0.3, 0.9] & [0.8, 0.9] & [0.5, 1] \\ [0.4, 0.8] & [0.5, 0.6] & [0.5, 0.8] & [1, 1] & [0.6, 0.8] \end{pmatrix}.$$

We check the possible robustness of $\boldsymbol{A}$ and find a robust matrix from $\boldsymbol{A}$ in the positive case.

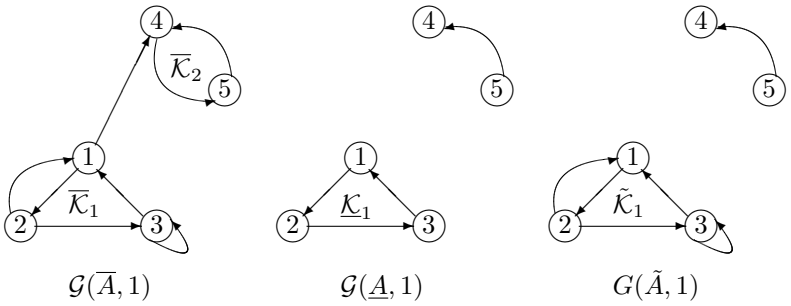

Figure 1: Threshold digraphs

Figure 1 describes the threshold digraphs $\mathcal{G}(\underline{A},1)$ and $\mathcal{G}(\overline{A},1)$ which follows directly from the given interval matrix. Moreover, there is described the threshold digraph $\mathcal{G}(\tilde{A},1)$ of the constructed matrix $\tilde{A}$.

We can see that $\mathcal{G}(\overline{A},1)$ contains two strongly connected components, namely $\overline{\mathcal{K}}_1$ with $V_{\overline{\mathcal{K}}_1}=\{1,2,3\}$, $\operatorname{per}\overline{\mathcal{K}}_1=1$ and $\overline{\mathcal{K}}_2$ with $V_{\overline{\mathcal{K}}_2}=\{4,5\}$, $\operatorname{per}\overline{\mathcal{K}}_2=2$. Thus $\operatorname{per}\overline{A}=2$, so $\overline{A}$ is not robust. Similarly $\underline{A}$ is not robust because $\operatorname{per}\underline{A}=\operatorname{per}\underline{\mathcal{K}}_1=3$. To check the possible robustness of $\boldsymbol{A}$ we have to check, whether the digraph $\mathcal{G}(\underline{A},1)/V_{\underline{\mathcal{K}}_2}$ is acyclic. Since that condition is satisfied, the given interval matrix is possibly robust. Moreover, we can compute the matrix $\tilde{A}$ according to (5).

$$\tilde{A}=\begin{pmatrix} 0.5 & 1 & 0.7 & 0.8 & 0.5\\ 1 & 0.5 & 1 & 0.5 & 0.6\\ 1 & 0.8 & 1 & 0.5 & 0.4\\ 0.5 & 0.6 & 0.3 & 0.8 & 0.5\\ 0.4 & 0.5 & 0.5 & 1 & 0.6 \end{pmatrix}.$$

As we can see on Figure 1, there is exactly one strongly connected component $\tilde{\mathcal{K}}_1$ with period equal to one in $\mathcal{G}(\tilde{A},1)$. Thus $\operatorname{per}\tilde{A}=1$ and $\tilde{A}$ is robust.

4.2 Universal robustness

We prove a necessary and sufficient condition for an interval matrix to be universally robust in this part.

Denote by $\mathcal{C}$ the set of all elementary cycles of length $\ell(c)>1$ in $\mathcal{G}(\overline{A},1)$. For a given $c\in\mathcal{C}$ let us define the matrix $A^c=(a^c_{ij})$ as follows

$$a^c_{ij}=\begin{cases}\overline{a}_{ij}, & \text{if } (i,j)\in c,\\ \underline{a}_{ij}, & \text{otherwise.}\end{cases} \tag{6}$$

Theorem 5. *An interval matrix $\boldsymbol{A}$ is universally robust if and only if $\mathcal{C}=\emptyset$ or the matrix A^c is robust for each $c\in\mathcal{C}$.*

Proof. Suppose that an interval matrix $\boldsymbol{A}$ is universally robust, $\mathcal{C}\neq\emptyset$ and $c\in\mathcal{C}$. Since $A^c\in\boldsymbol{A}$, the matrix A^c is robust.

For the converse implication suppose that there exists $A\in\boldsymbol{A}$ such that $\operatorname{per}A\neq 1$, i.e., there exists $\mathcal{K}\in\mathrm{SCC}^\star\,\mathcal{G}(A,1)$ such that $\operatorname{per}\mathcal{K}\neq 1$. Then there exists a cycle $c\in\mathcal{K}$ with length $\ell(c)>1$ in $\mathcal{G}(A,1)$, i.e., $\mathcal{C}\neq\emptyset$. It follows from $\mathcal{G}(A,1)\subseteq\mathcal{G}(\overline{A},1)$ that $c\in\mathcal{G}(\overline{A},1)$. Moreover, $c\in\mathcal{G}(A^c,1)$. Denote by $\mathcal{K}^c$ the strongly connected component of $\mathcal{G}(A^c,1)$ containing cycle c. Since $\mathcal{K}^c\subseteq\mathcal{K}$ and $\operatorname{per}\mathcal{K}\neq 1$, according to Lemma 1 we get $\operatorname{per}\mathcal{K}^c\neq 1$. Then $\operatorname{per}A^c\neq 1$ and by Theorem 3 the matrix A^c is not robust. □

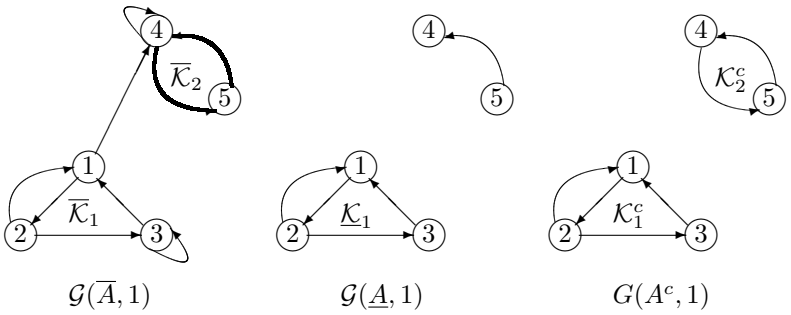

Figure 2: Threshold digraphs

Example 2. Let $\boldsymbol{A}$ be of the form

$$\boldsymbol{A} = \begin{pmatrix} [0.5,0.8] & [1,1] & [0.7,0.9] & [0.8,1] & [0.5,0.8] \\ [1,1] & [0.5,0.6] & [1,1] & [0.5,0.9] & [0.6,0.8] \\ [1,1] & [0.8,0.9] & [0.2,1] & [0.5,0.7] & [0.4,0.9] \\ [0.5,0.5] & [0.6,0.7] & [0.3,0.9] & [0.8,1] & [0.5,1] \\ [0.4,0.8] & [0.5,0.6] & [0.5,0.8] & [1,1] & [0.6,0.8] \end{pmatrix}.$$

We check the universal robustness of $\boldsymbol{A}$.

As we can see on Figure 2, $\mathcal{G}(\overline{A},1)$ contains two strongly connected components $\overline{\mathcal{K}}_1$ and $\overline{\mathcal{K}}_2$ with $\operatorname{per}\overline{\mathcal{K}}_1 = \operatorname{per}\overline{\mathcal{K}}_2 = 1$. Thus $\overline{A}$ is robust. Also $\underline{A}$ is robust because the only strongly connected component $\underline{\mathcal{K}}_1$ of $\mathcal{G}(\underline{A},1)$ has period equal to one. However, the robustness of matrices $\underline{A}, \overline{A}$ is not sufficient for the universal robustness of $\boldsymbol{A}$.

Denote by c the cycle $c = (4,5,4)$ of digraph $\mathcal{G}(\overline{A},1)$ and compute the matrix A^c according to (6). We obtain

$$A^c = \begin{pmatrix} 0.5 & 1 & 0.7 & 0.8 & 0.5 \\ 1 & 0.5 & 1 & 0.5 & 0.6 \\ 1 & 0.8 & 0.2 & 0.5 & 0.4 \\ 0.5 & 0.6 & 0.3 & 0.8 & 1 \\ 0.4 & 0.5 & 0.5 & 1 & 0.6 \end{pmatrix}.$$

The threshold digraph $\mathcal{G}(A^c,1)$ is described on Figure 2. As we can see, there is the strongly connected component $\mathcal{K}_2^c$ with period equal to 2 in $\mathcal{G}(A^c,1)$. Consequently $\operatorname{per} A^c \neq 1$. Thus A^c is not robust and $\boldsymbol{A}$ is not universally robust.

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Comparing monetary policy before and during crisis for Visegrád group countries

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Abstract. We focus on research of monetary policy mechanism of Visegrád group countries before and during crisis in this paper. Main objective of central banks function in V4 countries lies in maintaining price stability. For this purpose, inflation targeting regime is realized in a medium-term focus in V4, which means that there is a certain lag between a monetary policy operation and its influence on an inflation target. The monetary policy of each country can be described by suitable VAR model with appropriate variables. We suppose inflation, interbank interest rate and central bank key interest rate as appropriate variables. We would like to know if there is a continuity of the monetary policy before and after crisis in particular countries. We use one-dimensional and multi-dimensional calibration technique for this purpose. Based on estimated parameters of the one-dimensional calibration model with linear or quadratic calibration function, respectively, we can identify changes in monetary transmission in all countries during crisis. We can conclude on some similarities from estimated parameters of multi-dimensional calibration model, but not on the some transmission mechanism in all countries.

Keywords: one-dimensional calibration, multi-dimensional calibration, financial crisis, monetary policy.

JEL Classification: C32, E52
AMS Classification: 62P20, 62H12

1 Introduction

The paper focuses on contemporary problems of monetary policy. Monetary policy has gained a new wave of interest with the onset of financial crises. There is an assumption, that financial crisis is able to disrupt an effectiveness of monetary policy. Crisis can affect behaviour of consumers and firms within aggregate demand in sense of their lower sensitivity to interest rates. Financial instability can damage relationships between variables of transmission mechanism. Moreover, central banks often change their monetary rules during financial crisis. All this aspects determine a potential change in monetary policy effectiveness during financial crisis. Only few studies deal with an exact testing of size of this transformation and even fewer of them are addressed to Central Europe (for example Darvas [5], Égert et al. [6], Lyziak et al. [12]). That is why we compare monetary policy before and during crisis within impulse-response analysis and, furthermore, we calculate calibration models. One-dimensional calibrations help us to find in which way contemporary response of price level (i.e. during crisis) depends on monetary transmission before crisis. Multi-dimensional calibrations enable to conclude, how response of inflation on monetary shock in one country depends on a power of monetary policy in the other countries. Calculations are executed for Visegrád Four countries (V4), i.e. the Czech Republic, Hungary, Poland and Slovakia. These countries have similar economies, thus it will be obvious, whether monetary policy disrupting has identical course, or if it is affected by local terms. If particular responses are substantially different, we can conclude, that adopting of common currency will not be suitable, because common policy can lead to unpredictable impacts on price level. Moreover, we include Slovakia in research sample. Slovakia has adopted Euro in January 2009. This date can be identified with the beginning of the crisis. Therefore, we can watch plain effects of the crisis in the Czech Republic, Poland and Hungary and compare these plain influences with the Slovak case.

The basic objective of central banks in Visegrád Group is to achieve price stability. Therefore, the inflation target regime has been utilised in the implementation of monetary policy of all banks. Within the framework of this strategy, banks announce an inflation target and strive to fulfil it through adjusting their interest rates. Central bank base rate has key position among monetary policy instruments. The target is medium-term, so that an

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inflation rate forecast is drawn up and monetary policy rates are set on the basis of deviations from the target. An operational criterion is inter-bank market short-term interest rate. National Bank of Poland (NBP) implemented the inflation target strategy in 1999 and it follows a continuous target at the level of 2.5 % with a permitted deviation ± 1 %. Since 1998 the inflation targeting has been utilized by Czech National Bank (ČNB). The target is currently 2 % with a permissible fluctuation band of ± 1 percentage point. Hungarian National Bank (MNB) has worked under inflation targeting regime since 2001 with the present inflation target 3 % ± 1 %. National Bank of Slovakia (NBS), which implemented the inflation targeting strategy in implicit form in 2000, has entered to Eurosystem in 2009. Therefore, the European Central Bank (ECB) took over its mission with the target less than 2 %.

An essential assumption of effective monetary policy is an existence of functional transmission mechanism. There is not a common consensus about its behaviour. Transmission mechanism is a process, in which changes in settings of monetary policy interest rates work via changes of intermediary markets on final level of economy, particularly of inflation with variously long lags. Despite the fact that central banks operate with complex and elaborated models to achieve stable prices, the effectiveness of the inflation targeting is not completely proved. Many authors deal therefore with a modelling of this mechanism, in Central European countries e.g. Brčák & Burian [1], Krušec [10]. However, these papers apply standard techniques and do not reflect potential changes of model parameters with time. There are only few studies, which model potential structure transformation: Darvas [5] for the Czech Republic, Poland and Hungary; Lyziak et al. [12] for Poland during financial crisis. The aim of the present paper is to complement the above efforts and to study more thoroughly current changes in transmission mechanism. The paper focuses on contemporary monetary policy problems in Central Europe. VAR modelling is used: it is an appropriate method for analysis of monetary policy impact on inflation, because it works with lags and economic theory assumes a delay between central bank actions and inflation response. Calibrations is more sophisticated method for analysis of above said dependencies and similarities.

2 Data and Methods

For all countries three indicators were included: inflation, interbank interest rate and central bank key interest rate. Theoretical causality within this model is: base rate influences interest rate and following interest rate influences inflation. Sample data contain monthly observations in from January 2002 to August 2012 for Czech Republic and Hungary and to July 2012 for Poland and Slovakia. It was not possible to use quarterly indicators, such as GDP, because available number of observations would not be sufficient for an during-crisis period.

First variable used is central bank base rate *BR* taken from national central banks. In Slovakia it was National Bank of Slovakia key rate to the end of 2008 and rate of ECB since the time. Next included variable is *IR*, 3 month interbank interest rate from an identical source. In Slovakia there was a transition from BRIBOR to EURIBOR on January 2009. Last variable is an inflation rate *P* as an increase in CPI compared with the corresponding month of preceding year, with all items. Data sources were national statistical offices. We set the breakpoint for all countries as the October 2008. In this time the Chow test detects structural change in inflation progress for all V4 datasets.

One of calibration goals is to find a relationship between two measurements, which both are measured with some error. Thus calibration technique is suitable for this purpose, because we can consider impulse responses in the time as measurements with some inaccuracy. Magnitude of the measurement error corresponds to the variance estimate obtained from the VAR model.

Depend on a number of variables, which we measure, we consider one- or multi-dimensional calibration. One-dimensional calibration we use for comparing the behavior before and during crisis for each country. We suppose, that the impulse response before crisis is a realization of *n*-dimensional random vector *X* with a mean value μ and a covariance matrix $\sigma_x^2 I_n$ and the impulse response during crisis is a realization of *n*-dimensional random vector *Y* with a mean value ν and a covariance matrix $\sigma_y^2 I_n$. Calibration function, in a linear form $\nu = a1_n + b\mu$, describes relationship between real values. Thus we have model

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \left(\begin{pmatrix} \mu \\ \nu \end{pmatrix}, \begin{pmatrix} \sigma_x^2 I_n & 0 \\ 0 & \sigma_y^2 I_n \end{pmatrix} \right)$$

with constrains $\nu = a1_n + b\mu$. Because constrains are nonlinear due to a product of unknown parameters *b* and μ, we linearize the model using Taylor series expansion around b_0 and μ_0 and neglecting higher order terms. We get linear model

$$\begin{pmatrix} X - \mu_0 \\ Y \end{pmatrix} \sim \left(\begin{pmatrix} \delta\mu \\ \nu \end{pmatrix}, \begin{pmatrix} \sigma_x^2 I_n & 0 \\ 0 & \sigma_y^2 I_n \end{pmatrix} \right)$$

with constrains $0 = (1_n, \mu_0)\begin{pmatrix} a \\ b \end{pmatrix} + (b_0 \mathrm{I}_n, -\mathrm{I}_n)\begin{pmatrix} \delta\mu \\ \nu \end{pmatrix}$, where $\delta\mu = \mu - \mu_0$. We derive parameters estimators according [11]. Multi-dimensional calibration is used for describing how the impulse response of one country depends on other countries. The idea is similar as in one-dimensional calibration, but the derivation of unknown parameters estimators are more complicated. More information can be found in [13]. Calibration is not common method in econometrics, but it was also used for rating calibration [14].

3 Results and Discussion

Economic theory presumes that period of the most efficient transmission is 12–18 months (the theory stems from Milton Friedman research from 1950s and 1960s – e.g. [8]). Our results do not confirm this hypothesis: inflation responses on monetary shock within 1-4 months (see *Figure 2*). This finding does not correspond with the Friedman's neoclassical theory, but it is in accordance with finding low orders in others recent studies for V4 (e.g. Krušec [10], Darvas [5]). We do not conclude that transmission is a short-time process, because shocks persist for 12-37 months in our systems (we calculated duration of the responses in the *Figure 2* up to the instant when their absolute value is lower than 0.01).

Figure 1 shows a first part of transmission mechanism. Reactions of market short-term interest rates on monetary shock are compared for before-crisis- and during-crisis-period. We can see a positive initial response in the Czech Republic and Poland – this result is in accordance with the economic theory. PRIBOR reacts immediately in the Czech Republic and with smaller intensity during crisis. Interest rate response is faster and more powerful in Poland during crisis. Response of interest rate in Slovakia is inconclusive before crisis. The Slovak Republic has entered to the European Monetary Union (EMU) at the beginning of the crisis: a reaction of interest rate is then significant, but it has bad initial direction. Hungarian reaction is not consistent with theory, too. Initial response is negative here. In this respect, we can mention a small credibility of Hungarian National Bank, which had implemented a crawling peg bend to 2008 and it fights with an enormous indebtedness in Hungary at the moment.

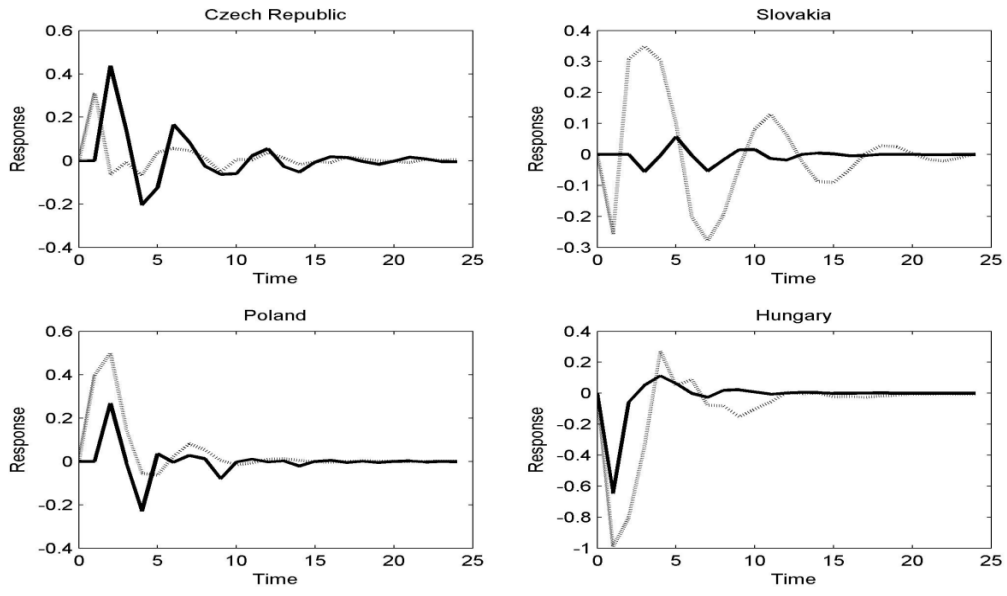

Figure 1 Responses IR on BR before (solid line) and during (dashed line) crisis

In *Figure 2* we can see responses of inflation, i.e. main goal of monetary policy, on monetary shock. All responses have correct negative initial direction, except the Slovak. Slovakia reacts negatively during crisis, but strong oscillations follow and persist for long period. Again, we can discuss the entrance to the EMU: the European transmission does not seem to be too successful in maintaining price stability in Slovakia. Hungarian case is unusual in comparison with *Figure 1* results. In this respect it is possible to consider only false causality on monetary policy shocks, (e.g. Walsh [15] explains this phenomenon). Then a main affecting of markets would

work via expectations of market participants. It is interesting to compare our results with those of Darvas [5]; according to him the monetary transmission is the most powerful in Poland, then in the Czech Republic and followed by Hungary. Our deductions are opposite: Hungary responses most powerfully, Slovak and Czech transmissions are weaker and Poland has the smallest reaction.

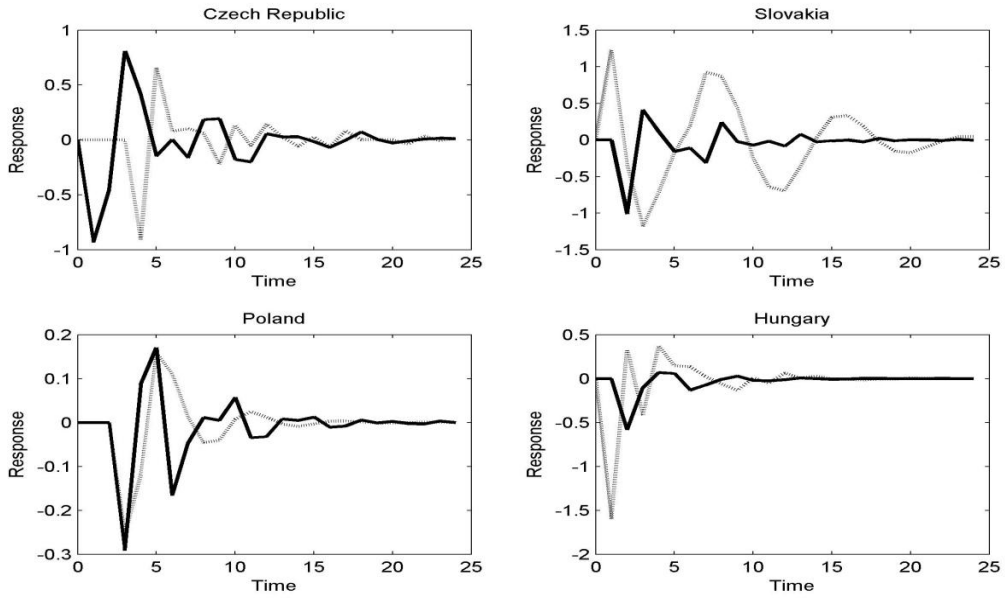

Figure 2 Responses P on BR before (solid line) and during (dashed line) crisis, on IR respectively for the Czech Republic

Results of one-dimensional calibrations are in *Tables 1* a *2*. We set manually lag in some cases, thus we can compare running of responses in better way. Shift parameters (a) are near to zero. It is more interesting to discuss slope parameters (b). It is not possible to calibrate Slovak responses, because results before and during crisis are too different. Parameter for the Czech Republic means, that response IR on monetary shock during crisis has two-thirds size of the before-crisis one. Similarly, response inflation depends from 88 % on reaction before crisis there. Responses are multiplied in Hungary, we can talk about some destabilizing of its economy. Reaction IR on BR is 2,5 times bigger than the before-crisis- one in Poland, however inflation responses in less powerful way.

Czech Republic (-1)		**Poland**		**Hungary**	
a	**b**	**a**	**b**	**a**	**b**
0,003343	0,666441	0,045429	2,515191	-0,05	2,493613

Table 1 Calibration of responses IR on BR before and during crisis

Czech Republic (+2)		**Poland**		**Hungary (-1)**	
a	**b**	**a**	**b**	**a**	**b**
0,017296	0,881575	0,000545	0,674156	0,071364	3,477904

Table 2 Calibration of responses P on BR before and during crisis

Table 3 contains results of three-dimensional calibrations of IR on BR. It is not possible to construct a calibration function for Hungary before crisis and for Slovakia during crisis, because they have too different responses from other states. Shift parameters are close to zero, thus they are not included in the tables. Result for the Czech Republic says, that response of its interest rate does not almost depends on Hungary, but it is positively correlated with Poland (it is more powerful – parameter 1,276). Slovak parameter is large, but Slovak response is very small (see *Figure 2*), hence it has minimal importance in calibration. This result is related with values for Poland before crisis. Poland is correlated with Czech Republic, but response is made only by part of the Czech one (parameter 0,784). Slovak before-crisis- results correspond to it. Parameters are very small.

Slovakia (resp. Eurozone) has intense oscillations with long durations during crisis hence it is not possible to calibrate it with other countries. Calibration function shows unexpected parameters for other states: although Hungary has negative response, its curve shape is similar to other. Therefore, Hungarian coefficients are negative. Other coefficients are negative too, because Hungarian responses are the most powerful. In this way calibration successfully describes monetary policy effects during crisis.

	Czech Republic	**Slovakia**	**Poland**	**Czech Republic**	**Poland**	**Hungary**
Czech Republic	x	-0,209	0,784	x	-0,476	-0,867
Slovakia	-4,775	x	3,742	-0,112	-0,053	-0,097
Poland	1,276	0,267	x	-2,102	x	-1,822
Hungary	0,020	0,004	-0,016	-1,154	-0,549	x

Table 3 Multidimensional calibration of IR on BR before and during crisis
Note: Before-crisis- calibrations are in left columns, the after-crisis- ones in right columns.

Results of three-dimensional calibration of P on BR are in *Table 4*. It is not possible to calibrate the Czech Republic before crisis, because it responses without lag alone of all countries. Results for other states are consistent: Slovak depends on amplified response of Hungary and Poland parameter is negative, because Poland has larger delay than Slovakia. Coefficients for Hungary and Poland are coherent with these values. Polish response P on monetary shock is very small during crisis hence its parameters in calibration are large (*Table 4*). Coefficients for the Czech Republic mean, that the CZ has larger lag than Slovakia and Hungary, thus it reacts reversely (negative parameters). The Czech Republic has the same delay as Poland (positive parameter). Slovakia and Hungary are very similar, but have a different sign.

	Slovakia	**Poland**	**Hungary**	**Czech Republic**	**Slovakia**	**Poland**	**Hungary**
Czech Republic	0,101	0,069	-0,052	x	-2,794	0,163	-1,881
Slovakia	x	-0,682	0,519	-0,358	x	0,059	-0,673
Poland	-1,465	x	0,761	6,117	17,091	x	11,506
Hungary	1,927	1,315	x	-0,532	-1,485	0,087	x

Table 4 Multidimensional calibration of P on BR before and during crisis
Note: Before-crisis- calibrations are in left columns, the after-crisis- ones in right columns.

Crisis affected monetary transmission in all V4. But effects of monetary policy differ substantially in particular countries, as we can see on one-dimensional calibrations. Under such conditions we can conclude, that an acceptance of common currency can lead to unanticipated and unwanted impacts on economies as it happened in Slovakia.

We found out in multi-dimensional calibrations that response IR on BR is close in the Czech Republic and Poland before crisis. Responses of the Czech Republic, Poland and Hungary are mutually explainable, because their responses on shocks have similar running, despite the fact that Hungarian BUBOR reacts negatively. Response P on BR is correlated in Slovakia and Hungary before crisis and Poland significantly enters to the calibration, too. We calibrated also responses P on BR during crisis, although Slovakia has positive initial reaction and large oscillations.

We can note that analysis of monetary policy effects on the basis of comparison with monetary policy in other countries is little explored area. Most studies compare responses only in a visual way (e.g. Boivin et al. [2]). One line of research compares monetary policy shocks using correlation analysis (Christiano et al. [4], Bagliano and Favero [1]). Calibrations are one possible course how to investigate better and more sophisticatedly similarities or dependencies in monetary policy power across states. However it is preferentially method for historical analysis, because, as it was shown, monetary policy impacts are varying in time. It is possible to use calibration method for estimation of unknown economic variables on the basis of historical parameters. For example we can try to estimate national interest rates in countries, which have already adopted the euro, and use them for more detailed analysis of those countries.

In general we adhere to conclusion, that effects of monetary policy on its targets are rather weak, as other studies found out - [7], [8]. Besides, Walsh [15] explains, that found causality between monetary policy interest rates and inflation can be only pseudo-causality, because central banks make macroeconomics predictions, so that their rates already incorporate future inflation development. Moreover, it is difficult to explain inflation by monetary policy shocks by Christiano et al. [4]. It does not mean, that monetary policy have no power. We only

say that transmission is very different in each country and we would like to focus on this phenomenon and its determinants in future.

4 Conclusions

This paper deals with contemporary problems of monetary policy. There is an ongoing debate, whether monetary policy is able to affect price level and whether it could be disrupted by financial crisis. There is a presumption, that financial crisis can change structure of economy, monetary rules and so disturb transmission mechanism. We focused on V4 countries. We calculated impulse-response functions and made calibration analysis with them, so that we could compare changes in monetary transmission in exact way in particular countries.

Monetary transmission has changed in all countries during crisis. But it has transformed in a different way in each country: the power of monetary policy has decreased in the Czech Republic, increased in Hungary, both decreased and increased in Poland and swung in Slovakia. It is not possible to find any relation between before-crisis- and during-crisis-transmission in Slovakia. We can conclude that its entrance to the EMU has led to marked change of monetary policy, in a negative way. An implementation of common currency in other V4 would not have desired impacts on economies. Another noticeable finding is that a first part of interest rate channel works incorrectly in Hungary. In this respect, we can discuss constant non-fulfillment of inflation target and low credibility of Hungarian central bank.

We were able to explain responses of particular country on monetary shock by responses of other states in multi-dimensional calibrations. We found significant dependencies in some cases, but they were varying in time, too. The most similar responses were: the response IR on BR in Poland before crisis, P on BR in Slovakia and Hungary before crisis. Exact comparison of dependencies of monetary policy effects on other countries is still little known part of economic science. One method for this comparison can be calibration analysis, but it is usable mainly for historical analysis.

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Coordinatewise Characteristics of Functional Data

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Abstract. It is shown that a number of natural coordinatewise summary characteristics for continuous functional data, namely the functions of moments, quantiles, and the centre of symmetry, need not be continuous. This holds even if they are defined and are finite at each coordinate separately. Sufficient conditions for the continuity of these characteristics are studied.

Keywords: Functional Data, Moments, Coordinatewise Median, Multivariate Symmetry.

JEL classification: C14
AMS classification: 62H12

1 Introduction

In the descriptive statistics of multivariate data are the summary characteristics of a probability distribution often considered in the coordinatewise sense. While, for instance, the expectation of a (finite-dimensional) random vector is already defined as the vector of the expectations of its components, the situation turns out to be more complicated when switching to continuous functions as a prime example of infinite-dimensional data. For probability measure P on a general measurable Banach space B one may utilize the concept of the Pettis integral and define the *barycentre* of P (Fremlin [3, Chapter 46]) as a generalized expectation. Pursuing this approach for B, the space of continuous real functions in t, x being a barycentre of P necessitates all the coordinates $x(t)$ being expectations of the corresponding marginal measures of P. However, the converse is not true and in general it is virtually impossible to determine if a measure P on B has a barycentre, even if all these marginal expectations exist.

Theoretical difficulties like these accompany almost every other summary statistic when trying to generalize it to a functional data framework. That, apart from the computational ease, is the main reason why most of the authors resort to coordinatewise characteristics as a reasonable alternative. For example, Ramsay and Silverman [7] extensively use the coordinatewise moments of functional variables, and various approaches to the assignment of quantiles to functional data rely on the coordinatewise median of functions. Nevertheless, a fact that has by in large been ignored is that there is usually no guarantee for these statistics to be continuous. We show in Sections 2, 3 and 4, respectively, that the coordinatewise moments, coordinatewise quantiles or the centre of symmetry of continuous functional data need not be continuous.

Throughout this paper, a number of examples illustrating these facts are demonstrated. Sufficient conditions for the continuity of these characteristics are derived in terms of sample paths behaviour. As a corollary, the continuity of these characteristics for P supported by uniformly Lipschitz continuous functions as well as for P being separable Gaussian processes is discussed.

2 Continuity of Moments

For a measurable space S equipped with a σ-algebra of subsets, denote by $\mathcal{P}\big(S\big)$ the class of all probability measures on S. Consider now two measurable separable Banach spaces S_1, S_2; S_1 being compact, for simplicity. Then by $\mathcal{C}\left(S_1, S_2\right)$ we understand the Banach space of continuous functions $x\colon S_1 \to S_2$ equipped with the uniform norm. For $P \in \mathcal{P}\big(\mathcal{C}\left(S_1, S_2\right)\big)$ and $t \in S_1$, $P_t \in \mathcal{P}\big(S_2\big)$ stands for the marginal distribution of P at t. In the following, all the examples concern probability measures from $\mathcal{P}\big(\mathcal{C}\left([-1,1], \mathbb{R}\right)\big)$, which will be shortened to $\mathcal{P}\big(\mathcal{C}\left([-1,1]\right)\big)$. However, the theoretical results are asserted in a more general way using continuous mappings between Banach spaces as described above.

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In this section, the conditions under which an expectation or higher order moments are continuous functions are investigated. To derive an initial example of what can go wrong we alter the construction used by Osius [6, Example 1] to the functional data setup to obtain a probability distribution on $\mathcal{C}([-1,1])$ with finite coordinatewise moments of all orders not forming a continuous function.

Example 1. Write $\ln\mathcal{N}\left(\mu,\sigma^2\right)$ for the usual log-normal distribution on $\mathbb{R}$ with parameters $\mu\in\mathbb{R}$ and $\sigma^2>0$. Denote $X\sim\ln\mathcal{N}(0,1)$ and for $t\in(0,1]$ set $Y_t\sim\ln\mathcal{N}\left(0,2t^{-2}\right)$; all these variables are independent of each other. Define X_t as the mixture of X and Y_t with the mixing proportions $1-\mathrm{e}^{-\frac{1}{t}}$ and $\mathrm{e}^{-\frac{1}{t}}$. Then, since the mixing proportion of Y_t vanishes as t is small, it is easy to see that

$$X_t \xrightarrow[t\to 0+]{} X \text{ weakly in } \mathcal{P}(\mathbb{R}).$$

Define now X_t for $t\in[-1,0)$ symmetrically, that is $X_t\sim X_{-t}$, and finally $X_0\sim X$. Then the system of probability measures induced by $\{X_t|\,t\in[-1,1]\}$ is evidently weakly continuous. Moreover, as the support of the law of each X_t is the positive halfline $(0,\infty)$, by a general result of Blumenthal and Corson [2, Theorem 2.1] there exists a probability measure $P\in\mathcal{P}\big(\mathcal{C}([-1,1])\big)$ with marginals P_t distributed as X_t for all $t\in[-1,1]$.

Finally, for any $s>0$

$$\mathrm{E}\,X_t^s=\begin{cases}\left(1-\mathrm{e}^{-\frac{1}{|t|}}\right)\mathrm{E}\,X^s+\mathrm{e}^{-\frac{1}{|t|}}\mathrm{E}\,Y_{|t|}^s=\left(1-\mathrm{e}^{-\frac{1}{|t|}}\right)\mathrm{e}^{\frac{s^2}{2}}+\mathrm{e}^{-\frac{1}{|t|}+\left(\frac{s}{t}\right)^2} & \text{for } t\neq 0,\\ \mathrm{e}^{\frac{s^2}{2}} & \text{for } t=0,\end{cases}$$

and the s-th moment of the marginals P_t is finite and discontinuous in t, as $\lim_{t\to 0}\mathrm{E}\,X_t^s=\infty\neq\mathrm{E}\,X_0^s$.

It is widely accepted that a weakly convergent sequence of random variables X_n converges also in the k-th mean sense if and only if the sequence $\left\{\|X_n\|^k\right\}_{n=1}^{\infty}$ is uniformly integrable, that is if

$$\lim_{M\to\infty}\sup_{n\in\mathbb{N}}\mathrm{E}\,\|X_n\|^k\,\mathbb{I}_{\left\{\|X_n\|^k>M\right\}}=0.$$

The following characterization provides a connection between the uniform integrability of the marginals of a measure on functions and the continuity of its coordinatewise moments.

Proposition 1. *Let $P\in\mathcal{P}(\mathcal{C}(S_1,S_2))$, $X\sim P$ and $k\in\mathbb{N}$. Then the mapping $t\mapsto\mathrm{E}\,\|X(t)\|^k$ is continuous on S_1 if and only if $\left\{\|X(t)\|^k\,\middle|\,t\in S_1\right\}$ is a uniformly integrable class of r.v.*

Proof. Notice that for $t_n\to t$ in S_1 the a.s. path continuity of X implies $P_{t_n}\to P_t$ weakly by the portmanteau theorem. The "if" part then follows directly from Osius [6, Lemma 1]. On the other hand for the "only if" part, consider a sequence t_n in S_1 violating the uniform integrability assumption. As S_1 is compact, t_n may be chosen convergent to $t_0\in S_1$ and employing again Osius [6, Lemma 1] this yields that $\|X(t)\|^k$ is discontinuous at t_0. □

For the next theorem, it is useful for a function $x\in\mathcal{C}(S_1,S_2)$ to define the *diameter* of x to be

$$\Delta x=\sup_{s,t\in S_1}\|x(t)-x(s)\|\in[0,\infty).$$

Theorem 2. *Let $P\in\mathcal{P}(\mathcal{C}(S_1,S_2))$, $X\sim P$ and $k\in\mathbb{N}$. If $\mathrm{E}\,(\Delta X)^k<\infty$ and $t_0\in S_1$ exists such that $\mathrm{E}\,\|X(t_0)\|^k<\infty$, then $\mathrm{E}\,\|X(.)\|^k\in\mathcal{C}(S_1,S_2)$.*

Proof. By the assumptions for each non-increasing sequence of measurable sets $A_{n+1}\subset A_n\subset\mathbb{R}$, $A_n\to\emptyset$ the dominated convergence theorem yields

$$\mathrm{E}\,(\|X(t_0)\|+\Delta X)^k\,\mathbb{I}_{A_n}\le 2^{k-1}\mathrm{E}\left(\|X(t_0)\|^k+(\Delta X)^k\right)\mathbb{I}_{A_n}\xrightarrow[n\to\infty]{}0. \tag{1}$$

Taking

$$A_n=\left\{x\in\mathcal{C}(S_1,S_2)\,\middle|\,\sup_{t\in S_1}\|x(t)\|^k>n\right\}$$

in (1) we have

$$\sup_{t\in S_1} \mathrm{E}\,\|X(t)\|^k \mathbb{I}_{\{\|X(t)\|^k>n\}} \le \mathrm{E}\,(\|X(t_0)\| + \Delta X)^k \mathbb{I}_{A_n} \xrightarrow[n\to\infty]{} 0,$$

the uniform integrability of the marginals. Proposition 1 now provides the assertion. □

Two interesting corollaries arise from Theorem 2, one for the measures on the class of a.s. (uniformly) equicontinuous functions, and the other for Gaussian processes.

Proposition 3. *Let $P \in \mathcal{P}(\mathcal{C}(S_1, S_2))$, $X \sim P$. If one of the following conditions holds*

- *the support of P consists of (uniformly) equicontinuous functions, i.e.*

$$\forall \varepsilon > 0\, \exists \delta > 0 \text{ such that } P(\|X(s) - X(t)\| < \varepsilon \text{ whenever } \|s - t\| < \delta) = 1, \tag{2}$$

  and $t \in S_1$ exists such that $\mathrm{E}\,\|X(t)\|^k < \infty$, or

- *$S_2 = \mathbb{R}$ and P is a separable Gaussian process,*

then $\mathrm{E}\,\|X(.)\|^k \in \mathcal{C}(S_1, S_2)$ for each $k \in \mathbb{N}$.

Proof. If the functions from P are equicontinuous, then $(\Delta X)^k$ is a.s. bounded since S_1 is compact. If P is Gaussian, then the path continuity of the functions yields $P(|X(t)| < \infty) = 1$ (Belyaev [1, Theorem 1]), which in turn implies according to Landau and Shepp [4] the finiteness of the last term in

$$\mathrm{E}\,(\Delta X)^k = \mathrm{E}\left(\sup_{s,t\in S_1} |X(t) - X(s)|\right)^k \le 2^k \mathrm{E} \sup_{t\in S_1} |X(t)|^k.$$

In both cases, Theorem 2 may be used now to conclude the proof. □

The following example shows that the conditions given in Theorem 2 are not necessary.

Example 2. For $n \in \mathbb{N}$ set the random function X from $P \in \mathcal{P}\big(\mathcal{C}([-1,1])\big)$ to have the support in the sequence of piecewise linear functions defined by

$$x_n(t) = \begin{cases} n & \text{for } t = t_n, \\ 0 & \text{if } |t - t_n| \ge 2^{-n}, \end{cases}$$

decreasing around the peak t_n linearly to zero. Set $P(X \equiv x_n) = \frac{c}{n^2}$ for $c = \frac{6}{\pi^2}$ and $t_n \in [-1, 1]$. Then $\mathrm{E}\,\Delta X = \sum_{n=1}^{\infty} n\frac{c}{n^2} = \infty$ and the assumptions of Theorem 2 are not satisfied.

If $t_n \equiv 0$ is chosen now, then the expectation function of P is evidently continuous on $[-1, 1] \setminus \{0\}$, but tends to infinity at zero. On the other hand, by setting $t_n = -1 + \sum_{m=1}^{n-1} 2^{-m+1} + 2^{-n}$ the intervals on which the functions x_n are positive are disjoint and for the expectation function we can write

$$\mathrm{E}\,X(t) = \begin{cases} \frac{c}{n} & \text{for } t = t_n, \\ 0 & \text{for } t = 0, 1 \text{ and } t = t_n + 2^{-n}, \end{cases}$$

and is defined linearly at each interval between these points. Hence, by a simple rearrangement of the peaks of the supporting functions we achieved a significant change in the course of the expectation function; if the peaks are spread in the interval $[-1, 1]$, the expectation function is continuous.

3 Continuity of Quantiles

Another coordinatewise characteristic for functions from $\mathcal{C}(S_1, \mathbb{R})$ studied in the literature is the coordinatewise median. Many authors consider it as a reasonable robust counterpart of the expectation and derive nonparametric procedures for functional data by means of median estimation. In particular, in the methodology based on the depth of functional data, the vast majority of applicable depth functionals rely on coordinatewise median as the centre of "generalized symmetry", cf Nagy et al. [5]. Nevertheless, just as for the function of moments studied in Section 2, in general there is no guarantee that a median (or quantiles) should be a function from $\mathcal{C}(S_1, \mathbb{R})$.

Example 3. Construct the distribution $P \in \mathcal{P}\big(\mathcal{C}\left([-1,1]\right)\big)$ as a mixture of $P_1, P_2 \in \mathcal{P}\big(\mathcal{C}\left([-1,1]\right)\big)$. For P_1, consider three independent random variables

$$I \sim -1 - \mathrm{Exp}\left(1\right), \quad J \sim 1 - \mathrm{Exp}\left(1\right), \quad K \sim \mathrm{Exp}\left(1\right),$$

where $\mathrm{Exp}\left(\lambda\right)$ is the exponential distribution with expectation $\lambda > 0$. Using these auxiliary variables, define the random function X from P_1 as

$$X(t) = \begin{cases} I & \text{for } t \in [-1,0], \\ (J-I)t^K + I & \text{for } t \in (0,1]. \end{cases}$$

Then $X \in \mathcal{C}\left([-1,1]\right)$ and the support of the marginals of P_1 is $(-\infty,-1)$ for $t \leq 0$ and $(-\infty,1)$ for $t > 0$.

Define P_2 as the distribution of the random function $\widetilde{X}(t) = -X(-t)$. Then $P_2 \in \mathcal{P}\big(\mathcal{C}\left([-1,1]\right)\big)$ and its support is marginally $(-1,\infty)$ for $t < 0$ and $(1,\infty)$ for $t \geq 0$. Thus, the marginals of P_1 and P_2 have disjoint supports and if P is chosen as an equal prior mixture of P_1 and P_2, it is easy to see that the coordinatewise (set-valued) median function takes the form

$$\mathrm{median}\left(P_t\right) = \begin{cases} \{-1\} & \text{for } t \in [-1,0), \\ [-1,1] & \text{for } t = 0, \\ \{1\} & \text{for } t \in (0,1]. \end{cases}$$

Apparently, no continuous selection can be drawn from this median set. Moreover, if we define the median uniquely as a 0.5-quantile of P_t at each t (see Definition 1 below), it is also not a function from $\mathcal{C}\left([-1,1]\right)$.

By setting the mixing proportions of P_1 and P_2 to $q \in [0,1]$ and $1-q$, the same result would be obtained for the (set-valued) q-th coordinatewise quantile.

Definition 1. Let $Q \in \mathcal{P}\left(\mathbb{R}\right)$ and $q \in [0,1]$. Then the q-the quantile of Q is defined as

$$x_q = \inf\left\{x \in [-\infty,\infty] \middle| \, Q\left((-\infty,x)\right) \leq q \text{ and } Q\left((-\infty,x]\right) \geq q\right\}.$$

In what follows, it turns out to be convenient to use the so-called *minimal l_∞ metric* between two probability measures $P, Q \in \mathcal{P}\left(S\right)$ defined as

$$l_\infty\left(P,Q\right) = \inf\left\{\eta > 0 \middle| \, P\left(A\right) \leq Q\left(A^\eta\right) \text{ for all } A \in \mathcal{S}\right\}, \tag{3}$$

where A^η is the (open) η-neighbourhood of the set $A \in \mathcal{S}$, i.e. $A^\eta = \{x \in S \mid \mathrm{dist}\left(x,A\right) < \eta\}$. Although not apparent from the definition, it can be shown that l_∞ is a metric indeed, hence it is symmetric. This metric is similar in construction, yet stronger than Prokhorov metric and it appears that a continuity of the marginals of a functional probability in l_∞ is sufficient for the continuity of the quantiles to hold.

Theorem 4. *Let $P \in \mathcal{P}\left(\mathcal{C}\left(S_1,\mathbb{R}\right)\right)$ such that the set of its univariate marginals $\{P_t \mid t \in S_1\} \subset \mathcal{P}\left(\mathbb{R}\right)$ is continuous in l_∞ metric on S_1. Then for each $q \in [0,1]$ the q-th quantile function defined at $t \in S_1$ as the q-th quantile of P_t is continuous on S_1.*

Proof. Fix $q \in [0,1]$. The existence and the uniqueness of the q-th quantile function is obvious from Definition 1. Thus, it is enough to show that for $s,t \in S_1$ such that $l_\infty\left(P_s,P_t\right) < \varepsilon$ the q-th quantiles of P_s and P_t lie close to each other. Observe the following fact for $Q \in \mathcal{P}\left(\mathbb{R}\right)$, its q-th quantile x_q and $a \in [-\infty,\infty]$:

- If $Q\left((-\infty,a)\right) \geq q$, then
  - either $Q\left((-\infty,a)\right) > q$ and hence immediately $a > x_q$,
  - or $Q\left((-\infty,a)\right) = q$ and a is an element of the set whose infimum is defined as the q-th quantile, that is $a \geq x_q$.
- If $Q\left((-\infty,a]\right) \leq q$, then similarly
  - either $Q\left((-\infty,a]\right) < q$ and $a < x_q$, or
  - $Q\left((-\infty,a]\right) = q$ and $a \geq x_q$.

Designate now the q-th quantile of P_s and P_t as x_s and x_t, respectively. Using the observation above, from (3) we yield

$$q \leq P_s\left((-\infty, x_s]\right) \leq P_t\left((-\infty, x_s + \varepsilon)\right),$$

implying $x_t \leq x_s + \varepsilon$, and by the symmetry of l_∞ for each $\delta > 0$

$$P_t\left((-\infty, x_s - (1+\delta)\varepsilon]\right) \leq P_s\left((-\infty, x_s - \delta\varepsilon)\right) < q$$

and $x_s - (1+\delta)\varepsilon < x_t$. Combining these results, for each $s, t \in [-1, 1]$ we may conclude

$$l_\infty\left(P_s, P_t\right) < \varepsilon \text{ implies } |x_s - x_t| \leq \varepsilon. \tag{4}$$

□

A natural question when the assumptions of Theorem 4 are satisfied is answered by the following proposition.

Proposition 5. *Let $P \in \mathcal{P}\left(\mathcal{C}\left(S_1, \mathbb{R}\right)\right)$ be such that (2) holds. Then for each $q \in [0, 1]$ the q-th quantile function is continuous on S_1. In particular, if P is concentrated on the (uniformly) C-Lipschitz continuous functions on S_1, that is $\varepsilon = C\delta$ can be chosen in (2), then the q-th quantile function is C-Lipschitz continuous on S_1.*

Proof. From (2) it holds immediately that for $|s - t| < \delta$

$$l_\infty\left(P_s, P_t\right) = \inf\left\{\eta > 0 \middle| P_s\left(A\right) \leq P_t\left(A^\eta\right) \text{ for all } A \text{ measurable}\right\} < \varepsilon.$$

The assertion of the theorem now follows from (4). □

Observe that for a Gaussian process its marginal medians are also its marginal expectations. Hence, by Proposition 3 its coordinatewise median is always continuous. However, already for the class of mixtures of Gaussian processes no analogy to Proposition 3 holds for quantiles in general. In fact, it is possible to construct a path-continuous mixture of two (degenerated) Gaussian process on $[-1, 1]$ such that the coordinatewise median set contains no continuous selection, see Nagy et al. [5] for a more detailed discussion.

4 Continuity of Centre of Symmetry

The last coordinatewise characteristic we will be dealing with is the centre of symmetry. It is not easy to define a reasonable notion of symmetry for functional data, as already in the finite-dimensional setup a great deal of ambiguity arises. However, if we adopt the principle that any appropriate symmetry in the functional case must be inherited by all finite-dimensional marginals, we may start with a set of symmetric marginal distributions $\left\{P_t \in \mathcal{P}\left(\mathbb{R}^d\right) \middle| t \in S_1\right\}$ and explore when the centre of symmetry of $P \in \mathcal{P}\left(\mathcal{C}\left(S_1, \mathbb{R}^d\right)\right)$ defined by the centres of its marginals P_t is continuous.

In this section we confine ourselves to the broadest notion of symmetry (cf Zuo and Serfling [8]) for finite-dimensional measures–the halfspace symmetry.

Definition 2. We say that $Q \in \mathcal{P}\left(\mathbb{R}^d\right)$ is *halfspace symmetric about* $x \in \mathbb{R}^d$ if

$$Q(H) \geq 0.5 \text{ for each closed halfspace } H \text{ with } x \text{ on its boundary.}$$

As for $d = 1$ every $Q \in \mathcal{P}\left(\mathbb{R}\right)$ is halfspace symmetric about its median, we may refer to the examples in Section 3 for the instances of distributions $P \in \mathcal{P}\left(\mathcal{C}\left(S_1, \mathbb{R}\right)\right)$ such that all its marginals are halfspace symmetric, but no continuous selection can be drawn from the coordinatewise median set. The common attribute of all these examples is the existence of $t \in S_1$ such that the (set-valued) median of P_t is not a singleton. This may be observed in Example 3 at $t = 0$, where the median set was a closed interval. It can be shown that this non-uniqueness of a median is necessary for a discontinuity of the function of the centre of symmetry. The proof of the following theorem can be found in Nagy et al. [5] and will not be presented here.

Theorem 6. *Let $P \in \mathcal{P}\left(\mathcal{C}\left(S_1, \mathbb{R}^d\right)\right)$ be such that for each $t \in S_1$ the marginal distribution $P_t \in \mathcal{P}\left(\mathbb{R}^d\right)$ is halfspace symmetric and is not concentrated only on a line in $\mathbb{R}^d$ on which it has two distinct medians. Then there exists a unique function $x \in \mathcal{C}\left(S_1, \mathbb{R}^d\right)$ such that P_t is halfspace symmetric about $x(t)$ for all $t \in S_1$.*

As a corollary, path-continuous (vector-valued) random processes, whose support of the marginals forms a connected set for each $t \in S_1$, can be marginally halfspace symmetric only about a function in $\mathcal{C}\left(S_1, \mathbb{R}^d\right)$. Even more could be said if we pursued this idea from the most general case of halfspace symmetry to some more restrictive alternative of marginal symmetry, e.g. central symmetry or elliptical symmetry as considered in Zuo and Serfling [8]. Nonetheless, the uniqueness of the (univariate) median assumption in Theorem 6 is satisfied in most of the applications to random processes or functional data. Thus, the continuity of the centre of symmetry for all other notions of multivariate symmetry follows immediately from Theorem 6 and the general results of Zuo and Serfling [8].

5 Conclusions

The main aim of this paper was to present the conditions that need to be satisfied in order to obtain reasonable coordinatewise summary statistics for functional data. If they are not fulfilled, the characteristics of a probability distribution on continuous functions may fail to be continuous. This phenomenon is interesting in many applications where functional data emerges, in particular in various economic models where the data is obtained as a random sample of continuous functions. In these cases it is often assumed that the model can be written in a form of a fixed continuous mean function with an additive random sample of centred noise functions. Under the assumptions as stated herein we have justified this approach and shown that it might be reasonable to consider mean curves, quantile curves or marginalwise estimates of the centres of symmetry computed from the data. From the future perspectives related to this topic, one could pinpoint the derivation of conditions under which additional properties remain inherited by the functional summary statistics, e.g. Lipschitz continuity or smoothness.

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Evaluating labour market flexibility in V4 countries

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Abstract. This contribution focuses on evaluation of labour market flexibility in the V4 countries. These countries have similar historical destiny but the economic transition led to new labour market institutions. Labour market outcomes are thus quite different in the last decades. Using a small closed DSGE model with search and matching frictions, the main structural labour market differences among these countries are revealed. These differences are related to the changes in separation rates, matching efficiency, match elasticity of unemployed and the bargaining power of the workers. The models are identified using Bayesian techniques. Comparing the moments of simulated data, we can conclude that the estimated models are able to replicate the main statistical properties of the observed data. The labour market in Hungary seems to be the most flexible regarding the labour market dynamics in the light of the nascent economic recovery. On the other hand, all examined economies are similar to the extent of the low wage bargaining power of the workers.

Keywords: search and matching model, Bayesian estimation, DSGE model, labour market flexibility, V4 countries.

JEL classification: C51, E24, J60
AMS classification: 91B40, 91B51

1 Introduction

Labour market flexibility is an important factor of labour market dynamics. It has direct consequences on vacancies creation, unemployment dynamics and on responses of the economy to exogenous shocks. The goal of this contribution is to reveal possible structural differences among the labour markets of Visegrad countries (V4): the Czech Republic, Slovakia, Hungary and Poland. For these purposes, a small closed DSGE model with search and matching frictions with time-varying demand elasticity and separation rates will be estimated. There is no unique measure of the labour market flexibility but we will focus on some key structural difference which might be connected with the flexible labour markets. In particular, we will investigate the differences in the estimated separation rates, matching efficiency, match elasticity of unemployed and the bargaining power of the workers. Moreover, the labour market flexibility is evaluated with respect to the length and amplitude of the responses of the economies to the selected labour market shocks. A review of studies dealing with the labour market flexibility in the Czech and Slovak Republic may be found in Němec [7]. Behar [3] found out that tax wedges and duration of benefits in the Central and Eastern European (CEE) countries were important factors of the poor labour market outcomes. On the other hand, he concluded that labour market policies and institutions in the CEE countries are generally more flexible than those in the rest of Europe. Twenty-seven OECD countries are investigated by Hobijn and Sahin [4]. Their results provide relatively big differences in job-finding rates and separation rates which may suggest the heterogeneity in the European labour markets. We are convinced that the DSGE approach may improve our knowledge of the labour markets in V4 countries.

2 Model

In this contribution, we shall use a simple search and matching model incorporated within a standard DSGE framework. This model was developed by Lubik [5] and is described in Němec [6] and [7]. We will thus present and comment only the main equations and the log-linearized version of the model. Compared to previously mentioned papers two important enhancements have been made. First, the

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price elasticity of demand is more explicitly taken as a time-varying parameter that allows us to capture the aggregate output dynamics in a more consistent and reliable way. Previous attempts to treat this parameter as a constant required formulating an exogenous autoregressive process for output dynamics. Second enhancement concerns the time-varying separation rate. Incorporating this feature may further distinguish various sources of dynamics on the labour market. A representative household maximizes its expected utility function

$$E_t \sum_{j=t}^{\infty} \beta^{j-t} \left[\frac{C_j^{1-\sigma} - 1}{1-\sigma} - \chi_j n_j \right], \tag{1}$$

where symbol E_t represents expectation operator conditioned by the information available in time t, C_j is aggregate consumption, $n_j \in [0,1]$ is a fraction of employed household members, $\beta \in (0,1)$ is the discount factor and $\sigma \geq 0$ is the coefficient of relative risk aversion. Variable χ_j represents an exogenous stochastic process which may be taken as a labour shock. The budget constraint is defined as $C_t + T_t = w_t n_t + (1-n_t)b + \Pi_t$, where b is unemployment benefit financed by a lump-sum tax, T_t. Variable Π_t represents profits from ownership of the firms, and w_t is wage. There is no explicit labour supply because it is an outcome of the matching process. The first-order condition is thus simply $C_t^{-\sigma} = \lambda_t$, where λ_t is the Lagrange multiplier on the budget constraint.

The labour market is characterized by search frictions captured by a standard Cobb-Douglas matching function $m(u_t, v_t) = \mu_t u_t^{\xi} v_t^{1-\xi}$, where unemployed job seekers, u_t, and vacancies, v_t, are matched at rate defined by $m(u_t, v_t)$. Parameter $0 < \xi < 1$ is a match elasticity of the unemployed, and μ_t is stochastic process measuring the efficiency of the matching process. Aggregate probability of filling a vacancy may be defined as $q(\theta_t) = m(u_t, v_t)/v_t$, where $\theta_t = \frac{v_t}{u_t}$ is a standard indicator of the labour market tightness. The model assumes that it takes one period for new matches to be productive. Moreover, old and new matches are destroyed at a separation rate, $0 < \rho_t < 1$, which corresponds to the inflows into unemployment and is considered as a time-varying parameter. The labour force is normalised to one and evolution of employment or equivalently employment rate, $n_t = 1 - u_t$, is given by $n_t = (1-\rho_t)\left[n_{t-1} + v_{t-1} q(\theta_{t-1})\right]$.

The model assumes monopolistic behaviour of the firm in each sub-market. Demand function of a firm is defined by $y_t = \left(\frac{p_t}{P_t}\right)^{-1-\omega_t} Y_t$, where y_t is firm's production (and its demand), Y_t is aggregate output, p_t is price set by the firm, P_t is aggregate price index and ω_t is time-varying demand elasticity. Production function of each firm is $y_t = A_t n_t^{\alpha}$, where A_t is an aggregate technology (stochastic) process and $0 < \alpha \leq 1$ introduces curvature in production. The firm controls the number of workers, n_t, number of posted vacancies, v_t, and its optimal price, p_t, by maximizing the inter-temporal profit function

$$E_t \sum_{j=1}^{\infty} \beta^{j-t} \lambda_j \left[p_j \left(\frac{p_j}{P_j}\right)^{-(1+\omega)} Y_j - w_j n_j - \frac{\kappa}{\psi} v_j^{\psi} \right], \tag{2}$$

subject to the employment accumulation equation and production function. Profits are evaluated in terms of marginal utility λ_j. The costs of vacancy posting is $\frac{\kappa}{\psi} v_t^{\psi}$, where $\kappa > 0$ and $\psi > 0$. For $0 < \psi < 1$, posting costs exhibit decreasing returns. For $\psi > 1$, the costs are increasing while vacancy costs are fixed for $\psi = 1$. The first order condition represents the evolution of current-period marginal value of a job, τ_t, and a link between the cost of vacancy, $\kappa v_t^{\psi-1}$ and the expected benefit of a vacancy in terms of the marginal value of a worker (adjusted by the job creation rate, $q(\theta_t)$) and a stochastic discount factor $\beta_{t+1} = \beta \frac{\lambda_{t+1}}{\lambda_t}$. Wages are determined as the outcome of a bilateral bargaining process between workers and firms. Both sides of the bargaining maximize the joint surplus from employment relationship:

$$S_t \equiv \left(\frac{1}{\lambda_t} \frac{\partial \mathcal{W}_t(n_t)}{\partial n_t}\right)^{\eta} \left(\frac{\partial \mathcal{J}_t(n_t)}{\partial n_t}\right)^{1-\eta}, \tag{3}$$

where $\eta \in [0,1]$ is the bargaining power of workers, $\frac{\partial \mathcal{W}_t(n_t)}{\partial n_t}$ is the marginal value of a worker to the household's welfare and $\frac{\partial \mathcal{J}_t(n_t)}{\partial n_t} = \tau_t$ is the marginal value of a worker to the firm. After straightforward adjustments are carried out, one can get the expression for the bargained wage. The model assumes that unemployment benefits, b, are financed by lump-sum taxes, T_t, where a condition of balanced budget holds, i.e. $T_t = (1-n_t)b$. Social resource constraint is thus $C_t + \frac{\kappa}{\psi} v_t^{\psi} = Y_t$. The technology shock A_t, the labour shock χ_t, the matching shock μ_t, the time-varying demand elasticity ω_t and the separation rate ρ_t are assumed to be independent $AR(1)$ processes (in logs) with coefficients ρ_i, $i \in (A, \xi, \mu, \omega, \rho)$ and autoregression residuals $\epsilon_t^i \sim N(0, \sigma_i^2)$. In the following log-linearized equations, the line over a variable

means its steady-state value. Steady-state values are derived simply from the non-linear equations. Initial steady-state values which are common for all economies are calibrated as follows: $\overline{\mu} = \overline{A} = \overline{\chi} = \overline{p} = \overline{P} = 1$, $\overline{\rho} = 0.03$, $\overline{\omega} = 10$, $\overline{\beta} = 0.98$. Country specific steady states for the Czech Republic $\overline{u} = 0.0831$, $\overline{v} = 0.0114$, for the Slovak Republic $\overline{u} = 0.1359$, $\overline{v} = 0.0055$, for Hungary $\overline{u} = 0.1027$, $\overline{v} = 0.009$, and for Poland $\overline{u} = 0.1418$, $\overline{v} = 0.002$. Remaining steady-states are computed using these values and the prior means of all parameters. The variables with a tilde represent the gaps from their steady-states. It should be mentioned, that the gaps are computed as log-differences, e.g. $\tilde{u} = \log u - \log \overline{u}$.

$$\tilde{\lambda}_t = -\sigma \tilde{C}_t \qquad \tilde{m}_t = \tilde{\mu}_t + \xi \tilde{u}_t + (1-\xi)\tilde{v}_t$$

$$\tilde{q}_t = \tilde{m}_t - \tilde{v}_t \qquad \tilde{\theta}_t = \tilde{v}_t - \tilde{u}_t$$

$$\tilde{n}_t = -\frac{\overline{u}}{1-\overline{u}}\tilde{u}_t \qquad \tilde{n}_t = \frac{1}{\overline{n}+\overline{vq}}\left[\overline{n}\tilde{n}_{t-1} + \overline{vq}(\tilde{v}_{t-1} + \tilde{q}_{t-1})\right] - \frac{\overline{\rho}}{1-\overline{\rho}}\tilde{\rho}_t$$

$$\tilde{y}_t = (-1-\overline{\omega})(\tilde{p}_t - \tilde{P}_t) - \overline{\omega}\log\left(\frac{\overline{p}}{\overline{P}}\right)\tilde{\omega}_t + \tilde{Y}_t \qquad \tilde{y}_t = \tilde{A}_t + \alpha\tilde{n}_t$$

$$(\psi - 1)\tilde{v}_t = \tilde{q}_t + E_t\left(\tilde{\beta}_{t+1} + \tilde{\tau}_{t+1}\right) - \frac{\overline{\rho}}{1-\overline{\rho}}\tilde{\rho}_t \qquad \tilde{\beta}_t = \tilde{\lambda}_t + \tilde{\lambda}_{t-1}$$

$$\tilde{\tau}_t = \frac{1}{\alpha\frac{\overline{y}}{\overline{n}}\frac{\overline{\omega}}{1+\overline{\omega}} - \overline{w} + (1-\overline{\rho})\overline{\beta}\overline{\tau}} \times \left[\alpha\frac{\overline{y}}{\overline{n}}\frac{\overline{\omega}}{1+\overline{\omega}}(\tilde{y}_t - \tilde{n}_t + \overline{\omega}\tilde{\omega}_t) - \overline{w}\tilde{w}_t + \overline{\tau}\overline{\beta}E_t\left((1-\overline{\rho})(\tilde{\beta}_{t+1} + \tilde{\tau}_{t+1}) - \overline{\rho}\tilde{\rho}_t\right)\right]$$

$$\tilde{w}_t = \frac{1}{\overline{w}}\left[\eta\left(\alpha\frac{\overline{\omega}}{1+\overline{\omega}}\frac{\overline{y}}{\overline{n}}(\tilde{y}_t - \tilde{n}_t + \frac{1}{1+\overline{\omega}\tilde{\omega}_t}) + \kappa\overline{v}^{\psi-1}\overline{\theta}\left((\psi-1)\tilde{v}_t + \tilde{\theta}_t\right)\right) + (1-\eta)\overline{\chi}\overline{C}^{\sigma}(\tilde{\chi}_t + \sigma\tilde{C}_t)\right]$$

$$\tilde{Y}_t = \frac{1}{\overline{C} + \frac{\kappa}{\psi}\overline{v}^{\psi}}\left(\overline{C}\tilde{C}_t + \kappa\overline{v}^{\psi}\tilde{v}_t\right)$$

$$\tilde{A}_t = \rho_A\tilde{A}_{t-1} + \epsilon_t^A \quad \tilde{\chi}_t = \rho_\chi\tilde{\chi}_{t-1} + \epsilon_t^\chi \quad \tilde{\mu}_t = \rho_\mu\tilde{\mu}_{t-1} + \epsilon_t^\mu \quad \tilde{\omega}_t = \rho_\omega\tilde{\omega}_{t-1} + \epsilon_t^\omega \quad \tilde{\rho}_t = \rho_\rho\tilde{\rho}_{t-1} + \epsilon_t^\rho$$

We have thus five shocks (ϵ_t^i) for four observed variables – $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ and $\tilde{Y}$). The model consists of 20 endogenous variables, five shocks and 13 parameters. Due to the fact that the price gaps, $\tilde{p}$ and $\tilde{P}$, are undetermined and out of importance in our model, they may be treated as zero variables.

3 Estimation results and model evaluation

The models for the V4 countries are estimated separately using the quarterly data sets covering a sample period from 1998Q1 to 2012Q4. The observed variables are real output (Y), hourly earnings (w), unemployment rate (u) and rate of unfilled job vacancies (v).[1] Parameters of the model are estimated using Metropolis-Hasting algorithm (two chains, 2000000 draws per chain with 80% draws burned-in) combined with Kalman filtering procedures that are necessary to evaluate likelihood function of the model (see An and Schorfheide [2]). All computations have been performed using Dynare toolbox for Matlab (version 4.3.2) developed by Adjemian et al. [1]. Table 1 reports the model parameters and the corresponding prior densities and calibrated (fixed) values. The priors and calibrated quantities are the same for all four economies. These quantities proceed from those used by Lubik [5] or Němec [7]. The standard deviations of the prior densities are rather uninformative. Table 2 presents the means and standard deviations of the posterior densities including the 90% highest posterior density intervals (HPDI). All generated chains converged to the stationary distribution. Parameters are mostly identified from the data. There are only two exceptions: parameter representing unemployment benefits, b, and the scale parameter in the vacancy posting costs, κ. This result is not surprising because we have incorporated two time varying parameters which are able to capture the influence of these non-identified parameters. This fact does not speak against the quality of our model. Both parameters are rather auxiliary parameters which do not

[1] The original data come from database of OECD. Unfilled job vacancies for Slovakia were taken from the Ministry of Labour, Social Affairs and Family of the Slovak Republic (SAFSR). The following variables were used: GDP - expenditure approach (chained volume estimates, national reference year, OECD), index of hourly earnings (manufacturing, 2005=100, OECD), registered unemployment (level, OECD), unfilled job vacancies (level, OECD and SAFSR). The time series are seasonally adjusted by the data providers using the TRAMO/SEAT procedure. The variables were transformed using logarithmic transformation and de-trended using Hodrick-Prescott filter with the smoothing parameter $\lambda = 1600$. The variables used are thus expressed as corresponding gaps. This approach is fully consistent with the log-linear equations.

contribute to the overall model dynamics. The results for the cross-correlations, sample moments and autocorrelation coefficients of the observed and simulated data are not presented here but the fit of all models is outstanding. Regarding our estimates we can see that the structural parameters of V4 countries are different although we could found some common patterns among two or three of them. As an example, take a look on the parameter σ. From this point of view households in the Czech Republic and Hungary are very similar. Slovak households are less risk averse and on the other hand Polish households are more risk sensitive. As for the parameters of persistence of shocks and their standard deviations, it is clear that the shocks hitting the economies and their labour markets did not either coincide or last for the same time period. Different observed history does not imply opposing dynamic properties of the economies (see Figure 2).

Description	Parameter	Density	Mean	Std. Dev.
Discount factor	β	fixed	0.98	–
Labour elasticity	α	fixed	0.67	–
Relative risk aversion	σ	gamma	1.00	0.50
Match elasticity	ξ	beta	0.70	0.10
Bargaining power of the workers	η	uniform	0.50	0.30
Unemployment benefits	b	beta	0.30	0.15
Elasticity of vacancy creation cost	ψ	gamma	1.00	0.50
Scaling factor on vacancy creation cost	κ	gamma	0.10	0.05
AR coefficients of shocks	$\rho_{\{\chi,A,\mu,\omega,\rho\}}$	beta	0.50	0.20
Standard deviation of shocks	$\sigma_{\{\chi,A,\mu,\omega,\rho\}}$	inv. gamma	0.05	∞

Table 1 Prior densities and description of estimated parameters

	CZE	SVK	HUN	POL
σ	0.4673	0.2149	0.4556	1.4005
	(0.2533; 0.6849)	(0.0989; 0.3261)	(0.1716; 0.7277)	(0.5410; 2.2139)
ξ	0.7422	0.8622	0.8740	0.6307
	(0.6911; 0.7936)	(0.8154; 0.9370)	(0.5069; 0.7634)	(0.5069; 0.7634)
η	0.0257	0.0418	0.2920	0.3451
	(−0.0196; 0.0697)	(-0.0196;0.0943)	(−0.0102; 0.6226)	(0.0961; 0.5934)
b	0.3003	0.3008	0.2998	0.3007
	(0.0537; 0.5291)	(0.0580; 0.5342)	(0.0599; 0.5361)	(0.0568; 0.5314)
ψ	2.1346	1.7226	0.9314	1.7422
	(1.3679; 3.0180)	(0.9624; 2.4648)	(0.5142; 1.4588)	(1.0964; 2.3904)
κ	0.1013	0.1012	0.1204	0.1006
	(0.0230; 0.1757)	(0.0239; 0.1754)	(0.0270; 0.2072)	(0.0236; 0.1755)
ρ_χ	0.6602	0.2308	0.7115	0.8095
	(0.4942; 0.8354)	(0.0679; 0.3878)	(0.5661; 0.8577)	(0.7007; 0.9192)
ρ_A	0.8418	0.4832	0.8059	0.7405
	(0.7511; 0.9374)	(0.3038; 0.6545)	(0.6991; 0.9175)	(0.6065; 0.8781)
ρ_μ	0.7282	0.8264	0.6350	0.5971
	(0.5689; 0.9050)	(0.6688; 0.9715)	(0.3859; 0.8817)	(0.3324; 0.8581)
ρ_ω	0.9056	0.8500	0.6903	0.8334
	(0.8474; 0.9670)	(0.7619; 0.9425)	(0.5495; 0.8337)	(0.7445; 0.9247)
ρ_ρ	0.6644	0.7436	0.6125	0.4593
	(0.4277; 0.8941)	(0.4578; 0.9617)	(0.4234; 0.8164)	(0.2197; 0.6984)
σ_χ	0.0082	0.0132	0.0149	0.0349
	(0.0068; 0.0095)	(0.0109; 0.0155)	(0.0083; 0.0232)	(0.0208; 0.0489)
σ_A	0.0081	0.0163	0.0087	0.0082
	(0.0068; 0.0093)	(0.0138; 0.0186)	(0.0074; 0.0100)	(0.0070; 0.0095)
σ_μ	0.0308	0.0348	0.0246	0.0333
	(0.0162; 0.0428)	(0.0145; 0.0503)	(0.0124; 0.0378)	(0.0132; 0.0560)
σ_ω	0.5021	0.4451	0.2511	0.2401
	(0.2975; 0.7164)	(0.2349; 0.6589)	(0.1145; 0.4209)	(0.1165; 0.3584)
σ_ρ	0.0367	0.0491	0.0618	0.0715
	(0.0146; 0.0579)	(0.0138; 0.0853)	(0.0376; 0.0846)	(0.0397; 0.0991)

Table 2 Posterior means and the 90% HPDIs

The main focus of this contribution is to evaluate the labour market flexibilities in V4 countries. Surprisingly, the bargaining power of the workers is almost zero in the Czech and Slovak Republic. The worker in Hungary and Poland are stronger but their power does not overweight the one of firms. From this point of view, all the labour markets might be assessed as relatively flexible because the wages are not set in contradiction with the productivity. Costs of vacancy posting are one of many signs of lower labour market flexibility. Estimated parameters ψ are greater than one in case of Czech, Slovak and Polish labour markets. These values mean increasing vacancy posting costs. Hungary may seem to have

constant costs implying good conditions in vacancy posting. Some kind of dispersion might be seen in the case of parameter ξ. Higher values in Slovakia and Hungary indicate higher sensitivity of successful matching process considering the changes in unemployment. The case of Poland shows relatively similar weight of unemployment and vacancies in the match pairing process.

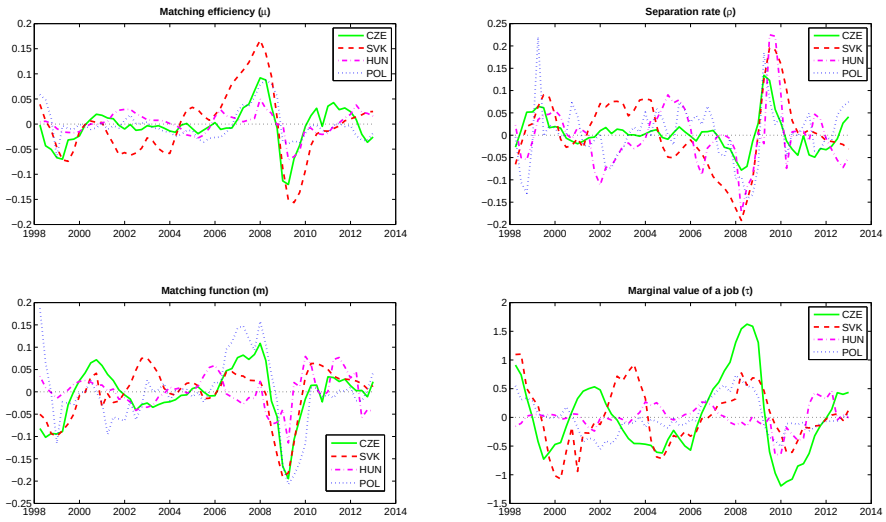

Figure 1 Trajectories of selected (smoothed) variables

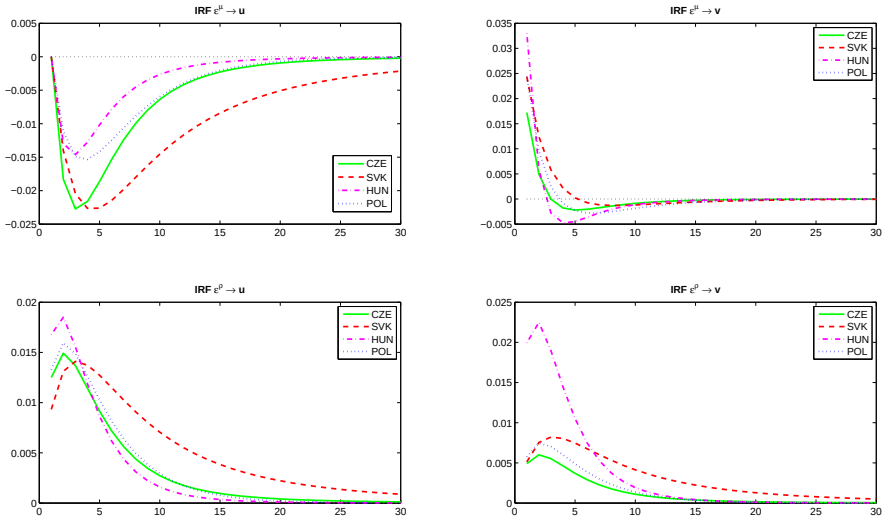

Figure 2 Selected impulse-response functions (IRFs)

Figure 1 shows historical development of selected smoothed variables. In particular, we can see the trajectories of the match efficiency, μ, of the separation rate, ρ, of the matching function, m, and of the marginal value of the jobs, τ. Separation rate as a time-varying parameter is an important feature in all economies and does not seem to be constant within the analysed period. Its variability plays a role in the labour market dynamics because its adjustment helps the economy to recover from the disequilibrium states caused by exogenous shocks. Czech labour market may be thus viewed as less flexible due the lower variability in separation rate. As for the matching efficiency, there were only little positive changes in the Czech Republic and Poland in the period from 2000 to 2007. Negative development in Slovakia in the period 2001-2004 may be connected with unfavourable labour market policies in this period. It should be noted, that the high negative correlation between μ and ρ (Czech Republic -0.87, Slovakia -0.84) is not a general feature of the model. We have found correlation of -0.71 in Hungary and -0.57 in Poland. High negative correlations support the evidence that the low match efficiency (i.e. a lower ability of the labour market to pair the vacancies with unemployed persons) on the labour market is accompanied by

higher separation rate that increase the flows from employment to unemployment. The V4 economies are similar in their reactions in matching function in the period of economic slowdown starting in 2008. But, this pattern is not so convincing in case of Hungary. Stability in marginal value of job in Hungary illustrates the institutional quality of Hungarian labour market. The rest of V4 countries suffered from the uncertainty with evaluation of workers and their contribution to the firms. This characteristic discourages the employees from new jobs creation and labour market flexibility is undermined. Figure 2 allows us to evaluate labour market flexibility as the ability of the economy to absorb exogenous shocks. We focus on match efficiency shock and the separation rate shock and their influence on unemployment gap and gap in vacancies. With regard to the amplitude and the speed of adjustment we could conclude that the short lasting responses are typical for the Hungary. Slovakia seems to be the least flexible. Moreover, the strong response of vacancies creation to the separation rate shock in Hungary indicates high ability of this economy to create new jobs as a result of increasing unemployment. The flows from one to another job are thus very flexible. Polish and Czech labour markets are very similar in their shock response dynamics.

4 Conclusion

All examined economies are similar to the extent of the low wage bargaining power of the workers and increasing vacancy posting cost (excluding the constant vacancy post in Hungary). As Lubik [5] pointed out in U.S. labour market, the lower bargaining power is accompanied by the increasing vacancy costs preventing excessive vacancy creation. But, our results show that this is not a general feature of the model. With respect to our results we are able to observe low bargaining power and constant vacancy costs (see estimates for Hungary). Although only a part of the results has been presented, it appears that the labour market in Hungary is the most flexible and the expected recovery of the economy may be accompanied with a quick return of the unemployment to its pre-crisis level. On the other hand, the Slovak labour market seems to be more rigid and the future development may be not too much positive. The estimated model provides a satisfactory description of employment flows in both economies, and is able to replicate observed data and some of its basic properties. Extending our analysis to the other CEE countries and the rest of EU countries will be further step of our research. We hope that this approach may offer a new way to compare and classify European labour markets. As stated in Němec [7], knowledge of the degree of labour market flexibility is important for considering the intended extents of future labour market reforms and for the evaluation of the reforms up to now.

Acknowledgements

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Competitiveness of Czech and Slovak NUTS 2 regions measured by AHP and equal importance method

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Abstract: Problem of regional competitiveness measurement is current in period when ways leading to economic growth are sought. There are many possibilities how to approach competitiveness measurement and assessment. One of them are multicriteria decision-making methods. Aim of this contribution is to determine mutual ranking of Czech and Slovak NUTS 2 regions in closed programming period of 2000 – 2006 with respect to selected quantitative criteria which are some of macro-economic indicators: gross domestic product, gross fixed capital formation, gross domestic expenditures on research and development, net disposable income, rate of employment, knowledge intensive services and patents. For this purpose two methods of multicriteria decision-making are used to determine weights of criteria: analytic hierarchy process which takes subjective opinions into consideration, and equal importance method which is unbiased.

Keywords: competitiveness, region, analytic hierarchy process, European Union

JEL Classification: C49, C61, O18, P48, R11, R58

AMS Classification: 90B50, 90C29, 62H99, 62P20, 91B50, 91B06

1 Introduction

The competitiveness has become quite a common term used in many professional and non-specialized publications. Nevertheless, evaluation of the competitiveness issue is not less complicated. Effectively analysed competitiveness means to be based on a defined concept of competitiveness. For evaluation of regional competitiveness, we face the problem of the basic concept and definition of competitiveness due to absence of a consistent approach of its definition. In the absence of mainstream views on the assessment of competitiveness, there is sample room for the presentation of individual approaches to its evaluation. In our paper we will examine the possibility of evaluation the competitiveness of the regions of Czech Republic and Slovakia at NUTS 2 (Nomenclature of Territorial Units Statistics) level in terms of equal importance method and analytic hierarchy process. The level of NUTS 2 regions for evaluation of competitiveness seems to be legitimate especially because of the fact that European Commission accents the level of regional units from aims of economic and social cohesion view and realization of structural aid in the EU member states. When making concept of suitable evaluation tools of national and regional competitiveness it is necessary to suggest not only difficult but also simple methods which enable quick evaluation of competitiveness by accessible tools. Database for our paper has been taken from OECD Regional Statistics – eLibrary system. Paper analysis includes last programming period (from 2000 to 2006) of European Union – in case of Czech Republic and Slovakia.

2 Evaluation of competitiveness

Creation of competitiveness evaluation system in terms of the EU is greatly complicated by heterogeneity of countries and regions and also by own approach to the original concept of competitiveness. Comparing instruments for measuring and evaluation of competitiveness in terms of the EU is not a simply matter. Evaluation of regional competitiveness is determined by the chosen territorial region level, especially in terms of the European Union through the NUTS – in our paper we apply NUTS 2 level, but we can also apply different NUTS level – e.g. NUTS 3 level. Evaluation of competitiveness in terms of differences between countries and regions should be measured through complex of economic, social and environmental criteria that can identify imbalanced areas that cause main disparities [5] and we can use various methods (e.g. DEA [4] and criteria for evaluation of competitiveness).

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3 Selected macroeconomic indicators

First represented entrance criterion is **rate of employment** in age group 20 – 64 years (ER). From the economic relevance rate of employment is important in accordance to number of economic active people in above mentioned age group. Employed population consists of those persons who during the reference week did any work for pay or profit for at least one hour, or were not working but had jobs from which they were temporarily absent.

Gross domestic product (GDP) was chosen as it is one of the most important macroeconomic aggregates which is simultaneously suitable basic for competitiveness assessment of the country, but also for the regional level, where also NUTS 2 regions belong. On the other hand we can also evaluated group of least development countries or regions [2]. It is obviously not always valid that with increasing level of GDP [11] (i.e. increasing efficiency of regions) also the rate of obtained competitiveness/competition advantage grows.

Gross domestic expenditures on research and development (GERD) are sources for further economic growth increasing as stimulation of basic and applied research creates big multiplication effects with long-term efficiency and presumptions for long-term economic growth in economics. R&D is defined as creative work undertaken on a systematic basis in order to increase the stock of knowledge, including human knowledge, culture and society and the use of this stock of knowledge to devise new applications.

Gross fixed capital formation (GFCF) due to international accounting is a basic part of gross capital (capital investments), in which is also the change of inventories and net acquisition of valuables included. According to ESA 95 methodology GFCF consists of the net assets acquisition minus decrease of fixed assets at residential producers during the time period plus certain increasing towards the value of non-produced assets originated as a consequence of production activity of producers or institutional units. It is estimated in purchase price including costs connected with instalment and other costs on transfer of the ownership. Fixed assets are tangible or intangible/invisible assets produced as the output from production process and are used in production process repeatedly or continuously during the one-year period. It is an index of innovating competitiveness which enables to increase production on modern technical base.

Knowledge intensive services (KIS) as % of total employment are among the fastest growing and dynamic sectors of the economy. Knowledge intensive services are characterized by high degrees of contact intensity and a high number of variants. Typical examples are professional business services like consulting, IT and marketing. Knowledge-intensive services are supplied mainly to final consumers, as public services (e.g. health) or private professional ones (consumer financial advice [10] or computer repair).

Net disposable income (NDI) is the result of current incomes and expenditures, primary and secondary disposal of incomes. It explicitly excludes capital transfers, real profits and loss from possession and consequences of the events as disasters. In contrast to gross disposable income, it does not cover fixed capital consumption. Disposable income (gross or net) is the source of expenditures on final consumption cover and savings in the sectors: governmental institutions, households and non-profit institutions for households.

Patents (PAT) are a key measure of innovation output, as patent indicators reflect the inventive performance of regions. Patent indicators can serve to measure the output of R&D, its productivity, structure and the development of a specific technology/industry. Among the few available indicators of technology output, patent indicators are probably the most frequently used. Patents are often interpreted as an output indicator; however, they could also be viewed as an input indicator, as patents are used as a source of information by subsequent inventors.

4 Used multicriteria decision-making methods

In this contribution we apply two multicriteria decision-making (MCDM) methods. The equal importance method is very simple and purely unbiased: all objects are regarded to be equally significant. On the other hand, the analytic hierarchy process takes subjective preferences (based on expert experience) into consideration.

4.1 Equal importance method

Equal importance method is used when there is no ordinal or cardinal information about evaluated objects. No object is more preferred over another one by the decision-maker; there is no information about importance of objects; objects are not ordered.

Let us have n objects which are considered to be equally important. Then weight of i-th object is given by:

$$w_i = \frac{1}{n}, i = 1,2,\ldots,n, \tag{1}$$

satisfying the normalization condition $\sum_{i=1}^{n} w_i = 1$. It is possible to write this result as a priority vector $w_{EIM} = (w_1, w_2, \ldots w_n)$.

4.2 Analytic hierarchy process

Analytic hierarchy process (AHP) is method which allows including both quantitative and qualitative criteria and is used to determine priorities (weights). Pair-wise comparisons matrices which entries are results of pair-wise comparisons are characteristic for this method, see e.g. [6], [7]. The mutual dependence of criteria is not considered in AHP. There are some ways how to allow for dependence, see e.g. [3].

The essence of pair-wise comparison is mutual measure of all pairs of considered elements. We compare criteria among themselves or alternatives with respect to given qualitative criterion. For numerical expression of intensity of relations between compared elements Saaty created nine-point scale [9], where 1 means equality and 9 extreme difference of importance.

Data obtained through pair-wise comparisons are inserted into the pair-wise comparison matrix *A*, its entries are signed generally a_{ij}. An $n \times n$ (square) matrix is created, see Figure 1.

$$\begin{array}{c} \\ \text{element } x_1 \\ \text{element } x_2 \\ \vdots \\ \text{element } x_k \end{array} \begin{array}{c} \begin{array}{cccc} \text{element } x_1 & \text{element } x_2 & \ldots & \text{element } x_k \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix} \end{array}$$

Figure 1 General pair-wise comparison matrix

Entries of the pair-wise comparison matrix represent estimation of weight ratio of two compared elements, i.e. of criteria or alternatives with respect to qualitative criterion. These weights are not known, they are calculated in the analytic hierarchy process. If a_{ij} is an element of pair-wise comparison matrix, $a_{ij} \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, w_i is wanted weight of the element x_i, w_j is wanted weight of the element x_j for all i and j, we can write:

$$a_{ij} = \frac{w_i}{w_j}, \; a_{ij} \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}, \tag{2}$$

$$a_{ji} = \frac{1}{a_{ij}}, \; a_{ij} \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}, \tag{3}$$

$$a_{ij} \cdot a_{ji} = 1 \text{ for all } i, j = 1, 2, \ldots, n. \tag{'3}$$

Formula (3) corresponds to one of the pair-wise comparison matrix characteristic – the reciprocity.

Consistency is characteristic of pair-wise comparison matrix which expresses how much individual pair-wise comparisons are mutually consistent. This characteristic can be expressed by the following formula illustrating transitivity of pair-wise comparisons:

$$a_{ij} = a_{ik} \cdot a_{kj}, \; i, j, k = 1, 2, \ldots, n. \tag{4}$$

We have to compute the eigenvector $w_{AHP} = (w_1, w_2, \ldots, w_n), \sum_{i=1}^{n} w_i = 1$ corresponding to the maximal eigenvalue $\lambda_{\max}$ of the pair-wise comparison matrix *A* to determine result element priorities of the given matrix. Eigenvector w_{AHP} contents information about result priorities.

$$Aw_{AHP} = \lambda_{\max} w_{AHP} . \tag{5}$$

Pair-wise comparison matrix is square, nonnegative and irreducible. These characteristics ensure existence of maximal eigenvalue $\lambda_{\max}$ and corresponding positive eigenvector [8].

It is possible to measure the consistency, respective inconsistency of pair-wise comparisons using consistency index $I_c(A)$ of pair-wise comparison matrix *A*:

$$I_c(A) = \frac{\lambda_{\max} - n}{n-1} . \tag{6}$$

In case of consistent pair-wise comparison matrix $I_c(A) = 0$.

As it follows from formula (6) the consistency index $I_c(A)$ depends on dimension of the matrix. Therefore the consistency ratio $CR_c(A)$ was implemented. It is defined as ratio of consistency index $I_c(A)$ and *random consistency index* $R_c(n)$ – mean value of $I_c(A)$ calculated for randomly generated positive reciprocal matrices of dimension *n*. Values of $R_c(n)$ are published e.g. in [9]. It is formulated as follows:

$$CR_c(A) = \frac{I_c(A)}{R_c(n)} . \tag{7}$$

Generally the maximal acceptable value of the consistency ratio is 10 %.

4.3 Synthesis

The above mentioned methods define the procedure of calculating weights of criteria or weights of alternatives with respect to a qualitative criterion. To rank alternatives it is necessary to calculate overall weights of alternatives. Technique of normalization is appropriate to determine weights of alternatives with respect to quantitative criterion.

If we term the value of *j*-th alternative with respect to maximizing criterion f_i as $e_j(f_i)$, then weight of *j*-th alternative with respect to this criterion $v_j(f_i)$ is:

$$v_j(f_i) = \frac{e_j(f_i)}{\sum_{j=1}^{n} e_j(f_i)} , \; i = 1, 2, \ldots, m. \tag{8}$$

The required result, i.e. weights of alternatives, we obtain through synthesis of information about weights of criteria and weights of alternatives with respect to given criterion. If weight of *i*-th criterion is w_i and weight of *j*-th alternative with respect to criterion f_i is $v_j(f_i)$, the overall weight E_j of *j*-th alternative is [8]:

$$E_j = \sum_{i=1}^{m} w_i \cdot v_j(f_i), \tag{9}$$

where *j*=1, 2, ..., *n*. On the basis of overall weights it is possible to rank evaluated alternatives from the best to the worst. Of course the best alternative gains the highest weight and vice versa.

5 Application

Our goal is to determine competitiveness of Czech and Slovak NUTS 2 regions ranking these regions according to achieved macroeconomic results in years 2000 – 2006. We apply above mentioned MCDM methods to determine weights of criteria.

We take 7 macroeconomic indicators in role of criteria into consideration: employment rate, gross domestic product, gross expenditures on research and development, gross fixed capital formation, knowledge intensive services, net disposable income and patent indicators. (In pair-wise comparison matrix and in priority vectors criteria are in this order.) All criteria are maximizing.

The evaluated objects (alternatives) are Czech and Slovak NUTS 2 regions: Praha, Střední Čechy, Jihozápad, Severozápad, Severovýchod, Jihovýchod, Střední Morava, Moravskoslezsko, Bratislavský kraj, Západné Slovensko, Stredné Slovensko, Východné Slovensko.

The pair-wise comparison matrix for criteria is following:

$$A = \begin{bmatrix} 1 & 1/8 & 1/3 & 1/5 & 1/3 & 1/4 & 3 \\ 8 & 1 & 7 & 5 & 8 & 3 & 9 \\ 3 & 1/7 & 1 & 1/3 & 2 & 1/2 & 5 \\ 5 & 1/5 & 3 & 1 & 4 & 1/2 & 5 \\ 3 & 1/8 & 1/2 & 1/4 & 1 & 1/3 & 4 \\ 4 & 1/3 & 2 & 2 & 3 & 1 & 5 \\ 1/3 & 1/9 & 1/5 & 1/5 & 1/4 & 1/5 & 1 \end{bmatrix}$$

Figure 2 Pair-wise comparison matrix A

According to (1) we calculate weights of criteria w_{EIM} = (0.143, 0.143, 0.143, 0.143, 0.143, 0.143, 0.143). We compute $\rho(A)$ = 7.51. By (6) and (7) and considering $R_c(7)$ = 1.35 [9] we gain $CR_c(A)$ = 0.06 < 0.1 with priority vector w_{AHP} = (0.039, 0.456, 0.088, 0.157, 0.064, 0.171, 0.025).

Applying (8) and (9) we obtain overall weights of NUTS 2 regions which are base for their ranking mentioned in Table 1 and Table 2:

Region/year	2000	2001	2002	2003	2004	2005	2006	average
Praha	1	1	1	1	1	1	1	1
Střední Čechy	3	3	3	3	3	3	3	3
Jihozápad	8	5	6	7	6	6	7	6
Severozápad	9	9	10	9	10	10	10	10
Severovýchod	5	6	5	4	4	5	5	5
Jihovýchod	4	4	4	5	5	4	4	4
Střední Morava	6	7	8	8	8	9	8	8
Moravskoslezsko	7	8	7	6	7	8	6	7
Bratislavský kraj	2	2	2	2	2	2	2	2
Západné Slovensko	11	10	9	10	9	7	9	9
Stredné Slovensko	10	11	11	12	12	11	11	11
Východné Slovensko	12	12	12	11	11	12	12	12

Table 1 Ranking of Czech and Slovak NUTS 2 regions by EIM

Region/year	2000	2001	2002	2003	2004	2005	2006	average
Praha	1	1	1	1	1	1	1	1
Střední Čechy	3	3	3	3	3	3	3	3
Jihozápad	6	4	5	5	5	5	5	5
Severozápad	9	9	9	7	9	10	10	9
Severovýchod	5	6	6	6	6	6	7	6
Jihovýchod	4	5	4	4	4	4	4	4
Střední Morava	7	7	7	8	8	8	8	8
Moravskoslezsko	8	8	8	9	7	7	6	8
Bratislavský kraj	2	2	2	2	2	2	2	2
Západné Slovensko	10	10	10	10	10	9	9	10
Stredné Slovensko	11	11	11	11	11	11	11	11
Východné Slovensko	12	12	12	12	12	12	12	12

Table 2 Ranking of Czech and Slovak NUTS 2 regions by AHP

For both methods, the first, second and third position does not change in given period – three most competitive regions are Praha, Střední Čechy and Bratislavský kraj. Oppositely, last two positions correspond to Stredné Slovensko and Východné Slovensko. Moravskoslezský kraj [1] has been changed own position during referencing period very significantly. From the methodological point of view, we would like to stress that our paper doesn't seek reasons of these changes inside of regions.

6 Conclusion

In this paper we applied two multicriteria decision making methods – equal importance method and analytic hierarchy process – in evaluation of regional competitiveness on NUTS 2 level in the Czech Republic and Slovakia. These methods were used to calculate weights of criteria and according to them ranking of regions was determined. One approach was purely unbiased and the second one included expert estimation about weight ratios of two compared criteria. Results of both methods were compared.

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Estimation of quantitative rating systems and application to bond rating prediction

Martina Novotná[1]

Abstract. The topic of this paper falls within the area of credit risk measurement and modelling, specifically the problem of bond rating models estimation. There are various approaches that can be used to develop an automated rating system, such as econometric models based on historical data on past defaults, neural networks or structural models. The approach in this paper is based on a database of rating decisions of a certified rating agency. Using a logistic regression technique, the model to bond rating prediction can be estimated. Ordinal logistic regression can be used to investigate the relationship between rating and various independent variables such as financial ratios, macroeconomic indicators or selected qualitative factors with a potential effect on rating. In comparison with multiple logistic regressions, this approach enables to consider the nature and ordinal ranking of rating assessment. The paper aims at using the ordinal regression on selected corporate bonds to build a bond prediction model based on Moody's external rating. The methodology will be explained in the first part of this paper, the next chapters are devoted to the model estimation and interpretation of results.

Keywords: Ordinal regression, multinomial logistic regression, bond rating, modelling, prediction.

JEL Classification: C35
AMS Classification: 62H12

1 Introduction

The aim of this paper is to estimate a bond rating model and investigate the relationship between selected non-financial companies' variables and a certified bond rating. The hypothesis is set in the way that some financial indicators or other variables can be good predictors for corporate bond rating. As some authors suggest, for example Altman and Katz [1], Ang and Patel [2], or more recently Ong [6], Servigny and Renault [7], Trueck and Rachev [9] or Waagepetersen [10], the variables can be both quantitative and qualitative indicators that can be relevant to bond rating. By the relationships analysis among all the variables and the rating category of corporate bonds, it is possible to identify the main areas and variables that should be of the highest interest.

The credit rating is an important indicator on financial markets and by issuing credit assessment, rating firms help to reduce the financial asymmetry resulting from the asymmetric information among issuers, investors and financial institutions. Thus, it is clear that understanding the rating system procedure and the most significant indicators of rating is a valuable tool for making investment decisions. As Witzany [11] says, there are various methods and approaches how credit ratings can be evaluated and how credit rating models can be estimated. In this paper, the focus will be on an econometric analysis that is used to evaluate the relationships among the actual credit ratings and a set of selected variables. The purpose of this analysis is to identify variables with the highest impact on bond rating and to estimate models for bond rating prediction. For this type of problem, various multivariate statistical methods can be used, such as discriminant analysis, logistic regression analysis, neural networks or classification trees. All of these methods stand on the knowledge of current rating and combine selected independent variables to get the final predictive models. The aim of this study is to examine and quantify relationships among bond rating and other relevant data by means of ordinal logistic regression.

The study is based on 169 cross-sectional data of European and U.S. non-financial companies from oil and gas, automobile, chemical and telecommunication industry. The next paragraph describes the methodology; results and classification ability of models will be assessed in the following chapters of this paper.

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2 Description of methodology

The ordinal regression analysis is an alternative approach to multinomial logistic regression or discriminant analysis that should be used when category outcomes are ordinal, as is suggested for example by Hosmer and Lemeshow [3], Tabachnik [8], Menard [4] or Norušis [5]. The advantage of ordinal analysis is that it respects the ordinal nature of the outcome. The multinomial logistic regression can be also used for this type of problem, however it does not take into account any ordering of the values of independent variable. In the following text, the method of ordinal logistic regression will be concisely described.

As mentioned in the above text, ordinal logistic models explicitly incorporate the ordering of the dependent variables. As Menard [4, p. 197] says, different models make different assumptions about whether the dependent variable is intrinsically ordinal or reflects an underlying continuous interval or ratio variable. Various models can be proposed for the analysis of ordinal dependent variables, such as the cumulative logit model, the continuation ratio logit model, the adjacent categories logit model or the stereotype model. Since the cumulative logit model is the most widely used logistic regression model, it will be explained in more detail in this chapter.

Menard [4, p. 197] describes the model in the form,

$$\eta = \alpha + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_K X_K, \text{ where} \quad (1)$$

η represents a measurement of an underlying continuous interval or ratio variable. It cannot be directly observed, but can be transformed into *Y*, an ordinal variable according to the following rules:

- Y = 1 (the first category) when η is less than some threshold value (θ_1),
- Y = 2 when $\theta_1 \leq \eta < \theta_2$, where θ_2 is the second threshold value,
- Y = 3 when $\theta_2 \leq \eta < \theta_3$, where where θ_3 is the third threshold value, and so on, until the last category (I) is reached, then
- Y = I when $\theta_{I-1} < \eta$.

The effect of the predictors is assumed to be the same in the equal slopes model for each category for *Y*, or for each range between two consecutive threshold values for η. As Menard says [4, p. 197], notation for the cumulative logit model varies. The intercept format can be expressed in the following form,

$$\ln[P\ (Y > i)/P(Y \leq i)] = \alpha_i + \beta_{i1} X_1 + \beta_{i2} X_2 + \cdots + \beta_{iK}, \quad (2)$$

and for the format with thresholds, we can use the equation,

$$\ln[P\ (Y \leq i)/P(Y > i)] = \theta_i - (\beta_{i1} X_1 + \beta_{i2} X_2 + \cdots + \beta_{iK} X_K). \quad (3)$$

In both formulas (2) and (3), symbols *k* denote the predictors, where k = 1, 2,…, K, and the subscripts *i* denote the categorical values of the dependent variable, where i = 1, 2, …, I.

The usual assumption in the cumulative logit model is that the coefficients β for the predictors are equal across logistic functions and across cases, resulting in equation [4, p. 203],

$$\ln[P\ (Y > i)/P(Y \leq i)] = a_i + b_1 X_1 + b_2 X_2 + \cdots + b_K, \quad (4)$$

where the coefficients *a* denotes to the estimates of the intercepts $\alpha_1, \alpha_2, \ldots, \alpha_{I-1}\ \alpha_i\ \alpha_i\ \alpha_i$ and coefficients *b* are the coefficients by which the predictors are multiplied to obtain the predicted values of the dependent variable.

3 Characteristics of the sample and data

The analyses are based on the data sample of 169 non-financial companies from US, Canada and European countries. Only firms with Moody's bond rating assessment have been considered in this study, the relevant data come from Moody's official websites[2], companies' annual reports and Yahoo! Finance websites[3] of business finance, stock market, quotes and news. For the reasons of calculations, original rating categories have been re-coded as presented in the Table 1. The first three highest categories have been merged together because of a small number of representative companies, which could negatively affect results and stability of models.

[2] http://www.moodys.com/
[3] http://finance.yahoo.com/

Rating category	Rating code	Number of cases	Marginal percentage
Aaa, Aa, A	4	41	24.3%
Baa	3	39	23.1%
Ba	2	60	35.5%
B	1	29	17.2%
Total		**169**	**100%**

Table 1 Sample structure

The selection of independent variables should be thoroughly considered, because the set of input variables can substantially affect results, specifically predictive ability and stability of final models. In addition to all the assumptions with respect to statistical procedures, the analysts should take into account also the practical point of view. The variables, or predictors should be easily obtained or calculated, should be based on publically available data and their use in the models should be clear. Only these conditions can enable a wider use of such tools and predictive models.

There is a variety of financial indicators that can be used in the analysis. The predictors should reflect profitability, activity, liquidity and capital structure of companies and all of them should have an evident relationship to rating. Basically, the data should be checked for multicollinearity. In this study, after assessment of multicollinearity and considering an economic rationality, the following nine variables have been included.

Financial variable	Symbol	Financial variable	Symbol
Equity to total assets	EQTA	Return on equity	ROE
Short term debt to total assets	STDTA	Current liquidity	CURR
Interest coverage	INTCOV	Days inventory outstanding	DAYSINV
Return on assets	ROA	Logarithm of total assets	LogTA
Return on capital employed	ROCE		

Table 2 Financial predictors

In addition to financial variables mentioned above, two qualitative independent variables will be considered in this study, such as sector and country. The companies fall within four industry sectors: Oil and gas (O&G), automobile (AUT), chemical (CHEM), and telecommunication (TEL) industry. The majority of companies come from O&G (60 %), following by TEL (16%), CHEM (13%) and AUT (11%).

The second categorical independent variable will be the country of each company. Totally, 21 countries will be covered in the analysis.

4 Modelling bond rating

The analysis aims at investigating the relationships between rating and selected variables, the main objective is to estimate models that can be used for bond rating prediction. The ordinal regression will be used on data sample to estimate bond rating models. In this case, the ordinal outcome is bond rating that can be ranked from the worst to the best investment quality of bonds (B → Aaa). As it is described in the third chapter, this study will consider totally four rating grades. The event of interest is observing a particular rating grade, or less. Ordinal model considers a set of dichotomies, for example grade (1) versus grades (2, 3 or 4), grades (1 or 2) versus (3 or 4) and grades (1, 2 or 3) versus grade (4). For rating, we model the following odds,

$$\theta_1 = \frac{prob\ (rating\ 1)}{prob\ (rating\ greater\ than\ 1)}, \quad \theta_2 = \frac{prob\ (rating\ 1\ or\ 2)}{prob\ (rating\ greater\ than\ 2)}, \quad \theta_3 = \frac{prob\ (rating\ \ 1\ or\ 2\ or\ 3)}{prob\ (rating\ greater\ than\ 3)}. \tag{5}$$

The last category does not have an odds associated with it since the probability of ranking up to and including the last grade is 1. By using the ordinal regression, we estimate just one set of regression coefficients compared to multinomial logistic regression.

4.1 Models with categorical independent variables

The first analysis will examine whether there is a significant relationship between rating and a sector. The model is specified by four dependent categorical variables (bond rating) and one independent categorical variable (sector). The companies fall within four sectors, AUT, CHEM, O&G and TEL. Parameter estimates for the model are shown in the table below (Tab 3).

		Estimate	Std. error	Wald	df	Sig.
Threshold	[GROUP4 = 1,00]	-1,716	,381	20,320	1	,000
	[GROUP4 = 2,00]	-,645	,361	3,193	1	,074
	[GROUP4 = 3,00]	1,079	,370	8,516	1	,004
Location	[Sector=AUT]	-,313	,552	,322	1	,570
	[Sector=CHEM]	-,011	,516	,000	1	,984
	[Sector=O&G]	-,876	,398	4,847	1	,028
	[Sector=TEL]	0[a]	.	.	0	.

Table 3 Parameter estimates

The estimates labeled thresholds are the α_j's, the intercept equivalent terms [5, p. 73]. The estimates referred to location are the coefficients for the predictor variable. The number of coefficients is one less than the number of categories of the variable. Sector TEL is the reference category and has a coefficient of 0. The probability of an event, or rating assignment, is defined in terms of cumulative probabilities. In this model, the logit function is applied. The form of logit function is $\ln\left(\frac{\gamma}{1-\gamma}\right)$, where γ is the cumulative probability. To compare observed and expected counts, the predicted probabilities for firms in each sector must be calculated. For example, for those in sector TEL, $\beta = 0$, prob (rating 1) $= \frac{1}{1+e^{1,716}} = 0{,}1524$. It means that the probability that the firm from sector TEL will be assigned rating 1 is 15,24%. The probability of rating 1 or 2 will be calculated as prob (rating 1 or 2) $= \frac{1}{1+e^{0,645}} = 0{,}3441$. Based on the cumulative probabilities, the estimated probabilities for each group can be calculated, for example the probability that the firm from sector TEL will be assigned rating 2 is 19,17%. The results for other rating groups are 40,22% (rating 3) and 25,37% (rating 4). From cumulative probabilities based on equations (5), the probabilities for each sector and rating group were similarly calculated. The summary of estimated probabilities, expected and observed counts is shown in the following table (Tab 4).

		Rating group				
		1	2	3	4	Total
AUT	Observed counts	2	7	5	4	18
	Expected counts	3,554	3,968	6,894	3,584	18
	Estimated probability	0,1975	0,2204	0,3830	0,1991	1
CHEM	Observed counts	2	6	10	5	23
	Expected counts	3,538	4,434	9,238	5,790	23
	Estimated probability	0.1538	0.1928	0.4017	0.2517	1
O&G	Observed counts	34	20	33	14	101
	Expected counts	30,458	25,848	32,162	12,532	101
	Estimated probability	0.3016	0.2559	0.3184	0.1241	1
TEL	Observed counts	3	6	12	6	27
	Expected counts	4,116	5,177	10,856	6,851	27
	Estimated probability	0,1524	0,1917	0,4021	0,2537	1

Table 4 Number of observed and estimated counts

Based on observed and estimated frequencies and using the Person and Deviance goodness-of-fit measures[4], the model fits well. To assess the overall fitting of the model, we compare the model with and without predictors. This test is based on the difference between the two log-likelihoods and confirms that the model with

[4] If the model fits well, the value of each statistic is small and the observed significance is large [5]. The significance of model is 0,273 (Pearson) and 0,278 (Deviance).

predictors is better than the model without predictors[5]. It means that including the category sector in the model has an important influence on bond rating prediction.

Finally, the other categorical independent variable, country, was included in the analysis. The overall results suggest that this additional category does not have any significant relationship to rating and the model without predictors is as good as with the predictors. For this reason, no more attention will be paid to this model and the category country will not be considered in further analysis.

4.2 Models with categorical and quantitative independent variables

In addition to categorical variable sector, nine quantitative variables were included in the model (see Tab. 2). Two types of link function were used, the cauchit and logit function. The cauchit function can be described in the form $\tan(\pi(\gamma - 0{,}5))$ and usually is applied on outcome with many extreme values [5]. The parameter estimates are attached in Annex. As it is evident from the table (Annex), the financial variables STDTA, INTCOV, ROA, ROCE, LogTA appear to be highly related to the rating, the estimates of parameters are statistically significant at .05 significance level.

	Cauchit	Logit
Group 1	51,2%	63,4%
Group 2	33,3%	0%
Group 3	68,3%	70%
Group 4	24,1%	31%

Table 5 Predicted rating category within groups

Both models fits well, the overall fitting tests show that both models are better than those without predictors. The total classification ability of both models does not differ substantially, it is 48,5% (cauchit) and 45,6% (logit). However, the classification ability within the rating groups is different, as it is evident in the table provided (Tab. 5). Although the logit model fails in the prediction of rating grade 2, it is much better in the prediction of other rating categories. Mainly, the classification ability to predict category 1 is much better for logit model. This is very important, because group 1 represents rating B that already has some speculative attributes. For this reason, logit model should be preferred to the cauchit model. In addition, we should also consider results of model fitting tests. As mentioned above, both models fit well, however when we compare Chi-squares of -2LogLikelihood and measures of strength of association (Pseudo R-squares)[6], the logit model achieves better results. It supports the first conclusion that the logit model is more suitable for bond rating prediction.

5 Conclusion

The topic of this paper is focused on the area of credit risk measurement and modelling, specifically the problem of bond rating models estimation. There are various approaches that can be used, for example econometric models based on historical data on past defaults, neural networks or structural models. In this paper, the econometric bond rating model is estimated using a database of rating assessments of a certified rating agency, Moody's. For this purpose, ordinal logistic regression was used to investigate the relationship between rating and various independent variables such as financial ratios, and selected qualitative factors with a potential effect on rating. In comparison with multiple logistic regressions, this approach enables to consider the nature and ordinal ranking of rating assessment.

The data sample covers non-financial companies from oil & gas, chemical, telecommunication and automobile industry. The dependent variable in the analysis was rating, specifically four selected categories of Moody's rating. The independent variables include nine quantitative variables reflecting profitability, activity, liquidity, interest coverage or a size of companies. Then, two qualitative indicators were used in the analysis. The results of the ordinal logistic regression suggest that the industry plays an important role in rating prediction, while the country is not a significant factor. Regarding the financial indicators, STDTA, INTCOV, ROA, ROCE and LogTA appear to be highly correlated to rating. Two types of link function were used, the cauchit and logit function. Both models fits well, the total classification ability of both models does not differ substantially, however the classification ability within the rating groups is different. Further analysis suggests that the logit

[5] The Chi-square is 8,144, df = 3, Sig. = 0,043.
[6] Cox and Snell, Nagelkerke, McFadden

model is more suitable for bond rating prediction. Its classification ability of a marginal rating category B is better and the results of model fitting tests confirm this conclusion.

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Annex

		Cauchit					Logit				
		Estimate	Std. error	Wald	df	Sig.	Estimate	Std. error	Wald	df	Sig.
Threshold	[GROUP4 = 1,00]	-2,267	1,093	4,300	1	,038	-2,679	1,117	5,754	1	,016
	[GROUP4 = 2,00]	-,946	1,052	,807	1	,369	-1,328	1,102	1,451	1	,228
	[GROUP4 = 3,00]	1,513	1,068	2,008	1	,156	,930	1,101	,714	1	,398
Location	EQTA	1,821	1,182	2,376	1	,123	1,507	1,183	1,623	1	,203
	STDTA	11,360	3,112	13,326	1	,000	8,787	2,753	10,188	1	,001
	INTCOV	,075	,026	8,671	1	,003	,091	,025	13,703	1	,000
	ROA	93,701	27,294	11,785	1	,001	72,226	25,234	8,192	1	,004
	ROCE	-75,496	22,140	11,628	1	,001	-57,498	20,651	7,752	1	,005
	CURR	,021	,015	2,104	1	,147	,020	,011	3,406	1	,065
	LogTA	-,436	,128	11,506	1	,001	-,432	,123	12,318	1	,000
	ROE	-,802	,946	,719	1	,397	-,590	,966	,373	1	,541
	DAYSINV	-,001	,003	,176	1	,674	-,001	,004	,061	1	,805
	[Sector=AUT]	-2,046	,700	8,554	1	,003	-1,829	,693	6,966	1	,008
	[Sector=CHEM]	-1,009	,663	2,312	1	,128	-1,012	,642	2,484	1	,115
	[Sector=O&G]	-1,994	,523	14,524	1	,000	-2,096	,487	18,490	1	,000
	[Sector=TEL]	0[a]			0		0[a]			0	

Project portfolio selection using multiple criteria decision tree

Maciej Nowak[1]

Abstract. Project portfolio selection is a classical example of a dynamic multiple criteria decision making problem under risk. Decision tree is a useful tool to formulate and solve such problem. Although a lot of efforts have been done to propose procedures for identifying efficient solutions of such problems, relatively little work is dedicated to final solution identification. In the paper a new procedure for a project portfolio selection problem is proposed. Our technique takes into account multiple criteria, risk and dynamics that characterize such problems. We use decision tree and interactive procedure for identifying the optimal strategy.

Keywords: project portfolio selection, multiple criteria methods, decision tree, interactive approach.

JEL Classification: C44
AMS Classification: 90C15

1 Introduction

The decision aiding techniques proposed for the project portfolio selection problem can be roughly classified into two main groups. The prior involves techniques targeted to a wide range of practitioners, the latter – advanced decision aiding techniques proposed within operational research. In most project management textbooks, scoring methods are proposed for prioritization of projects. On the other hand a large number of more sophisticated methods are proposed in literature, including the ones in which project portfolio selection problem is formulated as a combinatorial problem, in which the set of alternatives is very large. Various meta-heuristic techniques are proposed for solving it, e.g. ant colonies [2]. It is often pointed out that uncertainty must be taken into account while considering the project portfolio selection problem. It is an impulse to use methods based on stochastic and fuzzy approach. The prior is used, for example, by Gutjahr and Reiter [5]. Fuzzy methodology was applied, among others, by Ravanshadnia et al. [8] and Ahari et al. [1].

Works mentioned above ignore the dynamics of the project portfolio selection. In real-life, however, the portfolio construction is a "never-ending story". It is especially true for companies applying a so-called "project management style of business". Most of their revenues comes from implementation of various projects. The management has to make decisions affecting portfolio structure sequentially at different period of time. As the results are uncertain, the problem they face is a dynamic decision making problem under risk.

The decision tree is a useful tool to formulate and solve such problems. In classical single criterion approach expected value is used for evaluating various strategies. A multiple criteria decision tree was analyzed by Haimes et al. [6], who proposed a method for generating the set of efficient solutions. Frini et al. [3] solve the multi-criteria decision tree problem without generating the set of all efficient solutions. Their approach combines the advantages of decomposition with the application of multi-criteria decision aid (MCDA) methods at each decision node.

In this paper a multiple criteria decision tree is employed for solving a dynamic project portfolio selection problem. The procedure consists of two main steps. First, efficient strategies are identified. Next an interactive procedure is used to identify the final solution. The scenario of the dialog with the decision maker is based on the approach used in Interactive Multiple Goal Programming (IMGP) proposed by Spronk [9].

2 Project portfolio selection problem

Projects are assumed to be the main tool for implementing and achieving the strategic goals of organization. Intense global competition forces many organizations to look for new methods of management. Continuous innovation, renewal, and organizational learning are regarded to be vital for survival [4]. As a result flexible and project driven organizations are considered to be market leaders.

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There is general consensus that the key to success in managing a portfolio of projects is to choose the right projects at the right time [7]. Thus, the process of projects and programs selection is considered to be the main component of the portfolio management system. Is should be accompanied by the portfolio review procedure that is started periodically to identify the projects or programs which should be terminated.

While in the case of a single project, or even a program, the goal should be clearly defined, the situation changes when a project portfolio is analyzed. Commonly it is assumed that the portfolio should be constructed in such a way that maximizes the possibility to achieve organization's strategic objectives. This lead us directly to the conclusion that we are faced with a multiple criteria decision making problem.

The primary objective of any project is to increase the value of the company. Thus, for most managers financial criteria are the preferred method to evaluate projects. Net Present Value, Internal Rate of Return, Payback Period, and the Rate of Return are well-known measures that are used most often. Additionally, however, other issues should also be considered, including [7]:

- Alignment with strategic and tactical plans;
- Balance between maintenance projects and investment projects;
- Allocation of R&D expenditures and resources;
- Allocation of marketing expenditures and resources;
- Effective use of resources;
- Probability of delivering the project on time, within budget, and with the designed work scope;
- Nonfinancial benefits.

In this paper a dynamic multiple criteria project portfolio selection is analyzed. The analysis is carried out for the time horizon of T periods. In each period the project portfolio is reviewed, to evaluate it's consistency with strategic goals. As a result the top management may decide to start new projects, but also to terminate some others that are currently executed. Even if a company has free resources, it's sometimes better to decide not to start a new project and wait for a more profitable proposal. However, while the set of projects that are currently ready for implementation is clearly defined, it is not so easy to predict what new proposals will occur in the future.

Here we assume that projects that can be started in period 1 are specified. The duration of each of this projects is equal to the time horizon T. As the available resources are not enough to start all these projects, the company must select some of them. However, it is possible, that new proposals will appear in next periods. The company must decide whether to assign all free resources to the current proposals, or leave some part of resources unused for some time and wait for better alternatives that can occur later. We propose to use multiple criteria decision tree and interactive procedure to solve this problem.

3 Multiple criteria decision tree

3.1 Definitions and notation

Decision tree is a useful tool to formulate and solve a dynamic decision making problem under risk, when a series of decisions should be made at different periods of time in order to achieve one or several objectives. In this section basic concepts of a multiple criteria decision tree are presented.

Let us assume the following notation:

T – number of periods,

$\mathbf{D}^t$ – the set of decision nodes of period t (for $t \le T$) or the terminal nodes (for $t = T + 1$):
$\mathbf{D}^t = \left\{d_1^t, \ldots, d_k^t, \ldots . d_{n_d(t)}^t\right\}$

$\mathbf{A}_k^t$ – the set of decisions (alternatives) at node d_k^t of period t ($t = 1, \ldots, T$):
$\mathbf{A}_k^t = \left\{a_{1(t,k)}^t, \ldots, a_{i(t,k)}^t, \ldots . a_{n_a(t,k)}^t\right\}$

$\mathbf{E}_{k,i(t,k)}^t$ – the set of states of nature emerging from alternative $a_{i(t,k)}^t$:
$\mathbf{E}_{k,i(t,k)}^t = \left\{e_{1(t,k,i)}^t, \ldots, e_{j(t,k,i)}^t, \ldots, e_{n_e(t,k,i)}^t\right\}$

$p_{j(t,k,i)}^t$ – probability, that the state of nature $e_{j(t,k,i)}^t$ will occur; the following condition is fulfilled for each decision made at any node:

$$\sum_{j=1}^{n_e(t,k,i)} p_{j(t,k,i)}^t = 1 \text{ for all } t = 1, \ldots, T,\ k = 1, \ldots, n_d(t),\ i = 1, \ldots, n_a(t,k) \tag{1}$$

Ω_t – transition function defining the index of the decision or terminal node $d_{k'}^{t+1}$ which is achieved from the decision node d_k^t assuming the decision $a_{i(t,k)}^t$ is made and the state of nature $e_{j(t,k,i)}^t$ occurs:

$$k' = \Omega_t\left(d_k^t, a_{i(t,k)}^t, e_{j(t,k,i)}^t\right) \tag{2}$$

Let us define a strategy s_l as a decision made in the initial decision node, and decisions made in nodes that can be achieved as a result of the decisions made in previous stages. A partial strategy $s_{l(t,k,i)}^t$ is a part of the strategy involving decision made in the node d_k^t and decisions made in the nodes that can be achieved as a result of the decisions made in previous stages. Thus, $s_{l(t,k,i)}^t$ is a composition of a decision $a_{i(t,k)}^t$ and partial strategies $s_{l'(t+1,k',i')}^{t+1}$, where values of *k'* are defined by (2).

Let $\mathbf{F} = \{f_1,...,f_m,\ldots, f_M\}$ be the set of criteria functions, and $f_m\left(d_k^{T+1}\right)$ be the value of *m*-th criterion function obtained if the process achieves the terminal node d_k^{T+1}. In this work we assume that all criteria are maximized and expected values are used for evaluating decision alternatives with respect to criteria. The following recurrent formula can be used to calculate the expected value of the *m*-th criterion function obtained if a partial strategy $s_{l(t,k,i)}^t$ is applied:

$$g_m^t\left(s_{l(t,k,i)}^t\right) = \begin{cases} \sum_{j=1}^{n_e(t,k,i)} p_{j(t,k,i)}^t f_l\left(\Omega_t\left(d_k^t, a_{i(t,k)}^t, e_{j(t,k,i)}^t\right)\right) & \text{if } t = T \\ \sum_{j=1}^{n_e(t,k,i)} p_{j(t,k,i)}^t g_l^{t+1}\left(s_{l'(t+1,k',i')}^{t+1}\right) & \text{otherwise} \end{cases} \tag{3}$$

where *k'* is defined by (2).

3.2 Identification of the efficient strategies

A partial strategy $s_{l(t,k,i)}^t$ dominates a partial strategy $s_{l'(t,k,l)}^t$ if for each criterion the following condition is satisfied:

$$g_m^t\left(s_{l(t,k,i)}^t\right) \geq g_m^t\left(s_{l'(t,k,q)}^t\right), \text{ where } q = 1, \ldots, n_a(t,k) \tag{4}$$

with strict inequality for at least one criterion. A strategy $\hat{s}_{l(t,k,i)}^t$ is efficient if it is not dominated by any other partial strategy $s_{l'(t,k,q)}^t$ ($q = 1, \ldots, n_a(t, k)$, $l' = 1, \ldots, n_s(t, k, q)$).

Let us assume that the partial strategy $s_{l(t,k,i)}^t$ is efficient, and $s_{l'(t+1,k',i')}^{t+1}$ are partial strategies such that *k'* is defined by (2) for $j = 1, \ldots, n_e(t, k, i)$. It can be shown, that if the partial strategy $s_{l(t,k,i)}^t$ is efficient, then all partial strategies $s_{l'(t+1,k',i')}^{t+1}$ are also efficient. Thus, the efficiency of $s_{l'(t+1,k',i')}^{t+1}$ is a necessary condition for the efficiency $s_{l(t,k,i)}^t$. This relation can be used to simplify the identification of the set efficient strategies.

The procedure that can be used for identifying all efficient strategies works as follows:

1. Start from the last period: $t = T$; identify partial efficient strategies for all decision nodes of period *T*.
2. Go to the previous period: $t = t - 1$.
3. For each decision node of period *t*, identify strategies satisfying the necessary condition for efficiency (take into account efficient partial strategies for all decision nodes of period $t + 1$).
4. For each decision node of period *t* identify strategies satisfying the sufficient condition for efficiency – compare strategies pairwisely using formula (4).
5. If $t > 1$ – go to 2, otherwise: stop the procedure.

In step 3 partial strategies consisted of the decisions made in node d_k^t and all combinations of efficient partial strategies identified for nodes achieved from the decision node d_k^t are identified. The efficient partial strategies identified for the decision node of period 1 are the efficient strategies for the whole decision process.

4 An interactive procedure for identifying the final solution

In real-world problems the set of the efficient strategies is usually large. Here we propose to use a simple interactive procedure for the selection of the final solution. In each iteration the potency matrix is generated and presented to the decision maker. It consists of two rows: the first groups the worst (pessimistic), and the second – the best (optimistic) values of criteria attainable independently within the set of efficient strategies. The decision maker is asked whether pessimistic values are satisfactory. If the answer is yes, he/she is asked to make a final choice. Otherwise, the decision maker is asked to express his/her preferences defining values, that the criteria should achieve, or at least indicating the criterion, for which the pessimistic value should be improved.

Let $\mathbf{S}^{(l)}$ be the set of strategies analyzed in iteration l, and $\mathbf{P}^{(l)}$ be the potency matrix:

$$\mathbf{P}^{(l)} = \begin{bmatrix} \underline{g}_1^{(l)} & \cdots & \underline{g}_m^{(l)} & \cdots & \underline{g}_M^{(l)} \\ \overline{g}_1^{(l)} & \cdots & \overline{g}_m^{(l)} & \cdots & \overline{g}_M^{(l)} \end{bmatrix} \tag{5}$$

where: $\underline{g}_m^{(l)} = \min_{s_q \in \mathbf{S}^{(l)}} \{g_m(s_q)\}$, $\overline{g}_m^{(l)} = \max_{s_q \in \mathbf{S}^{(l)}} \{g_m(s_q)\}$.

The procedure for identifying the final solution works as follows:

1. $l = 1$, $\mathbf{S}^{(1)} = \hat{\mathbf{S}}$.
2. Identify potency matrix $\mathbf{P}^{(l)}$.
3. Ask the decision maker whether he/she is satisfied with the pessimistic values. If the answer is *yes*, go to (8).
4. Ask the decision maker whether he/she would like to define the aspiration levels for criteria. If the answer is *no* – go to (6).
5. Ask the decision maker to specify aspiration levels $\tilde{g}_m^{(l)}$ for $m = 1, \ldots, M$. Identify the set of strategies satisfying decision maker's requirements:

$$\mathbf{S}^{(l+1)} = \left\{ s_l : s_l \in \mathbf{S}^{(l)}, g_m(s_l) \geq \tilde{g}_m^{(l)} \text{ for } m = 1, \ldots, M \right\} \tag{6}$$

   If $\mathbf{S}^{(l+1)} = \varnothing$ – report it to the decision maker and go to (4), otherwise go to (7).
6. Ask the decision maker to indicate the index m of the criterion, for which the pessimistic value is unsatisfactory. Identify the set of strategies for which the value of the m-th criterion exceeds the current pessimistic value $\underline{g}_m^{(l)}$:

$$\mathbf{S}^{(l+1)} = \left\{ s_q : s_q \in \mathbf{S}^{(l)}, g_m(s_q) > \underline{g}_m^{(l)} \right\} \tag{7}$$

7. Assume $l = l + 1$ and go to (2).
8. Ask the decision maker to indicate the index m of the criterion that should achieve the optimistic value, identify the final solution as a strategy maximizing the value of m-th criterion.

5 Application of the procedure in project portfolio selection

Let us consider the following example. The company reviews it's project portfolio twice a year. At the beginning of the year three projects A, B, and C are considered. The company does not have sufficient resources to implement all these projects – it is possible to start only two of them. Thus three alternatives are considered in the first period: (1) implement A and B, (2) implement A and C, (3) implement B and C, According to the company's analysts, there is relatively high probability, that in the middle of the year two new proposals (D and E) will occur. It is assumed that the company will be able to move part of the resources to one new project, but this will decrease the profit made on projects chosen initially. The decision tree describing the problem under consideration is presented on fig. 1.

The first decision made in node A means that A and B projects are selected, the second – A and C are selected, and the last one – B and C are selected. Regardless of the decision made in the first period, three states of nature can occur: (1) projects D and E are ready to start (probability = 0.42), (2) only project D is ready (probability = 0.28), (3) only project E is ready (probability = 0.18), (4) neither D, nor E are ready (probability = 0.12). In the first case three decision can be made: (1) start project D (terminal nodes B1, F1, and J1), (2) start project E (terminal nodes B2, F2, and J2), (3) – do not start neither D nor E (terminal nodes B3, F3, and J3). If the second state of nature occurs, it is possible to start project D (terminal nodes C1, G1, and K1) or not (terminal nodes C2, G2, and K2). Similarly, if the third state of nature occurs, the company can either start project E (terminal nodes

D1, H1, L1) or not (terminal nodes (D2, H2, and L2). Terminal nodes E1, I1, M1 mean that neither D, nor E is ready to start in period 2.

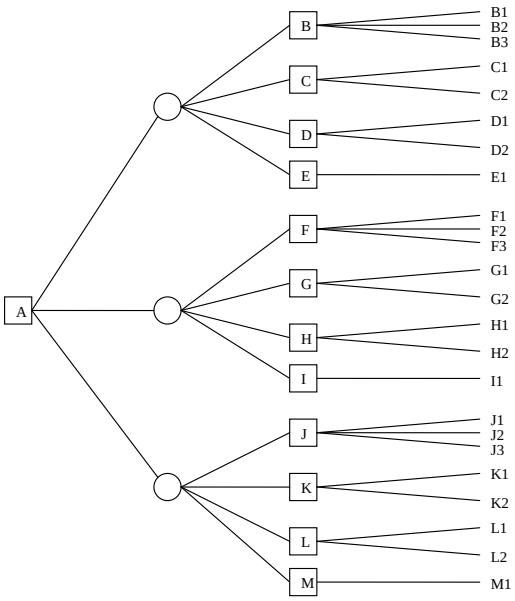

Figure 1 Decision tree of the problem

Three criteria are used to evaluate the project portfolio:

- f_1 – Net Present Value;
- f_2 – the percentage increase in sales of from new products ;
- f_3 – the percentage increase in sales in a new market;

The expected values of criteria obtained for each scenario are presented in table 1.

Terminal node	f_1	f_2	f_3	Terminal node	f_1	f_2	f_3	Terminal node	f_1	f_2	f_3
B1	120	10	2	F1	130	15	3	J1	125	20	5
B2	100	5	3	F2	95	10	6	J2	80	15	8
B3	105	0	2	F3	100	5	3	J3	95	10	5
C1	120	10	2	G1	130	15	3	K1	125	20	5
C2	105	0	2	G2	100	5	3	K2	95	10	5
D1	100	5	3	H1	95	10	6	L1	80	15	8
D2	105	0	2	H2	100	5	3	L2	95	10	5
E1	105	0	2	I1	100	5	3	M1	95	10	5

Table 1 Criteria values under the considered scenarios

First, the set of efficient strategies is identified. We start with period 2. For each decision node we analyze partial strategies, which in this case are defined by just one decision made in the last period. For example for the decision node B we see that the partial strategy $s^2_{1(2,B,1)}$ leading to the terminal node B1dominates strategy $s^2_{1(2,B,3)}$ leading to the terminal node B3. In this step 6 partial non-efficient strategies are identified. Thus, in the next stem, when we move to period 1 we must analyze only 12 strategies (instead of all 36) in order to identify the set of efficient strategies. Finally we state that 6 strategies are efficient (table 2).

	Strategy	f_1	f_2	f_3
1.	A(2) – F(1) – G(1) – H(1) – I(1)	121,0	12,0	3,0
2.	A(2) – F(1) – G(1) – H(2) – I(1)	120,1	12,9	3,5
3.	A(2) – J(1) – K(1) – L(1) – M(1)	116,0	17,0	5,0
4.	A(2) – J(1) – K(1) – L(2) – M(1)	113,3	17,9	5,5
5.	A(2) – J(2) – K(1) – L(1) – M(1)	97,1	14,9	6,3
6.	A(2) – J(2) – K(1) – L(2) – M(1)	94,4	15,8	6,8

Table 2 The set of efficient strategies

At the beginning of the dialog with the decision maker the potency matrix $\mathbf{P}^{(1)}$ is presented to him (table 3).

Value	f_1	f_2	f_3
pessimistic	94,4	12,0	3,0
optimistic	121,0	17,9	6,8

Table 3 Potency matrix $\mathbf{P}^{(1)}$ presented to the decision maker in iteration 1

Let us assume that the decision maker states that the pessimistic value of criterion f_1 is not satisfactory, but does not define any aspiration level. In such case, strategy no. 6 for which criterion f_1 obtains the worst value is rejected and the new potency matrix is constructed. Assuming that in next iterations the decision maker asks for improving criteria f_2, f_3 and again f_1, in iteration no. 5 the potency matrix $\mathbf{P}^{(5)}$ presented in table 4 is obtained.

Value	f_1	f_2	f_3
pessimistic	113,3	17,9	5,5
optimistic	116,0	17,0	5,0

Table 4 Potency matrix $\mathbf{P}^{(5)}$ presented to the decision maker in iteration 5

As the decision maker finds pessimistic values satisfactory, he is asked to select the criterion that should be maximized. Assuming that his answer is f_1, the strategy no. 3 is identified as a final solution. Thus, projects B and C should be selected in period 1. Then, if project D will be ready to start, it should be included into the portfolio, otherwise, the company should not change the portfolio.

6 Conclusions

Various criteria are usually taken into account in project portfolio selection. Moreover, in most organizations portfolio construction is a continues process. As a result this problem can be formulated as a dynamic multiple criteria decision making under risk. In this paper an interactive technique based on the decision tree is proposed for such problems. The procedure uses expected value criterion for comparing alternate strategies with respect to criteria. In future work the decision maker's attitude to risk will be taken into account by applying stochastic dominance rules.

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Scheduling Problem $1||g_i(C_i)$ as a Shortest Path Problem

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Abstract. This paper studies the scheduling problem $1||g_i(C_i)$ where g_i are non negative and non decreasing functions. Several instances of this problem are NP-hard. Literature presents a dynamic programming algorithm to solve this problem. This work shows, that the task to find an optimum operation order for $1||g_i(C_i)$ means to find a shortest path in a digraph. Application of Dijkstra algorithm and clever computer implementation reduce the computation time and memory used.

Keywords: scheduling, dynamic programming, digraph, shortest path.

JEL Classification: C610
AMS Classification: 90B35, 68R10, 90C39

1 Scheduling problem 1 || $\Sigma g_i(C_i)$

Scheduling problem $1||g_i(C_i)$ is a single machine problem with n jobs $J_1, J_2, \dots, J_n$. Each job consists of one operation. Denote by p_i the processing time of job J_i. Additional data can be given for every job J_i, e.g. due date d_i, weight w_i etc. To construct a feasible schedule for a scheduling problem means to determine for every job J_i the beginning time B_i and the completion time C_i such that $(C_i - B_i) = p_i$ and such that no two jobs are processed in the same time what means that – $\langle B_i, C_i) \cap \langle B_j, C_j) = \emptyset$ for $i \neq j$.

There is a lot of feasible schedules for a scheduling problem. The goal is to create a schedule which is the best from some point of view – which minimizes a objective function $f(C_1, C_2, \dots, C_n)$. Classification of scheduling problems uses notation $1||f$ for single machine problems with objective function $(C_1, C_2, \dots, C_n)$. Moreover, some job characteristics or constraints can be given. If several such job characteristics are given the corresponding scheduling problem is denoted by $1|\beta_1\beta_2\beta_3 \dots|f$ where $\beta_1, \beta_2, \beta_3, \dots$ are symbols describing the given characteristics. For example if the precedence relation is given to the set of jobs, this problem is indicated by the symbol $1|prec|f$.

The objective function $f(C_1, C_2, \dots, C_n)$ which is nondecreasing with respect to all variables C_i is called regular. It holds for a regular objective:

$$\text{if } C'_1 \leq C_1,\ C'_2 \leq C_2,\ \dots, C'_n \leq C_n \text{ then } f(C'_1, C'_2, \dots, C'_n) \leq f(C_1, C_2, \dots, C_n).$$

It is proved (see [2], [3], [5], [6] or [7]), that it suffices to search an optimal schedule of the problem $1||f$ with regular objective function $f(C_1, C_2, \dots, C_n)$ among schedules, where the jobs $J_1, J_2, \dots, J_n$ are processed one by one in the order given by a permutation $\pi[1], \pi[2], \dots, \pi[n]$ - in operating sequence $J_{\pi[1]}, J_{\pi[2]}, \dots, J_{\pi[n]}$. For simplicity, we write $[i]$ instead of $\pi[i]$, for two different permutations we write $[i]$, $[i]'$.

Suppose we are given a problem $1||\sum T_i w_i$ where $T_i = \max\{0,\ C_i - d_i\}$ is the tardiness of the job J_i and w_i is the penalty which has to be paid for unit of time-delay of the job J_i. The goal of optimization is to find a schedule which minimizes the total penalty for all delayed jobs.

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Although the problem $1||\sum T_i w_i$ seems to be very simple, it is known that it is NP-hard. The $1||\sum T_i$ problem has received an enormous amount of attention in the literature. For many years its computational complexity remained open, until its NP-hardness was established by 1989.

In this paper we will discuss a generalization of the problems $1||\sum T_i$ and $1||\sum T_i w_i$ as a problem $1||\sum g_i(C_i)$, where $g_i(C_i)$ are arbitrary nonnegative and non-decreasing function of variables C_i. It follows from the properties of functions $g_i(C_i)$, that $f(C_1, C_2, \dots C_n) = \sum g_i(C_i)$ is regular objective function. Therefore, it suffices to find an optimum operating sequence jobs $J_{[1]}, J_{[2]}, \dots, J_{[n]}$ to determine an optimal schedule.

2 Polynomial solvable cases

Theorem 1. If all the functions $g_i(\,) = g(\,)$ are the same, then the permutation $[i]$ for which

$$p_{[1]} \le p_{[2]} \le \cdots \le p_{[n]}$$

determines the optimal order of processing of the jobs for problem $1||\sum g_i(C_i)$.

Proof. Suppose we are given a schedule S determined by permutation $[i]$. Suppose that $p_{[j]} > p_{[j+1]}$ for some j. Let us define a permutation $[i]'$ as follows:

$$[i]' = [i] \text{ for } i \neq j, \quad i \neq j+1$$
$$[j]' = [j+1]$$
$$[j+1]' = [j]$$

It holds for the given schedule S determined by permutation $[i]$:

$$C_{[i]} = \sum_{k=1}^{i} p_{[k]}$$

Denote by S' the schedule defined by permutation $[i]'$, denote by $C'_{[i]'}$ corresponding completion times. It holds

$$C'_{[i]'} = \sum_{k=1}^{i} p_{[k]'}$$

It is easy to see, that $C'_{[i]'} = C_{[i]}$ for $i \neq j$ and

$$C'_{[j]'} = C'_{[j+1]'} - p_{[j+1]'} = C_{[j+1]} - p_{[j]} = C_{[j]} + p_{[j+1]} - p_{[j]} < C_{[j]}$$

Therefore

$$g(C_{[i]}) = g(C'_{[i]'}) \text{ for } i \neq j \text{ and } g(C'_{[j]'}) \le g(C_{[j]}) \text{ what implies } \sum_{i=1}^{n} g(C'_{[i]'}) \le \sum_{i=1}^{n} g(C_{[i]}).$$

Remark 1. Let $g_i(C_i) = w_i C_i + z_i$. Then $\sum_{i=1}^{n}(w_i C_i + z_i) = \sum_{i=1}^{n} w_i C_i + \sum_{i=1}^{n} z_i$. Since $\sum_{i=1}^{n} z_i$ is a constant therefore schedule is optimal with respect to $\sum g_i(C_i)$ if and only if it is optimal with respect to $\sum w_i C_i$. This case is well known from the scheduling theory (see [2], [3], [5], [6] or [7]) which proves, that the sufficient condition for optimality of a schedule given by permutation $[i]$ is:

$$\frac{p_{[1]}}{w_{[1]}} \le \frac{p_{[2]}}{w_{[2]}} \le \cdots \le \frac{p_{[n]}}{w_{[n]}}.$$

3 General case

3.1 Solution by dynamic programming

Denote by $\boldsymbol{J} = \{J_1, J_2, \dots, J_n\}$ the set of all jobs, let p_i be the processing time of job J_i, let $V = 2^{\boldsymbol{J}}$ be the set of all finite subsets of set $\boldsymbol{J}$ and let $p = \sum_{j=1}^{n} p_j$ be the total processing time of all jobs from $\boldsymbol{J}$. Let $U \in \boldsymbol{J}$ i.e. U is a finite subset of the set of all jobs $\boldsymbol{J}$. Suppose we have an optimal order of jobs given by the permutation $[i]$ such that all jobs from the set U are sheduled behind jobs from the set $U^C = \boldsymbol{J} - U$.

The processing of the first job from the set U can start in the time $q_U = p - \sum_{j\in U} p_j$ no matter how the jobs from U are sequenced. If the order $[i]$ is optimal then the operating sequence of set U provided that it begins time q_U has be optimal regardless of the operating sequence of jobs from the set U^C.

Let $G(U)$ be the minimum of $\sum_{i\in U} g_i(C_i)$ provided that processing of jobs from the set U begins at the time q_U. If the job $J_j \in U$ is the first job of an optimal order of the set U, this job begins at the time q_U and ends at the time $(q_U + p_j)$. In this case is

$$G(U) = g_j(q_U + p_j) + G(U - \{j\}).$$

We will search the optimal order among all such orders, where the first job is some job from the set U followed by the optimal order of jobs from the set $U - \{j\}$. Therefore

$$G(U) = \min_{j\in U}\{g_j(q_U + p_j) + G(U - \{j\})\} \quad (1)$$

Let us have a set U containing i jobs. If we know the values $G(U - \{j\})$ for all sets of the type $U - \{j\}$ where $j \in U$, we can calculate the value $G(U)$ from equation (1).

It is immediately seen, that $G(\emptyset) = 0$ for empty set, $G(\{j\}) = g_j(p)$ for single element sets $\{j\}$ and $G(\boldsymbol{J})$ is equal to the optimal value of the criterial function $\sum g_i C_i$ for the entire set of jobs $\boldsymbol{J}$. Therefore if we can calculate the value of the function $G(U)$ for the empty set and every single element set, we can calculate $G(U)$ using the equation (1) step by step for all two-, three-, etc. to n-element sets.

Algorithm of dynamic programming for searching the optimal schedule for the problem $1||\sum g_i(C_i)$:

Step 1. Put $G(U) = 0$ for the empty set $U = \emptyset$.
Step 2. Put $G(\{j\}) = g_j(p)$ and $X(\{j\}) = j$ for $j = 1, 2, \dots, n$. Put $= 1$.
Step 3. While $k < n$ do:

{Put $k = k + 1$
For every k-element subset $U \subseteq \boldsymbol{J}$ put
$G(U) = \min_{j\in U}\{g_j(q_U + p_j) + G(U - \{j\})\}$
$X(U) = \text{argmin}\{g_j(q_U + p_j) + G(U - \{j\})\}$
}

Step 4. Construct optimal order of the jobs:

Put: $U = \boldsymbol{J}$, $i = 1$, $[1] = X(\boldsymbol{J})$.
While $i < n$ do:
{ $[i + 1] = X(U - \{[i]\})$
$U = U - \{[i]\}$
$i = i + 1$
}

This algorithm reduces the computational complexity of full search $O(n!)$ to $O(2^n)$.
For $n = 30$ it holds $n! \doteq 2{,}7.\,10^{32}$, that is currently computationally unfeasible, but $2^n \doteq 10^9$, which is still on the limits of computability by current computer technology.

3.2 Solution by methods of graph theory

Let's construct a digraph $D = (V, H, c)$ with vertex set V of all finite subsets of the set $\boldsymbol{J}$ including the empty set $\emptyset$ and the set . The set H of arcs of digraph D is defined as follows:

$$(U, W) \in H \text{ if and only if } \exists\, J_k \in \boldsymbol{J}\text{-}U \text{ such that } W = U \cup \{J_k\}$$

The cost of the arc (U, W), where $W = U \cup \{J_k\}$, is defined as $c(U, W) = g_k(q_U)$.

Theorem 2. Let $m(\emptyset, \boldsymbol{J})$ be any path from the vertex $\emptyset$ to the vertex $\boldsymbol{J}$ in the digraph D. Then this path is determined by the sequence of vertices of the type

$$\emptyset,\ \ \emptyset \cup \{J_{[n]}\},\ \ \{J_{[n]}\} \cup \{J_{[n-1]}\},\ \ \{J_{[n]}, J_{[n-1]}\} \cup \{J_{[n-2]}\}, \ldots,\ \ \{J_{[n]}, J_{[n-1]}, \ldots, J_{[2]}\} \cup \{J_{[1]}\} \tag{2}$$

and its length equals the value of objective function $\sum g_{[i]}(C_{[i]})$, corresponding to operation order:

$$J_{[1]}, J_{[2]}, \ldots, J_{[n-1]}, J_{[n]} \tag{3}$$

On the other hand, if (3) is any operation order, then (2) is an oriented path with the length $\sum g_{[i]}(C_{[i]})$ in digraph D.

Proof. Every path $m(\emptyset, \boldsymbol{J})$ in digraph D has the form (2), where $[i]$ is a permutation.
On the other hand, any operation order of vertices (2) of digraph D determines a path in D for arbitrary permutation $[i]$. Arcs of this path are

$$(\emptyset,\ \ \emptyset \cup \{J_{[n]}\}), \qquad (\{J_{[n]}\},\ \{J_{[n]}\} \cup \{J_{[n-1]}\}), \quad \ldots\ldots\ldots\ldots\ , (\{J_{[n]}, J_{[n-1]}, \ldots, J_{[2]}\}, \{J_{[n]}, J_{[n-1]}, \ldots, J_{[2]}\} \cup \{J_{[1]}\})$$

Costs of these arcs are

$$g_{[n]}(q_{\{\emptyset,\}}),\ g_{[n-1]}\left(q_{\{J_{[n]}\}}\right), \quad g_{[2]}\left(q_{\{J_{[n]}, J_{[n-1]}, \ldots, J_{[3]}\}}\right), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ , g_{[1]}\left(q_{\{J_{[n]}, J_{[n-1]}, \ldots, J_{[2]}\}}\right)$$

Therefore the length of the path (2) determined by the operation order $[i]$ is (costs of arcs are counted in reverse order):

$$\sum_{i=1}^{n} g_{[i]}\left(q_{\{J_{[n]}, J_{[n-1]}, \ldots, J_{[i+1]}\}}\right).$$

For operation order $J_{[1]}, J_{[2]}, \ldots, J_{[n-1]}, J_{[n]}$ it holds $q_{\{J_{[n]}, J_{[n-1]}, \ldots, J_{[i+1]}\}} = C_{[i]}$, and therefore the path length (2) equals to $\sum_{i=1}^{n} g_{[i]}(C_{[i]})$ – i.e. to the value of objective function of order (3).

Corollary 3. To find the optimal order (3) of jobs for the objective function $\sum g_{[i]}(C_{[i]})$ means to find the shortest $(\emptyset, \boldsymbol{J})$ -path (2) in digraph D.

We will use Dijkstra's algorithm to find the shortest $(\emptyset, \boldsymbol{J})$-path in the digraph D. This is a point – to point shortest path algorithm. It stops just after it finds the shortest $(\emptyset, \boldsymbol{J})$ −path. It does not examine any other longer paths. This enables us not to search the shortest paths to all vertices – i.e. to examine the values $G(U)$ for all subsets, but only for those, whose distance from the vertex $\emptyset$ is less or equal to the optimum. This approach may make time savings for some objective functions of considered type. Moreover, good implementation, that will keep only the verticess with finite label in the memory, can save operating memory.

3.3 Implementation

Algorithm of dynamic programming for searching the optimal schedule for the problem $1||\sum g_i(C_i)$ with usage of Dijkstra's algorithm:

Step 1. Put $G(\emptyset) = 0$. We assume that $G(V)$, $X(V)$ are undefined for each nonempty set $V \in \boldsymbol{V}$.
Put $\mathcal{E} = \{\emptyset\} = \{\{\,\}\}, q_\emptyset = p = \sum_{j=1}^{n} p_j$.
(Attention! $\mathcal{E}$ is not empty set. $\mathcal{E}$ contains one element – the empty set)
Step 2. Find an element R with lowest value $G(R)$ in set $\mathcal{E}$ and put $\mathcal{E} = \mathcal{E} - \{R\}$.
Step 3. If $R = \boldsymbol{J}$, STOP.
Step 4. For all sets of type $W_j = R \cup \{j\}$, where $j \notin R$ do:
If $W_j \notin \mathcal{E}$, put $\mathcal{E} = \mathcal{E} \cup \{W_j\}$, $G(W_j) = G(R) + g_i(q_R)$, $X(W_j) = j$, $q_{W_j} = q_R - p_j$
If $W_j \in \mathcal{E}$ and $G(W_j) > G(R) + g_i(q_R)$, put $G(W_j) = G(R) + g_i(q_R), X(W_j) = j$
GOTO STEP 2

The final order of jobs is determined by backtracking of labels $X(V)$ from the last vertex $\boldsymbol{J}$ of the shortest $(\emptyset, \boldsymbol{J})$ −path in digraph D to its first vertex $\emptyset$.

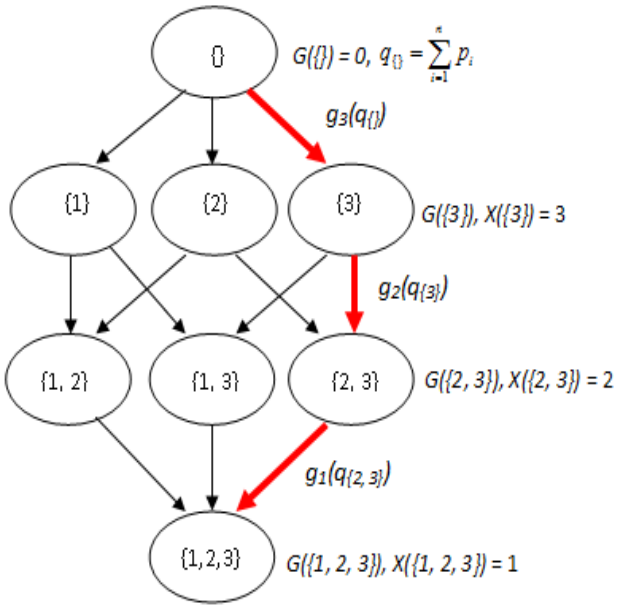

Figure 1 Digraph D

Just formulated algorithm is illustrated in figure 1 for an instant with 3 jobs. All vertices of corresponding digraph D are drawn in this picture. Each vertex V has labels $G(V)$ and $X(V)$. Label $G(V)$ will represent after algorithm stops the minimum of $\sum_{i \in V} g_i(C_i)$ provided that processing of jobs from the set V begins at the time q_V. During the run of algorithm it will be equal to upper bound of this minimum. Label $X(V)$ represents recently added job.

Application of Dijkstra algorithm ensures that there is no need to create all possible vertices in computer memory. A vertex V is created only if it achieves a finite label for the first time and inserted into the set $\mathcal{E}$. Vertices are created until the vertex $\boldsymbol{J}$ is extracted from the set $\mathcal{E}$ - at this moment the computation is over. Shortest $(\emptyset, \boldsymbol{J})$ −path is represented by red color.

The optimum operation order will be created as follows.
The last vertex of shortest $(\emptyset, \boldsymbol{J})$ −path is $= \{1, 2, 3\}$. Label $X(\{1, 2, 3\}) = 1$, that means the first job of optimal schedule will be the job 1. Now we have jobs 2 and 3 left. $X(\{2, 3\}) = 2$, so the second job of the optimal schedule will be the job 2 and the last one will be the job 3. The final order of jobs is $J_1,\ J_2,\ J_3$.

Representation of set of jobs:
Any subset V of $\boldsymbol{J}$ can be represented by a n-digit binary number having i-th digit equal 1 if and only if $J_{i \in} V$.
Functions g_i were implemented in two ways:

1. as linear functions with random generated coefficients for each job
2. as functions $T_i w_i = \max \{C_i - d_i, 0\} * w_i$ for each job J_i
3. as functions $F_i w_i = (C_i - r_i) * w_i i$ for each job J_i

We have implemented the set $\mathcal{E}$ by these two data structures to enhance computing speed:

- Binary heap
  - it is a heap data structure represented by binary tree
  - each vertex is characterized by priority
  - root vertex has the highest priority (in this case the lowest value $G(V)$)
  - structure is used to find the element with minimal value $G(V)$, that is the root element from the heap
  - enhancement from searching complexity $O(n)$ to $O(1)$
- Treap
  - it is binary searching tree with heap-ordered vertices
  - each vertex has a key (decimal number representing jobs in set *V*) and random generated priority for heap-ordering
  - structure is used to find out, whether vertex with the given set of jobs exists or should be created
  - enhancement from searching complexity *O(n)* to *O(log n)*

3.4 Results

In the following table you can find results of this algorithm with implemented linear functions g_i:

1. without heap and treap data structures (**none**)
2. with treap data structure (**with T without H***)*
3. with treap and heap data structures (**both**)

Result are compared for instances with various number of jobs from the point of view of computation time and from the point of view of percentage of created vertices.

Data structure:		none		with T without H		both	
Jobs n	Number of all vertices $= 2^n$	Created vertices	Time (min)	Created vertices	Time (min)	Created vertices	Time (min)
10	1024	79,88%	00:00,4	89,16%	00:00,2	25,00%	00:00,0
11	2048	85,74%	00:06,8	79,83%	00:01,8	24,80%	00:00,0
12	4096	68,80%	00:35,8	69,24%	00:10,8	28,59%	00:00,0
13	8192	68,10%	05:03,4	62,45%	01:21,4	28,43%	00:00,1
14	16384			64,31%	08:51,5	30,27%	00:00,1
15	32768			55,29%	34:55,6	32,90%	00:00,2
18	262 144					29,33%	00:03,0
20	1 048 576					28,48%	00:21,7
21	2 097 152					27,16%	00:52,5
22	4 194 304					27,83%	01:44,0
23	8 388 608					27,96%	03:49,9
24	16 777 216					27,70%	07:48,6

Table 1 Computation results for linear functions g_i.

Best results were obtained by using both data structures treap and heap. Because we can find an optimal solution faster by using both structures, more jobs can be scheduled in admissible time.

		g_i linear		$g_i = T_i w_i$	
Jobs n	Number of all vertices $= 2^n$	Created vertices	Time (min)	Created vertices	Time (min)
10	1024	25,00%	00:00,0	38,87%	00:00,0
13	8192	28,43%	00:00,1	14,42%	00:00,0
15	32 768	32,90%	00:00,2	5,88%	00:00,0
18	262 144	29,33%	00:03,0	1,53%	00:00,0
20	1048576	28,48%	00:21,7	0,38%	00:00,0
24	16 777 216	27,70%	07:48,6	0,04%	00:00,1
27	134 217 728			0,63%	00:04,9
31	2 147 483 648			0,01%	00:00,8

Table 2 Comparison of computation results for linear functions g_i, and $g_i = T_i w_i = \max\{0,\ C_i - d_i\}.w_i$. Both heap and treap data structures were used.

Instances with linear functions g_i are polynomial solvable problems. They are the worst cases for this approach since there is a lot of subsets $V \subseteq \boldsymbol{J}$ with the same or similar values $G(V)$. Therefore they were used to find out the size of surely solvable instances. Even thought we are able to solve instances with 24 jobs.

Algorithm had better results for functions g_i implemented as $T_i w_i$. It could solve problem with up to 31 jobs. The limit 31 jobs results from chosen representation of subsets $V \subseteq \boldsymbol{J}$ as 32 – bit numbers since we did not hope that we would succeed to solve larger instances. Maybe, a 64 – bit representation would solve slightly larger tasks but the problems remains still NP-hard and computation time will grow rapidly with increased number of jobs.

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Capital mobility – econometric approach

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Abstract. Capital is one of the most important productive inputs. In the conditions of globalization it is characterised by a certain mobility degree, which can be a significant factor influencing an economic growth. In general, a more open capital account implies a higher productive performance than economies with restricted capital mobility. But this relation is not straightforward, and there is an empirical evidence that for weaker economies a high degree of capital mobility is not desirable.
To measure a degree of capital mobility, a Feldstein – Harioka hypothesis is usually applied: Under perfect capital mobility, domestic savings and investment rates should be uncorrelated. In econometric models, a possible long-run relation between savings and investment is studied by the help of the concept of cointegration. Newly, an importance of comprising foreign direct investment into the cointegration equation is documented. Relevant econometric analysis of capital mobility in Czech Republic and Slovakia is presented.

Keywords: Capital mobility, econometric models, cointegrated time series, panel data

JEL Classification: E20, C22, C51
AMS Classification: 62J02

1 Introduction

Capital is one of the most important productive inputs. In the conditions of globalization it is characterised by a certain mobility degree, which can be a significant factor influencing an economic growth. To measure a degree of capital mobility, a Feldstein – Harioka hypothesis is usually applied: Under perfect capital mobility, domestic savings and investment rates should be uncorrelated. In econometrics models, a possible long-run relation between savings and investment is studied by the help of the concept of cointegration. Newly, an importance of comprising foreign direct investment into the equation is documented. After subtracting FDI – outflows from savings, an estimated coefficient of savings rate reflects more precisely the extent to which domestic savings is used to finance domestic investment.

Economic background is briefly described, econometric models follow, an interpretation of estimated parameters with respect to capital mobility is explained. Based on a panel data set, an econometric analysis of capital mobility in Czech Republic and Slovakia is presented.

2 Capital Mobility

Capital as one of the most important productive inputs can be seen as an old one, exploited by those who control the powers of government, and a new one, related to firms prepared to invest in some other country. New capital is characterised by a certain capital mobility, a degree of which is influencing an economic growth.

Capital mobility is defined as ability of the private funds to move across national boundaries in pursuit of higher returns. This mobility depends on the absence of currency restriction on the inflows and outflows of capital. In a historical survey opening his text, Taylor [8] reminds readers of the fact that late-nineteenth century capital markets were relatively well integrated under the classical gold standard. A disintegration and imperfect capital mobility relates to inter-war period, especially after 1929; the postwar period shows gradually increasing capital market integration. Measurements of capital mobility are also interesting in the context of the macroeconomic trilemma [6] . Having typically desirable, yet contradictory, objectives:

1. to stabilize the exchange rate;
2. to enjoy free international capital mobility,;
3. to engage in a monetary policy oriented toward domestic goals,

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only two out of the three objectives can be mutually consistent. That is why policymakers must decide which one to give up.

Common experience shows that the capital mobility is appreciated in different ways according to the level of an economic development of a country. As it is stated in [2], economies with a more open capital account show out a higher productive performance than economies with restricted capital mobility. But, a positive influence of an open capital account emerges only after a country has achieved a certain degree of economic development. Developing countries tend to be acceptors of capital and the conditions are supported by their governments that attract investment.

To measure a degree of capital mobility, different approaches appear in the literature. In [2], two alternative measures of extent of capital mobility are presented, both constructed during the 90 – ties.

First, it is the NUYCO (NUmber-of-Years-with-COntrol) index studying a relationship between capital controls and economic performance. A higher value denotes a higher degree of capital controls. Second, the Quinn´s index of capital mobility; a higher value denotes a higher degree of capital mobility. Both indices explained e. g. in [7].

Another possibility is to use econometric models studying saving – investment correlation and is known as a Feldstein – Harioka hypothesis [3]; they argued that if there is perfect capital mobility, we should observe low correlation between domestic investment and savings what related to a relation

$$\left(\frac{I}{Y}\right)_t = \alpha + \beta\left(\frac{S}{Y}\right)_t + u_t \tag{1}$$

with I for investment, S savings and Y for GDP. Finding that the parameter β is close to unity means that changes in domestic saving passed through almost fully into domestic investment what implies an imperfect international capital mobility. Under perfect capital mobility, domestic savings and investment rates should be uncorrelated.

3 Econometric measurements

Under a savings – investment dynamics a simple correlation of both variables and also the regression equation (1), represent a temporary short-run phenomenon. Long-run relations between time series can be seen through the theory of cointegration (e.g. [4]). Cointegration in fact, is a long-run relation which can be represented by an error-correction model (ECM). ECM has a long-run and a short-run parts; validation of a long-run relation between investment and savings than means a low capital mobility, and vice versa.

So, instead of relation (1), a formulation is proposed in [6] in the form

$$\Delta\left(\frac{I}{Y}\right)_t = \alpha + \beta\Delta\left(\frac{S}{Y}\right)_t + \gamma\left[\left(\frac{S}{Y}\right)_{t-1} - \left(\frac{I}{Y}\right)_{t-1}\right] + \delta\left(\frac{S}{Y}\right)_{t-1} + u_t\,. \tag{2}$$

In the long – run (2) tends to an equilibrium level

$$\alpha + \gamma\left(\frac{S}{Y} - \frac{I}{Y}\right) + \delta\frac{S}{Y} = 0\quad. \tag{3}$$

If δ=0 in (3), $\left(\frac{S}{Y} - \frac{I}{Y}\right)$ is a time – series which is stationary around $\frac{\alpha}{\gamma}$ and relevant cointegration vector is (1, -1)´. If $\delta \neq 0$, $\left(\frac{S}{Y} - \frac{I}{Y}\right)$ is a non – stationary variable and cointegration vector is (1+δ/γ, -1).

In case of an open economy, a low values of β and γ are expected. On the other hand, a capital immobility should be reflected in results as high values of β, γ, and α, δ being near to zero.

Nevertheless, weak points of the above approach are criticised, e.g. in [9]. He sees that one problem with the conventional way of gauging capital mobility based on the correlation between domestic savings and domestic

investment lies with the inclusion of foreign direct investment (FDI) in the latter. His findings suggest that capital is remarkably more mobile in both developed as well as developing countries when FDI is excluded from domestic investment of the recipient country. After subtracting FDI – outflows from savings, an estimated coefficient of savings rate reflects more precisely the extent to which domestic savings is used to finance domestic investment.

Instead of (1), it is more precisely formulated

$$\left(\frac{I-FDI}{Y}\right)_t = \alpha + \beta\left(\frac{S}{Y}\right)_{t-1} + u_t \quad . \tag{4}$$

or

$$\left(\frac{I-FDI}{Y}\right)_t = \alpha + \gamma\left(EG\left(\frac{S}{Y}\right)_{t-1}\right) + u_t \,, \tag{5}$$

where EG is the globalization index compiled by Dreher ([1]). It is a weighted index comprising economic, social and political integration. In his text, the highest value refers to U.S.A. and equals 6.48. For the Czech Republic (27th place) it is 3.80, Slovakia (48th) has 3.10. Comprising EG factor lead to unsignificant results in our computations, that is why the form is dropped and a long – run counterpart to (2) is formulated on the base of (4) as

$$\Delta\left(\frac{I-FDI}{Y}\right)_t = \alpha + \beta\Delta\left(\frac{S}{Y}\right)_{t-1} + \gamma\, ecm_{t-1} + u_t \tag{6}$$

with $ecm_t = \left(\frac{I-FDI}{Y}\right)_t - \hat{\alpha} + \hat{\beta}\left(\frac{S}{Y}\right)_{t-1}$. Here again, lowest value of β parameter speaks in favor of highest degree of capital mobility.

4 Application

While GDP, investment and savings are followed annually as well as quarterly, foreign direct investment (FDI) occur in Eurostat statistic tables as annual only. Besides, no FDI values are available before 1997 and temporarily also not after 2011. This fact determined the set of observation to be assembled not only for the Czech Republic but also for the Slovakia; for that matter, Czech and Slovak economies used to be compared rather often, as e.g. in [5] . Treating the doubled number of observations as a panel by the pooled regression makes the estimated results more efficient, the trade off is a common result without a possibility to distinguish between both economies. The mean value of FDI represents roughly 30 % of the mean investment, on the other hand, there is an apparent volatility in FDI outflows, what makes the FDI idea incorporated in equation (6) rather interesting. All variable are measured in millions of Euro, computations are performed by the help of PcGive software and summarised in Table 1. Statistical significance of estimated parameters is given by relevant t-probability.

Short-run	$\hat{\beta}$	**t-prob**	$\hat{\gamma}$	**t-prob**	$\hat{\delta}$	**t-prob**	**R^2**
Eq. (1)	0.787	0.002					0.41
Eq. (4)	0.589	0.000					0.45
Long-run							
Eq. (2)	0.453	0.004	0.328	0.000	-0.015	0.871	0.32
Eq. (6)	0.139	0.238	0.558	0.000			0.25

Table 1. Results of computations

Evidently, long-run characteristics show (by the help of parameter $\hat{\beta}$) a higher capital mobility as it is in short-run equations. But, the near zero value of $\hat{\beta}$ in equation (6) is accompanied by a low value of coefficient of

determination what calls the result into question. That is why (6) cannot be interpreted as speaking in favour of very high capital mobility. Equation (2) brings a qualitatively other message: parameter $\hat{\delta}$ is statistically zero what does not support an evidence of high mobility of capital; it matches the rather medium values of $\hat{\beta}$ and $\hat{\gamma}$. The result $\hat{\beta}$ = 0.453 makes CR and Slovakia comparable with South Korea ($\hat{\beta}$ = 0.46) , as it is presented in the list of Asia economies in [7].

Conclusions

The responses to a demand for measurement of capital mobility are econometric models based on investment – savings relations.

In the process of capital mobility, Czech Republic and Slovakia do not exhibit a very high capital mobility. Using variant econometric models, the long-run approach represented by the error correction concept, relates to higher mobility of capital as the short-run models do. The capital mobility is appreciated in different ways according to the level of an economic development of a country; that is why there is no straightforward evaluation of the found results. There is an evidence in relevant literature, that both countries are comparable with South Korea, as for the crucial parameter of an econometric model.

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CO 2 emissions, energy consumption and economic growth in the Visegrad Group countries: a panel data analysis

Monika Papież [1],

Abstract. This paper examines causal relationships between carbon dioxide emissions, energy consumption and economic growth using panel vector error correction modelling techniques based on the panel data for the Visegrad Group countries over the period 1992–2010. Panel cointegration tests show the existence of long-run relationships among carbon dioxide emissions, energy consumption and economic growth. The long-run equilibrium indicates that energy consumption has a positive and statistically significant impact on emissions. However, the results obtained cannot confirm the Environmental Kuznets Curve (EKC) hypothesis for the Visegrad Group countries. The results of panel short-run Granger causality tests reveal the existence of bidirectional causality between CO2 emissions and economic growth. Also, the short-run dynamics suggests unidirectional causality from energy consumption to carbon dioxide emissions and from energy consumption to economic growth in the Visegrad Group countries. The findings indicate that there is no causality running from carbon dioxide emissions to energy consumption and from economic growth to energy consumption.

Keywords: Carbon dioxide emissions, energy consumption, economic growth, panel cointegration.

JEL Classification: C33, Q43, Q50, Q56
AMS Classification: 91B76, 62M10

1 Introduction

The Visegrad Group originated in 1991 after the dissolution of the Communist Bloc and consisted of the Czech Republic, Slovakia (after the dissolution of Czechoslovakia in1992), Hungary and Poland. Up till 1990 the economy of these countries was centrally planned, and in 1992 they had low gross domestic product per capita. Poland was the country with the lowest GDP per capita (2936 in constant 2000 US$) and the Czech Republic was the country with the highest GDP per capita (4756 in constant 2000 US$). For comparison, GDP per capita in the European Union countries was 14660 in constant 2000 US$. These countries also exhibited a very high energy intensity of the economic activity in 1992. Energy consumption in relation to GDP in Slovakia equalled to 337 (kg of oil equivalent per $1,000 GDP in constant 2005 PPP), in Poland 331, in the Czech Republic 292, and in Hungary 215. In the European Union countries energy consumption in relation to GDP equalled to 162 (kg of oil equivalent per $1,000 GDP in constant 2005 PPP). Numerous reforms introduced in the period 1992 - 2010 resulted in rapid economic growth and reductions in the energy intensity of the economy. Currently the most important areas of the Visegrad Group activity embrace culture, environmental protection, infrastructure, transportation, tourism, and energy.

The dynamic changes which took place in the Central European countries pose a question whether and to what extent economic growth influenced the environment. One of the methods of measuring the impact of economic development on the environment is the Environmental Kuznets Curve (EKC). It indicates the relationship between GDP per capita and the 'demand' for clean environment. Additional variables taken into consideration in this relationship can include other factors indicating environmental degradation. The Environmental Kuznets Curve (EKC) hypothesis states that the level of environmental degradation will rise to a certain point and then will drop (it can be presented by a curve shaped as an inverted U).

The aim of the paper is to verify the Environmental Kuznets Curve (EKC) hypothesis for the Visegrad Group countries based on the panel data over the period 1992 – 2010. Additionally, causal relationships between carbon dioxide emissions, energy consumption and economic growth will be examined using panel vector error correction modelling techniques. The paper is structured in the following way: Section 2 presents brief literature review, Section 3 presents methodology, data, the discussion of the methods and results are given in Section 4, while Section 5 contains the main conclusions.

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2 Brief literature review

The relations between economic growth and environmental pollution have been intensely analysed empirically over the past two decades. These analyses can be divided according to causal relations examined in them. Most of them focus on the analysis of short-term causal relations and long-term relations between energy consumption, economic growth and CO2 emission. The analyses have been conducted for single countries (e.g. France (Ang [2]), the United States (Soytas et al. [22]), China (Wang et al. [23]), and Turkey (Halicioglu [6], Ozturk and Acaravci [12]) as well as for groups of countries (e.g. BRIC (Pao and Tsai [18]), Central America (Apergis and Payne [3]), ASEAN (Lean and Smyth [9])) with the use of panel data sets Numerous analyses of relations between environmental pollution, energy sources prices and economic growth included additional factors (for example, foreign direct investment proposed by Pao and Tsai [19] or the financial development suggested by Sadorsky [20], Śmiech and Papież [21], Al-mulali and Che Sab [1], Papież and Śmiech [13], [14]).

3 Methodology

The first step in the estimation of dynamic panels is to test whether the variables contain unit roots. For this study three panel unit root tests were chosen, namely Levin–Lin–Chu (LLC) test (Levin et al. [8]), Im, Pesaran and Shin (IPS) test (Im et al. [7]), and Fisher-type tests using Augmented Dickey–Fuller (Fisher-ADF) (Maddala and Wu [11]; Choi [5]). The null hypothesis of those three panel unit root tests states that the panel series has a unit root (nonstationary).

If the variables contain a unit root, the second step is to test whether there exists any long-run equilibrium relationship between the variables using the Pedroni panel cointegration test (Pedroni [15], [17]). To test for the null hypothesis of no-cointegration against the cointegration in the panel, Pedroni [15] developed seven cointegration statistics. Four of them are based on pooling the residuals of the regression along the within-dimension (panel test) of the panel. The other three are based on pooling the residuals of the regression along the between-dimension (group test) of the panel. The equation for Pedroni cointegration tests can be formulated as follows:

$$\ln CO2_{it} = \beta_{0i} + \beta_{1i} \ln EC_{it} + \beta_{2i} \ln GDP_{it} + \beta_{3i} \ln^2 GDP_{it} + u_{it}, \tag{1}$$

where $i = 1,\ldots,4$ denotes the country and $t = 1,\ldots,T$ denotes the time period. Equation (1) describes the Environmental Kuznets Curve (EKC). The variables in equation (1) denote respectively: carbon dioxide emissions per capita (CO2), energy consumption per capita (EC), and GDP per capita (GDP).

If a long-run relationship between the variables is found, the third step is to estimate the equation of long-run relationship. To obtain the long-run estimates for the cointegrating relationship, the panel fully modified ordinary least squares (FMOLS) is used, the method developed by Pedroni [16], [17]. The FMOLS procedure accommodates the heterogeneity that is typically present, both in the transitional serial correlation dynamics, and in the long-run cointegrating relationships.

If a long-run relationship between the variables is found, the final step is to estimate a panel vector error correction model in order to determine Granger causal relations between the variables. The Granger causality test is based on the model with a dynamic error correction term. The panel VECM can be written as follows (Apergis and Payne [3], [4]):

$$\begin{aligned}\Delta \ln CO2_{it} = {} & \alpha_{1i} + \sum_{j=1}^{p} \beta_{11ij} \Delta \ln CO2_{it-j} + \sum_{j=1}^{p} \beta_{12ij} \Delta \ln EC_{it-j} + \sum_{j=1}^{p} \beta_{13ij} \Delta \ln GDP_{it-j} \\ & + \sum_{j=1}^{p} \beta_{14ij} \Delta \ln GDP2_{it-j} + \theta_{1i} ECT_{it-1} + \varepsilon_{1it},\end{aligned} \tag{2a}$$

$$\begin{aligned}\Delta \ln EC_{it} = {} & \alpha_{2i} + \sum_{j=1}^{p} \beta_{21ij} \Delta \ln CO2_{it-j} + \sum_{j=1}^{p} \beta_{22ij} \Delta \ln EC_{it-j} + \sum_{j=1}^{p} \beta_{23ij} \Delta \ln GDP_{it-j} \\ & + \sum_{j=1}^{p} \beta_{24ij} \Delta \ln GDP2_{it-j} + \theta_{2i} ECT_{it-1} + \varepsilon_{2it},\end{aligned} \tag{2b}$$

$$\Delta \ln GDP_{it} = \alpha_{3i} + \sum_{j=1}^{p} \beta_{31ij} \Delta \ln CO2_{it-j} + \sum_{j=1}^{p} \beta_{32ij} \Delta \ln EC_{it-j} + \sum_{j=1}^{p} \beta_{33ij} \Delta \ln GDP_{it-j} + \sum_{j=1}^{p} \beta_{34ij} \Delta \ln GDP2_{it-j} + \theta_{3i} ECT_{it-1} + \varepsilon_{3it}, \quad (2c)$$

$$\Delta \ln GDP2_{it} = \alpha_{4i} + \sum_{j=1}^{p} \beta_{41ij} \Delta \ln CO2_{it-j} + \sum_{j=1}^{p} \beta_{42ij} \Delta \ln EC_{it-j} + \sum_{j=1}^{p} \beta_{43ij} \Delta \ln GDP_{it-j} + \sum_{j=1}^{p} \beta_{44ij} \Delta \ln GDP2_{it-j} + \theta_{4i} ECT_{it-1} + \varepsilon_{4it}, \quad (2d)$$

where $i = 1,\ldots,N$ denotes the country, $t = 1,\ldots,T$ denotes the time period, p is the optimal lag length(s) determined by the SBC, ECT term is the residuals from the panel FMOLS estimation of the Eq. (1), and the error term ε_{it} is assumed to be i.i.d with a zero mean and constant variance.

4 Data and empirical results

The analysis of causal relationships between carbon dioxide emissions, energy consumption and economic growth was based on the annual panel data for the Visegrad Group countries (the Czech Republic, Hungary, Poland, and Slovakia) over the period 1992 – 2010. The following variables were chosen: carbon dioxide emissions in metric tons per capita (CO2), energy consumption in kg of oil equivalent per capita (EC) (data taken form the International Energy Agency), and GDP per capita in constant 2000 US$ (GDP) (data taken from the World Bank Development Indicators). The summary statistics are presented in Table 1.

Variable		Czech Republic	Hungary	Poland	Slovakia
CO2	Min	10.50	4.86	7.31	6.12
	Mean	11.68	5.64	8.08	7.22
	Max	12.73	6.08	9.06	8.55
EC	Min	3756.3	2209.6	2317.5	2959.4
	Mean	4098.9	2346.8	2496.1	3272.6
	Max	4488.1	2544.8	2679.0	3482.0
GDP	Min	4754.4	3707.1	2936.2	4059.4
	Mean	6198.7	4785.0	4612.7	5991.2
	Max	8042.0	5947.2	6574.3	8545.6

Table 1 Summary statistics, 1992-2010

In the panel data analysis, the panel unit root test must be taken into consideration first in order to identify the stationary properties of the relevant variables. Thus, Table 2 presents the results of the panel unit root tests (Levin–Lin–Chu (LLC), Im, Pesaran and Shin (IPS) test, and Fisher-type tests using Augmented Dickey–Fuller (Fisher-ADF) for each variable. It can be seen from Table 2 that each variable is integrated of order one, i.e. I(1), which meets the requirements of the cointegration test.

Variable	LLC	IPS	Fisher- ADF
lnCO2	-1.13628	-0.09570	8.07581
lnEC	-1.26301	-1.19207	11.4000
lnGDP	-1.25857	1.38311	5.32871
lnGDP2	-0.96798	1.55955	5.29957
Δ lnCO2	-6.60337 ***	-6.08031 ***	45.0505 ***
Δ lnEC	-5.41123 ***	-6.11110 ***	45.0172 ***
Δ lnGDP	-2.80443***	-2.37235 ***	21.8439 ***
Δ lnGDP2	-2.77632***	-2.37467 ***	21.6868 ***

Note: ***, **, * indicate statistical significance at 1, 5 and 10 percent level of significance, respectively

Table 2 Results for panel unit root tests

Given that each of the variables contains a panel unit root, the existence of a long-run relationship between the variables is examined using the Pedroni panel cointegration test (Pedroni [15]). Pedroni's testing approach requires the estimation of the Eq. (1) for each cross-sectional unit. The results for the panel cointegration tests are presented in Table 3. The results allow us to reject the null hypothesis of no cointegration for the tests at 5% or higher significance level apart from the panel and group rho-Statistic and panel v-Statistic. On the other hand, three Pedroni test statistics which do not allow us to reject the null hypothesis are negligible for short periods (Pedroni, [17]). That is why it can be assumed that the variables in Eq. (1) are panel cointegrated.

Test Statistics	Statistics	Weighted Statistics
Panel v-Statistic	0.888622	0.426102
Panel rho-Statistic	0.017867	-0.362482
Panel PP-Statistic	-1.728883 **	-2.731571 ***
Panel ADF-Statistic	-3.170544 ***	-3.486382 ***
Group rho-Statistic	0.447631	
Group PP-Statistic	-2.680004 ***	
Group ADF-Statistic	-3.832084 ***	

Notes: ***, **, * indicate statistical significance at 1, 5 and 10 percent level of significance, respectively.

Table 3 Pedroni results for panel cointegration tests

Once the cointegration relationship is established, the next step is to estimate the long-run parameters of the model (1) by using the FMOLS technique. The FMOLS corrects the standard OLS for bias induced by the endogeneity and serial correlation of the regressors (Lee [10]). The results from the panel FMOLS estimations are illustrated in Table 4. For each country, all the coefficients are significant at the 5% level or higher. The results for individual countries indicate that for all countries an increase in energy consumption per capita affects the increase in carbon dioxide emissions per capita. The results show that a 1% increase in energy consumption per capita increases CO2 emissions per capita by 0.52% in Slovakia, by 0.87% in the Czech Republic, by 1.10% in Poland and by 1.67% in Hungary.

The results of the analysis of the relationship between GDP and CO2 emissions, however, are not conclusive. It is worth noting that only for Poland and Hungary the coefficients have the expected sign, that is are the same as the EKC hypothesis suggests: $\beta_2 > 0$ and $\beta_3 < 0$. In case of the Czech Republic and Slovakia the coefficients have the reverse signs for variables lnGDP and lnGDP2. The results, different from those given in Ang [2], Apergis and Payne [3], [4] and Lean and Smyth [9], are contrary to the EKC hypothesis. However, similar results (the reverse coefficient) were obtained by Wang et al. [23], who used panel cointegration techniques based on the panel data for 28 provinces in China over the period 1995–2007. Thus, it is advisable to conduct a detailed analysis of a reverse impact of GDP on CO2 emissions in those countries.

	Czech Republic	Hungary	Poland	Slovakia	PANEL
lnEC	0.87 (6.96) ***	1.67 (8.74) ***	1.10 (17.15) ***	0.52 (2.09) *	1.04 (17.46) ***
lnGDP	-8.09 (-2.34) **	17.54 (3.93) ***	2.05 (2.46) **	-8.28 (-2.38) **	0.80 (0.84)
lnGDP2	0.44 (2.25) **	-1.07 (-4.03) ***	-0.13 (-2.64) **	0.46 (2.29) **	-0.07 (-1.07)
Intercept	32.00 (2.16) **	-83.29(-4.32)***	-14.48(-3.78)***	34.94 (2.47) **	-7.71 (-1.73) *
Adjusted R-squared	0.794	0.735	0.965	0.826	

Notes: ***, **, * indicate statistical significance at 1, 5 and 10 percent level of significance, respectively.

Table 4 The panel cointegration coefficients

The long-run panel cointegration equation (1) can be written as:

$$\ln CO2 = \underset{(-1.73)}{-7.71} + \underset{(17.46)}{1.04}\ln EC + \underset{(0.84)}{0.80}\ln GDP - \underset{(-1.07)}{0.07}\ln^2 GDP\,, \tag{3}$$

where the numbers in parentheses denote the values of t-statistics.

For the panel of countries as a whole, all the coefficients have the expected sign, but only the coefficient of lnEC is significant at the 1% level. So, the results for the panel indicate that a 1% increase in energy consumption per capita is associated with an increase in carbon dioxide emissions per capita of 1.04%. On the other hand, CO2 emission is positively related to GDP and negatively related to GDP2, although the coefficients of lnGDP and lnGDP2 seem not to be significantly different from zero. Most probably, the lack of significance

of the coefficients of lnGDP and lnGDP2 results from two different forms of Eq. 3 (one for Poland and Hungary and one for the Czech Republic and Slovakia). The results obtained indicate the need to extend the analysis to the remaining post socialist countries.

The existence of panel long-run cointegration relationship between CO2 emissions, energy consumption and economic growth allowed us to estimate a panel vector error correction model using Eq. (2a-2d). The estimated panel-based VECM allows us to investigate both short-run and long-run Granger causality. Table 5 presents the results of the panel short-run and long-run Granger causality analysis.

Dependent variable	**Source of causation (independent variables)**			
	Short run			**Long run**
	Δ lnCO2	**Δ lnEC**	**Δ lnGDP& Δ lnGDP2**	**ECT**
	F-statistics			**t-statistics**
Δ lnCO2		5.393 **	15.803 ***	-8.693 ***
Δ lnEC	0.013		1.169	2.419 **
Δ lnGDP	5.511 **	5.858 **		-7.774 ***
Δ lnGDP2	5.628 **	6.239 **		- 8.043 ***

Notes: ***, **, * indicate statistical significance at 1, 5 and 10 percent level of significance, respectively.

Table 5 Panel Granger causality results.

The results of panel short-run Granger causality tests show the existence of bidirectional causality between CO2 emissions and economic growth in the Visegrad Group countries at 5% significance level. Also, the short-run dynamics suggests unidirectional causality from energy consumption to carbon dioxide emissions and from energy consumption to economic growth. It indicates that an increase in energy consumption will lead to an increase in CO 2 and an increase in energy consumption will lead to a decrease in economic growth. The results indicating a reverse relation between energy consumption and economics growth can be explained by a decrease in energy intensity of the economy. The results presented in Table 5 indicate that there is no causality running from carbon dioxide emissions to energy consumption and from economic growth to energy consumption. The estimated coefficients of ECT are statistically significant at least the 5% level in each equation. Apart from the equation for energy consumption, the coefficients of the ECT are negative, which implies that all three variables dynamically interact to restore long-run equilibrium whenever there is a deviation from the cointegrating relationship.

5 Conclusions

The results of the analysis of four Visegrad Group countries indicate that it is not possible to conclusively verify the Environmental Kuznets Curve (EKC) hypothesis based on the panel data over the period 1992–2010. The long-run panel cointegration equation reveals statistical significance only between carbon dioxide emissions and energy consumption. Unfortunately, the analysis did not allow us to identify the impact of economic growth on carbon dioxide emissions. The results of the panel data analysis may result from a different character of the countries taken into consideration. A detailed analysis of the relations in particular countries revealed that the Environmental Kuznets Curve (EKC) hypothesis can be confirmed for Poland and Hungary. However, in case of the Czech Republic and Slovakia the results obtained are contrary to the Environmental Kuznets Curve (EKC) hypothesis. Thus, it can be concluded that it is justified to conduct further detailed analyses of the impact of GDP on CO2 emissions in the four countries in question as well as other post socialist countries.

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Asset Allocation by Maximizing the Sortino Ratio utilizing Differential Evolution Solution Procedure

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Abstract. Decision making on the allocation of portfolio assets can be supported by solving of the mathematical models based on the measures of risk level, as well as on measures of portfolio performance. Sometimes it is problematic to find corresponding solutions of some models that deal with portfolio performance measurement because of their substandard structures. Evolutionary algorithms can be used to address such problematic groups of problems, which are an alternative to traditional ways of solving problems of mathematical programming. One of the portfolio performance techniques is the Sortino ratio. The paper presents the approach that allows maximizing Sortino ratio by differential evolution. The aim was to show the suitability of evolutionary algorithms in the field of financial modeling due to the rapid achievement of real-time suboptimal solutions. That approach was applied to the assets contained in the Down Jones Industrial Index.

Keywords: Sortino ratio, portfolio optimization, portfolio selection models, differential evolution.

JEL Classification: C02, C61, G11
AMS Classification: 91B28, 90A09, 90-08

1 Introduction

The basic problem in portfolio theory is the selection of an appropriate mix of assets in a portfolio in order to maximize portfolio expected return and subsequently to minimize portfolio risk. Decision process on the selection of the portfolio assets can be supported by mathematical models based on maximizing the portfolio performance measurement techniques e.g. Sharpe ratio, Treynor ratio, Jensen's alpha, Information Ratio, Sortino ratio, Omega function and the Sharpe Omega ratio ([6], [7], [8], [9], [12], [14], [15], [18]) that are aimed to determine the allocation of the available resources in the selected group of assets. This paper presents the possibility of computing the Sortino ratio that can be difficult in general due to its substandard structures of performance level (the nature of objective function) and therefore the use of standard techniques seems to be relatively complicated. The alternative is the use of evolutionary techniques. One of alternative procedures may include the principle of differential evolution that can be considered to be effective tool for measure of performance in financial modeling [13]. The analyze were provided on assets included in the Down Jones Industrial Index and its historical data[4] published from July 1st 2001 to April 1st 2012 on weekly basis were used.

2 Sortino Ratio

The *Sortino ratio* is based on the known *Sharpe ratio*, but though both ratios measure an investment's risk-adjusted returns, they do so in significantly different ways. The formula for the Sharpe Ratio based on standard deviation (simply the square root of variance) allows to measure a non directionally-biased measurement of volatility to adjust for risk and therefore penalizes both upside and downside volatility equally. This concept has been critized and many investors prefer its modification known as Sortino Ratio. Instead of using standard deviation, the Sortino Ratio uses downside semi-variance and penalizes only those returns falling below a user-specified rate. This is a measurement of return deviation below a minimal acceptable rate. By utilizing this value, the Sortino Ratio only penalizes for “harmful” volatility.

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[4] http://finance.yahoo.com/ (2012)

Sortino ratio is calculated by the formula [16]:

$$SR = \frac{E(R_P) - MAR}{\sqrt{\frac{1}{T} \sum_{\substack{t=1 \\ R_{Pt} < MAR}}^{T} (R_{Pt} - MAR)^2}} \tag{1}$$

where

T represents the number of periods,

MAR denotes the minimum acceptable return,

R_{Pt} denotes the return of portfolio in t-th period, where $t = 1, 2, \ldots T$,

$E(R_P)$ denotes the expected return of portfolio.

The formulation of the portfolio selection problem based on Sortino ratio includes the maximization of performance measurement function (1). The model deals with the variables $w_1, w_2, \ldots w_d$ (where d represents the number of assets) that represent the weights of each asset in the portfolio. Corresponding problem can be formulated as follows [16]:

$$\max \; SR(\mathbf{w}) = \frac{\mathbf{w}^{\mathbf{T}}\mathbf{E} - MAR}{\sqrt{\frac{1}{T} \sum_{\substack{t=1 \\ \mathbf{w}^{\mathbf{T}}\mathbf{r}_t < MAR}}^{T} (\mathbf{w}^{\mathbf{T}}\mathbf{r}_t - MAR)^2}} \tag{2}$$

subject to:

$$\mathbf{w}^{\mathbf{T}}\mathbf{e} = 1 \tag{3}$$

$$\mathbf{w} \geq 0 \tag{4}$$

where

T represents the number of periods,

MAR denotes the minimum acceptable return,

$\mathbf{r_t}$ denotes the return of portfolio assets in t-th period, where $t = 1, 2, \ldots T$,

$\mathbf{E}$ denotes the excpected returns of portfolio assets.

3 Differential Evolution

Differential evolution (introduced by Price and Storn [17]) belongs to the class of evolutionary techniques that are considered to be effective tools that can be used to search for solutions of continuous non-linear problems, where is hard to use traditional mathematic methods (e.g. [1], [2], [3], [4], [5]). Differential evolution involves a search on population of individuals (np – number of individuals), where each individual represents one candidate solution for the given problem that is represented by parameters of individual (d – number of parameters of individual). Associated with each individual is also the fitness, which represents the relevant value of objective function. The steps of the algorithm are described in e.g. [11], [19], where it is also possible to find recommended values for control parameters.

If we want to apply the algorithm of differential evolution for solving problems of portfolio selection based on performance measurement, it is necessary to consider the following factors: *selection of an appropriate representation of individual, setting of the control parameters, transformation of unfeasible solutions.*

Selection of an appropriate representation of individual. The population $P^{(0)}$ is randomly initialized at the beginning of evolutionary process according to the rule:

$$\mathrm{P}_{(i)}^{(0)} = w_{i,j}^{(0)} = \frac{rnd\langle 0,1\rangle}{\sum_{j=1}^{d} w_{i,j}^{(0)}} \quad i = 1,2,...np \;\; j = 1,2,...d \tag{5}$$

that ensure that the total weights of portfolio is equal to one. Each individual is then evaluated with the fitness (given by function $SR(\mathbf{w})$).

Setting of the control parameters. Differential evolution is controlled by a special set of parameters. Recommended values for the parameters are usually derived empirically from experiments ([10], [11], [19]):

d – dimensionality. Number of parameters of individual is equal to number of assets.

np – population size. Number of individuals in population. Recommended setting is $5d$ to $30d$, respectively $100d$, in case the optimized function is multimodal ([10], [19]).

g – generations. Represent the maximum number of iteration (g is also stopping criterion).

cr – crossover constant, $cr \in \langle 0,1\rangle$. The value of cr was set on the base of statistical experiments.

f – mutation constant, $f \in \langle 0,1\rangle$. The value of f was set on the base of statistical experiments.

Transformation of unfeasible solutions. To ensure the feasibility of solution we use the following rule: if $w_j^{test} < 0$, then $w_j^{test} = rnd\langle 0,1\rangle$ and $\mathrm{P}_{(i)}^{test} = \mathrm{w}_{i,j}^{test} = \frac{w_j^{test}}{\sum_{j=1}^{d} w_j^{test}} \quad i = 1,2,...np \;\; j = 1,2,...d$.

Presented approach was also used to solve portfolio selection problem with Omega function performance measure [13].

4 Empirical Results

The portfolio analysis was based on index Dow Jones Industrial, which is one of the major market indexes, as well as one of the most popular indicators of the U.S. market. It is a stock market index, and one of several indices created by Wall Street Journal editor and Dow Jones & Company co-founder Charles Dow. It was founded on May 26, 1896. It is an index that shows how 30 large publicly owned companies based in the United States have traded during a standard trading session in the stock market (4M Co., Alcoa Inc., American Express Co., AT&T Inc., Bank of America Corp., Boeing Co., Caterpillar Inc., Chevron Corp., Cisco Systems Inc., Coca-Cola Co., DuPont, Exxon Mobil Corp., General Electric Co., Hewlett-Packard, Home Depot Inc., IBM, Intel, Johnson & Johnson, JPMorgan Chase & Co., Kraft Foods Inc., McDonald's Corp., Merck & Co. Inc., Microsoft Corp., Pfizer Inc., Procter & Gamble Co., Travelers Cos. Inc., United Technologies Corp., Verizon Communications Inc., Wal-Mart Stores Inc., Walt Disney Co.). Data are processed weekly for the period July 1st 2001 to April 1st 2012. A total of 559 data were analyzed.

The input parameter of MAR (the target of required rate of return) was set to 0.055.

The algorithms were implemented in MATLAB 7.1. All the experiments were run on PC INTEL(R) Core(TM) 2 CPU, E8500 @ 3.16 GHz, 3.25 GB RAM under Windows XP. The control parameters were set to: f = 0.1, cr = 0.2, np = 3000 and g = 2000 (according to [13]), and obtained value of the Sortino Ratio in all simulations was 0.1182.

Based on model results, recommended allocation is to invest assets in McDonald's companies at the rate 82.95 % and Caterpillar Inc. at the rate 17.05 %. According to result achieved, it is not recommended to invest to other companies since values of weights (variables) are equal to 0 %.

Mentioned problem was also solved using the Risk Solver Platform V.12.0, with the result equal to 0.1165, which is a smaller value compared to the value computed of Sortino ratio (the difference is 0.0017).

5 Conclusion

Various sets of performance measurement tools can be used to assist the investor to allocate capital over the number of assets. One of them is known Sortino ratio, which computability is difficult due to its structure and therefore the use of standard techniques to maximize it can be relatively complicated. Based on it, we use

differential evolution from the group of evolutionary techniques that enables to determine good real-time solution. The quality of results is comparable with results obtained by professional software. Based on showed results, it can be stated the suitability of presented approach.

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Method for Solving Robust Optimization Models

Jan Pelikán[1], Maria Kobzareva[2]

Abstract. Robust optimization is a modeling tool, which is designed to solve problems with uncertain data set. This kind of problem is a frequently solved problem in many fields, and it is more appropriate for real-life applications, because in practice data are hardly deterministic, but tend to change unexpectedly. The paper introduces robust optimization methodology for knapsack problem and suggests a possibility of solving portfolio optimization problem with uncertain data set using robust optimization model. To solve the problem we use robust linear programming. The problem takes into account the fact that share profits could not be the same as predicted, but real value of share profit could be less. In this paper we present the method, which transforms this problem into sequence of linear problems. The algorithm is programmed in VBA for MS Excel, which is connected with Lingo and uses it to solve the problem; results are returned to MS Excel. The application is described on a case study and provided with mathematical models and its detailed description. The paper also presents computation experiments and its results.

Keywords: robust optimization, data uncertainty, robustness, portfolio optimization problem

JEL Classification: C44
AMS Classification: 90C15

1 The portfolio selection problem and its robust formulation

Let us introduce portfolio optimization problem, where we have to select shares from a given number of shares, respecting the condition that the value of selected shares will not exceed given investment limit K. The aim is to maximize future profit of selected shares, future profit of shares are given. The problem is that the real share profit may be less than predicted value. Deviation of the estimation of future share profit from its predicted value can be expressed by probability distribution of the profit, but for our robust formulation we will use only estimation of future decrease in profit from the predicted value of future profit.

Let us define n as a number of shares, K – limit of total expenses on selected shares, for share i, the price of share is a_i, expected profit of share i is c_i and d_i is possible decrease in share profit. Binary variable x_i defines the decision to choose share i, when x_i=1, or decision not to choose the share in case x_i=0.

The basic model of portfolio selection where each share can be selected only once is described below:

$$\max_x \sum_{i=1}^{n} c_i x_i, \; s.t. \; \sum_{i=1}^{n} a_i x_i \le K, \;\; x_i \in \{0,1\} \; i = 1,2,...,n. \tag{1}$$

No possible decreases of future profit of shares are considered in the model (1).

Now we are looking for a solution which unlike the solution of the model (1) reflects uncertainty of the future profit of shares. But the decrease in profit does not need to occur at all shares, only at some shares. We will assume a decrease in profit at most in Γ shares, where integer $\Gamma \in \{0,1,...,n\}$ is a given **protection level**. As we do not know for which shares the profit will decrease we will assume that profit of Γ shares with highest d_i will decrease. According to this assumption we will create robust optimization portfolio model which gives us a robust solution of the problem. Robust solution will consider limited number of share profit decrease, denoted as

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Γ, defining the maximum number of shares that could meet the decrease in profit value. The prediction of total loss in share profits is given as a maximum loss for all subsets of Γ shares. The purpose of the case study is to find so called robust optimal solution under given circumstances, which respects unexpected decrease in future share value.

1.1 Mathematical model of robust portfolio selection problem

We denote a set of shares with a decrease in profit value S, so the robust portfolio model is (2), (3).

$$\max_x \ (\sum_{j=1}^{n} c_j x_j - \max_S \sum_{j\in S} d_j x_j), \quad S \subseteq \{1,2,...,n\}, \ |S| \le \Gamma, \tag{2}$$

$$s.t. \ \sum_{j=1}^{n} a_j x_j \le K, \ \ x_j \in \{0,1\} \ j = 1,2,...,n. \tag{3}$$

Γ is maximum number of shares which profit will decrease from value c_j on value $c_j - d_j$.

This robust optimization problem can be formulated alternatively as the following model (4), (5), where binary variable y_j denotes that share j belongs to subset S, in case y_j=1, y_j=0 otherwise.

$$\max_x \ (\sum_{j=1}^{n} c_j x_j - \max_y \sum_{j=1}^{n} d_j x_j \ y_j), \tag{4}$$

$$s.t. \ \sum_{j=1}^{n} y_j \le \Gamma, \ \sum_{j=1}^{n} a_j x_j \le K, \ \ x_j, y_j \in \{0,1\} \ j = 1,2,...,n. \tag{5}$$

The presented model is two step optimization problem, to solve this problem we can use the method presented in [1].

1.2 Method for solving robust optimization problem

The method considers basic robust problem presented as

$$F = \min_{x\in P}(\sum_{j=1}^{n} c_j x_j + \max_S \sum_{j\in S} d_j x_j), \ s.t. \ x \in P, S \subseteq \{1,2,...,n\}, \ |S| \le \Gamma \tag{6}$$

where a set P is any of the subset $\{0,1\}^n$.

Comment 1.

If the maximization problem is defined as maximization problem (2), (3) then we apply algorithm for minimization problem presented as follows:

$$-F = \min_{x\in P}(\sum_{j=1}^{n} (-c_j) x_j + \max \sum_{j\in S} d_j x_j), \ S \subseteq \{1,2,...,n\}, \ |S| \le \Gamma, P = \left\{ x \text{ binary}; \ \sum_{i=1}^{n} a_i x_i \le K \right\}. \tag{7}$$

The solution of (6) is based on the following proposition 1.

Proposition 1 [1]**.**

The assumption is $d_1 \ge d_2 \ge ...d_n \ge d_{n+1} = 0.$ The optimal solution is

$$F = \min_{l=1,2,...,n+1} G_l, \text{ where } \ G_l = \Gamma d_l + \min_{x\in P} (\sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{l} (d_j - d_l) x_j). \tag{8}$$

We can formulate algorithm that solves robust optimization problem (6) according the proposition 1.

Algorithm 1[1]**:**

Step 1. Solve the problem and calculate values of G_l, l= 1, 2, …, n+1 as follows:

$$G_l = \Gamma d_l + \min_{x \in P} (\sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{l} (d_j - d_l) x_j). \tag{9}$$

Step 2. Calculate the value $F = \min\{G_1, G_2, ..., G_{n+1}\} = G_{l*}$. The optimal robust solution is x_l*, which is optimal solution for problem G_l*.

Comment 2.

The problem (7) and (9) are linear optimization problems with binary variables.

Comment 3.

The value Γ close to zero indicates an optimistic forecast of the future profit of shares and conversely high value Γ means the pessimistic estimation of the shares profit.

1.3 Generalization of the method for robust portfolio optimization problem with integer variables

Now we will reformulate algorithm for solution of portfolio selection model with integer variables, so in (2), (3) or (4), (5) are the integrality condition for x_j instead of binary conditions.

We will solve the portfolio selection problem with the variable $\bar{x}_k$, which is integer and it holds $0 \le \bar{x}_k \le s$, so the maximal number of k-th share is s in the solution. So we can express value $\bar{x}_k = \sum_{i=0}^{s} x_{k+i}$, where x_i are binary variables. The solution is based on the Proposition 1 and Algorithm 1.

Proposition 2.

Now we suppose:

$$d_1 \ge d_2 \ge ... \ge d_k = d_{k+1} = ... = d_{k+s} \ge d_{k+s+1} \ge ... \ge d_n \ge d_{n+1} = 0.$$

Then $G_k = G_{k+1} = ... = G_{k+s}$.

Proof.

For i=0, 1, …, s consider

$$d_{k+i} = d_k \ \ and \ \ (d_{k+1} - d_{k+i}) = (d_{k+2} - d_{k+i}) = ... = (d_{k+i} - d_{k+i}) = 0.$$

Then

$$G_{k+i} = \Gamma d_{k+i} + \min_{x \in P} (\sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{k} (d_j - d_{k+i}) x_j + (d_{k+1} - d_{k+i}) x_{k+1} + (d_{k+2} - d_{k+i}) x_{k+2} + ... + (d_{k+i} - d_{k+i}) x_{k+i}) =$$

$$= \Gamma d_k + \min_{x \in P} (\sum_{j=1}^{n} c_j x_j + \sum_{j=1}^{k} (d_j - d_{k+i}) x_j) = G_k. \tag{10}$$

Comment 4.

This proposition gives us a possibility to solve robust portfolio optimization problem with integer conditions for variables x_i. To solve problems G_l, we can replace binary conditions for x_i by integrality conditions. The upper bounds for x_i can be added.

2 Case study

Let us solve the problem with investment limit K=50 000 and set of 23 shares, from which a subset has been selected. The data subset was obtained from web pages http://finance.yahoo.com/, where we used path investment, stocks, earning dates to collect necessary data. Each share has a current price, which was stated in the beginning of April 2013 (the time the experiments were made), estimated profit of the share and possible decrease in profits. The price a_i of share i, the profit c_i and decrease of the profit d_i are given in the Tab. 1. Protection level is set $\Gamma=3$, meaning the solution will take into account decrease in profit of maximum three shares.

№	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
a_i	2999	10400	1633	6825	1625	1791	1189	998	2253	3934	3554	3608
c_i	237	271	45	72	30	1	10	-13	42	44	62	61
d_i	64	31	15	14	14	13	12	12	11	9	9	9
№	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	
a_i	3918	4901	2543	768	3134	1398	3299	5272	168	283	1808	
c_i	102	91	26	10	8	12	3,7	58	-1	-8	23	
d_i	7	6	6	5	3	2	2	2	0	0	0	

Tab. 1: List of shares and their parameters

We will use the algorithm described above and solve linear program and calculate G_1, G_2,..., G_{n+1}, results are shown in the Tab. 2. Beside the optimal value G_1 in the table you can also find the corresponding solution x, which denotes the share selection.

l	G_l	**solution *x***
1	912	1 1 1 1 1 0 0 0 1 0 1 1 1 1 0 1 0 0 0 1 0 0 1
2	978	1 1 1 1 1 0 0 0 1 0 1 1 1 1 0 1 0 0 0 1 0 0 1
3	**994**	**1 1 1 1 1 0 0 0 1 0 1 1 1 1 0 1 0 0 0 1 0 0 1**
4	994	1 1 1 1 1 0 0 0 1 0 1 1 1 1 0 1 0 0 0 1 0 0 1
5	994	1 1 1 1 1 0 0 0 1 0 1 1 1 1 0 1 0 0 0 1 0 0 1
6	993	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
7	992	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
8	992	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
9	991	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
10	987	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
11	987	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
12	987	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
13	977	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
14	971	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
15	971	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
16	963	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
17	947	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
18	939	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
19	939	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
20	939	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
21	919	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
22	919	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
23	919	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1

Tab. 2: Steps of the algorithm and optimal robust solution, Γ = 3

As you can see in Tab. 2 maximum robust solution value is F=max{ G_1, G_2,…, G_{n+1}}=max {912, 978, 994, 994, 994, 993, 992, 992, 991, 987, 987, 987, 977, 971, 971, 963, 947, 939, 939, 939, 919, 919, 919} = 994 with the

corresponding share selection x=(1 1 1 1 1 0 0 0 1 0 1 1 1 1 0 1 0 0 0 1 0 0 1). The following profit takes into account decrease in profits of maximum three shares (considering the highest value of profit loss).

In further experiments we'd change protection level to Γ from 0 to 23, to see how the change affects the solution. The results are provided in the table below. The first row obtains the solution for Γ = 0, which means we consider data to be static and no future decrease in profit is expected, the objective function in this case is at its maximum. Then as you can see, solution remains the same, should we take into account decrease in profits of maximum 0 to 4 shares. Maximum robust solution value though decreases from 1040 to 980, which is given by consideration of decrease in profits of more shares in every further step. When we consider Γ to be 5 to 23, the solution has changed, now robust value of the solution remains constant for every Γ>12 and is 919. This means that after a certain point it is not effective to change Γ, as number of selected shares is less than number of shares that could meet possible decrease in profit.

Γ	***F***	**Solution *x***
0	1104	1 1 1 1 1 0 0 0 1 0 1 1 1 1 0 1 0 0 0 1 0 0 1
1	1040	1 1 1 1 1 0 0 0 1 0 1 1 1 1 0 1 0 0 0 1 0 0 1
2	1009	1 1 1 1 1 0 0 0 1 0 1 1 1 1 0 1 0 0 0 1 0 0 1
3	994	1 1 1 1 1 0 0 0 1 0 1 1 1 1 0 1 0 0 0 1 0 0 1
4	980	1 1 1 1 1 0 0 0 1 0 1 1 1 1 0 1 0 0 0 1 0 0 1
5	969	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
6	960	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
7	951	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
8	942	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
9	935	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
10	929	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
11	923	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
12	921	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
13	919	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
14	919	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
15	919	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
16	919	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
17	919	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
18	919	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
19	919	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
20	919	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
21	919	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
22	919	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1
23	919	1 1 1 0 1 0 0 0 1 1 1 1 1 1 1 0 0 1 0 1 0 0 1

Tab. 3: Optimal robust solutions of the portfolio selection problem for different value of Γ.

3 Conclusion

This paper offers possibility to solve portfolio optimization problem with uncertain share values using robust optimization tool. The problem takes into account the fact that share profits could not be the same as predicted, but real profit share value could be less. The described potential loss is considered only with a few shares, their number is a given by parameter Γ, the protection level. When considering the parameter Γ we take into account maximum possible loss in profits for given shares. In this paper we present the method, which transforms the problem into sequence of linear problems. The presented method is described on a case study, providing with computation experiments.

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Quantification of multiplication activity of government on product

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Abstract. Article deals with quantification measures realized by government aiming to influence economy´s efficiency. Firstly the quantification of the multiplication effect of governmental expenditure on the amount of created product. The second part is the description of the effect of taxation on the level of created product. In conclusion the effects of above mentioned elements influencing the amount of product are compared.

Keywords: government, multiplication effect, governmental expenditure, tax, product.

JEL Classification: E12, E16
AMS Classification: 20G40

1 Introduction

The simple Keynesian model as one of the models that are used to determine the equilibrium output in the economy or the determination of its changes. In this model, there is a balance or change through the aggregate expenditure. The model is based on the assumption that the aggregate expenditures are stimulus of growth the product and conversely, growth of product significantly affects the growth of aggregate expenditures.

This model is also called model with line 45° or also a model with a multiplier. First, will be discussed about government interference, which use the effect of government expenditures on the equilibrium product,, to influence the equilibrium product. Then it will then be characterized the influence of taxes on household incomes, to the consumption function and the multiplier effect on the equilibrium level of the product.

Keynesian model and the determination of the product

The basic principle of the Keynesian model [10], beginning from the simple multiplier to the three sector model, is that a determination of the income is based on the flow condition of the goods market in short-run equilibrium [9]. The assumptions for the determination of the equilibrium product in three sector economy model are as follows:

- Rigid prices, doesn´t affect an aggregate demand.
- The supply of capital is sufficient, can be produced the same output which is demanded.
- The labor offer is sufficient, can be produced the same output which is demanded by the fixed nominal wage rate.
- All nominal variables are the real variables.
- The economy is closed.

From the assumption of the sufficient capital and the sufficient labor offer, we can deduce, that the production gap is existing in this economy, it means that real product Y is below the potential product Y*. Within the economic model, we abstract from:

- The difference between the gross national product and the national income.
- The wear of the capital, it means the difference between the gross and net national product.
- The difference between the gross and net investment, we will call about investment expenditures [8].

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The Keynesian assumption posits [1, p.256; 5] those prices are sticky, which implies that output is driven by demand. When prices are constant, inflation is nil-and expected to remain nil-and the nominal and real interest rates are the same.

The character of the Keynesian economic model is reflected as the insufficient of the aggregate demand, demand-limited economy. To determine the equilibrium product of Three-sector model is necessary to determine the real aggregate expenditures on product.

The real aggregate expenditures on product are called AD (or AE) and they represent the sum of the household consumption expenditures C and the company investment expenditures I and government expenditures for the purchase goods and services G. The definition of aggregate demand [2, p. 69; 11] has to be augmented to include government purchases of goods and services-purchases of military equipment and services of bureaucrats, for instance. Thus we have:

$$AD = C + I + G \tag{1}$$

1.1 Total expenditure assuming the absence government expenditure

If there is no delay, we assume that the planned consumption and investment will implement, there exist an equilibrium condition of balance of production and planned consumption and planned investment, it means:

$$Y = C + I \tag{2}$$

where: Y is product,
C is a consumption, i.e. household consumption expenditures,
I are planned investment expenditures;

This means an equivalent with equilibrium condition, i.e. the equality of the savings and investments in planned amount:

$$S = I \tag{3}$$

where:

$$S = Y - C \tag{4}$$

and S are the savings, i.e. function of the savings.

Suppose that P the price level is known and the variables Y, S, C and I are in monetary state. Investment I depends on the interest rate r, they are autonomous and independent on income. The consumption function (2) is a fiction of income:

$$C = f(Y) \tag{5}$$

and the function of savings is from the relation (3) and is a function of income as well (because the income is the sum of consumption and savings).

$$S = Y - f(C) = S(Y) \tag{6}$$

Then sum of the marginal propensity to consume c=dC/dY and the marginal propensity to savings s=1-c=dS/dY on dependence to income is equal to one. Constant parts of variables C and I can be dislocated and marked as autonomous expenditures.

Aggregate demand (effective demand) contains only requirements of households on consumer goods C and investment goods required by the business community I, by assuming, there is no government economic activity and no foreign trade. If the aggregate demand is lower than would be by the full-employment, then the savings from the income by the full-employment are higher than planned investment. So, Y > C + I, then S = Y - C > I. If we think about opposite side, i.e. aggregate demand is higher than production by the full-employment, then there is a deficit in savings (lack of savings).

The equilibrium in Keynesian model requires the situation when there is no surplus neither lack of aggregate demand, i.e. there doesn´t exist the difference between planned savings and investments. If there is a surplus or deficit of aggregate demand, it means equilibrium of goods market. The surplus or deficit of aggregate demand

has an influence on employment. The unemployment in Keynesian model is created from the inflexible wages in down-direction or by influence of existed non-equilibrium state.

1.2 The influence of government activity - The government sector

If the model includes the government expenditures for purchasing the goods and taxation then the aggregate demand contains expenditures, equation (1). The state of equilibrium becomes when the income is equal to total expenditures:

$$Y = AD \tag{6}$$

If the government expenditures G are higher than decrease of sum C+I, can be eliminated the unemployment or can be increased low equilibrium level of employment.

The surplus of aggregate demand can be caused by a sudden interest to investing or by the government side and then the preservation of equilibrium is accompanied by a rise of price level. The situation of price increasing can be explained by comparing the static equilibrium states.

$$Y = Y_d + T \tag{8}$$

The government interventions by the government expenditures and taxation cause two changes in the Keynesian model. At first, to the aggregate expenditures can be addend government expenditures G. Secondly, we make a difference between the total income Y and available income Yd, which is the volume of taxation T, and then the total income is calculated:

$$Y = Y_d + T \tag{9}$$

By the influence of taxation is necessary to adjust consumption function to match the planned consumption from the available income: consumption function is a function of available income C=C(Yd) and the marginal prosperity to consumption c = dC/d Yd. That is why the savings are the function of available income: S=S(Yd), where S(Yd) = Yd - C(Yd) and the marginal prosperity to savings is s = dS/d Yd.

Impact of government activity - a multiplier of government expenditure and tax multiplier

In the Keynesian model, all investments (I) are autonomous, and if they are added autonomous consumption expenditure of households C_A, their sum makes the autonomous spending $A_A = C_A + I$. Also we assume that government spending G and taxation T are autonomous, given by the external factors and not dependent on the level of income. Then the equilibrium condition on the goods market can be expressed as follows:

$$Y = C_A + C(Y_d) + I + G \tag{10}$$

If we substitute for Y = Yd + T, the condition of equilibrium on the goods market is in the form:

$$Y_d = C_A + C(Y_d) + I + G - T \tag{11}$$

In the model of the multiplier, condition of the equilibrium (Y = C + I) determines the equilibrium level of disposable income Yd, and also the total income Y. The solution depends on the shape of consumption function. This balance condition can also be written using saving function: S (Yd) = Yd - C (Yd). If we denote by Sg as government saving, i.e. supplement to non-public savings S (Yd) is calculated:

$$S_g = T - G \tag{12}$$

then the equilibrium condition (Y = C + I) takes shape:

$$S(Y_d) + S_g = A_A, \tag{13}$$

i.e. the new version of the relationship between savings and investment. The values A_A and Sg are given hard outside, so the equilibrium level of disposable income is given by the above relationship and hence the total income Yd = Y + T.

If the government savings or taxes are changed, how the income will change? In the basic situation the government spending are at a given level G and taxation at the level T, then the income will be fixed at the equilibri-

um level by the above two relations. Suppose that the government increases its spending by that amount ΔG or taxing by the amount ΔT.

There are several situations:

1. The government changes government spending about ΔG and ΔT=0, the government surplus Sg decreases. If we want to find the change of total income, we derive the relationship $Y_d = C_A + C(Y_d) + I + G - T$ according to G:

$$\frac{Y_d}{dG} = \frac{dC}{dY_d}\frac{dY_d}{dG} + 1 \tag{14}$$

and

$$\frac{dY_d}{dG} = \frac{dY_d}{dG} \tag{15}$$

therefore:

$$\frac{dY_d}{dG} = \frac{1}{1-c} = \frac{1}{s} \tag{16}$$

When expressed as a change of income ΔY:

$$\Delta Y = \frac{\Delta G}{s} \tag{17}$$

It is given by the multiplier resulting from increasing of government spending ΔG. Additional government spending is multiplied by multiplier l / s> 1. Influence of multiplier government expenditure on size of product can be graphically represented as shown below [4, 6].

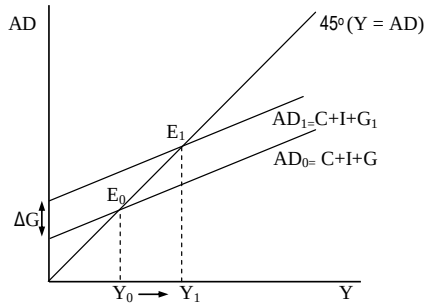

Figure 1 The multiplier effect of government expenditure

2. The government changes taxes ΔT and ΔG=0, the government surplus Sg increases. If we want to find change of total income, we derive again that relation according T:

$$\frac{Y_d}{dT} = \frac{dC}{dY_d}\frac{dY_d}{dT} - 1 \tag{18}$$

and

$$\frac{dY_d}{dT} = \frac{dY_d}{dT} + 1 \tag{19}$$

therefore:

$$\frac{dY_d}{dT} = -\frac{1}{1-c} + 1 = -\frac{c}{1-c} = -\frac{1-s}{s} \tag{20}$$

When expressed as a change of income ΔY:

$$\Delta Y = \frac{1-s}{s}(-\Delta T) \tag{21}$$

where:

$$\frac{1-s}{s} \langle \frac{1}{s} \tag{22}$$

Valid: income decreases due to higher taxes (see figure below) and vice versa due to influence of decreases of taxes the income increases.

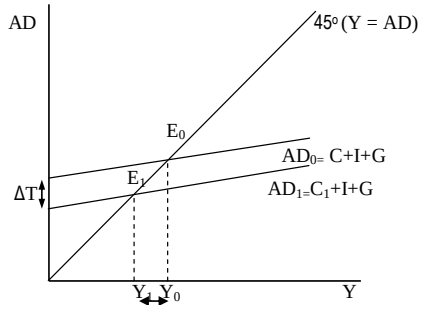

Figure 2 The multiplier effect of government expenditure

The multiplier effect caused by a decrease in taxes is multiplied by a small factor (1 - s) / s. The multiplier may be smaller than the unit, since (1 - s) /s <1, where s >1/2. This is due to the fact that part of the income resulting from the reduction of taxes, i.e. c(-ΔT) = (1 - s) (-ΔT) is spent immediately, but the remaining part is saved.

Government use budgets to control and record their fiscal affairs. A budget shows, for a given years, the planned expenditures of government programs and the expected revenue from tax system. The budget typically contains a list of specific programs (education, welfare, defense, etc.), as well as tax sources (individual income tax, social-insurance taxes, etc.) [3; 7, p.288; 11].

If the government pursues a balanced state budget, government spending and revenues are in balance (G = T). Therefore, equation (25) gives: Y = (A / s) + G. The equilibrium level of income, given by multiplier is increased exactly by the amount of government expenditure for the purchase of goods. Generally, we can assume that for any consumption function and any level of government expenditure and taxation, public expenditure increased by ΔG, which corresponds to the same increase in taxation ΔT = ΔG. Therefore, from the above we calculate increasing of income:

$$\Delta Y = \frac{\Delta G}{s} - \frac{1-s}{s}\Delta T = (\frac{1}{s} - \frac{1-s}{s})\Delta G = \Delta G \tag{26}$$

From this follows that offset the increase in government spending and taxation leads to income growth just by the amount of increase in government expenditure on purchase of goods.

2 Summary

Based on the analysis above about the effect of government on the economy through government expenditure and taxes can be concluded that for the multiplier effects caused by changes government expenditure or tax changes apply: if the marginal propensity to savings in the equilibrium level of output is equal to s (0 <s <1), then the growth government expenditure increases equilibrium income by a multiple (1 / s) of increment ΔG and decline taxes increases income about less multiple.

The two situations may be combined in various proportions. The real income level corresponding to a state of equilibrium can be calculated, if known specific shape of consumption or saving function. If we have a linear consumption function: C = cYd where the marginal propensity to consume is constant c in all incomes, then equation (1) gives:

$$Y_d = cY_d + A_A + G - T \tag{23}$$

and

$$Y = Y_d + T \tag{24}$$

then applies:

$$Y = \frac{A_A + G - T}{s} + T = \frac{1}{s}(A_A + G) - \frac{1-s}{s}T \tag{25}$$

Thus, the linear multiplier l / s applies to autonomous expenditures (A_A as G), but only for the part (1 - s) taxes.

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Min-max model for the network reduction problem

Štefan Peško[1]

Abstract. We deal with solving a NP-hard problem that occurs in reduction of transport networks. The reduction means to omit some edges. There are two requirements imposed on the resulting reduced network: First, the upper bound of the total length of reduced network is known. Second, the distance between arbitrary two points in reduced network should not exceed q-multiple of that in original network, where q should be as small as possible. There are many applications of just mentioned problem e.g. when designing a bus line system in a town. We present a constraint programming formulation and an exact solution using MiniZinc model.

Keywords: network reduction problem, min-max model, constraint programming, MiniZinc

JEL classification: C51, R48
AMS classification: 15A06, 90C48

1 Introduction

In practice, one can often meet different decision problems how to choose a reduction of transport network. The common features of all of them is that a subgraph fulfilling some constraint has to be found for the given graph. There are many applications of mentioned problems e.g. when designing a bus line system in a town. We have found interesting application in biology Hassan [4], where the acyclic network considered there has two attributes in each arc: time and length, with a time limit on the shortest length path.

Černá, et al. [3] studied the family of subgraphs not lengthening some important trip more that for the given percentage. Czimmermann [2] studied the computational complexity of the problem Admissible Lengthening of Important Routes and showed that the problem is NP-hard for any lengthening parameter $q > 1$. Přibyl [5] deals with a single bus route design problem where a goal of reduction of a network is to minimize the mean walking distance of passengers between the nearest stops of bus route.

We deal with solving the case of min-max reduction of transport network where are two requirements imposed on the resulting reduced network:

1. The upper bound of the total length of reduced network T is known;
2. The distance between arbitrary two points in reduced network should not exceed q-multiple of that in original network.

The goal is to find omitted edges of transport network so that q should be as small as possible. This minimal value q^* gives maximal lengthening of any minimal path in reduced network.

2 Mathematical formulation

We deal with the following mathematical formulation of network reduction problem: Suppose we are given a weighted graph $G = (V, E, d)$ and a number T greater than spanning tree of graph G. The goal is to find number $q, q \geq 1$ and a connected spanning subgraph $G_q = (V, E_q, d_q)$ of G with minimal value of parameter q such that the length of subgraph G_q is at most T and the distance between each pair of vertices of removed edge is at most q-times of the length of this edge.

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More precisely, we find a connected spanning subgraph $G_q = (V, E_q, d_q)$ such that

$$\begin{aligned} q &\rightarrow \min && (1)\\ \sum_{e \in E_q} d(e) &\leq T, && (2)\\ d_q(u,v) &= d(u,v) \qquad \forall \{u,v\} \in E_q, && (3)\\ d_q(u,v) &\leq q \cdot d(u,u) \qquad \forall \{u,v\} \in E - E_q, && (4)\\ E_q &\subseteq E, && (5)\\ q &\geq 1. && (6) \end{aligned}$$

The model (1)–(6) is an original model inspired by [3]. Note that for a real problem an expected value of q is about 1.3.

The idea of constraint programming (CP) is to solve problems by simply stating constraints (conditions, properties) which must be satisfied by a solution of the problem. We show how it is possible to formulate and solve our problem (1)–(6) via CP.

3 MiniZinc models

MiniZinc [1] is a language designed for specifying constrained optimization and decision problems over integers and real numbers. We first define used parameters and domains and then we formulate a model for computing a distance matrix in a weighted graph. Next, we will formulate our reduction problem which is based on computing the distance matrix.

3.1 MiniZinc model for distance matrix

The goal of this model is to find a distance matrix $X = (x_{ij})$ where x_{ij} is the length of the shortest $i-j$ path in graph $G = (V, E, d)$. For any connected graph G it is possible to solve this as following original LP problem:

$$\begin{aligned} \sum_{i \in V} \sum_{j \in V: i<j} x_{ij} &\rightarrow \max && (7)\\ x_{ik} + x_{kj} &\geq x_{ij} \qquad i \in V, j \in V, k \in V : i \neq j, i \neq k; j \neq k, && (8)\\ x_{ij} &= x_{ji} \qquad i \in V, j \in V : i < j, && (9)\\ x_{ij} &\leq d_{ij} \qquad \{i,j\} \in E. \end{aligned}$$

For formulation of problem (7)–(10) in MiniZinc we will use following notation:

Parameters

- $nVertix$ – number of vertices,
- $nEdge$ – number of edges,
- $infinity$ – satisfactory big integer number,
- $V = \{1, \ldots, nVertix\}$ – set of vertices,
- $E = \{1, \ldots, nEdge\}$– set of order of edges,
- $Edges[]$ matrix of weighted edges of size $nEdge \times 3$ where k-th edge $\{i, j\}$ weighted by integer d_{ij} is represented by k-th row with $Edges[k, 1] = i, Edges[k, 2] = j, Edges[k, 3] = d_{ij}$.

Variables

- $X[]$ – matrix of distances of size $nVertix \times nVertix$ with domain $\{0, \ldots, infinity\}$,
- *total_cost* – sum of distances.

Note that variable *total_cost* gives us the sum of all distances between any distinct vertices of graph. It is used as variable equal to the result of goal function of model. We use two constraints as follows:

- **C1**: Since $x_{ij} \leq d_{ij}$ and $x_{ij} = x_{ji}$ for each edge $\{i, j\}$ we can write in MiniZinc:

```
constraint
  forall(e in E)(
      X[Edges[e,1], Edges[e,2]] <= Edges[e,3]
      /\
      X[Edges[e,1], Edges[e,2]] = X[Edges[e,2], Edges[e,1]]
  );
```

- **C2**: The feature of the distance matrix is that it is a metric on the set V and so can write:

```
constraint
  forall(i in V, j in V, k in V where i!=j /\ k!=i /\ k!=j)(
      X[i,k] + X[k,j] >= X[i,j]
  );
```

Diagonal elements of distance matrix are zeros, but this feature will satisfy our requirement by default since variables $X[i, i]$ are declared in model only and domain for distances is set $\{0, \ldots, infinity\}$.

3.2 MiniZinc model for min-max network reduction

We show how to utilize the MiniZinc model for distance matrix as a base for our network reduction problem. We will extend above used notation:

Parameters (continue)

- T – upper bound of length of reduced network.

Variables (continue)

- $q100$ – integer coefficient of extension is equal $100 * q$,
- $Y[]$ – binary matrix of size $nEdge$ where $Y[k] = 1$ if k-th edge is in reduced network and $Y[k] = 0$ otherwise,
- $Z[]$ – matrix of flows of size $nEdge \times 2$ where $Z[k, 1]$ is flow $Edges[k, 1] \rightarrow Edges[k, 2]$ and $Z[k, 2]$ is flow $Edges[k, 2] \rightarrow Edges[k, 1]$.

Interpretation of the variable *total_cost* is unchanged. Next we will assume that the goal variable $q100$ is in range $100..150, (1 \leq q \leq 1.5)$ because all parameters and variables must be integer. If an optimum q is greater then 1.5 then constraints (2)–(6) are unsatisfiable. The objective function is in the form of a fraction

$$\frac{total\_cost}{100 + q100}, \tag{11}$$

where the numerator is maximized and the denominator is minimized and so the fraction is maximized.

We use six constraints as follows:

- **C4**: The maximum length of reduced network is given by the constraint (2):

```
constraint
  sum(e in E)(Edges[e,3]*Y[e]) <= T;
```

- **C5**: In reduced network we choose some edges only and so corresponding constraint C1 must be modified:

```
constraint  % upper bounds and symmetry
  forall(e in E) (
      X[Edges[e,1], Edges[e,2]] <= max(Edges[e,3], (1-Y[e])*infinity)
      /\
      X[Edges[e,1], Edges[e,2]] = X[Edges[e,2], Edges[e,1]]
  );
```

- **C6**: Constraint **C2** above triangulation inequality remains valid:

```
constraint
  forall(i in V, j in V, k in V where i!=j /\ k!=i /\ k!=j)(
      X[i,k] + X[k,j] >= X[i,j]
  );
```

- **C7**: A feasible reduction of edge $\{i, j\}$ is possible only if exists a vertex k so that $x_{ij} = x_{ik} + x_{kj}$:

```
constraint
  forall(e in E) (
      Y[e] = 0 ->  exists(k in V diff {Edges[e,1],Edges[e,2]})
              (X[Edges[e,1],Edges[e,2]] = X[Edges[e,1],k] + X[k,Edges[e,2]])
  );
```

- **C8**: The feasible extension of minimum i-j path is possible if constraints (3)–(4) are valid

```
constraint
  forall(e in E) (
      (Y[e] = 1 ->  X[Edges[e,1], Edges[e,2]] = Edges[e,3])
      /\
      (Y[e] = 0 ->  X[Edges[e,1],Edges[e,2]] <= q100 * Edges[e,3] div 100)
       /\
      (Edges[e,3] < infinity -> X[Edges[e,1],Edges[e,2]] >= Edges[e,3])
  );
```

- **C9**: The reduced network is at least spanning tree with $|V| - 1$ edges:

```
constraint
  sum(e in E) (Y[e]) >= nVertex-1;
```

- **C10** The spanning tree is constructed using flows from every (non-root) vertex to root 1:

```
constraint
  sum(e in E) (bool2int(Edges[e,1]=1)*Y[e]*Z[e,1])=nVertex-1
  /\
  forall(i in 2..nVertex) (
      sum(e in E) (bool2int(Edges[e,1]=i)*Y[e]*Z[e,1]
          +bool2int(Edges[e,2]=i)*Y[e]*Z[e,2]) =
      sum(e in E) (bool2int(Edges[e,2]=i)*Y[e]*Z[e,1]
          +bool2int(Edges[e,1]=i)*Y[e]*Z[e,2])-1
  );
```

4 Ilustrative example

Let us show a small example of transportation network modelled by the weighted graph $G = (V, E, d)$ that is pictured on the left of the Figure 1. Assume that the transport company wants to save around 50% of total length 134 units of network so it will tolerate maximum length of network 70 units i.e. $T = 70$. We used gecode [6] solver library for solving constraint programming. After computing via

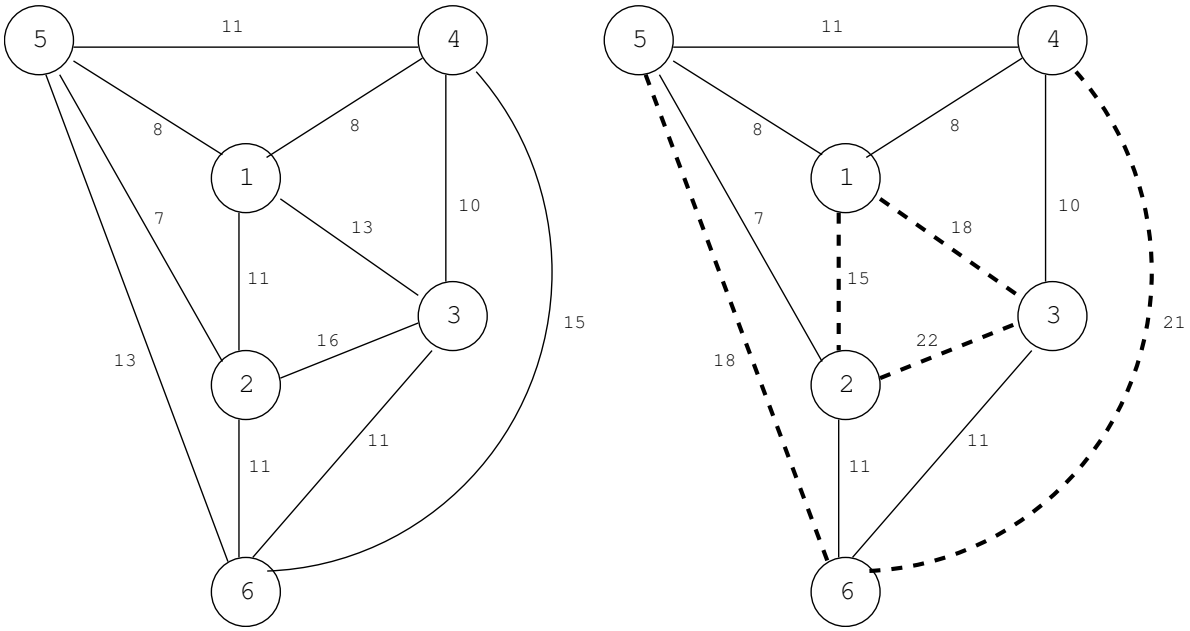

Figure 1 Original and reduced transportation networks

our model we got a solution, reduced network which is represented as the connected spanning subgraph $G_q = (V, E_q, d_q)$ with required properties. Dashed edges pictured on the right of the Figure 1 can be removed from original network. Passengers could tolerate maximal 1.4-multiple excess of travel time i.e. $q = 1.4$. For removed edges $\{i, j\}$ applies extension q_{ij}: $\{1, 2\}, 1.36; \{1, 3\}, 1.38; \{2, 3\}, 1.38; \{5, 6\}, 1.38$ and $\{4, 6\}, 1.4$.

5 Conclusion

MiniZinc model does not dictate how to solve the problem. First experiments with solver gecode show that we can really solve only small instances with number of vertices $10 - 30$. We observed that rate and time of solution is dependent on choice of parameter infinity. We got the best result for

$$infinity = 1.5 \cdot \max\{d(i, j) | i \in V, j \in V\} + 1,$$

where $d(i, j)$ is distance between vertices i and j in graph G.

We hope that solving real instances with $40 - 60$ vertices will be possible after using some solver for parallel search of constraint programming on grids.

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Multivariate stochastic orderings consistent with preferences and their possible applications

Filomena Petronio[1], Sergio Ortobelli[2], Tomáš Tichý[3]

Abstract. In this paper we propose multivariate stochastic orderings of risk positions that are consistent with the preferences of investors. Since investors maximize an expected state-dependent utility function, we analyze stochastic orderings that take into account of possible investors' benchmarks. We begin by analyzing the links among continua stochastic dominance orders, survival and inverse stochastic dominance rules. Then we tie together the consistency-isotonicity of risk and reward multivariate measures with classical orderings and we show how risk/variability multivariate measures are used to obtain non dominated choices. In second analysis we discuss an extension of the classical orderings using probability functionals that satisfy an opportune identity property and the basic rules of the theory of integral stochastic orders. In the discussion we propose several examples that clarify the main developments of the proposed analysis. In particular we describe new multivariate orderings that are consistent with a particular ordering of preferences. We discuss how to better distinguish and classify the multivariate choices using the fractional integral properties. We draw upon a recent classification of risk measures and orderings that is based on the theory of probability metrics. Employing the techniques of probability metrics theory, we then present further extensions and representations of probability functionals which are consistent with classic and dual orderings. Finally, we analyze some potential applications in finance of the new orderings.

Keywords: Multivariate preferences, Moment ordering, Concordance ordering

JEL classification: G02, C44, G11
AMS classification: 60E15, 90B50, 91B06

1 Introduction

The fundamentals of utility theory under uncertainty conditions have been developed by von Neumann and Morgenstern in [1]. Many improvements and advancements of the theory have been proposed. In particular, Karni [2] and Schervisch et al. [3] have emphasized that investors' choices are strictly dependent on the possible states of the returns. Thus, investors have generally state-dependent utility functions. Moreover, as shown by Castagnoli et al. [4], the state-dependent utility and the target-based approaches are equivalent. Thus, the more appealing benchmark approach is a generalization of the classic von Neumann and Morgenstern approach.

In this paper we introduce multivariate orderings consistent with investors' preferences and we classify them, distinguishing several categories associated with different classes of investors. Moreover, we show how we can use multivariate risk measures and orderings consistent with some preferences to determine investors' optimal choices. We are interested in the economic use of probability functionals to optimize choices for a given order of investors' preferences. In particular we generalize the concept of univariate FORS orderings, risk and reward measures in the multivariate framework (see Ortobelli et al. [5], [6], [7]). FORS probability functionals and orderings generalize those found in the literature (see, among others, Shaked et al. [8]) and are strictly related to the theory of choice under uncertainty and to theory

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of probability functionals and metrics, (see Rachev [9] and Stoyanov et al. [10]). While the new orderings serve to further characterize and specify the investors' choices/preferences, the new risk measures should be used either to minimize the risk or to minimize its distance from a given benchmark.

The paper is organized as follows. In Section 2 we introduce multivariate FORS orderings. The last section summarizes some possible financial applications.

2 Multivariate FORS measures and orderings

In this section we introduce FORS multivariate measures and orderings. Recall that the most important property that characterizes any probability functional associated with a choice problem is the consistency with a stochastic order. In terms of probability functionals, the consistency is defined as: X dominates Y with respect to a given order of preferences $\succ$ implies $\mu(X,Z) \leq \mu(Y,Z)$ for a fixed arbitrary benchmark Z (where $X, Y, Z \in \Lambda$ that is a non-empty space of real valued random variables defined on $(\Omega, \Im, P)$). Since an univariate FORS measure induced by of preferences $\succ$ is any probability functional $\mu : \Lambda \times \Lambda \to R$ that is consistent with a given order of preferences $\succ$ we can similarly define multivariate FORS measures.

Definition 1. We call FORS measure induced by a preference order $\succ$ any probability functional $\mu : \Lambda \times \Lambda \to R$ (where Λ a non-empty set of real-valued n-dimensional random vectors defined on the probability space $(\Omega, \Im, P)$) that is consistent with a given order of preferences $\succ$ (that is, if X dominates Y with respect to a given order of preferences $\succ$ implies $\mu(X,Z) \leq \mu(Y,Z)$ for a fixed arbitrary benchmark Z).

Example 1. Suppose the random vector X dominates Y with respect to the usual multivariate order. Then for any fixed benchmark $x \in R^n$

$$G_X(x) := -\int_{A_x} dP_X \leq G_Y(x)$$

where $A_x = \{y \in R^n | \exists i \leq n : y_i > x_i\}$ with $P_X(y) = P(X_1 \leq y_1, \cdots, X_n \leq y_n) = F_X(y_1, \cdots, y_n)$. Thus the functional $G_X(x)$ associated to the random vector X is consistent with the usual multivariate order.

As for the FORS measures we can easily extend the definition of multivariate FORS ordering developed in see Ortobelli et al. [5] and [6].

Definition 2. Let $\rho_X : A \to \bar{R}$ (with compact and convex $A \subseteq \bar{R}^n$) be a bounded variation function, for every n-dimensional random vector X belonging to a given class Λ. Assume that $\forall X, Y \in \Lambda, \rho_X = \rho_Y$, a.e. on A iff $X \overset{d}{=} Y$. If, for any fixed $\lambda \in A$, $\rho_X(\lambda)$ is a FORS measure induced by an ordering $\succ$, then we call FORS orderings induced by $\succ$ the following new class of orderings defined for every $(\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \geq 1$, $\forall X, Y \in \Lambda_\alpha = \left\{ X \in \Lambda : \left| \int_A \prod_{i=1}^n |t_i|^{\alpha_i - 1} d\rho_X(t_1, \ldots, t_n) \right| < \infty \right\}$ we say that X dominates Y in the sense α-FORS ordering induced by $\succ$, in symbol:

$$X \underset{\succ,\alpha}{FORS}\, Y \quad \text{if and only if} \quad \rho_{X,\alpha}(u) \leq \rho_{Y,\alpha}(u) \quad \forall (u) \in A \tag{1}$$

where

$$\rho_{X,\alpha}(u_1, \ldots, u_n) = \begin{cases} \dfrac{1}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_{a_1}^{u_1} \cdots \int_{a_n}^{u_n} \prod_{i=1}^n (u_i - t_i)^{\alpha_i - 1} d\rho_X(t_1, \ldots, t_n) \text{ if } \alpha_i \geq 1 \wedge \exists i : \alpha_i > 1 \\ \\ \rho_X(u_1, \ldots, u_n) \quad \text{if} \quad \alpha_i = 1; \quad \forall\, i = 1, \ldots, n \end{cases} \tag{2}$$

$a_i = \min\{t_i / (t_1, \ldots, t_i, \ldots, t_n) \in A\}$. Besides we call ρ_X FORS measure associated with the FORS ordering of random vectors belonging to Λ. We say that ρ_X generates the FORS ordering.

Example 2. From the previous example, the functional $G_X(x)$ generates a FORS ordering. So the measure associated to the α-FORS ordering is:

$$G_X^{(\alpha)}(u) = \frac{1}{\prod_{i=1}^{n}\Gamma(\alpha_i)} \int_{-\infty}^{u_1} \dots \int_{-\infty}^{u_n} \prod_{i=1}^{n} (u_i - t_i)^{\alpha_i - 1} dG_X(t_1, \dots, t_n).$$

Example 3. Consider the survival multivariate function associated with the vector X, $M_X(x_1, \dots, x_n) = -P(X_1 > x_1, \dots, X_n > y_n)$. It generates the upper orthant FORS order (see Shaked et al. [8]). So the measure associated to the α-FORS ordering is:

$$M_X^{(\alpha)}(u) = \frac{1}{\prod_{i=1}^{n}\Gamma(\alpha_i)} \int_{-\infty}^{u_1} \dots \int_{-\infty}^{u_n} \prod_{i=1}^{n} (u_i - t_i)^{\alpha_i - 1} dM_X(t_1, \dots, t_n).$$

Example 4. Consider the cumulative multivariate function associated with the vector X, $P_X(y) = P(X_1 \leq y_1, \dots, X_n \leq y_n) = F_X(y_1, \dots, y_n)$. It generates the lower orthant FORS order (see Shaked et al. [8]). So the measure associated to the α-FORS ordering is

$$F_X^{(\alpha)}(u) = \frac{1}{\prod_{i=1}^{n}\Gamma(\alpha_i)} \int_{-\infty}^{u_1} \dots \int_{-\infty}^{u_n} \prod_{i=1}^{n} (u_i - t_i)^{\alpha_i - 1} dF_X(\mathbf{t}) = E\left(\prod_{i=1}^{n} (u_i - X_i)_+^{\alpha_i - 1} \right) / \prod_{i=1}^{n} \Gamma(\alpha_i).$$

2.1 Properties and characteristics of multivariate FORS orderings

As for the univariate FORS measures and orderings (see Ortobelli et al. [5], [6]) using the properties of the fractional integral we can distinguish: *dual orderings*; *limited/unlimited orderings*, *survival orderings*; *risk/variability orderings*; *static/dynamic orderings*; *several levels of ordering*, etc.

Given a FORS orderings the associated FORS measure ρ_X defined on the class of the random vectors Λ, then we get the following extensions and implications[1]:

1. For every $\alpha >> v \geq \mathbf{1}$ (i.e., $\alpha_i > v_i \geq 1$ with $i = 1, \dots, n$) and $\alpha, v \in R^n$ $X \underset{\succ,v}{FORS}\ Y$ implies $X \underset{\succ,\alpha}{FORS}\ Y$ and we can write

$$\rho_{X,\alpha}(u_1, \dots, u_n) = \frac{1}{\prod_{i=1}^{n}\Gamma(\alpha_i - v_i)} \int_{a_1}^{u_1} \dots \int_{a_n}^{u_n} \prod_{i=1}^{n} (u_i - t_i)^{\alpha_i - v_i - 1} \rho_{X,v}(t_1, \dots, t_n) dt_1 \dots dt_n$$

2. We define the survival FORS orderings saying that $X \underset{\succ,sur\alpha}{FORS}\ Y$ if and only if $\overline{\rho}_{X,\alpha}(t) \leq \overline{\rho}_{Y,\alpha}(t)$ $\forall t \in A \subseteq R^n$ where

$$\overline{\rho}_{X,\alpha}(u_1, \dots, u_n) = \begin{cases} \dfrac{1}{\prod_{i=1}^{n}\Gamma(\alpha_i)} \int_{u_1}^{b_1} \dots \int_{u_n}^{b_n} \prod_{i=1}^{n} (t_i - u_i)^{\alpha_i - 1} d\rho_X(t_1, \dots, t_n) \ if\ \alpha_i \geq 1 \wedge \exists i : \alpha_i > 1 \\ -\overline{\rho}_X(u_1, \dots, u_n) \quad if \quad \alpha_i = 1;\ \forall i = 1, \dots, n \end{cases}$$

[1] For any $\alpha, \beta \in R^n$, we generally point out

- $\alpha \geq \beta$ if $\alpha_i \geq \beta_i$ $\forall i$
- $\alpha >> \beta$ if $\alpha_i > \beta_i$ $\forall i$
- $\alpha > \beta$ if $\alpha_i \geq \beta_i$ $\exists i : \alpha_i \geq \beta_i$

$b_i = \sup\{t_i/(t_1,\dots,t_i,\dots,t_n) \in A\}$ and for every $\alpha >> v \geq 1, \quad \alpha, v \in R^n$, we obtain the analogous formula

$$\overline{\rho}_{X,\alpha}(u_1,\dots,u_n) = \frac{1}{\prod_{i=1}^{n}\Gamma(\alpha_i - v_i)} \int_{u_1}^{b_1} \dots \int_{u_n}^{b_n} \prod_{i=1}^{n} (t_i - u_i)^{\alpha_i - v_i - 1} \bar{\rho}_{X,v}(t_1,\dots,t_n) dt_1 \dots dt_n$$

3. We say that X dominates Y in the sense of α FORS variability (or uncertainty) ordering induced by $\succ$ (i.e. $X \underset{\succ,unc\,\alpha}{FORS} Y$) if and only if $\rho_{\pm X,\alpha}(u) \leq \rho_{\pm Y,\alpha}(u)\ \forall u \in A$ (i.e., $X \underset{\succ,\alpha}{FORS} Y$ and $-X \underset{\succ,\alpha}{FORS} Y$).

4. Suppose $|\rho_X(b)| < \infty$, and/or $|\rho_X(a)| < \infty$ for every vector X belonging to Λ. Then we can extend ρ_X on all $\overline{R}^n$ assuming $\rho_X(u) = \rho_X(a)$, $\forall u \leq a$, i.e., $a_i = \inf\{t_i/(t_1,\dots,t_i,\dots,t_n) \in A\}$ and $\rho_X(u) = \rho_X(b)$, $\forall u \geq b$, i.e. $b_i = \sup\{t_i/(t_1,\dots,t_i,\dots,t_n) \in A\}$.

   Moreover, we say X unbounded α FORS dominates Y if and only if $\rho_{X,\alpha}(u) \leq \rho_{Y,\alpha}(u)$ for every $u \in \overline{R}^n$ where we define

$$\rho_{X,\alpha}(u) = \frac{1}{\prod_{i=1}^{n}\Gamma(\alpha_i)} \int_{-\infty_1}^{u_1} \dots \int_{-\infty_n}^{u_n} \prod_{i=1}^{n} (u_i - t_i)^{\alpha_i - 1} d\rho_X(t)$$

   If ρ_X is monotone, then unbounded $\underset{\succ,\alpha}{FORS}$ order implies bounded $\underset{\succ,\alpha}{FORS}$ order.

5. For any monotone increasing FORS measure ρ_X associated with a FORS ordering, induced by $\succ$ the opposite of, the left inverse, $-\rho_{X,}^{-1}(x)$ generates itself a FORS ordering.

Moreover, as for univariate FORS ordering (see Ortobelli et al. [5] and [6]) if a vector X is preferable to vector Y for a multivariate FORS ordering we can define a class of utility functional for which the expected utility functional of X should be greater than the expected utility functional of Y.

Proposition 1. ***Representation with utility functionals.*** *Suppose ρ_X is a FORS measure associated with a FORS ordering $\succ$ on a given class of random vectors Λ. Then,*

$$\forall X, Y \in \Lambda_{(a)} = \left\{ X \in \Lambda : \left| \int_A \prod_{i=1}^{n} |t_i|^{\alpha_i - 1}\, d\rho_X(t_1,\dots,t_n) \right| < \infty \right\},$$

$X \underset{\succ,\alpha}{FORS} Y$ *if and only if*

$$\int_A \phi(u) d\rho_X(u) \geq \int_A \phi(u) d\rho_Y(u)$$

for every ϕ belonging to

$$W^{\alpha} = \left\{ \phi(x) = -\int_{x_1^+}^{b_1} \dots \int_{x_n^+}^{b_n} \prod_{i=1}^{n} (s_i - x_i)^{\alpha_i - 1} d\tau(s) | \tau \right.$$
$$\text{is a } \ \sigma - \text{finite positive measure } \ \text{in } \ A \subseteq R^n \ \ \text{t.c. } \ \forall X \in \Lambda_{(a)}$$
$$\left. \text{the function } \ \prod_{i=1}^{n} (s_i - x_i)^{\alpha_i - 1} \text{is } \ d\tau(s) \times d\rho_X(X) \ \ \text{integrable in } \ A \times A \right\} \quad (3)$$

Besides for every $v >> \alpha \geq \mathbf{1}$ (i.e. $v_i > \alpha_i \geq 1$) $\phi_v \in W^v$ if and only if there exists a function $\phi_\alpha \in W^\alpha \prod_{i=1}^{n}$ such that

$$\phi_\alpha(x) = \int_{x_1^+}^{b_1} \dots \int_{x_n^+}^{b_n} \prod_{i=1}^{n} (s_i - x_i)^{v_i - \alpha_i - 1} \phi_\alpha(s) ds_1, \dots, ds_n. \quad (4)$$

The proof of this proposition follows as the application of the fractional integral properties and Fubini theorem. In addition, the multivariate fractional integral can be seen as a particular multivariate transform, then starting by a multivariate FORS ordering we get different levels of multivariate FORS orderings.

Theorem 2. *Suppose* $|b_i| < +\infty$ $(\forall i \leq n)$ *and* $\rho_X^{(1)} : A \to R$ *is a FORS1 measure associated with a FORS1 ordering* $\succ$ *defined on a class of random vectors* Λ*. If* $\rho_X^{(1)}$ *is a bounded and monotone function, then the probability functional* $\rho_X^{(2)} : [1,p]^n \to R$ *defined by* $\rho_X^{(2)}(u_1,\ldots,u_n) = \rho_{X,u}^{(1)}(b_1,\ldots,b_n)$ *points out a FORS2 measure (induced by* $\succ$*) on the class of random vectors*

$$\Lambda_p = \left\{ X \in \Lambda / p = min(p_1,\ldots,p_1) > 1 : \left| \rho^{(1)}_{X,(p_1,\ldots,p_1)}(b_1,\ldots,b_n) \right| < +\infty \right\}.$$

In addition $\rho_X^{(2)}$ *is associated with the following new FORS2 ordering induced by the previous one* $\succ$ *defined for every* $\alpha \geq \mathbf{1}$ $(\alpha_i \geq 1)$*, and for every*

$$X, Y \in \Lambda_{p,\alpha} = Z \in \Lambda_p : \left| \int_1^p \ldots \int_1^p \prod_{i=1}^n t_i^{\alpha_i - 1} d\rho_z^2(t_1,\ldots,t_n) \right| < \infty :$$

$$X \underset{\succ,\alpha}{FORS} Y \quad \textit{if and only if} \quad \rho_{X,\alpha}^{(2)}(u) \leq \rho_{Y\alpha}^{(2)}(u) \quad \forall u \in [1,p]^n . \tag{5}$$

To prove this theorem we need to use the multivariate Mellin trasform that identifies the common multivariate distribution when the second level of functionals are identical. Thus even the multivariate orderings are uniquely determined by an identity property and a consistency property that are used in the proof of the theorem analogously to the univariate case.

2.2 Example of a FORS ordering at a second level

Typical examples are the classic orderings.

Moments orderings: Consider the lower orthant FORS order with FORS measure associated $\rho_X(y) = F_X(y_1,\ldots,y_n) = P(X_1 \leq y_1,\ldots,X_n \leq y_n)$. In this case the measure associated to the α-FORS ordering is

$$F_X^{(\alpha)}(u) = \frac{E\left(\prod_{i=1}^n (u_i - X_i)_+^{\alpha_i - 1} \right)}{\prod_{i=1}^n \Gamma(\alpha_i)}$$

Consider the family Λ of random vectors below bounded from the vector $a = (a_1,\ldots,a_n)$ and above bounded from the vector $b = (b_1,\ldots,b_n) \in R^n$. These random vectors admits all the finite moments. We obtain a second level of ordering of vectors belonging to Λ considering as associated measure

$$\rho_X(v) = F_X^{(v)}(b) = \frac{E\left(\prod_{i=1}^n (b_i - X_i)^{v_i - 1} \right)}{\prod_{i=1}^n \Gamma(v_i)},$$

for any $v = (v_1,\ldots,v_n)$ t.c. $(v_i \geq 1)$ $\forall i$.

Thus we deduce that $\rho_{X,\alpha}(u)$, for $\alpha = (\alpha_1,\ldots,\alpha_n)$ with $\alpha_i \geq 1$ and $\forall u = (u_1,\ldots,u_n) \in (1,+\infty)^n$, is given by

$$\rho_{X,\alpha}(u) = \frac{1}{\prod_{i=1}^n \Gamma(\alpha_i)} \int_1^{u_1} \ldots \int_1^{u_n} \prod_{i=1}^n (u_i - t_i)^{\alpha_i - 1} d\rho_X(t). \tag{6}$$

3 Concluding remarks

FORS orderings can be used to extend many results of the theory of integral stochastic orderings that can be used to solve several financial problems. Moreover, there exists a parallel between the theory of probability metrics and FORS measures as suggested by Ortobelli et al. [6] and [7]. In particular, by ordering probability metrics we include further stochastic orderings and viceversa the stochastic orderings could induce an ordering in probability distances and metrics. For these reasons multivariate FORS orderings and measures can be used in different financial applications for example in the problems: of replicating some prefixed benchmarks; of the valuation of the efficiency of the choices. In the first problem risk managers minimize a distance of their financial expositions by some prefixed benchmarks. So using an opportune ideal probability metric (see Rachev [9]) we can create an opportune FORS ordering to optimize the financial choices. In the second problem of testing multivariate efficiency financial institution should test the efficiency of their portfolios in order to classify them and proposing the best opportune investments to the right category of clients.

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Vehicle routing problem with a private fleet and common carriers – the variety with a possibility of sharing the satisfaction of demand

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Abstract. The delivery of goods from a depot to individual consumers is one of the most frequently solved problems in transport logistics. When the individual demands are smaller than the vehicle capacity, it is possible to formulate this task as the well-known vehicle routing problem. Apart from their private fleet of vehicles the managers may, in substantiated cases, consider using an external common carrier. This problem is called the vehicle routing problem with a private fleet and a common carrier (VRPPC) in the existing literature.
The purpose of this contribution is to formulate a new variety of this problem, which is interesting from the point of practice and which was formulated on the basis of an authentic case of a local logistic firm. Let us suppose that it is possible to consider, with each individual customer, meeting his demand by means of one´s own private fleet or by means of a common carrier, but also by sharing both these ways. In the contribution a mathematical model of this VRPPC variety has been developed. In the final part of the study a particular authentic example of the above problem and its prime solution by means of very simple heuristics is presented.

Keywords: vehicle routing, logistics, private fleet, common carrier

JEL Classification: C65, L87
AMS Classification: 90B06

1 Introduction

These days reducing the cost of logistics as well as maintaining the level of satisfying customers´ needs has become a dominant trend in company practice. A number of companies try to use external mediation by means of other companies, usually well specialised for that purpose. In case of the logistic services the term "third party logistic" (3PL) has become familiar. In professional literature it is possible to find a number of various definitions for the term 3PL, for example „the use of external companies to perform logistics functions that have traditionally been performed within an organization" [6] or „an external organization that performs all or part of a company's logistics functions" [4]. Business activities in the field of 3PL are developing quickly and are affecting virtually all the main fields of logistics, such as transport, docking and control of stock, consolidation, distribution, cross-docking, packaging and labelling products for automatic identification and such like. In this contribution we will try and deal with utilizing an external provider for a specific case of provision of transport services.

Nevertheless, the decision made about the provision of a specific service by means of a third party cannot be reduced only to an isolated decision making procedure of the type "either – or" meaning "make-or-buy" for a particular service. In most cases it is necessary to carry out a more detailed analysis of the cost incurred by outsourcing such a service not only in relation to the cost of other activities provided by the company but also in relation to a potential variability of demands for the volume of this service [10].

A number of distribution companies providing, among other things, transportation services for their customers, have to tackle the need of reducing the cost, for example, by reducing the capacity of their own company transport fleet to the bottom level of the variable demands of their customers. External carriers are engaged to cover the remaining demand. These distribution companies have to, within the process of meeting their customers´ demands, deal not only with the issues of scheduling and routing their own vehicles, but they also have to deal with the decisions about what parts of the transport orders are to be implemented by means of external independent carriers.

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In the following text we will attempt to deal with the deliveries of goods to customers by means of round trips from one centre (VRP). The company´s own fleet is available and to satisfy the needs of some selected customers it is possible to use, wholly or partially, the services of an external carrier. As far as we know and as far as it becomes obvious form the following chapter, professional literature has not yet dealt with this VRP variety. An authentic problem observed in a particular local logistic company was the inspiration to this study.

2 The well-known VRP varieties

The Vehicle Routing Problem (hereinafter only VRP) is often defined as finding the optimum route from one or more depots to customers when, at the same time, respecting some limiting conditions. This problem has been studied since the year 1959, when this problem was described as generalization of a travelling salesman problem (TSP). The following examples are the most frequent varieties of VRP:

1) The Capacitated Vehicle Routing Problem (CVRP). We deliver the goods to the customer by vehicles, and each of the vehicles has its own maximum capacity. The capacitated vehicle routing problem is usually formulated in the following way:
   Let $G = (V, H)$ be a non-oriented graph with a set of vertexes $V = \{v_0, v_1, v_2, \ldots, v_n\}$ representing the customers (or cities), where the vertex v_0 represents the depot from which the customers are served, H is a set of paths (transport routes). For each path h_{ij} that connects vertex v_i with vertex v_j, the costs $c_{ij} \geq 0$ are known for $\{v_i, v_j\} \in H$. It is necessary to deliver the quantity of goods (units) b_i to every customer i, assuming $b_0 = 0$. Let us have m of identical vehicles with carrying capacity Q, deployed, at the beginning, in the depot, and they all can be used to satisfy the needs of the customers. It is necessary to find a set of routes with the lowest cost (one route for each vehicle) in such a way that the carrying capacity is not exceeded, the start and end of the route is in the vertex v_0, and each vertex of the route $V\backslash\{v_0\}$ is visited just once and just by one vehicle.
2) The Vehicle Routing Problem with Time Windows (VRPTW), represents a situation when the individual customers are only allowed to be served within a limited time window. VRPTW is a generalization of VRP which can be looked at as a classical VRP combination task of planning times in which it is possible to serve customers. It is necessary to optimize the fleet of vehicles and assign each vehicle to customers including the order in which they are to be served so that the costs are minimized. The times in which customers can be served are the limiting conditions here.
3) The Vehicle Routing Problem with Pick-up and Delivery (VRPPD). The customer can return some goods to the depot. VRPPD reflects the reality in cases when not only the necessity of transporting the goods from the depot to the customer is assumed but the transportation of the goods from the customer to the depot is also admitted.
4) The Multiply Depot Vehicle Routing Problem (MDVRP). Here the supplier supplies goods from multiple depots.
5) The Split Delivery Vehicle Routing Problem (SDVRP). Here the customer may be served by a few various vehicles.
6) The Stochastic Vehicle Routing Problem (SVRP). Here some data, such as the number of customers, their demands, the time of service or the time of the trip from one customer to the other one, are random quantities.
7) The Dynamic Vehicle Routing Problem (DVRP). Here VRP is formulated dynamically, i.e. the customers´ demands may occur randomly during the time of service (at random time, these are not known at the beginning of the process).
8) The Open Vehicle Routing Problem (OVRP). Here the vehicles do not need to return to the depot (the circle is not completed).

In the above classic varieties of VRP the task is dealt with by means of the company´s own fleet and we look for the minimum costs for this fleet. The task where we have a possibility (we get an offer) of serving some customers by a third party is an interesting extension of this problem. If we know the offer related to serving some customers (not necessarily all of them), we can propose the VRP task in a variant way and try to reduce the costs of the solution in which we serve all the customers ourselves in such a way that we let some of them be served by a third party (see e.g. [3]). This problem is called the integrated transportation planning problem (ITPP, see [5]) or 3PL (Third Party Logistics, see [1]), or vehicle routing problem with private fleet and common carrier (VRPPC, see [2]) and it is dealt with in general, for various types of a contractual relation with a third party for PDTW version of VRP (combination of PDVRP and VRPTW).

In professional literature only few articles are known that deal with VRP extended by subcontracting. As far as we are aware, in all the known cases, the issue concerns the mutually independent requirements related to the service, i.e. delivering the less-than-truckload loading, while the division of these requirements is not permitted. All these approaches focus on minimizing the overall costs of the customer service.

Some initial approaches (Savelsberg and Sol [8] or Stumpf [9]) considered a possibility of subcontracting all the pre-arranged routes of vehicles. This procedure can be applied in practice for such cases in which it is possible to hire an entire vehicle from a subcontractor.

The basic VRPPC model was, as far as we know, modelled mathematically by Chu [3]. In this publication a heuristic construction procedure is proposed based on the calculation of savings that is further improved by intra-route and inter-route exchanges. Chu [3], and similarly Bolduc et al. [2], considered the direct transfer of the overall requirements of the individual selected customers to a subcontractor. Each particular sale of an individual requirement to an external freight carrier is implemented independently of the method of performing other requirements.

Krejevska and Kopfer [5] consider VRPPD where, unlike the previously stated authors, three types of subcontracting are applied. Apart from the company´s own homogeneous limited fleet the requirements may be met by means of a subcontractor on the basis of: 1) the payment for a trip calculated per covered kilometres, 2) the payment for daily use of a vehicle, where the only limitation is the pre-arranged maximum route length of this vehicle, and 3) the payment for a particular order, i.e. for the provision of meeting a particular requirement.

3 Mathematical model of VRPPC with shared satisfaction of demand

In all the above cases concerning the use of a subcontractor it is not admissible to divide the requirement of a particular customer into the part met by the customer´s own fleet and the part met by the external carrier. From the practical point of view, however, this restriction may be a considerably limiting factor. In practice, the only requirement of a customer is to have all his/her requirements met on schedule, regardless of the fact whether this is to be done in one shipment or in, say, two part shipments. In fact, he/she may not be interested whether his/her requirement had been met by the supplier´s own vehicles, by a hired external carrier or by means of a combination of the above cases.

It is customary that the payment to the external carrier is directly related to the size (number of units) of the requirement to be met. If we serve customers within a limited area which, from the point of view of the external carrier, falls into one tariff zone, then the payment to an external carrier is not dependent on the shipping distance but only on the size of the transported load. In such a case it may be advantageous, within the planned existing trip of the company´s own vehicle, to partly satisfy another customer, whose total requirement would otherwise exceed the parameters of this route. Only the remaining part of this requirement may then be met by an external carrier. We assume that this way significant savings can be achieved as opposed to the situations where part shipments are not admissible.

In the following part of this chapter we will attempt to formulate a mathematical VRPPC model which admits the above mentioned division of the customer´s requirement into the part met by the customer´s own fleet and the part met by an external carrier (subcontractor).

Let us assume that $R = \{1,2,\ldots, n\}$ is a set of the indexes of the customers to be served. The vertex with the 0 index is the depot from which the vehicles depart and the set $I = R \cup \{0\}$ is a set of customers including the depot. To meet the customers´ requirements we have the set $K = \{1,2,\ldots,m\}$ of our own fleet available where each vehicle k bears the fixed costs f_k and it has the capacity Q_k. The costs of the k-th vehicle on the route between customers i and j are c_{ijk}. The requirements of customer i are labelled as q_i. Let us further assume that C is the fixed cost charged by the external carrier for the transportation of 1 unit of the shipment sent to any customer.

Let us define the following variables:

$$x_{ijk} = \begin{cases} 1 \ldots \text{if the vehicle } k \text{ moves from customer } i \text{ to customer } j \\ 0 \ldots \text{otherwise} \end{cases}$$

y_{ik} ..... the number of units transported by the company´s own vehicle k to customer i

z_i ..... the number of units z from the requirement of customer i served by an external carrier.

u_{ik} ..... the upper bound on the load of vehicle k upon leaving customer i .

The mathematical model of the VRPPC with the shared satisfaction of demand can then be formulated as follows:

$$\text{minimize} \quad z = \sum_{k=1}^{m} f_k + \sum_{k=1}^{m} \sum_{i=0}^{n} \sum_{j=0}^{n} c_{ijk}\, x_{ijk} + C \sum_{i=1}^{n} z_i \tag{1}$$

subject to

$$\sum_{\substack{j=0 \\ j \neq i}}^{n} x_{jik} = \sum_{\substack{j=0 \\ j \neq i}}^{n} x_{ijk} \qquad (i = 0, 1, \ldots, n;\ k = 1, 2, \ldots, m) \tag{2}$$

$$\sum_{k=1}^{m} \sum_{j=1}^{n} x_{0jk} = \sum_{k=1}^{m} \sum_{i=1}^{n} x_{i0k} \leq m \tag{3}$$

$$z_i + \sum_{k=1}^{m} y_{ik} = q_i \qquad (i = 0, 1, \ldots, n) \tag{4}$$

$$q_i \sum_{j=0}^{n} x_{jik} \geq y_{ik} \qquad (i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, m) \tag{5}$$

$$\sum_{i=1}^{n} y_{ik} \leq Q_k \qquad (k = 1, 2, \ldots, m) \tag{6}$$

$$u_{ik} - u_{jk} + Q_k\, x_{ijk} \leq Q_k - q_j \qquad (i, j = 1, \ldots, n, i \neq j;\ k = 1, 2, \ldots, m) \tag{7}$$

$$x_{jik} \in \{0;\ 1\} \qquad (i, j = 0, 1, \ldots, n;\ k = 1, 2, \ldots, m) \tag{8}$$

$$z_i \geq 0 \qquad (i = 1, 2, \ldots, n) \tag{9}$$

$$y_{ik} \geq 0 \qquad (i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, m) \tag{10}$$

$$u_{ik} \geq 0 \qquad (i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, m) \tag{11}$$

The objective function (1) minimizes the total costs that are incurred by servicing customers by the company´s own vehicles (2nd term of the function) and that are charged by an external carrier for the distribution of the relevant number of units (3rd term). Also included are the fixed costs of the company´s own fleet, but these are constant and have no impact on the final solution, only on the final objective function value.

The conditions (2) indicate that the vehicle arriving at the customer also departs from him/her (i.e. it neither starts nor finishes the trip there).

The condition (3) specifies that no more than *m* vehicles depart from the depot and they return back into it.

The conditions (4) ensure meeting the requirements of each customer by a random combination of the delivery by the company´s own vehicles or by an external carrier.

The conditions (5) provide the arrival of the vehicle at the customer, as long as this vehicle meets the non-zero part of the customer´s requirement.

The conditions (6) are the capacitated conditions of vehicles.

The conditions (7) prevent emerging such round trips that would not include the depot.

The conditions (8), (9), (10) and (11) are obligatory conditions determining the domain of definition of the applied variables.

4 Simple heuristic algorithm

A very simple heuristic algorithm has been drawn up for the purposes of solving the formulated problem in case of a particular logistic company. The proposed procedure is based on the principle of creating clusters of customers for the individual vehicle routes by means of a sweep algorithm (see e.g. [7]), when the number of customers is first reduced by those customers the visit of whom, in case of applying the company´s own vehicles, is not cost-effective. This reduction can be done in two steps. First of all, let us eliminate those customers to whom it does not pay off to send vehicles, simply because of the fact that the distance is too big, from the circle of customers served by their own vehicles, i.e. let us calculate the so called limit of the effective distance of a customer from the depot (Step 1). Consequently, this distance shall be reduced further on the basis of a simple estimate of the length of one round trip (Step 2). When creating clusters by means of the sweep algorithm we assign

the individual customers to a given cluster by means of rotating a half line going out from the place of the depot until the capacity of the vehicle is full (Step 3).

The rotation of the half line creates some sort of sectors in the served area and each such sector defines one cluster of customers (see Figure 1). By solving a TSP task for each sector created in such a way we will determine the length of a round trip of the company´s own vehicle (Step 4).

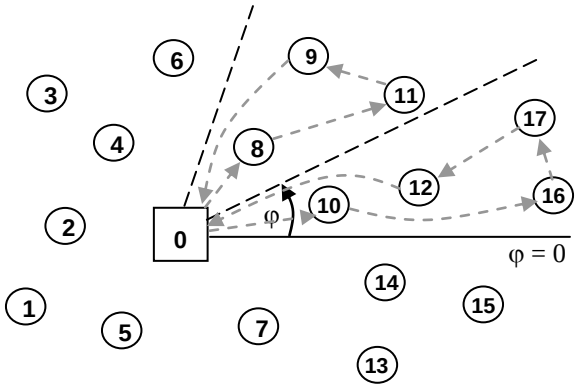

Figure1 Principle of the sweep algorithm

The outlined procedure can be formalized in the following way:

Let *C* be the fixed costs charged by the external carrier for the transport of 1 unit of load to any customer, $c^{(km)}$ be the costs for the trip of the company´s own vehicle calculated for the distance of 1 km and *K* be the capacity of the company´s own vehicle (we assume a homogenous fleet of vehicles).

Step 1: Set the limit of the effective distance of the customers from the depot $d_{max}^{(1)}$ for the service of the company´s own vehicles, when the following applies: $2\, d_{max}^{(1)}\, c^{(km)} \leq K * C$. Then:

$$d_{max}^{(1)} \leq \frac{K * C}{2\, c^{(km)}} \tag{11}$$

Eliminate all the customers *i* for whom $d_{0i} \geq d_{max}^{(1)}$ is valid. Identify the final set of the remaining customers as *I´*.

Note: Setting the limit $d_{max}^{(1)}$ is based on the consideration of the maximum distance for the customer to be served if he/she was satisfied by the so called shuttle trip, i.e. by means of a fully loaded vehicle only to this customer.

Step 2: Specify the limit of the effective distance of the customers from the depot to the value $d_{max}^{(2)}$ in the following way:

$$d_{max}^{(2)} = \frac{d_{max}^{(1)}}{(1 + \sin\frac{\pi}{p})} \tag{12}$$

where *p* is the estimated number of the routes set by means of the following relationship: $p = \lceil (\sum_{i \in I'} q_i) \,/\, K \rceil$, where $\lceil x \rceil$ represents the entire part of number *x*.

Eliminate all the customers *i* for whom $d_{0i} \geq d_{max}^{(2)}$ is valid. Label the final set of the remaining customers as *I´´*.

Note: Setting the limit $d_{max}^{(2)}$ is based on the consideration of the maximum length of a particular trip in the created sector and it depends on the overall number *p* of the estimated round trips, i.e. on the number of the created sectors. The higher the value *p*, the "narrower" the created sectors will be and the more compact the final route will be – its length will be close to the length of one shuttle trip from the depot to the most remote customer in the given sector. And, on the contrary, the smaller the value *p*, the "wider" the individual sectors will be, and the estimated length of the final route will grow. This will result in the reduction of the limit of the effective distance of the customers from the depot. The relation (12) is derived from a simplified estimate of the length of the round trip as the length of the circumference of an isosceles triangle formed by both the sides of the given sector (length *d*) and the external

connecting line (of the length $2d \sin(\pi/p)$. The length of the circumference of this triangle, i.e. the estimate of the length of the route, is: $2d + 2d \sin(\pi/p) = 2d\,(1 + \sin(\pi/p))$.

Step 3: Create the individual clusters of customers from the set *I''* by gradually rotating the half line and "sweeping" the customers in the chosen direction from the set initial position of the half line. In case the half line touches the first customer whose requirement already exceeds the capacity of the vehicle, we proceed on the basis of the size of the requirement of this customer according to one of the following three procedures:

a) if the existing capacity of the vehicle **is** used up to at least r_1 per cent, we will finish setting up the given cluster and, starting with this particular customer, we start creating a new cluster;

b) if the existing capacity of the vehicle **is not** used up to at least r_1 per cent, we will assign the customer to the cluster with the requirement corresponding to the unused capacity of the vehicle. If the remaining requirement of this customer is higher than r_2 per cent from the average requirement of the served customers, we, starting with this customer and his/her remaining requirement, will start creating a new cluster;

c) otherwise the customer will be assigned to the cluster with the requirement corresponding to the unused capacity of the vehicle and the customer´s remaining requirement will be **met by an external carrier**.

Step 4: For each created cluster of customers solve the TSP and set the length of the round trip of the company´s own vehicle.

Step 5: Calculate the costs of the service provided by an external carrier for the customers eliminated in Steps 1 and 2 and for the parts of the requirements of the customers pursuant to *Step 3 c)*.

Step 6: The total costs of the service are given by the sum of the costs of the trips carried out by the company´s own vehicles pursuant to *Step 4* and the costs of the service provided by an external carrier pursuant to *Step 5*.

It is obvious that the above heuristic method might be improved further and its effectiveness might be made more accurate i.e. depending on the values of the parameters r_1 a r_2. This may become the subject of a subsequent research study.

5 Computational results

The proposed procedure was tested on the real data gained from a particular logistic company operating in the region of West Bohemia. The data from the deliveries carried out within one week were used. In case of these distributions the company decided about the handover of the individual customers´ requirements to the external carrier intuitively. The routes of the company´s own vehicles were drawn up by a dispatcher based on his/her previous experience, or in some cases by means of a simple, freely accessible route planner.

The heuristic procedure from Chapter 4, in its simplest form, was applied to solve the task. The values of the algorithm parameters were set as follows: $r_1 = 95\%$, $r_2 = 50\%$. The gained values of the final costs of the solution, originally carried out by the company, and of the solution gained by our heuristic algorithm are summarized by the below Table 1.

The cost of the service on the individual days:	Day 1	Day 2	Day 3	Day 4	Day 5	Total
Original solution	3 280*	35 732	30 557	10 269	22 320	*102 158*
Solution by means of the heuristic algorithm	10 080	21 072	21 374	9 552	19 592	*81 670*
Savings	*-6 800*	*14 660*	*9 183*	*717*	*2 728*	**20 488**

*Note: * due to a small volume of the orders all the requirements were handed over to an external carrier*

Table 1 Results for the solved task

As is obvious from Table 1, apart from the first day, when the volume of the orders was very low and in the original solution the company handed all these orders over to an external carrier, various positive savings were achieved in all the other cases. The total overall savings accounted for approximately 20%. Higher savings were achieved on the days when the overall volume of the orders was higher. On the basis of the above facts it may be concluded that especially in these more complicated cases the proposed heuristic algorithm provides a significantly better solution than the original plan of the company deliveries as drawn up by the dispatchers.

6 Conclusion

In this contribution we dealt with a topical and important issue of goods delivery from the depot to customers formulated as VRP by means of the company´s own fleet of vehicles and with the possibility of using an external carrier (VRPPC). All the known, published cases of the solutions of this issue do not admit the division of a particular customer´s requirement into the part provided by the customers´ own fleet of vehicles and the part provided by an external carrier. The originality of the presented contribution consists in the formulation of the VRPPC with a possibility of sharing the satisfaction of demand of the customer into the part provided by the customer´s own fleet of vehicles and the part provided by an external carrier. A very simple heuristic method is proposed here to solve this task. Some results achieved by this heuristic algorithm are presented on the basis of the existing data of a local logistic company.

It might be appropriate, within a potential subsequent study, to carry out more extensive tests of the presented algorithm, including a possibility of improving the algorithm further and making it more accurate. At the same time, it might be interesting to verify the possibility of modifying other more sophisticated optimization techniques meant to solve the presented VRP variety.

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Model of Origin of Regional Disparities

Pavel Pražák[1]

Abstract. Paper deals with a mathematical model that describes a regional allocation of investment and indirectly it can be also used to clarification of origin of regional disparities. For simplicity it considers only two regions. Model is formulated for a linear production function with a constant output-capital rate. It is formulated as an optimal control problem that can be solved with the help of Pontryagin maximum principle. Paper also concentrates to the interpretation of the model and its optimal solution.

Keywords: investment, allocation, region, optimal control, Pontryagin maximum principle.

JEL classification: R11, C61
AMS classification: 49J15

1 Introduction and Assumptions

Most countries are usually divided into smaller regions that are autonomous in many spheres of theirs activities. Such a division (that usually has historical roots) is necessary for the administration of the given country. However, the existence of regions also brings problems that are solved in the framework of regional development. Problems joined with the regional development are receiving attention to many politicians, economists and also the general public currently. For instance, the allocation of public funds among regions to support economic growth, interregional redistribution or social capital formation belongs to main decisions of the central governments. Two main reasons for the interest in developing regions in the Czech Republic can be found: first the increasing regional disparities after 1989 and secondly the access of the Czech Republic into the European Union, cf [2] or [10]. One of the important problems of the regional development is to find relevant reasons of economic regional disparities, see e.g. [3]. The aim of this paper is to present a model that tries to explain the origin of regional diparities. Our considerations are mainly based on publication [9], which follows the articles [5], [1].

Consider a planning time horizon T, $T \in \mathbb{R}^+$, and an economic unit divided into regions. For simplicity, assume that it is comprised only of two regions, denoted by i, $i \in \{1, 2\}$. Each of these regions has an output $Y_i(t)$, where $i \in \{1, 2\}$, that is a function of time t, $t \in [0, T]$. The output of each region is an increasing function of regional capital stocks $K_i(t)$, where $i \in \{1, 2\}$ and $t \in [0, T]$. It means that the growth of capital K_i in the given region i, $i \in \{1, 2\}$, will cause a growth of the output Y_i of this region. Consider further that the increase in the stock of capital in a region i, $i \in \{1, 2\}$, is caused by increasing of the investment in that region. Suppose also that the investment funds that involve the total savings of the population of the economic unit are pooled into the central agency and then they are allocated to regions. Now one can put the following question. How to arrange the optimal allocation of investment fund in each of the two regions?

2 Model

Total output Y of the economic unit is a function of time t, $t \in [0, T]$, and it can be expressed as the sum of regional outputs

$$Y(t) = Y_1(t) + Y_2(t). \tag{1}$$

Assume that each of the regional output Y_i is determined as a product of the regional capital stock K_i and the constant regional output-capital ratio b_i, $b_i > 0$, $i \in \{1, 2\}$. Thus for $i \in \{1, 2\}$ and $t \in [0, T]$ we

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have

$$Y_i(t) = b_i K_i(t). \tag{2}$$

In each region i, $i \in \{1, 2\}$, saving S_i is proportional to regional income Y_i, thus for $t \in [0, T]$ it is possible to write

$$S_i(t) = s_i Y_i(t), \tag{3}$$

where s_i, $s_i \in (0, 1)$, is a constant saving ratio of region i. If (2) is substituted into (3) we gain

$$S_i(t) = g_i K_i(t), \tag{4}$$

where $g_i = s_i b_i$ is a constant growth rate of region $i \in \{1, 2\}$ and $t \in [0, T]$. It means that regional saving S_i is also proportional to regional capital stock K_i. National investment Z is the sum of regional savings S_i, $i \in \{1, 2\}$, and if we use (4) it can be written as follows

$$Z(t) = g_1 K_1(t) + g_2 K_2(t). \tag{5}$$

Let u, $u \in [0, 1]$, be the proportion of national investment that is allocated to region 1. This parameter, which can be a function of time t generally, is called allocation parameter, cf [8]. It means that $1 - u$, $1 - u \in [0, 1]$, is the proportion of investment that is allocated to region 2. Assume also that there is neither external investment to the given economic unit nor shift of capital from one region to another, then the regional accumulation of capital is described by

$$\dot{K}_1(t) = u(t) \cdot Z(t), \tag{6}$$
$$\dot{K}_2(t) = (1 - u(t)) \cdot Z(t). \tag{7}$$

In the latter system of equations zero depreciation of capital is assumed. If (5) is substituted to (6) and (7) the system can be rewritten as

$$\dot{K}_1(t) = u(t) \cdot (g_1 K_1(t) + g_2 K_2(t)), \tag{8}$$
$$\dot{K}_2(t) = (1 - u(t)) \cdot (g_1 K_1(t). \tag{9}$$

The initial states of regional stock of capital will be denoted as $K_1(0) = K_1^0$ and $K_2(0) = K_2^0$, respectively.

Consider that the aim of the government (or economic planner) is to choose the control variable $u(t)$ so as to maximize total income $Y(T)$ of the country at the given terminal time T, see [5]. Since $Y(T) = b_1 K_1(T) + b_2 K_2(T)$, cf (1) and (2), the problem facing the government is to allocate investment funds so as to find

$$\max\{b_1 K_1(T) + b_2 K_2(T) |\ u(t) \in [0, 1]\}, \tag{10}$$

subject to (8) and (9). While the allocation parameter $u(t)$, $u(t) \in [0, 1]$, is a control variable of the given optimal control problem (8) – (10), functions $K_1(t)$ and $K_2(t)$ are state variables of this problem. Recall that if the objective function is comprised only with the terminal component the optimal control problem is called Mayer's problem.

3 Necessary conditions for optimal solution

To determine optimal allocation of investment Pontryagin maximum principle will be used, for more details see [6]. This principle allows to find necessary conditions for optimal solution of the problem (8) – (10). The Hamiltonian associated with the problem is given by

$$\begin{aligned} H(K_1, K_2, u, p_1, p_2) &= p_1 u(g_1 K_1 + g_2 K_2) + (1 - u)(g_1 K_1 + g_2 K_2) \\ &= (up_1 + (1 - u)p_2))(g_1 K_1 + g_2 K_2), \end{aligned} \tag{11}$$

where we omitted variable t denoting time and p_i, $p_i = p_i(t)$, $i \in \{1, 2\}$, are adjoint functions of the optimal control problem corresponding to K_i. At every point t where $u(\cdot)$ is continuous the adjoint equations can be written as

$$\dot{p}_1(t) = -\frac{\partial H}{\partial K_1} = -(up_1 + (1 - u)p_2)) \cdot g_1 \tag{12}$$

and

$$\dot{p}_2(t) = -\frac{\partial H}{\partial K_2} = -(up_1 + (1-u)p_2)) \cdot g_2 \tag{13}$$

with the transversality conditions

$$p_1(T) = b_1 \text{ and } p_2(T) = b_2. \tag{14}$$

Adjoint variables p_1 and p_2 are auxiliary variables that can be used to find a decision criterion for optimal solution to the problem. They can be interpreted as shadow prices of capital in the given region, see [6]. Maximum principle yields

$$H(\widehat{K}_1, \widehat{K}_2, u, p_1, p_2) \leq H(\widehat{K}_1, \widehat{K}_2, \widehat{u}, p_1, p_2), \ u \in [0,1], \tag{15}$$

where $\widehat{K}_1$ and $\widehat{K}_2$, respectively, are optimal states of regional stock of capital and $\widehat{u}$ is optimal control. Since the Hamiltonian (11) is linear in the control variable u relation (15) can be rewritten as

$$(u - \widehat{u})(p_1 - p_2)(g_1\widehat{K}_1 + g_2\widehat{K}_2) \leq 0, \ u \in [0,1]. \tag{16}$$

Now it is possible to see that the optimal policy is a bang-bang solution which consist (in some combinations) of the following phases:

- $\widehat{u}(t) = 1$ if $p_1(t) > p_2(t)$,
- $\widehat{u}(t) = 0$ if $p_1(t) < p_2(t)$,
- $\widehat{u}(t)$ is not defined if $p_1(t) = p_2(t)$.

3.1 Adjoint variables

Applying adjoint equations (12) and (13) we gain

$$\frac{\dot{p}_1(t)}{\dot{p}_2(t)} = \frac{g_1}{g_2}.$$

If we integrate this relation use transverzality conditions (14) and rearrange the result we get

$$p_2(t) = \frac{g_2}{g_1}p_1(t) + \frac{b_1 b_2}{g_1}(s_1 - s_2). \tag{17}$$

Now we can construct a phase diagram of the system (12), (13). Since $b_i > 0$, $i \in \{1,2\}$, and $p_i(\cdot)$,

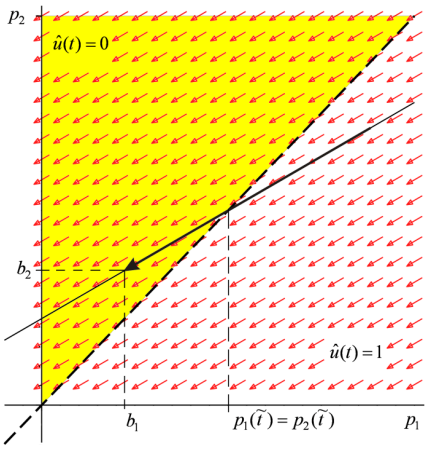

Figure 1 Phase diagram for system (12), (13) of adjoint variables p_1, p_2. The example of a trajectory for parameters $b_1 < b_2$ and $g_1 > g_2$.

$i \in \{1,2\}$, are decreasing functions, cf (12) or (13), value of adjoint functions are positive, which can be

formally written as $p_i(t) > 0$ for all $t \in [0, T]$. This observation allows us to construct a direction diagram, see figure 1. If we denote

$$k = \frac{g_2}{g_1},\ q = \frac{b_1 b_2}{g_1}(s_1 - s_2)$$

then (17) can be rewritten as

$$p_2 = kp_1 + q,$$

which is an equation of a line in the phase diagram, see figure 1. Depending on parameters $g_1 = s_1 b_1 > 0$ and $g_2 = s_2 b_2 > 0$ the slope k is either $k > 1$ or $0 < k \leq 1$. According to the allocation criterion we already know that $\widehat{u}(t) = 0$ whenever $p_1 < p_2$ and $\widehat{u}(t) = 1$ whenever $p_1 > p_2$, see figure 1.

3.2 Optimal Policy

For further analysis we modify (17). Since $g_i = s_i b_i$, $i \in \{1, 2\}$, the relation (17) can be written as

$$p_1(t) - p_2(t) = \frac{1}{s_2 b_2}[(s_1 b_1 - s_2 b_2)p_2(t) + b_1 b_2 (s_2 - s_1)].$$

(i) If

$$s_1 b_1 > s_2 b_2 \text{ and } s_2 > s_1$$

or

$$s_1 b_1 > s_2 b_2 \text{ and } s_2 = s_1$$

or

$$s_1 b_1 = s_2 b_2 \text{ and } s_2 > s_1$$

then always $p_1(t) - p_2(t) > 0$. It means that $\widehat{u}(t) = 1$ for all $t \in [0, T]$ and all investment is allocated in the first region. The second region does not receive any investment from the central fund. Note that all three cases are valid provided that $b_1 > b_2$. It means that the output-capital ratio is higher in the first region. Thus the first region has a higher productivity of capital and investment in this region is more efficient. Note also that at the first and the third case the first region has smaller rate of saving, which means that this region produce less sources for investment but the rate of consumption is higher.

(ii) If $s_1 b_1 < s_2 b_2$ and $s_2 < s_1$ or $s_1 b_1 < s_2 b_2$ and $s_2 = s_1$ or $s_1 b_1 = s_2 b_2$ and $s_2 < s_1$ then always $p_1(t) - p_2(t) < 0$. It means that $\widehat{u}(t) = 0$ for all $t \in [0, T]$ and all investment is allocated in the second region. The situation is the same as in (i) but the regions change their role.

(iii) If

$$s_1 b_1 > s_2 b_2 \text{ and } s_2 < s_1$$

then the difference $p_1(t) - p_2(t)$ can change its sign. It means that there can exist an instant $\tilde{t}$, $\tilde{t} \in [0, 1]$, when the direction of allocation change from one region to the other, cf figure 1. The instant $\tilde{t}$ is characterized by $p_1(\tilde{t}) = p_2(\tilde{t})$. At figure 1 it is represented by the intersection of the axis of the first quadrant and the trajectory of adjoint variables.

- If $b_1 < b_2$ then $\widehat{u}(t) = 1$ for $t \in [0, \tilde{t}]$ and $\widehat{u}(t) = 0$ for $t \in [\tilde{t}, T]$. The value of the instant $\tilde{t}$ is derived in subsection 3.3 and is given by the relation (20). In this case, the saving rate in the first region is higher than in the second region, but the region is less productive since its output-capital ratio is lower then in the second region. It meas that there arises a relatively more resources for investment, but the investment is less effective than in the second region. The optimal scenario is as follows: first for $t \in [0, \tilde{t}]$ invest only into the first region then switch and for $t \in [\tilde{t}, T]$ invest only to the second region.
- If $b_1 \geq b_2$ then $\widehat{u}(t) = 1$ for all $t \in [0, T]$, which means that all investment is directed into the first region that has higher productivity.

(iv) If $s_1 b_1 < s_2 b_2$ and $s_2 > s_1$ the situation is similar to (iii) but the regions change their role. The possible switching time is given by (21).

(v) If

$$s_1 b_1 = s_2 b_2 \text{ and } s_2 = s_1$$

both regions have the same parameters. They offer the same conditions and one is indifferent where to invest optimally.

3.3 Derivation of Switching Time

Let us derive the value for switching time $\tilde{t}$ that is characterized by the relation $p_1(\tilde{t}) = p_2(\tilde{t})$. Using (17) we find

$$p_2(\tilde{t}) = b_1 b_2 \frac{s_1 - s_2}{s_1 b_1 - s_2 b_2}. \tag{18}$$

Now consider scenario (iii) of subsection 3.2. In this case $\widehat{u}(t) = 0$ for $t \in [\tilde{t}, T]$ and adjoint equation (13) can be written as

$$\dot{p}_2(t) = -g_2 p_2(t)$$

with the terminal condition $p_2(T) = b_1$. This is a linear differential equation and its solution is

$$p_2(t) = b_2 e^{s_2 b_2 (T-t)},\ t \in [\tilde{t}, T]. \tag{19}$$

Substituting from (18) into the latter solution (19) and solving for $\tilde{t}$ we gain

$$\tilde{t} = T - \frac{1}{s_2 b_2} \ln\left(b_1 \frac{s_1 - s_2}{s_1 b_1 - s_2 b_2} \right), \tag{20}$$

provided that $\tilde{t} \in (0, T)$, otherwise the switching time does not exist. Similarly we can proceed in scenario (iv) of subsection 3.2. In this case we gain

$$\tilde{t} = T - \frac{1}{s_1 b_1} \ln\left(b_2 \frac{s_1 - s_2}{s_1 b_1 - s_2 b_2} \right). \tag{21}$$

4 Conclusion

According to the observations given in the previous section we can conclude that the optimal policy depends on the relative values of the saving rate and the output-capital ratio. It was shown that in most cases investment would be directed in the more productive region. There is one exception. If the saving rate of the less productive region is large enough with respect to the region with higher productivity, the optimal policy consists of two regimes with one switch: at the beginning of the process all investment are directed to the less productive region and then all investment are directed into the region with higher productivity. To briefly summarize we can introduce the following table

	$b_1 > b_2$	$b_1 < b_2$ and $s_1 > s_2$		$b_1 < b_2$	$b_1 > b_2$ and $s_1 < s_2$
$s_1 b_1 > s_2 b_2$	$\widehat{u}(t) = 1, t \in \mathbf{I}$	$\widehat{u}(t) = 1, t \in \mathbf{I}_1$ $\widehat{u}(t) = 0, t \in \mathbf{I}_2$	$s_1 b_1 < s_2 b_2$	$\widehat{u}(t) = 0, t \in \mathbf{I}$	$\widehat{u}(t) = 0, t \in \mathbf{I}_1$ $\widehat{u}(t) = 1, t \in \mathbf{I}_2$

where we denoted $\mathbf{I} = [0, T]$, $\mathbf{I}_1 = [0, \tilde{t}]$ and $\mathbf{I}_2 = [\tilde{t}, T]$.

Within the framework of the presented model it was shown that it is more effective to invest into the region with higher productivity. If all investment is received only by the region with higher productivity and the other region does not receive anything this latter region can fall behind. Between the regions can arise economic disparities in this way. Recall that regional disparities are usually understood as differences in the socio-economic development of regions that are the result of some inequalities. Economic disparities can lead to undesirable phenomenons: the less productive region can lose its competitiveness and regions can lose their cohesion. Due to the undesirable consequences of disparities states intervenes through regional policy to provide regional development. This is the reason why it could be useful to consider another models either with different dynamic or with different objective function, see [1],[4] or [7]. The alternative objective function would take into consideration also another aims of regional development. For instance it could minimize a measure of regional disparities. Some of possible measures of regional disparities are given in [10].

We suppose that the presented model can be improved at least in two ways. First, instead of two regions it could be useful to generalize the given model and consider an economic unit that is divided into n regions, where $n \in \mathbb{N}$, $n > 2$. Second, instead of linear production function (2) it could be useful to consider a nonlinear neoclassical production function. These two ideas can become bases for the future development of the model.

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The role of centres in supply network information distribution

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Abstract. Nowadays companies are presented as parts of supply network, where individual company does not compete as a unique independent entity. The network is represented as a system set of interrelated nodes (companies) and linkages (ties, relationships). Based on new findings, information flows are depicted by increased importance especially regarding the potential of information sharing. The flows are represented as potential set of logistic indicators shared by supply network companies.
Considered supply network is based on information flow and cooperation due to sharing of collected data. Research is focused on determination of centres in supply network through calculating and analysing their network characteristics such as centrality, connectivity etc. Main aim of the paper is to explore if key managing company (leader of the supply network) is identical to network centre with respect to connectivity and centrality features. Results show the real role of centres in supply network information distribution. Paper is one of the solutions of grant research MEB 061008: Cross border cooperation – B2B marketing and distribution in Austria and Czech Republic.

Keywords: supply network, information flow, fuzzy sets, centrality measures.

JEL Classification: D85, L14, L60
AMS Classification: 03B52, 90B10

1 Introduction

In recent years the study of supply chains and networks has become central to many disciplines. With universal availability of information and with supply chains that extend around the world, we operate in an environment where every company has become embedded in supply networks [9] and the supply chain is often represented as a network [4].

The supply chain system could be seen as a business enterprise with a high level of data transaction. Supply chain notion is based on flow paradigm where are observed the main three flows: material flow, information flow and financial flow. As a matter of fact, a well-organized information system (or information flow) is a foundation for a proper material flow in the supply chain [7]. Business researchers found that in a cooperative relationship, companies interact frequently and share meaningful information [1]. The information flow is represented as a potential set of logistic and financial indicators shared by nodes.

With respect to customer oriented economy paradigm as a leader of the supply network is most frequently determined last B2B customer (It is last up-stream company that designed a finished product. For example automobile factory hold the position of leader in automotive supply network.). The position of leader frequently means the key position in the network with respect to the management but not from the point of view of network structure. Structural dimensions of networks emphasize the information value and control advantages provided by structural position of the company in the network [5]. Dimensions such as network centrality, network size, and network density qualitatively and quantitatively measure the structure of networks [11]

The truly central position in networks is according to Barabasi [2] reserved for nodes that are simultaneously part of many large clusters. In the supply chain context, companies with network advantages as hubs (sellers to those with many alternative suppliers) or authorities (buyers from those with alternative customers) are companies that must be highly agile to survive in very competitive markets [4].

2 Methodology

The paper is focused on information flow and potential of sharing information in networks. Paper is concentrated on supplier-customer relationships not on inner information flow.

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Main aim of the paper is to explore if key managing company is identical to network centre with respect to connectivity and centrality features. We compare results of basic nodal and network characteristics and information links characteristics based on fuzzy sets.

Data source and collection. Data used for network construction came from questionnaire research performed in six year period (years 2008 – 2013). This research [12] was focused especially on four main groups of producers companies: food industry, building, machine industry, consumer goods industry. Over 190 companies were involved in this research and only micro companies (companies with less than 5 employees) did not participate in this research. The mentioned questionnaire research provided very important findings in the field of identification and evaluation of used logistic and financial indicators in both essential indicators dimensions: Evaluation of suppliers (indicators were mostly focused on evaluation of "perfect delivery" and "profile of supplier") and customers (especially concentrated on this two aspects: "perfect performed order" and "warranty and returns") [13].

Measurement. We conduct perceptual measurement which provides the foundation of network characteristics for analysis of network centrality. Two groups of characteristics are used:

a) Basic nodal and network characteristics

- Overall network density (D). The density of a network is simply the average value of the binary entries. It is a ratio of total number of links and the total number of possible links [14].
- Point connectivity gives matrix in which row i and column j gives the local point connectivity between all pairs of nodes in a network. The value is precisely the maximum number of nodes independent paths from i to j.
- Degree centrality (C_D) is defined as the number of links adjacent to upon a node (i.e., the number of links that a node has).
- Closeness centrality (C_C). The farness is the sum of the distance from each node to all others in the network. For a given network with nodes $v_1 \ldots v_n$ and maximum closeness centrality c_{max}, the network closeness centralization measure is $\sum(c_{max} - c(v_i))$ divided by the maximum possible value, where $c(v_i)$ is the closeness centrality of node v_i [8].
- Information centrality (C_I) of a node i is defined by the harmonic mean of all the information measures between i and all other vertices in the network. The information measure between two vertices i and j is the inverse of the variance of the weighted function [15].
- Core/periphery model calculates the influence measure between every pair of nodes by using the model which helps identify core and periphery parts of network, see [3], [17].
- Scores of Hubs and authorities centrality of the network are based on natural generalization of eigenvector centrality [10].

b) *Information links characteristics*.

The information flow in network is composed by links between individual customer and supplier (supplier-customer information link). Information links are obtained by questionnaire research and divided to:

- Supplier information link reflects potential amount of useful data that supplier could evaluate and obtain information flow control. It is based on set of average of the customer indicators measured by supplier. Measured data form this customer database system and are analysed in long term periods. Supplier information link is described by fuzzy set $A: X \rightarrow [0,1]$ with triangular membership function μ_A where parameters $a = 0$, $c = 0$. Parameter b is average of customer indicators [16].
- Customer information link is defined as a potential amount of useful data that customer could evaluate and obtain information flow control. It is based on set of averages of the supplier indicators measured by customer. Measured data forms supplier database system and are analysed in long term periods. Customer information link is described by fuzzy set $B: Y \rightarrow [0,1]$ with triangular membership function μ_B where parameter $a = 0$, $c = 0$. Parameter b is average of supplier indicators (similar to definition of μ_A).

The intersection of fuzzy sets is used for integration of supplier and customer information links to supplier-customer information link. The intersection of two fuzzy sets A and B denoted by $A \cap B$ is a fuzzy set C of $X \cup Y$ with the membership function $\mu_{A \cap B}$ defined by:

$$\mu_{A \cap B}(x) = min\{\mu_A(x), \mu_B(x)\}, \forall x \in X \cup Y. \quad (1)$$

It may be observed as the fuzzy intersection defined by Zadeh [19]. It is a particular case of the above defined intersection when $X = Y$.

The difference between customer link and supplier link is represented by **link similarity level (*S*)**. The measure is computed as the height of a fuzzy set C, where hgt(C) is the supremum (maximum) of the

membership grades of *C*. Higher similarity level in the interval <0,1> (maximum $S = 1$) implies smaller level of difference.

$$\text{hgt}(C) = \sup \mu_C(x),\ x \in X \cup Y \tag{2}$$

A fuzzy set *C* is normal if $\text{hgt}(A) = 1$. In other words, there is an x for which $\mu_C(x) = 1$. The set that is not normal is called subnormal. Such a set *C* is possible to normalize [6] by using the normalization function norm(*C*). It is defined for all $x \in X$:

$$\text{norm}(C): \qquad \mu_C(x) \mapsto \frac{\mu_C(x)}{\text{hgt}(C)} \tag{3}$$

The value for comparison of **strength of** individual **information link (*Z*)** is denoted by measure *Z* where value $Z \in C$ if $\mu_C(Z) = \text{hgt}(C)$. The higher value of *Z* (where $Z = <0,1>$, maximum $Z = 1$), the higher information flow potential between two companies is realized.

3 Results

Considered network is composed of 19 companies (denoted by *C*+number of company) linked by supplier customer relationships. Members of the network are mainly producers of automotive components. Major part of network is strictly oriented on B2B customer needs. The company determined as a C200 functions as a key control point in the network. Member C200 is a car manufacturer completed a product. First line surrounding of the company is strictly monitored and information flow between every company in the first line and key company reaches maximum level of observing (all important indicators are determined and evaluated). Control information flow and material flow are antagonistic oriented. If the strictly customer oriented network is presumed, it is possible to see one specific control information flow from customer to its suppliers and suppliers of the suppliers.

Information are shared between the companies in both directions (outcoming and incoming), feedback information is always dispatch. Only the control information flow is used for description of network structure. For that reason, the considered network is depicted as a directed and disconnected graph consists of 19 nodes and 21 links (two of them are reciprocal – control information flows from both companies are on the same level).

The results presented in the paper are divided into two groups: basic nodal and network characteristics determination, information links characteristics computing.

3.1 Basic nodes and networks characteristics

The overall network density (*D*) is calculated as $D = 0{,}0673$ ($\sigma = 0{,}2505$). It means the proportion of all possible links that are actually presented covers only about 6,73 % of potential network connections. The network is also sparse, where the links better integrate information exchanges and are better controlled by centres. Observed supply network suggests by its topology more chain than highly connected network. This is characteristic feature of the networks composed by automotive industry companies. Companies are concentrated on cooperation with only several key suppliers that are permanently audited. Alternative suppliers are used only in case of key supplier failure.

The point connectivity is measure that shows if there are many different pathways that connect two nodes. The connectivity results are represented by the matrix with source (row) and receiver (column) of information. With regard to low density in the network there are not many alternatives (links) how to reach other nodes. It shows high vulnerability of considered network.

The degree centrality (C_D) is conceptually the simplest measure of centrality. The degree is interpreted in terms of the immediate risk of a node for catching information whatever is flowing through the network. The companies C083 and C200 ($C_{DC083} = 6$; $C_{DC200} = 5$) have the highest value of out-degree centrality. In current case at macro level, the out-degree centralization of whole network is 28,086 % and in-degree 4,630 % of these theoretical maximums. It means high positional advantages equally distributed in the network (100 % is represented by the Freeman star with most centralized or most unequal possible network for any number of nodes).

The closeness centrality (Freeman farness) explains why the company is more powerful than others in network when this company is closer to more companies than any other. Companies that are able to reach other companies at shorter path lengths (measured as geodesic path) or that are more reachable by other companies at shorter path lengths have favored positions. Since the information network is directed, separate closeness and

farness can be computed for sending and receiving. According to the results, the companies C200, C106 and C083 reach the highest inFarness (*FI*) of the whole network, FI_{C200} = 307,0; FI_{C106} = 307,0; FI_{C083} = 306,0. Otherwise they have the lowest score of outFarness (*FO*), FO_{C200} = 42,0; FO_{C083} = 39,0; FO_{C106} = 55,0. These companies easy reach other companies at shorter paths (low *FO*), but they are harder reached by other companies.

Valente, Foreman [18] offer centrality measure using average of reversed distances, where **integration** is based on inward links and the **radiality** is based on outward ties. Integration and radiality are also reversed (Lower integration value indicates higher distance and lower closeness.). The results obtained by Valente closeness centrality measures are very similar to values mentioned above – companies with the lowest integration and the highest radiality are C200, C106 and C083 (I_{C200} = 0,5; I_{C106} = 0,5; I_{C083} = 0,556). To the contrary, the highest integration (*I*) is reached by companies C060, C016, C143 (I_{C060} = 1,333; I_{C016} = 1,278; I_{C143} = 1,167), so they are easier to reached by other companies. Summary statistics on the distribution of the integration and radiality measures are also calculated. It is also reflected by in-centralization (10,25 %) and out-centralization (63,02 %) measures. The out-distances are more unequally distributed than in-distances.

The information centrality (C_I) provides a more complex norming of the distances from each node to each other, and it summarizes the closeness centrality of each node by the harmonic mean of its distances to the others. The results of information centrality (the highest centrality have companies C083, C200, C016; C_{IC083} = 0,62; C_{IC200} = 0,57; C_{I016} = 0,53) are depicted in Table 1. The strong triad (companies C083, C200, C016) has the best information centrality position in the network.

Company	**C013**	**C016**	**C027**	**C032**	**C049**	**C060**	**C061**	**C076**	**C080**	**C083**	**C086**	**C106**	**C140**	**C143**	**C144**	**C148**	**C151**	**C165**	**C200**
C_I	**0,53**	0,36	0,31	0,04	0,05	0,05	0,45	0,40	0,40	**0,62**	0,38	0,51	0,25	0,27	0,38	0,43	0,27	0,32	**0,57**

Table 1 Information centrality

Core/periphery model identifies which nodes belong to the core and which belong to the periphery. The nature of the model is based on idea that core nodes are adjacent to other core nodes, core nodes are adjacent to some periphery nodes, and periphery nodes do not connect with other periphery nodes [3]. The model is able to find out how much the real structure approximates the ideal structure (Freeman star).

We test three discrete models based on fitness measures (CORR model: starting fitness = 0,335; Density model: starting fitness = 1,255 and Hamming model: starting fitness = 0,849). All models give similar results (so we present CORR model). Starting parameters were: data type = positive, fitness measure = CORR (The fit function is the correlation between the permuted data matrix and an ideal structure matrix consisting of ones in the core block interactions and zeros in the peripheral block interactions; this value is maximized.), number of iterations: 50, population size: 100.

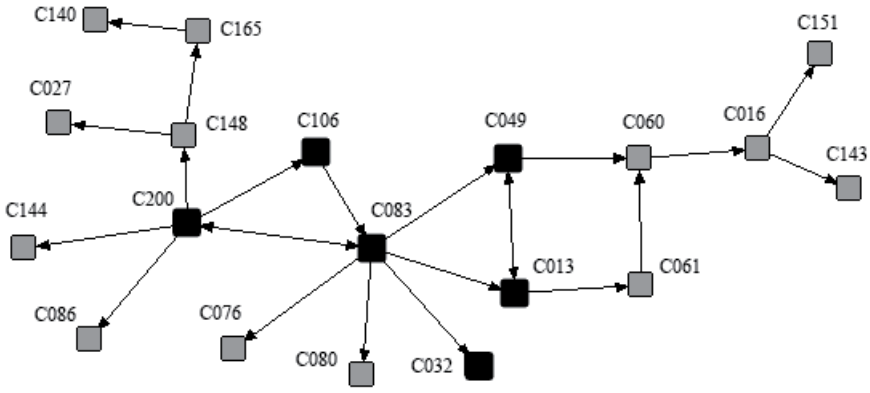

Figure 1 Core/periphery network model

The results of the Core/Periphery Class Memberships in blocked adjacency matrix are following (see Figure 1, core nodes are depicted in black colour, periphery in grey colour): Core (C013, C032, C049, C083, C106, C200) and Periphery (C016, C027, C060, C061, C076, C080, C086, C140, C143, C144, C148, C151, C165).

Hubs and authorities centrality uses two scores for each node, a hub score (C_H) and an authority score (C_A). A high hub node points to the many good authorities and a high authority actor receives from many good hubs. The results show that the highest hub value has company C083 (C_H = 0,951), this company takes centrality

advantages as a hub. Companies C013 (C_A = 0,462), C049 (C_A = 0,458), C032 (C_A = 0,374), C200 (C_A = 0,374), C076 (C_A = 0,374) and C080 (C_A = 0,374) are authorities concentrated around the hub - company C083.

3.2 Information links characteristics

All information links between suppliers and customers in network are described by fuzzy sets. Each link between two companies is represented by fuzzy set of supplier information link μ_A and customer information link μ_B with triangular membership function. The aggregation is based on intersection of the fuzzy sets pairs, see (1). The result of intersection is the fuzzy set of supplier-customer information link. In case of reciprocity links (for example between company C200 and C083) are fuzzy sets $\mu_{A\cap B}$ of link C200→C083 and $\mu_{A\cap B}$ C083→C200 normalized by (3). Then the intersection is used to aggregate them. There are 21 links $l_1 \ldots l_n$ in considered supply network. After intersection we compute with (2) similarity level (S) for each of these 21 links and the strength of information links (Z). The results of information links characteristics computing are presented in Table 2 where high values are depicted bold.

Links	**C140→C165**	**C165→C148, C148→C200**	**C027→C148**	**C144 →C200**	**C086→C200**	**C106→C200**	**C083 →C106**	**C200↔C083**	**C032 →C083**	**C080→C083**	**C049→C083, C076→C083**	**C013→C083**	**C013→C049**	**C049↔C013**	**C060→C061**	**C061→C013**	**C016→C060**	**C151→C016**	**C143→C016**
S	0,75	0,56	**0,99**	0,50	0,90	0,82	0,90	**0,96**	0,85	0,72	0,86	**0,94**	**0,95**	0,87	**0,92**	0,80	0,78	0,56	0,79
Z	0,58	0,56	0,23	0,50	**0,90**	**0,82**	0,70	**0,76**	0,62	0,52	**0,77**	0,69	0,63	0,71	0,72	0,65	0,66	0,44	0,61

Table 2 Information links characteristics

Link similarity level (*S*) is determined as an opposite value of difference between total amount of information shared and observed by individual supplier and customer (the value is computed for every link in network). In case of reciprocal links, the situation is similar. Reciprocal links are located between companies C49 and C13; C200 and C83 (see Table 2). Almost no difference is between C27 and C148 (S = 99 %), the information flow in this link has similar strength in both directions. Link between C200 and C83 (one of the reciprocity links) reaches similarity level 96 %, the three of four links with high similarity level are in surrounding of the company C13 (C13→C49, C13→C83).

Strength of individual link (*Z*) represents strength of information flow in one link. The maximum of computed Z values is achieved by link C86→C200 (Z = 0,9). Three of four highest Z values are reached by links incorporating the company C200 (Z from 0,90 to 0,76). We can deduce that C200 is possible to determine as a key control point or leader of the network. Almost all nodes in mentioned links belong to the core, these nodes send and receive extensive amount of information. By contrast, the links C27→C148 (Z = 0,23) and C151→C16 (Z = 0,44) achieve very low results.

4 Conclusion

The role of centres in information distribution process is determined by network characteristics. The research is provided on network (19 nodes, 21 links) where information sharing and information flow are observed. For network structure description are used basic nodal and network characteristics. Observed supply network suggests by its topology the low density of the network. With regard to sparse topology and low point connectivity the network tends to high level of vulnerability.

Network and nodal centralities are computed based on these measures: the degree centrality, the closeness centrality, the information centrality. The in-centrality measures are on the low level with high equally distribution of information. The out-centrality measures reach higher level and it reflect tendency to existence of less centres. The best centrality position and structural advantage in network have companies: C083, C200. Their role and structural advantage can be transformed into power which is exerted by direct bargaining and interchange.

The network structure is divided into two parts: core (6 companies) and periphery (13 companies), for this process the Core/periphery models are used. The core part includes hub and authorities, the companies which represent the centres of information receiving and sending.

For computation of information links characteristics are two measures created: the link similarity level and the strength of link. The highest link similarity level is possible to find between companies which share the same amount of information from both directions. These links are between C027 and C148; C200 and C83. Link similarity could be useful for information flow control and setting of information system (both connected

companies usually observe same indicators). The higher information flow potential is calculated as the strength of link. It belongs to links which are connected to centres (C200, C83). These findings mainly correspond to previous results of nodal and structured network characteristics. The results of considered network show that key managing company (C200) is identical to network centre with respect to nodal, network and information links characteristics.

The most important role of centres is to design and control the network structure and performance. For this reason, observing of information flow is crucial for obtaining and maintaining of competitive advantage. Advantages of proposed information links characteristics with respect to the roles of centres are their focus on relationships instead of nodal point of view.

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Volatility combination prediction in the presence of structural breaks

Peter Princ[1]

Volatility indicates a period of time series connected with high variability or growing variance. The behavior of the stock market Prague Stock Exchange through its official index PX was analyzed. Forecast combinations generated from the same GARCH model over different estimation windows and across various error distribution assumptions were examined. We outline relevant method for improving volatility forecast accuracy in the uncertainty of the timing and the size of structural breaks. We undertake an out-of-sample forecasting exercise to analyze how instabilities in the unconditional variance affect the forecasting performance according to generally preferred loss functions based on volatility proxy. Smaller samples are generally more efficient to model breaks in volatility process; nevertheless, parameter estimation on the sample sizes smaller than 500 yields negatively biased estimates. Our results suggest that forecasting combination schemes outperform predictions implemented on recursive forecasting scheme.

Keywords: volatility, combination forecast, GARCH

JEL Classification: C58
AMS Classification: 91B84

1 Introduction

Volatility indicates a period of time series that is associated with high variability or increasing variance. This phenomenon plays an important role in the modeling and analysis of financial time series. Volatility is a key element of procedures that assess the overall risk of financial assets. Predictions from volatility models are used in the investment decision-making processes for capturing the potential risk of potential investment portfolio, in the analysis of VaR models or models of options.

Financial time series are typified by volatility clustering, leptokurtic distributions and so-called leverage effect [1]. Such observations led to the wider use of the models with conditional variance. Engle [6] suggested in his pioneering work to model time-varying conditional variance as autoregressive conditional heteroscedasticity (ARCH) process with lagged random errors. The latter work has demonstrated the usefulness of higher-level ARCH process in dynamic nature representation of the conditional variance. Generalized ARCH model (GARCH) proposed by Bollerslev [2] is based on the infinite ARCH specification term that reduces the number of estimated parameters from infinity to two. Both ARCH and GARCH models are efficient to describe the volatility clustering in financial time series, however fail to model the leverage effect. Another issue that GARCH model cannot fully capture is the leptokurtosis. It means that probability distribution of the process is more kurtic with thicker tails than with normal distribution. The way how to overcome these problems is the use of random errors from non-normal distribution.

The application of the conditional variance on stock indices of stock markets in the Central Europe has been mentioned in few publications. Vošvrda et al. [15] examine the properties of the Czech PX-50 index and find significant variance in the time series of stock returns. Gilmore and McManus [8] reported a time-varying volatility in major stock indices in the Central Europe. Both studies assume conditional normality in equity returns and do not

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investigate empirical distribution of the random errors. Vošvrda and Žikeš [16] confirmed the validity of conditional heteroscedasticity and non-normal returns for central European equity markets.

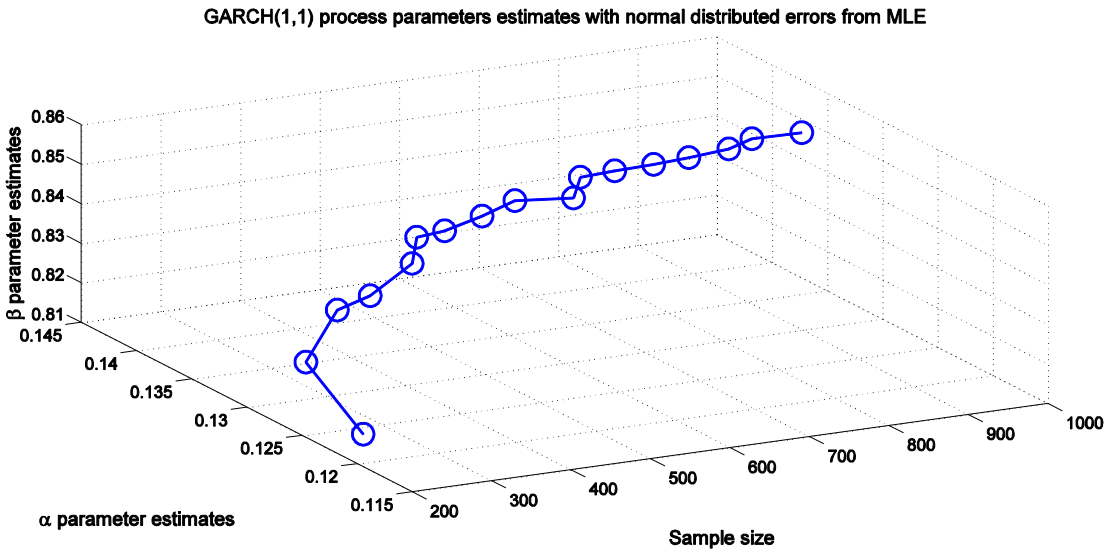
The remainder of this paper is organized as follows. Section 2 provides an econometric methodology used for volatility modeling and forecasting. Section 3 discusses prediction results and section 4 concludes.

2 Econometric methodology

We find a plethora of comparing studies that investigate the quality of GARCH models in the literature, nevertheless relatively little attention has been paid to the role of nonstationarity and structural breaks. Commonly used researcher's attitude to the issue of structural breaks incorporates the non-existence of structural breaks and therefore the existence of the stabilized conditional heteroscedasticity process that do not evolve over time. Diebold [5] noted in his comment to IGARCH paper of Engle and Bollerslev [7] that structural breaks can confound persistence in the estimation of GARCH models. Other researchs by Hendry [9], Lamoureux and Lastrapes [11] as well as more recent work by Mikosch and Starica [14] and Hillebrand [10] demonstrated that structural breaks in the unconditional volatility cause the sizable upward biases in the parameter estimates of GARCH process and lead to the persistence. The nature of such bias behavior results in so called "spurious almost-integration" outlined by Hillebrand [10] who found that neglecting structural breaks in the estimation process of GARCH model leads to tracking change defect in the unconditional variance and produces predictions that under- or overestimate volatility in the terms of longer sample sizes.

2.1 Small sample size and its bias

The question of adequate sample size for maximum likelihood estimation (MLE) in the field of volatility models is not rigorously responded in the current literature. For the purpose of MLE parameter estimates identification in small sample sizes let us consider the estimation procedure of GARCH(1,1) process with increasing sample size that starts at 200 and proceeds to 1000 observations. Figure 1 shows parameter estimates of such process with normal error distribution. It is evident that parameter estimates are negatively biased for small sample sizes.



Note: GARCH(1,1) parameters estimates are the means of MLE from Monte Carlo. For each step 1000 time series were generated with properties: $\sigma_t^2 = 0.00016 + 0.14\varepsilon_{t-1}^2 + 0.85\sigma_{t-1}^2$ where error term is distributed according to a normal distribution. The sample size is increased in each step by 50 observations. Simulated time series parameter are similar to estimation results from full-sample size (Table 2).

Figure 1 GARCH(1,1) process parameter estimates with normal distributed errors from MLE

Referring to figure 1 as the simplest example, the parameter estimation in the small sample size is more complex in the case with many GARCH parameters or under different error distribution assumptions and the latter involves employment of the quasi maximum likelihood method (QMLE). Variance-covariance matrix in the lines of Bollerslev and Wooldridge [3] to account for departures from normality was estimated. The smallest sample size for the estimation procedure in our empirical study of PX returns is determined to 500 observations.

2.2 GARCH(1,1) model

GARCH(1,1) model proposed by Bollerslev [2] is formulated as

$$w_t = \varepsilon_t \qquad \varepsilon_t = \sigma_t e_t \tag{1}$$

$$\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2, \tag{2}$$

where the conditional volatility σ_t^2 of w_t is set on the information known at the time *t-1*and e_t is distributed iid with zero mean and unit variance. The covariance-stationary condition for GARCH(1,1) process is $\alpha + \beta < 1$ and the level of persistence is given by $\alpha + \beta$. Parameter estimates from QMLE are consistent and asymptotically normal under conditions investigated by Ling and McAleer [11]. Similarly to Rapach et al. [13], we apply GJR-GARCH(1,1) model for supporting the evidence of leverage effect in PX returns, however, this model is substantially beaten by GARCH(1,1) and therefore we do not mention these results in our paper.

2.3 Forecasting models

The full-sample is divided into the in-sample (1500 observations) and the out-of-sample (599 observations). The aim of this study is to examine the usefulness of various forecasting schemes designed for accommodation nonstationary behavior in the financial returns conditional variance. The considered forecasting schemes are:

- GARCH(1,1) recursive window model. Sample size grows in each step by one observation. This model is used as a benchmark model.
- GARCH(1,1) 1500 rolling window model. In each step sample size consists of 1500 observations. We construct the GARCH(1,1) 1250/1000/750/500 rolling window models in the same manner, where the number of observations differs for each model.
- Combination rolling window. This is the average of 5 individual forecasts from rolling window models.
- Combination CM. This is the average of the GARCH(1,1) recursive window and the GARCH(1,1) 500 rolling window forecasts. This combination forecast is constructed in the manner of Clark and McCracken [3].
- Combination 3SS. This is the average of the three GARCH(1,1) rolling window forecasts with smallest sample sizes.
- Combination 3SS+R. This is the average of the three GARCH(1,1) rolling window forecasts with smallest sample sizes and the GARCH(1,1) recursive window forecasts.

2.4 Forecasting evaluation

Volatility prediction from the GARCH(1,1) model is expressed as

$$\hat{\sigma}_{t,t+q}^2 = (\hat{\alpha} + \hat{\beta})^q (\hat{\sigma}_t^2 - \upsilon) + \upsilon, \qquad \text{where } \upsilon = \frac{\omega}{1-\alpha-\beta} \tag{3}$$

is the unconditional variance and q is equal to 1. The value of σ_t^2 is subtracted from the model for the first prediction and in consequent predictions σ_t^2 is substituted by $\hat{\sigma}_t^2$. The volatility predictions were compared in all schemes by MSE metric:

$$MSE = \frac{1}{T}\sum_{t=1}^{T}(SR_{t,t+q} - \hat{\sigma}_{t,t+q}^2)^2, \tag{4}$$

where $SR_{t,t+q} = \sum_{j=1}^{q} w_{t+j}^2$ are squared returns used as a proxy for unobservable volatility. Such proxy is sufficient for volatility predictions classification.

3 Empirical results

3.1 Data

We use daily stock price index PX from Prague Stock Exchange from 3 January 1995 to 10 May 2013. In order to eliminate the nonstationarity in selected financial times series, we compute the differences between the logarithms of the prices on adjacent days

$$w_t = \ln z_t - \ln z_{t-1} = \ln \frac{z_t}{z_{t-1}}, \tag{5}$$

where z_t represents the closing price of the particular daily stock index in period *t*. The sample period covers financial crisis in the years 2007-2008 that is characterized by the turmoil on financial markets. Figure 2 shows PX returns and kernel density plot and table 1 reports descriptive statistics for the PX returns. The full-sample exhibits strong evidence of skewness and kurtosis. According to figure 2 we are not able to describe PX returns by normal distribution.

	Observations	Mean	Std. Deviation	Skewness	Kurtosis	Minimum	Maximum
PX returns	2099	-0.00005	0.01634	-0.52318	16.24383	-0.16185	0.12364

Table 1 Descriptive statistics for daily PX returns

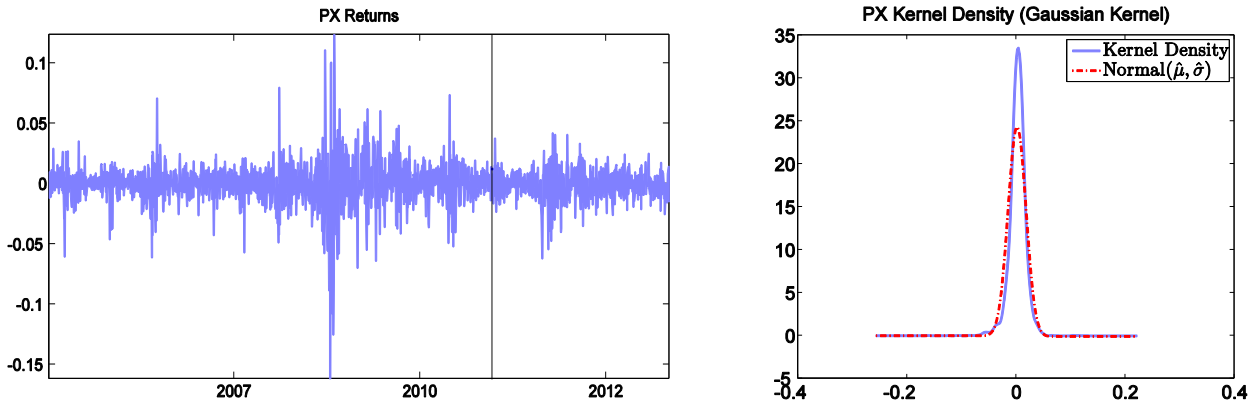


Figure 2 PX returns and PX Kernel Density against normal distribution(0,1)

3.2 Full sample results

Table 2 shows the full-sample GARCH(1,1) parameter estimates with normal, Student's t, general error distribution (GED) and skew Student's t distribution. The level of persistence is about 0.98 that suggests long persistence of the conditional variance. The level parameter estimates of the conditional variance is close to null and the parameter estimates of kurtosis and skewness statistics are in the accordance with descriptive statistics in table 1 and figure 2. The PX returns are most accurately described by the GARCH(1,1) model with skew Student's t distribution in consonance with the log likelihood value at the optimum of QMLE. Log likelihood value suggests the usefulness of kurtic distributions that surpass normal distribution.

Model Normal	Model Student t	Model GED	Model Skew t

	Model Normal	Model Student t	Model GED	Model Skew t
ω	0.0000	0.0000	0.0000	0.0000
	(0.0000)	(0.0000)	(0.0000)	(0.0000)
α	0.1517	0.1347	0.1434	0.1345
	(0.0191)	(0.0162)	(0.0173)	(0.0166)
β	0.8362	0.8483	0.8410	0.8478
	(0.0154)	(0.0149)	(0.0150)	(0.0154)
τ	-	7.9541	1.5264	8.2035
		(1.1634)	(0.0682)	(1.2100)
λ	-	-	-	-0.1178
				(0.0372)
LL	6200.4	6236.3	6224.3	6242.9

Note: The quasi maximum likelihood parameter estimates with robust standard errors in parentheses from the model estimation with different error distribution assumptions. LL means the log likelihood value at the optimum.

Table 2 Results from full sample estimation

3.3 Out-of-sample results

The out-of-sample consists of last 599 observations (12/20/2010 – 5/9/2013). Table 3 presents the out-of-sample volatility forecasting results for each error distribution. The table reports MSE ratio for individual forecasting schemes to the benchmark represented by recursive forecasting scheme. The ratio below unity thus means that the forecasting scheme is more adequate than benchmark.

Distribution		Normal	Students t	GED	Skew Students t
Forecasting scheme	Recursive	1	1	1	1
	Combination RW	0.9962	0.9973	0.9983	0.9973
	Combination CM	0.9959	0.9957	0.9969	0.9953
	Combination 3SS	0.9939	0.9954	0.9969	0.9955
	Combination 3SS+R	0.9952	0.9963	0.9975	0.9964
	1500 RW	0.9999	1.0004	1.0007	1.0004
	1250 RW	1.0008	1.0010	1.0017	1.0011
	1000 RW	1.0011	1.0035	1.0056	1.0039
	750 RW	**0.9906**	**0.9945**	**0.9940**	0.9954
	500 RW	0.9936	**0.9945**	0.9957	**0.9944**

Note: Values in the table are ratios of MSE forecasting method indicated on the left to the selected benchmark method represented by recursive forecasting scheme. Bold entry denotes the forecasting scheme with the lowest MSE.

Table 3 Out-of-sample one step-ahead volatility forecasting results

Figure 2 helps us explain the out-of-sample volatility forecasting results in table 3. The full sample as well as the in-sample and the out-of-sample include several jumps in the unconditional volatility mean. Therefore it is difficult to determine the optimal sample size for volatility forecasting. Smaller samples are generally more efficient to model breaks in volatility process; nevertheless, parameter estimation on the sample sizes smaller than 500 yields in the negatively biased estimates as was showed in section 2.1.

The smaller sample sizes (750 RW, 500 RW) and combinations forecasting schemes outperform recursive volatility forecasting scheme with different error distributions and prove the appropriateness of using combination forecasting schemes based on different sample sizes in the case of nonstationary unconditional variance.

4 Conclusion

This paper has given an account of and the reasons for the widespread use of combination forecasts based on different sample sizes. We consider combining forecasts generated from the same GARCH(1,1) model across different estimation windows as well as various error distributions. Different prediction schemes and their combinations were applied to the PX returns.

Our results are similar to those in Rapach et al. [13], that analyze volatility prediction of stock returns with GJR-GARCH(1,1) across different forecasting combination schemes. Combination forecasting schemes constructed from different window sizes beat forecasts from the GARCH(1,1) recursive model. These findings help us establish the usefulness of combination forecasting schemes designed to accommodate potential nonstationarity in the unconditional variance. However, more research on this topic needs to be undertaken before the issue of sufficient sample sizes for combinations forecasts in the field of volatility forecasting will be more clearly understood. Taken together, these results suggest that forecasting combination schemes outperform predictions implemented on recursive forecasting scheme.

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A Problem of Delay Management in Transport

Vladimír Přibyl[1], Jan Černý[2]

Abstract. Delay management is an important part of transport theory, mainly in the field of public transport as one can see from the book by A. Schöbel [4].
The main part of this theory is focused on "real time" management, i.e. on the handling of particular delay, which appeared in public transport. The presented paper, on the contrary, deals with the planning phase, which may be thought minor, but not negligible branch of the theory.
The first part of the paper is an introduction The second one presents the general problem and the third the simulation model of delay, caused either by the fact that each running time of a vehicle between two subsequent stops is a random variable supposed to have a beta distribution, but the departures of vehicles from stops are deterministic or by some unlikely event, such as a tree falling on the road, the bus fault.
The fourth part of the paper analyses the impact of departure choice to the delay propagation in the particular case of linear PERT-like activity graph representing a sequence of bus journeys served by the same vehicle. The main characteristics are the mean value of delay or the probability of exceeding some limit of delay. An asymptotic behavior of delay is observed as well.

Keywords: public transport, random running time, beta distribution, delay propagation, asymptotic behavior.

JEL Classification: C41, R42
AMS Classification: 60J20, 90B06

1 Introduction

Theory of public transport delay management consists of two parts (see [4]).The major part deals with real time management, i. e. solution of problems connected with current delays. The minor, but not negligible, part focuses on the planning phase, i.e. on the robust scheduling which prevents the long time propagation of the delay.

The robustness of vehicle scheduling against delay was studied e.g. in [1] [2] or [3]. Prof. M. Nonato from the University of Ferrara emphasizes the role of Time Space Networks as a modeling tool as well in a seminar presentation.

The present paper is focused on a simulation model, enabling to observe the situation when a timetable is given together with the vehicle scheduling. The user of the model can observe

- development of delay, caused by the fact that the real running time is random but the arrivals and departures are deterministic in the timetable – from this point of view the model is a generalization of the GOP-1 model [6],
- propagation of delay, caused by an unexpected, unlikely event, such as a tree falling on the road, the bus fault or broken traffic lights.

2 Problem Formulation

In the sequel, the bus transport terminology is used although it is obvious that all next results are more general. I.e. they can be applied to almost all sectors of transport, both passenger and freight, if they follow a fixed timetable.

It is assumed that the set of bus journeys with their timetable is given. Each bus journey can be divided into subsequent *pieces of work*. A piece b of bus work represents a movement of bus between two stops $s_b \in S$, $z_b \in S$ where S is the given set of important stops, with the departure d_b from s_b and the arrival a_b to z_b, all following to

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the valid timetable. That does not mean that there are no intermediate stops between s_b and z_b. but no stop between s_b and z_b enables passenger bus changing. Of course, it may happen that a bus journey consists of only one piece of work.

The set B of all bus pieces of work included into the problem can be partially ordered by the relation "possible passenger change from b' to b ". It will be denoted "$b' \rightarrow b$ ". According to the relation "$\rightarrow$" the set B can be stratified into several layers B_1, B_2, …:

- $b \in B_1 \Leftrightarrow$ no b' exist such that $b' \rightarrow b$.
- $b \in B_2 \Leftrightarrow ((b' \rightarrow b) \Rightarrow (b' \in B_1)) \wedge (b \notin B_1 \cup B_2)$
- $b \in B_3 \Leftrightarrow ((b' \rightarrow b) \Rightarrow (b' \in B_1 \cup B_2)) \wedge (b \notin B_1 \cup B_2)$
- …etc.

Naturally, $b' \rightarrow b$ is valid also when b is the direct continuation of b' within the service of the same vehicle. Therefore, instead of "possible passenger change from b' to b " it can be said "possible continuation of passenger trip by b after b' ".

In the sequel, it will be assumed that $R(d_b) = d_b$ for all $b \in B_1$, except of the case of an unexpected, unlikely event.

The value $a_b - d_b$ represents timetable net running time of b. Of course, real values of $R(d_b)$ and $R(a_b)$ may be random and different of d_b and a_b required by the timetable and therefore the real net running time $R(b) = R(a_b) - R(d_b)$ is a random variable.

Assume that some passengers arriving by the bus service b to the stop z_b want to change for another bus piece of work b' with the timetable departure $d_{b'}$ and the real departure $R(d_{b'})$ from $s_{b'} = z_b$. Since the arrival $R(a_b)$ does not represent the time when the passengers from b are ready to board b', it is not enough to ask that $R(a_b) = R(d_b) + R(b) \leq R(d_{b'})$. It is necessary to add some time T necessary for alighting b and walking to the boarding platform of b'. The random variable $X(b) = R(b) + T$ is said *random gross running time of b from* s_b to z_b. The timetable gross running time is denoted x_b. Then it is enough to meet the constraint

$$R(d_b) + X(b) \leq R(d_{b'}) \tag{1}$$

for the passenger transfer link-up between b and b'.

However, (1) is not the only constraint for $R(d_{b'})$. The real departure $R(d_{b'})$ is not allowed to be sooner than $d_{b'}$ required by the timetable, i.e. it is necessary to meet the constraint

$$d_{b'} \leq R(d_{b'}) \tag{2}$$

Of course, the real departure time $R(d_{b'})$ is the minimum possible value fulfilling both (1) and (2).

Obviously, it may happen that both b and b' belong to the same bus journey and that b' immediately follows (= is adjacent to) b. Then the additional time T represents the time, necessary for alighting some passengers and boarding the others and if necessary for some other manipulations (e.g. loading the bikes).

If a bus journey consists of a sequence of pieces of work b_1, b_2 …, then it can be considered a linear PERT-like graph with the edges b_1, b_2 …, having the durations $X(b_1)$, $X(b_2)$, ….

In a general case there is a set B of bus pieces of work belonging to a set of several bus journeys. Moreover, if there are some compulsory linkups between pairs of bus pieces of work for passenger bus changing, then a branched PERT-like graph with the edge set B can be used as a model. Of course, such a graph with linkups for bus changing can be linear in a particular case when the object of study consists only of a subsequent bus journeys.

Similarly, as in the "classic" book [5], it is assumed that any $X(b)$ is a Beta distributed variable with the distribution interval (min(b), max(b)) and the parameters $\alpha(b)$ and $\beta(b)$. Naturally, from these parameters the mode mode(b) of $X(b)$ can be calculated. In practice, min(b), max(b) and mode(b) are optimistic, pessimistic and the most probable durations of b respectively.

However, there is a significant difference between the present case and the "classic" calculation of total duration of two adjacent activities. Suppose that the graph consists of b and b' only. In the "classic" case the total duration $Y(b, b') = X(b) + X(b')$. On the contrary, due to (1) and (2)

$$d_b + Y(b,b') = \max\{d_b + X(b); d_{b''}\} + X(d_{b''}) \tag{3}$$

Therefore, the simulation model presented in the section 3 should reflect it.

3 Simulation Model

Description of the model contains three subsections. The first one focuses on the input data. The second one outlines the structure and functioning of the model. The third one describes the output values and relations between them the model enables to observe.

3.1 Input Data

The model requires input of the following data (the denotations are explained in subsection 1.1):

- the set of important stations S,
- the file characterizing the set B of bus pieces of work by means of vectors $(s_b, z_b, \min(b), \max(b), \alpha(b), \beta(b), d_b)$ for all $b \in B$,
- the tolerable delay $\tau \geq 0$,
- (optionally) one element $b \in B_1$ and an unexpected delay δ at the departure of b.

The all types of input data are quickly accessible because they may be changed for each run of the model.

3.2 Structure and Functioning of the Model

It is supposed that all random running times $X(b)$ for all $b \in B$ are mutually independent. The model operates with discrete form of the random variable $X(b)$. The discretization step Δ can be easily changed in the model in order to change the accuracy of the calculations. It's clear, that this parameter strongly influence the computational time.

Structure of the model corresponds to the layers $B_1, B_2, \ldots$

First, the layer B_1 is elaborated. For each $b \in B_1$ it is put $R(d_b) = d_b$ (or $d_b + \delta$ if it is given). Then the distribution of the random variable $R(a_b) = R(d_b) + X(b)$ is calculated.

Second, the layer B_2 is elaborated. For each $b \in B_2$ it is found $R(d_b) = \max\{\{d_b\} \cup \{R(a_{b'}): b' \rightarrow b\}\}$. Then the distribution of the random variable $R(a_b) = R(d_b) + X(b)$ is calculated.

Then, the layers $B_3, B_4 \ldots$ etc. are elaborated the analogous way.

For each $b \in B$, the following parameters are calculated:

- probability of delay $P\{R(d_b) - d_b > 0\}$,
- probability of delay exceeding τ $P\{R(d_b) - d_b > \tau\}$,
- mean delay $E\{R(d_b) - d_b\}$.

3.3 Output Results

The model is able to find e.g.:

- time behavior of delay in normal operation,
- propagation of unexpected delay,
- dependence of the previous two on the choice of departure times d_b in the timetable.

4 Results

As it is said above, functioning of the model was tested on a special case of a linear graph of consecutive bus pieces of work b, b', … forming the set B. Each of them is assumed to have the same random duration $X(b)$, $X(b')$, … with the given equal Beta distribution. The parameters of Beta distribution were set to: $\min(b) = 20$ minutes, $\max(b) = 30$ minutes, $\alpha(b) = 2$ and $\beta(b) = 4$. For these parameters, the mode(b) = 22,51, median(b) = 23,14 and mean value mean(b) = 23,34. Discretization step Δ was set to 0,1 min. The timetable gross running times x_b of all consecutive bus pieces of work b were assumed to be equal for each run of the model, so the timetable departure times d_b were easily calculated as multiples of x_b. Calculations were carried out for timetable gross running times varying from 20 to 29 minutes. It's clear, that with given parameters of the beta distribution the probability $P\{X(b) = 20\} = P\{X(b) = 30\} = 0$. It emerges from this, that probability of delay $P\{R(d_b) - d_b > 0\}$ equals to 1 for the timetable gross running time $x_b = 20$ and 0 for $x_b = 30$.

As it is mentioned above, the model operates in discrete way. Naturally, all the results generated by the model are discrete. On the contrary all graphs stated in this section are continuous in order to depict the trends rather than exact discrete values. The axes denoted "layer" in some graphs stated below represents the number of elaborated layer B_n. In the case of tested linear graph it simply represent the number of consecutive pieces of bus work.

4.1 Probability of Delay

The first important parameter observed on the model is probability of delay $P\{R(d_b) - d_b > 0\}$. Both graphs presented on **Figure 1** depict the course of it. The first (= the left) one shows the time progress of probability of delay for several selected timetable gross running times x_b. On the horizontal axis there are the numbers of layers, on the vertical one the probabilities. On the bottom are shown types of lines corresponding to the timetable gross running times: dotted curve x_b = 22 min., shortly dash curve 23, long 24, dash-and-dot 25 and full x_b = 26 min. One can see e.g. that the probability of delay for x_b = 23 min tends asymptotically to 1 and much quicker is the same true for x_b = 22. On the other hand choices of x_b = 25 or 26 tend to values close to 0,25 or 0,1.

The second (= right) one depicts the same model results from the opposite point of view. It shows the dependence of the delay probability on the timetable gross running time for selected model layers: full curve the 1st, long dash-and-dot the 5th, etc. An interesting fact is evident (and retrospectively from the left part of Figure 1 as well: After, say, 15th layer one can observe a significant break around the mean(b) = 23,34: under it the delay probability tends to 1, over mean(b) it tends to a value significantly less than 1.

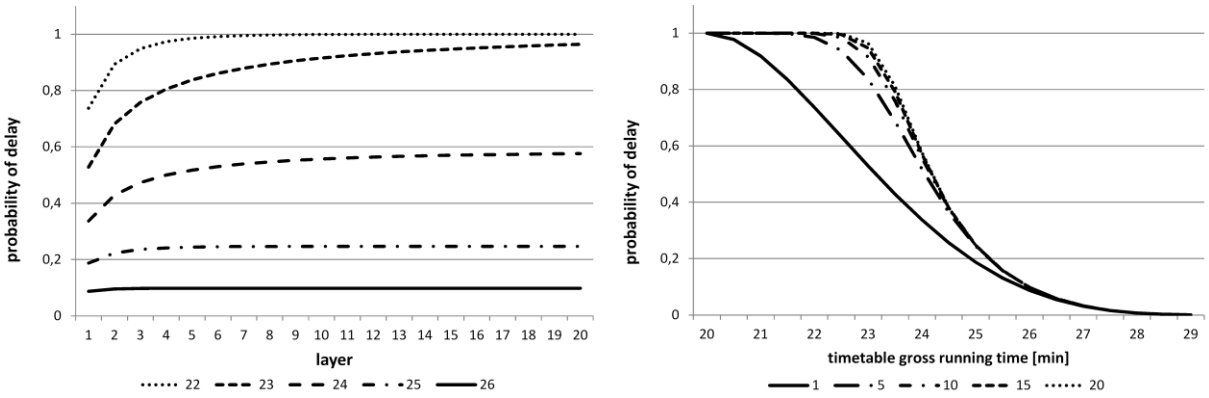

Figure 1 Probability of nonzero delay

4.2 Mean Value of Delay

The pair of graphs on **Figure 2** depicts by the similar way as in case of those on **Figure 1** the values of mean delay for the same selected timetable gross running times x_b in different model layers. The values of delay probability shown on graphs on **Figure 1** seems to be relatively high, but the graphs on **Figure 2** show, that the high probability of delay doesn't mean high value of mean delay. For example in case of x_b = 24 minutes the probability of delay nearly equals to 0,6 but the mean delay is less than 2 minutes in the 20th model layer.

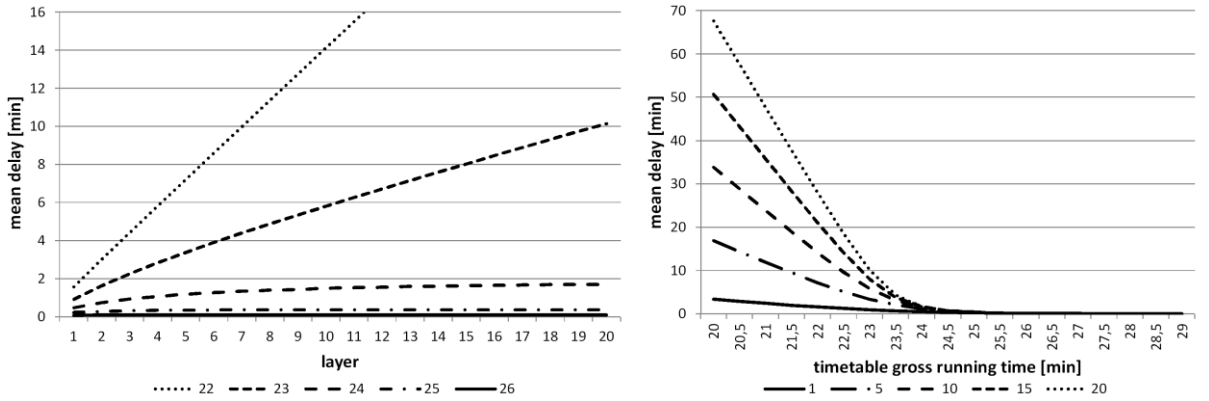

Figure 2 Mean delay

4.3 Tolerable Delay

Suppose that a delay not exceeding the given value $\tau > 0$ (e.g. $\tau = 1$ of 5 min.) is *tolerable*. Then it is interesting to know what time progress has the probability of delay exceeding τ, i.e. the value $P\{R(d_b) - d_b > \tau\}$. **Figure 3** depicts it by thicker lines, comparing with the former probabilities, corresponding de facto $\tau = 0$ (dotted lines). The left part corresponds to $x_b = 23$ and "23 + 1" means that $\tau = 1$ etc. The right part corresponds to $x_b = 24$. One can easy observe that (again) a break. Left: asymptotically tend to 1, right: tend to much smaller values.

The right part evokes the following conjecture:

Conjecture 1. For $x_b >$ mean(b) the asymptotic behavior of $P\{R(d_b) - d_b > \tau\}$ is similar as the one of delay probability (i.e. for $\tau = 0$) corresponding to $x_b' = x_b + \gamma\tau$ where $\gamma < 1$ is a coefficient.

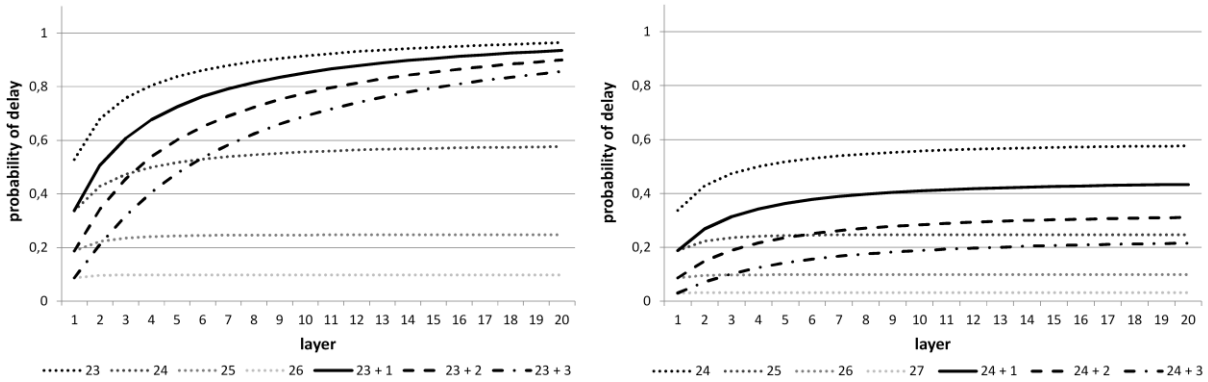

Figure 3 Probability of delay > τ

4.4 Delay after a "Breakdown"

In this context, "breakdown" means an unexpected, unlikely event, causing a delay of δ minutes of the first departure in the model. Of course, it need not be the real first departure of the bus, but only the processes after the event are interesting for the modeling.

Figure 4 depicts the behavior of delay probability (left part) and mean delay (right part), both for $x_b = 22$, 23, 24, 25 and 26. A break between values 23 and 24 is obvious. It manifests itself an interesting regularity, which can be formulated as another conjecture:

Conjecture 2. A) If $x_b <$ median(b) then the probability of delay asymptotically tends to 1 with respect to increasing time (= the layer number in the model), regardless of whether it is $\delta = 0$ or $\delta > 0$. **B)** If $x_b >$ mean(b) then the probability of delay asymptotically tends to a value significantly smaller than 1 with respect to increasing time (= the layer number in the model), regardless of whether it is $\delta = 0$ or $\delta > 0$.

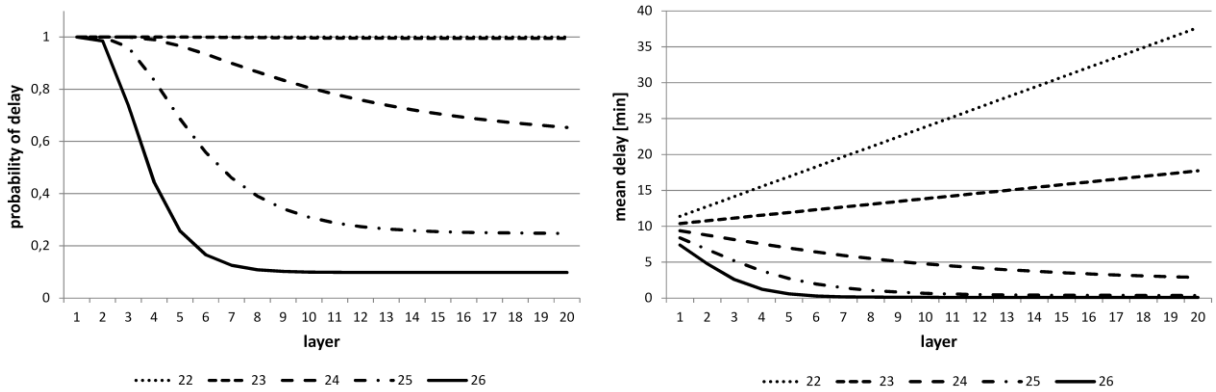

Figure 4 Model behavior in case of unexpected initial delay $\delta = 10$ minutes

4.5 Challenges for Mathematicians

In this context several challenges for mathematicians arose. First of all, if $b1$, $b2$, … are subsequent duties with the timetable departures x_{b1}, x_{b2}, …, then the discrete stochastic process $R(d_{b1})$, $R(d_{b2})$, … is neither Markov process, nor stochastic process with independent increments. Is that an absolutely unknown process? If yes, what are its basic properties? Anyhow, what is asymptotical behavior of $ER(d_{bn})$?

Further, can be the abovementioned conjectures mathematically proved or rejected? The authors hope that both of these challenges will be accepted and based on them interesting mathematical results will be achieved.

5 Conclusion

The paper first outlines problems connected with delay propagation through an activity graph representing bus pieces of work.

Afterwards it shows how the necessity of waiting for the timetable departure, if the bus is ready to depart sooner, complicates mathematical formulation of the problems.

Then it presents the main topic of the paper - an original simulation model and focuses on results and observations obtained from the model.

Finally some challenges for mathematicians are formulated.

Expected future research, in addition to the above mentioned challenges, will probably focus beyond the tree-like model, which can lead to interesting complications. E.g. a delay "coming from two or more directions" may be a new feature of the expected models.

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Convex Maximization with Special Feasible Regions: Computational Experiments

Miroslav Rada [1], Michal Černý[2]

Abstract. The paper deals with a special case of convex maximization — the convex maximization over a projection of a hypercube. An important example of a problem in this class is quadratic maximization with a positive semidefinite matrix.

The benefit of this special case is that if the projection of the hypercube is singular, then the number of examined extremal points (vertices) of the hypercube is reduced. The projection of the hypercube is actually a well-known geometric structure — a zonotope — and the presented case of convex maximization is reduced to the vertex enumeration problem for the zonotope.

The main result of this paper is an implementation of a recent compact output-polynomial algorithm for vertex enumeration problem for zonotopes and its comparison with the well-known Reverse search algorithm. Both algorithms share the same theoretical properties: output-polynomiality ensures that every vertex will be examined in time polynomial in the output size, and compactness allows us to solve the problem without the necessity to store already examined vertices. The experiments show that the recent method outperforms the Reverse Search algorithm.

Keywords: 0-1 convex quadratic maximization, reverse search algorithm, incremental enumeration, zonotope

1 Motivation

Our motivation is similar to the motivation of [1]. Let us consider the following problem: we are to maximize a given positive semidefinite (PSD) quadratic form over 0-1 variables. Formally, given a PSD matrix $H \in \mathbb{R}^{m\times m}$ for some m, the problem can be written as

$$\max\left\{x^{\mathsf{T}}Hx : x \in \{0,1\}^m\right\}. \tag{1}$$

Our initial, say a bit naïve, approach to the problem could be to traverse all the 0-1 solutions and choose the best. It consists of 2^m evaluations of the quadratic form $x^{\mathsf{T}}Hx$, which is clearly a very time-consuming task.

But what if we possess some additional information about the problem? For example, assume we know that the rank of H is d and assert d to be fixed and $d < m$. Every PSD $m \times m$ matrix H of rank d can be decomposed to $H = G^{\mathsf{T}}G$, where G is a matrix of size $m \times d$. With such a decomposition, we can substitute $y = Gx$ and the problem (1) becomes

$$\max\left\{y^{\mathsf{T}}y : y = Gx, x \in \{0,1\}^m\right\}. \tag{2}$$

The problem (2) differs from the problem (1) in some important aspects. First, the dimension of the feasible region $\left\{y \in \mathbb{R}^d : y = Gx, x \in \{0,1\}^m\right\}$ is d instead of m,[1] since the feasible points of (2) are images of the feasible points of (1) under the linear mapping $x \mapsto Gx$. Second, some of the feasible points of (2) can be expressed as a convex combination of other points, and hence don't have to be tested

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[1] *Dimension* stands for *affine dimension*.

for optimality of $y^\mathsf{T} y$. Note also that multiple feasible points of (1) can be mapped to a single point of (2). Hence, if we are able to identify "extremal" feasible points of (1), meaning that they aren't convex combinations of other points, we are able to save some work, and as we will see in Theorem 4 later, the amount of saved work can be very significant depending on d and its relation to m (the case $d \ll m$ is especially interesting).

Note the problem (1) is equivalent to the problem with a continuous feasible region $\{x \in \mathbb{R}^m : 0 \leq x \leq 1\}$ and also to the case of bounded integer variables and its continuous version

$$\{x \in \mathbb{R}^m : \underline{x} \leq x \leq \overline{x}\}. \tag{3}$$

For the continuous feasible region, note also that the optimal point must lie in a vertex.

2 Goal and structure of the paper

From the computational point of view, the problem with the feasible region (3) (which is actually an m-dimensional cube) is the most interesting. In fact, the image of a cube under a mapping of form $x \mapsto Gx$ defines a well-known geometric object — *a zonotope*.

For zonotopes, algorithms are known for solving common geometric tasks such as face enumeration and volume computation. The problem (1) can be clearly reduced to the vertex enumeration problem for zonotopes — one can traverse vertices one by one and evaluate the objective function for each of them. Hence we can restrict ourselves to enumerating vertices.

The goal of the paper is to implement two promising algorithms for vertex enumeration for zonotopes and compare them empirically. Both algorithms have similar general worst-case complexity-theoretic properties: output-polynomiality and compactness (see Section 4). However, there is an open question whether there exist some interesting classes of problems for which one of the algorithms outperforms the other or vice versa.

The rest of the paper is **structured** in the following way. In Section 3, we introduce some notions concerning zonotopes and show the combinatorial equivalence of zonotopes with their dual structures — arrangements of hyperplanes. In Section 4, we discuss some complexity-theoretic properties of algorithms for solving geometric tasks and provide a brief overview of work related to this paper. Section 5 describes our two algorithms for vertex enumeration in detail. Finally, in Section 6 we implement the algorithms, generate some test cases and discuss our results.

3 Preliminaries

Let us start with some notation and definitions. For a vector $a \in \mathbb{R}^d$, the symbol D_a is a diagonal $d \times d$ matrix with the elements of a on the diagonal. The symbol S^i is defined as $S^i := \{-1, 1\}^i$. The operation $A \dot{+} b$ for some $A \subset \mathbb{R}^d$ and $b \in \mathbb{R}^d$ is called *Minkowski sum*[2] and is defined as $A \dot{+} b := \{a + \lambda b : \ a \in A, \ 0 \leq \lambda \leq 1\}$.

The terms *vertex*, *face of a polytope*, *normal cone* and other geometric notions are understood in their usual way, see [7] in case of ambiguities.

Zonotopes. In Section 2, we introduced the notion of a zonotope as an image of a cube under a linear mapping. However, for the description of the algorithms it will be more useful to define the zonotope in terms of *generators*.

Definition 1. Let a list of m vectors $g_1, \ldots, g_m$ in $\mathbb{R}^d$ be given and let t be a vector in $\mathbb{R}^d$. *A zonotope* is the set

$$Z := \{t\} \dot{+} g_1 \dot{+} \cdots \dot{+} g_m. \tag{4}$$

The vector t is called *shift* of the zonotope and the vectors $g_1, \ldots, g_m$ are *generators* of the zonotope. The $(m+1)$-tuple $(t; g_1, \ldots, g_m)$ is referred to as a *representation* of the zonotope.

The following lemma joins the problems from Section 1 with the zonotope in Definition 1.

[2] Actually, the Minkowski sum is usually defined with two sets as its operands. However, for the purposes of this paper, it is advantageous to define it for a set as the first operand and a vector as the second operand.

Lemma 1. *Let a representation $(t; g_1, \ldots, g_m)$ of a zonotope $Z \subset \mathbb{R}^d$ be given. Let also an m-dimensional cube $K = \{\psi \in \mathbb{R}^m : \underline{\psi} \leq \psi \leq \overline{\psi}\}$ be given, where $\underline{\psi} \leq \overline{\psi} \in \mathbb{R}^m$. Let $p(K)$ be the image of K under the linear mapping $p : y \mapsto Gy$ for some G.*

Then the following holds: if $(g_1 \cdots g_m) = GD_{(\overline{\psi}-\underline{\psi})}$ and $t = G\underline{\psi}$, then $Z = p(K)$.

For a proof see [3].

Observe that zonotopes have some interesting properties. Every zonotope is a centrally symmetric set. Furthermore, every face of a zonotope Z is a zonotope generated by some generators of Z, in particular, every edge of a zonotope Z is parallel to some of the generators of Z.

The combinatorial structure of the set of vertices of a zonotope is independent on its shift. In fact, every vertex can be obtained as a 0-1-combination (and the shift is added to each combination) of generators of the zonotope.[3] Hence, for enumeration of vertices we can omit the shift of the zonotope completely and work only with the generators. Thus we can w.l.o.g. represent the zonotope just by the list of generators. We set $G := (g_1 \cdots g_m)$ for a given list $g_1, \ldots, g_m$. From now on, when we speak about a zonotope, we assume that the zonotope is given by G. This matrix is called a *generator matrix*.

Furthermore, the combinatorial structure of the set of vertices is also independent of the lengths of generators. Given a vertex v of a zonotope Z, a full-dimensional normal cone C corresponds to v. The cone C is the set of objective functions, for which the vertex v is an optimal solution over Z. A simple argument following from the simplex method implies that the normal cone for v has the form

$$C(s) := \{\xi \in \mathbb{R}^d : D_s G^{\mathsf{T}} \xi \geq 0\}, \tag{5}$$

where $s \in S^m$ and its i-th element s_i is chosen such that $v - s_i g_i \in Z$ holds.[4] This normal cone is clearly independent of the lengths of generators.

The family of hyperplanes $A(G) := \{\{\xi \in \mathbb{R}^d : \xi^{\mathsf{T}} g_i = 0\} : i \in \{1, \ldots, m\}\}$ forms an *arrangement of hyperplanes*, which is a subdivision of $\mathbb{R}^d$ into regions of the form (5). A full-dimensional set $C(s)$ is called *cell* of the arrangement. The vector s is called *sign vector*. If $C(s)$ is a cell of an arrangement, we say that s *generates* $C(s)$. We also say that s *generates* a vertex corresponding to $C(s)$.

Clearly, one vertex of Z can be assigned to a sign vector s such that $C(s)$ is full-dimensional, and vice versa.

Lemma 2. *Let a matrix $G \in \mathbb{R}^{d \times m}$ be given. There is one-to-one correspondence between the vertices of the zonotope represented by G and the cells of the arrangement $A(G)$.*[5]

Corollary 3. *To solve* (2)*, it is sufficient to enumerate all sign vectors generating full-dimensional cells of $A(G)$.*

The transformation "vertex enumeration → cell enumeration" does not bring any complexity advantages. However, such a transformation is a natural way to introduce the notion of sign vectors allowing us to describe algorithms in a very simple way.

4 Algorithmic properties

Algorithms in combinatorial geometry sometimes face the problem that the output size can be very large compared to the input size. This holds in particular for enumeration algorithms. By Theorem 4, this is also the case of zonotopes.

Theorem 4 ([7])**.** *Consider a zonotope in $\mathbb{R}^d$ with m generators. Then the following inequalities for the number of its facets (denoted f_{d-1}) and for the number of its vertices (denoted f_0) hold:*

$$f_0 \leq 2\sum_{i=0}^{d-1} \binom{m-1}{i}, \qquad f_{d-1} \leq 2\binom{m}{d-1}. \tag{6}$$

Furthermore, there are zonotopes for which the bounds (6) are tight.

[3] Observe that the 0-1-combinations are actually the vertices of the cube in Lemma 1.

[4] The element s_i determines the direction for the generator g_i in which an edge leads from v, actually.

[5] In fact, a stronger version of this lemma holds. Similarly to cells, one can define normal cones of dimensions lower than d. Then the one-to-one correspondence exists between all faces of a zonotope and the set of normal cones of all dimensions.

The bounds (6) are satisfied as equalities for zonotopes with no d generators linearly dependent. As a corollary, no general "fast algorithm" can exist neither for vertex enumeration nor facet enumeration, since the number of vertices or facets can be superpolynomial in m and d.

However, there are still some "desirable" properties of enumeration algorithms. Namely, one can measure computation time per unit of output. In case of vertex enumeration, if an algorithm ensures that the time complexity of outputting one vertex is polynomial in size of G, we speak about an algorithm *polynomial in the output size* or about an *output sensitive* algorithm.

Another interesting property of enumeration algorithms is *compactness*. An algorithm is compact if the space complexity of the algorithm is polynomial in the input size. In particular, it means that the algorithm generates its output without the necessity to remember previously enumerated elements.

Let us end this section with some references. There are several results on vertex enumeration of zonotopes. In [4], an algorithm for enumerating all faces (not only vertices) of zonotopes is described. However, the algorithm is not compact. The paper [3] describes an algorithm with a capability to enumerate vertices of a zonotope. The algorithm is neither output-sensitive nor compact. On other hand, it can also solve the volume computation problem and the facet enumeration problem simultaneously with the vertex enumeration with (almost) no additional computational effort.

For us, the algorithms from [2] (and its improvements [6, 1]) and [5] are the most interesting. They are both compact and output-sensitive.

5 Reverse Search Algorithm and Incremental Enumeration Algorithm

Both algorithms lean on building and traversing a tree of cells of arrangements. The main difference between the algorithms, affecting computational complexity, is the combinatorial structure of the trees. Though both algorithms generate trees of the same heights, the widths of the trees may differ significantly.

The reverse search algorithm [2] is a general meta-algorithmic frame for solving various enumeration problems on graphs. It assumes that two main ingredients are given: *an adjacency oracle* and a *forward search procedure*.

In fact, the reverse search algorithm starts in a root of a tree. In case of cell enumeration, it starts in one randomly chosen cell. Say that it is the "all-plus" cell in terms of the sign vector. The RevSearchEnu procedure (line {1} in Algorithm 1) is run for this initial cell.

In its course the algorithm tries to flip every "+" sign in the actual sign vector ({3}). If a new sign vector generates a new cell (the adjacency oracle tests whether such a flip is feasible), the algorithm performs the forward search to determine whether the preceding cell of the new cell is its parent cell ({5}). If so, the algorithm runs RevSearchEnu (line {6}) for the new cell, otherwise the new cell is thrown away, since it is ensured that it will be visited (or that it was already visited) from another branch of the tree.

We recall that the isCell procedure is performed as checking full-dimensionality of a normal cone, which is easily done with a linear program. The ParentCell procedure is performed in the original formulation as solving $\mathcal{O}(m)$ linear programs. However, we use an improvement from [6, 1] and process ParentCell as simple ray-shooting.

The incremental enumeration algorithm [5] works a bit different. The i-th level of the cell tree corresponds to vertices of the zonotope $Z_i := \{0\} \dotplus g_1 \dotplus \cdots \dotplus g_i$, which is represented by the first i generators of the generator list. Actually, it adds hyperplanes into the arrangement one by one and asks whether the addition of a hyperplane divides an existing cell. If all hyperplanes were inspected, the algorithm outputs the current cell. See Algorithm 2.

The algorithm is initially run for the empty sign vector. At each step (at the i-th level of recursion), the current sign vector has length i and determines a vertex of Z_i. If the level i is the last level m (line {2}), the algorithm outputs the corresponding sign vector; otherwise the algorithm prolongs the current sign vector by ± 1 and for both signs its checks whether they generate vertices of Z_{i+1}. If so, the recursion is run for the corresponding prolonged vector. The isCell procedure used in this algorithm is the same as the one used in the reverse search algorithm. In fact, one isCell is sufficient to be called in each IncEnu, see detailed description in [5].

Comparison of the algorithms. From a theoretical point of view, for each output cell of the

arrangement (or: each vertex of the zonotope), the maximum number of LPs to be solved is m for both algorithms. The worst-case time complexity of both algorithms is $\mathcal{O}(|C|\ m\ \mathrm{lp}(m,d))$, where $\mathrm{lp}(m,d)$ is the time necessary for solving a linear program with m constraints and dimension d. (Actually, the theoretical upper bound for time complexity of solving a linear program depends on the bit-size of the input data. The interested reader should interpret the symbol $\mathrm{lp}(m,d)$ in this sense.) The symbol C stands for the set of all cells of an arrangement.

Both algorithms can be improved by some refinements. For example, the dimension and the number of generators can be reduced by 1. We believe that all refinements introduced in [6, 1], important for a practical implementation in high-speed solvers, can be applied to both algorithms. Here we shall compare the "pure" versions of the algorithms, where their combinatorial properties are more transparent.

```
{1} Function RevSearchEnu(c):
{2}    output c
{3}    for each h such that c[h] = 1:
{4}       e = c; e[h] = −1
{5}       if isCell(e): if ParentCell(e) == c:
{6}          RevSearchEnu(e)
{7} end.
```

Algorithm 1 Reverse search enumeration

```
{1} Function IncEnu(c):
{2}    if Length(c) == m:
{3}       output c
{4}    else:
{5}       if isCell((c 1)): IncEnu((c 1))
{6}       if isCell((c −1)): IncEnu((c −1))
{7} end.
```

Algorithm 2 Incremental enumeration

6 Implementation and experiments

General aspects of implementation. Both algorithms were implemented in Python 2.7. The module `gurobipy` is used as a linear programming solver, providing capability for a fast implementation of the isCell procedure.

The incremental enumeration algorithm is implemented in a deterministic way. The reverse search algorithm chooses the initial cell randomly. Furthermore, the algorithm uses perturbation to resolve possible degeneracy. The perturbation vector is chosen randomly, too. Occasionally, the reverse search algorithm can fail due to numeric instability, for example in case of almost parallel generators. Similar problems are avoided in the incremental enumeration algorithm for a price of increased complexity by a factor $\varepsilon \in [1,2]$. Further details on the implementation are available from authors.

Data for the experiments. We are aware of several aspects that may influence the performance of the algorithms. First of all, the *degeneracy* of the input zonotope can influence the shapes and widths of the cell trees. Degeneracy depends on the structure of linear dependencies among generators of the input zonotope. A zonotope is *nondegenerate* if no d-tuple of generators is linearly dependent. Such a zonotope attains the bounds for the number of vertices from Theorem 4. Every vertex of such a zonotope is incident with "many" edges.

We will generate these cases randomly for some dimensions and some numbers of generators.

On the other hand, there are *simple* zonotopes. They are quite rare, see [7]. We will use *permutahedra* as representatives of simple zonotopes. Simple zonotopes have the property that each vertex is incident with exactly d edges (i.e. as least as possible), which can be advantageous for the reverse search algorithm.

Additionally, some more or less degenerate cases are generated, see Table 1. The shortcut "d.r." stands for *degeneracy ratio*. For details on generation of input data, please contact authors by e-mail.

There are more factors influencing the performance of the algorithms. For example, it is the initial ordering of generators, the starting cell and the mechanism of determining the parent cell (which influences the reverse search). These aspects deserve further theoretical elaboration.

Results. As stated in Section 5, both algorithms rely on linear programming. The remaining computational effort is negligible compared to the time consumed by solving the linear programs. Hence, it is natural to compare the algorithms in terms of the number of linear programs to be solved. Table 1 shows the results for individual generated instances. The incremental enumeration algorithm outperforms the reverse search algorithm for all test cases.

For the purposes of more detailed comparison, we call the ratio between the number of LP by reverse search (the first number in each cell of the table) and number of LP by incremental enumeration (the

d.r.	nondegenerate		0.33	0.66	permutahedron
d/m	20	50/45/35/25	20	20	$d(d-1)/2$
3	4634/2320/382	81261/39302/2452	3775/2322/376	3168/1648/274	12/10/6
4	2.4e4/1.0e4/2320	7.5e5/3.0e5/28470	22392/8926/2058	8366/3463/854	81/61/24
5	1.1e5/3.3e4/10072	1.9e6/6.6e5/1.1e5	1.0e5/3.2e4/9764	84196/27804/7974	641/403/120
6	3.7e5/8.8e4/33328	1.4e6/3.8e5/1.1e5	2.7e5/7.8e4/2.6e4	68608/19493/6894	5641/2970/720
7	8.7e5/1.9e5/87592	n.a.	7.2e5/1.6e5/6.7e4	2.9e5/5.3e4/2.9e4	5.4e4/2.4e4/5040
8	1.9e6/3.4e5/1.9e5	n.a.	1.8e6/3.0e5/1.7e5	70681/13686/7068	6.1e5/2.2e5/4.0e4

Table 1 Numbers of LP necessary by reverse search/incremental search/number of vertices

second number in each cell) to be the *performance ratio*.

Two trends seem to display in the results. First, the performance ratio rises with d, independent of degeneracy or other influences. Second, the more degenerate problems have lesser performance ratio.

The influence of the number of the generators we have not observed. However, it may be harder to recognize due to nondeterministic character of the implementation of reverse search algorithm.

7 Conclusion

We faced the problem of vertex enumeration of zonotopes. This geometric problem was motivated by a particular optimization problem, namely the convex maximization over an image of a high-dimensional cube in a low-dimensional space under a linear mapping. We implemented two algorithms with reasonably promising theoretical properties. The recent incremental algorithm [5] clearly outperformed the classical reverse search approach [2] for all test cases. The two trends observed in Results paragraphs of Section 6 deserve attention for future research. In particular, the main line of further research should give a theoretical proof of what has been observed empirically.

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Fuzzy preference matrix with missing elements and its application to ranking of alternatives

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Abstract: A fuzzy preference matrix is the result of pairwise comparison a powerful method in multi-criteria optimization. When comparing two elements, the decision maker assigns the value between 0 and 1 to any pair of alternatives representing the element of the fuzzy preference matrix. Here, we investigate relations between transitivity and consistency of fuzzy preference matrices and multiplicative preference ones. The obtained results are applied to situations where some elements of the fuzzy preference matrix are missing. We propose a new method for completing fuzzy matrix with missing elements called the extension of the fuzzy preference matrix. We investigate some important particular cases of fuzzy preference matrix with missing elements. Consequently, by the eigenvector of the transformed matrix we obtain the corresponding priority vector. Illustrative numerical examples are supplemented.

Keywords: AHP, decision analysis, fuzzy sets and systems.

JEL Classification: C44

AMS Classification: 90B50, 90C29, 91B06

1 Introduction

In many DM problems, procedures have been established to combine opinions about alternatives related to different points of view. These procedures are often based on pairwise comparisons, in the sense that processes are linked to some degree of preference of one alternative over another, see [2, 3], [5], [6], or [10]. The aim of this contribution is to investigate some important particular cases of fuzzy preference matrix with missing elements.

2 Pairwise comparison matrix – fuzzy preference matrix

Pairwise comparison matrices and fuzzy preference matrices are well known tools of multicriteria decision making (MCDM). An $n \times n$ pairwise comparison matrix $\boldsymbol{A} = \{a_{ij}\}$, with positive elements $a_{ij} > 0$ for all $i,j = 1,2, \dots, n$, is called *multiplicative reciprocal* (*m-reciprocal*) if:

$$a_{ij} \cdot a_{ji} = 1, \text{ for all } i, j, \tag{1}$$

where a_{ij} are the positive entries of *A*. The matrix $\boldsymbol{A} = \{a_{ij}\}$ is *multiplicative consistent (m-consistent)*, if

$$a_{ik} = a_{ij} \cdot a_{jk} \text{ for all } i, j, k. \tag{2}$$

The multiplicative transitivity (*m*-transitivity) is associated with the *m*-consistency as follows. A positive $n \times n$ matrix $\boldsymbol{A} = \{a_{ij}\}$ is *multiplicative-transitive (m-transitive)*, if

$$\frac{a_{ik}}{a_{ki}} = \frac{a_{ij}}{a_{ji}} \cdot \frac{a_{jk}}{a_{kj}} \text{ for all } i, j, k. \tag{3}$$

An $n \times n$ *fuzzy preference matrix* $\boldsymbol{B} = \{b_{ij}\}$ with the elements from the unit interval [0;1] is called *additive reciprocal* (*a-reciprocal*), if

$$b_{ij} + b_{ji} = 1, \text{ for all } i, j = 1,2, \dots, n, \tag{4}$$

where b_{ij} are entries of preference matrix $\boldsymbol{B} = \{b_{ij}\}$.

Matrix $\boldsymbol{B} = \{b_{ij}\}$ is called *additive transitive* (*a-transitive*) [9], if

$$b_{ik} - 0.5 = (b_{ij} - 0.5) + (b_{jk} - 0.5) \text{ for all } i, j, k. \tag{5}$$

If $\boldsymbol{B}$ is *a*-reciprocal and *m*-transitive, then $\boldsymbol{B}$ is called *additive consistent* (*a-consistent*).

For interpretation and some properties of the above mentioned pairwise comparison matrices and fuzzy preference matrices, see [2,4,6,9].

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The extent of m-consistency of a nonnegative m-reciprocal $n \times n$ matrix $\boldsymbol{A}$ is given by the *m-consistency index* $I_{mc}(\boldsymbol{A})$ defined in [7] as

$$I_{mc}(\boldsymbol{A}) = \frac{\rho(A) - n}{n-1}, \tag{6}$$

where $\rho(\boldsymbol{A})$ is the spectral radius of $\boldsymbol{A}$ (particularly, the principal eigenvalue of $\boldsymbol{A}$).

Proposition 1. [7] If $\boldsymbol{A} = \{a_{ij}\}$ is an $n \times n$ positive m-reciprocal matrix, then $I_{mc}(\boldsymbol{A}) \geq 0$. Moreover, $\boldsymbol{A}$ is m-consistent if and only if $I_{mc}(\boldsymbol{A}) = 0$.

The rank of the alternatives in $X = \{x_1, x_2, ..., x_n\}$ is determined by the vector of weights $\boldsymbol{w} = (w_1, w_2, ..., w_n)$, with $w_i > 0$, for all $i = 1,2,...,n$, satisfying $\boldsymbol{A}\boldsymbol{w} = \rho(\boldsymbol{A})\boldsymbol{w}$ such that $\sum_{i=1}^{n} w_i = 1$. This vector is called the (normalized) *priority vector of* $\boldsymbol{A}$. Since the element of the priority vector $w_i > 0$ is interpreted as the relative importance of alternative x_i, the alternatives in X are ranked by their relative importance, see [7,8].

Let $\boldsymbol{B} = \{b_{ij}\}$ be an a-reciprocal $n \times n$ matrix with $0 < b_{ij} < 1$ for all i and j. The *a-consistency index* $I_{ac}(\boldsymbol{B})$ of $\boldsymbol{B}$ is defined as

$$I_{ac}(\boldsymbol{B}) = I_{mc}(\boldsymbol{A}), \text{ where } \boldsymbol{A} = \{\phi(b_{ij})\}, \tag{7}$$

where $\phi(t) = \frac{t}{1-t}$, for $0 < t < 1$. By (7) and Theorem 3 in [6], matrix $\boldsymbol{B}$ is a-consistent if and only if $I_{ac}(\boldsymbol{B}) = 0$. The corresponding priority vector $w^{ac} = (w_1^{ac}, w_2^{ac}, \dots, w_n^{ac})$ determining the ranking of alternatives $x_i, i = 1,2, \dots n$, is given by the characteristic equation $\phi(\boldsymbol{B})w^{ac} = \rho(\phi(\boldsymbol{B}))w^{ac}$.

The *a-transitivity index* $I_{at}^{\sigma}(\boldsymbol{B})$ of $\boldsymbol{B}$ is defined as

$$I_{at}^{\sigma}(\boldsymbol{B}) = I_{mc}(\boldsymbol{A}_{\sigma}), \text{ where } \boldsymbol{A}_{\sigma} = \{\varphi_{\sigma}(b_{ij})\}, \tag{8}$$

where $\varphi_{\sigma}(t) = \sigma^{2t-1}$, for $t \in [0,1]$ and $\sigma > 1$ characterizes the given evaluation scale $[1/\sigma ; \sigma]$. By (8) and Theorem 4 in [6], $\boldsymbol{B}$ is *a-transitive* if and only if $I_{at}^{\sigma}(\boldsymbol{B}) = 0$. The ranking of alternatives $x_i, i = 1,2, \dots n$ is given by the priority vector $w^{at} = (w_1^{at}, w_2^{at}, \dots, w_n^{at})$ which is calculated from the characteristic equation $\varphi_{\sigma}(\boldsymbol{B})w^{at} = \rho(\varphi_{\sigma}(\boldsymbol{B}))w^{at}$.

To provide a consistency measure independently of the dimension n of the matrix $\boldsymbol{B}$, T. Saaty in [7] proposed the consistency ratio, here, we shall call it *m-consistency ratio*. This is obtained by taking the ratio I_{mc} to its mean value R_{mc}, i.e. an arithmetic average over a large number of positive m-reciprocal matrices of dimension n. By a similar way we define also the *a-consistency ratio*, i.e.

$$CR_{mc} = I_{mc}/R_{mc}, \quad \text{eventually, } CR_{ac} = I_{ac}/R_{mc}. \tag{9}$$

In practical DM problems a-consistency of a positive a-reciprocal matrix $\boldsymbol{B}$ is "acceptable" if $CR_{ac}(\boldsymbol{B}) < 0.1$, see [7,8]. Similarily, a-transitivity of a positive a-reciprocal matrix $\boldsymbol{B}$ is "acceptable", if $CR_{at}^{\sigma}(\boldsymbol{B}) < 0.1$.

3 Fuzzy preference matrix with missing elements

In many decision-making procedures we assume that experts are capable of providing preference degrees between any pair of possible alternatives. However, this may not be always true, which makes a missing information problem. A missing value in a fuzzy preference matrix is not equivalent to a lack of preference of one alternative over another. A missing value can be the result of the incapacity of an expert to quantify the degree of preference of one alternative over another. In this case he/she may decide not to guess the preference degree between some pairs of alternatives. It must be clear that when an expert is not able to express a particular value a_{ij}, because he/she does not have a clear idea of how the alternative x_i is better than alternative x_j, this does not mean that he/she prefers both options with the same intensity. The DM could be also bored by evaluating too many pairs of alternatives. To deal with these situations, in the following we introduce the incomplete fuzzy preference relation matrix. Here, we use different approach and notation comparing to [1], or [4].

Let $n > 2$, $I = \{1,2,...,n\}$ be the set of indices, $I^2 = I \times I$ be the Cartesian product of I, i.e. $I^2 = \{(i,j) | i,j \in I\}$. Here, we consider reciprocity of pairwise preference relation as a natural requirement. Therefore, we shall consider only a-reciprocal fuzzy preference relations, particularly, a-reciprocal fuzzy preference matrices.

Let $K \subseteq I^2$, $K = \{(i_1,j_1),(i_2,j_2),...,(i_q,j_q)\}$ be the set of couples (i,j) of indexes such that there exists a pairwise comparison value a_{ij}, $0 \leq a_{ij} \leq 1$, evaluated by an expert. By K' we denote the symmetric subset of pairs to K, i.e. $K' = \{(j_1,i_1),(j_2,i_2),...,(j_q,i_q)\}$, obtained by the expert, too. By reciprocity, each subset $K \subseteq I^2$ can be expressed as follows

$$K = L \cup L' \cup D, \tag{10}$$

where L is the set of couples of indexes (i,j) of given elements a_{ij} and D is the diagonal of the $n\times n$ matrix, i.e. $D = \{(1,1),(2,2),...,(n,n)\}$, where $a_{ii} = 0.5$ for all i. Here, by reciprocity, the expert is able to quantify both a_{ij} and a_{ji}. The eventual elements a_{ij} with $(i,j)\in I^2$, $(i,j)\notin K$ are called *missing elements*.

Now we define the *fuzzy preference matrix* $\boldsymbol{B}(K) = \{b_{ij}\}_K$ *with missing elements* as

$$b_{ij} = \begin{cases} a_{ij} & \text{if} \quad (i,j)\in K, \\ - & \text{if} \quad (i,j)\notin K. \end{cases} \tag{11}$$

The missing elements of matrix $\boldsymbol{B}(K)$ are denoted by dash "-". On the other hand, the elements given by experts are denoted by a_{ij} where $(i,j)\in K$. By reciprocity, it is sufficient that the expert quantifies only elements a_{ij} where $(i,j)\in L$, such that $K = L\cup L'\cup D$. In section 5 we shall investigate two special important cases of L, particularly, $L=\{(1,2);(2,3);...,(n\text{-}1,n)\}$ and $L=\{(1,2);(1,3);...,(1,n)\}$.

4 Extension of fuzzy preference matrix with missing elements and its consistency/transitivity

In this section we shall deal with the problem of finding the values of missing elements of a given fuzzy preference matrix so that the extended matrix is as much a-consistent/a-transitive as possible. In the ideal case the extended matrix would become a-consistent/a-transitive. The idea of this extension is based on the "least square method". Let us start with the a-consistency.

Let $K\subseteq I^2$, let $\boldsymbol{B}(K) = \{b_{ij}\}_K$ be a fuzzy preference matrix with missing elements. The matrix $\boldsymbol{B}^{ac}(K)=\{b_{ij}^{ac}\}_K$ called an *ac-extension of* $\boldsymbol{B}(K)$ is defined as follows

$$b_{ij}^{ac} = \begin{cases} b_{ij} & \text{if} \quad (i,j)\in K, \\ \dfrac{v_i^*}{v_i^*+v_j^*} & \text{if} \quad (i,j)\notin K. \end{cases} \tag{12}$$

Here, $v^* = (v_1^*, v_2^*, ..., v_n^*)$ called the *ac-priority vector with respect to K* is the optimal solution of the following optimization problem:

$$(\mathbf{P_{ac}}) \qquad d_{ac}(\boldsymbol{v}, \boldsymbol{B}(K)) = \sum_{(i,j)\in K} (b_{ij} - \frac{v_i}{v_i+v_j})^2 \to \min; \tag{13}$$

subject to

$\sum_{j=1}^{n} v_j = 1$, $v_i \geq \varepsilon > 0$ for all i=1,2,...,n. (Here, $\varepsilon > 0$ is a preselected sufficiently small number.)

Moreover, matrix $\boldsymbol{B}^{at}(K) = \{b_{ij}^{at}\}_K$ called the *at-extension of* $\boldsymbol{B}(K)$ is defined as follows

$$b_{ij}^{at} = \begin{cases} b_{ij} & \text{if} \quad (i,j)\in K, \\ \max\{0, \min\{1, \frac{1}{2}(1+nu_i^* - nu_j^*)\}\} & \text{if} \quad (i,j)\notin K. \end{cases} \tag{14}$$

Here, $u^* = (u_1^*, u_2^*, ..., u_n^*)$ called the *at-priority vector with respect to K* is the optimal solution of the following optimization problem

$$(\mathbf{P_{at}}) \qquad d_{at}(\boldsymbol{u}, \boldsymbol{B}(K)) = \sum_{(i,j)\in K} (b_{ij} - \tfrac{1}{2}(1+nu_i - nu_j))^2 \to \min; \tag{15}$$

subject to

$\sum_{j=1}^{n} u_j = 1$, $u_i \geq \varepsilon > 0$ for all i=1,2,...,n. ($\varepsilon > 0$ is a preselected sufficiently small number.)

Notice that in general, the optimal solution $u^* = (u_1^*, u_2^*, ..., u_n^*)$ of (P_{at}) does not satisfy the following condition

$$0 \leq \tfrac{1}{2}(1+nu_i^* - nu_j^*) \leq 1, \tag{16}$$

meaning that $\boldsymbol{B} = \{b_{ij}\} = \{\frac{1}{2}(1+nu_i - nu_j)\}$ is a fuzzy preference matrix. That is why in the definition of at-extension of $\boldsymbol{B}(K)$ we use formula (14) securing that all elements b_{ij} of $\boldsymbol{B}$ belong to the unit interval [0,1]. In the next section we shall derive necessary and sufficient conditions securing that (16) is satisfied at least for some special cases.

5 Special cases of fuzzy preference matrix with missing elements

For a complete definition of a reciprocal fuzzy preference $n\times n$ matrix we need $N=\frac{n(n-1)}{2}$ pairs of elements to be evaluated by an expert. For example, if n=10, then N=45, which is a considerable amount of pairwise comparisons. We ask that the expert would evaluate only "around n" pairwise comparisons of alternatives which seems to be a reasonable amount. In this section we shall deal with two important particular cases of fuzzy preference matrix with missing elements where experts evaluate only n-1 pairwise comparisons of alternatives.

Let $K\subseteq I^2$ be a set of indexes given by an expert, $\boldsymbol{B}(K)=\{b_{ij}\}_K$ be a fuzzy preference matrix with missing elements. Moreover, let $K = L\cup L'\cup D$. In fact, it is sufficient to assume that the expert evaluates only the matrix elements of L, e.g. $b_{12,}\, b_{23,}\, b_{34,\ldots,}\, b_{n-1,n}$.

5.1 Case L={(1,2);(2,3);...;(n-1,n)}

Here, we assume that the expert evaluates the following n-1 elements of the fuzzy preference matrix $\boldsymbol{B}(K)$: b_{12}, b_{23},...,$b_{n-1,n}$. First, we investigate the ac-extension of $\boldsymbol{B}(K)$. We can prove the following result.

Proposition 2. Let L={(1,2);(2,3);...,(n-1,n)}, $0 < b_{ij} < 1$ with $b_{ij} + b_{ji} = 1$ for all $(i,j)\in K$, $K = L\cup L'\cup D$, and L' = {(2,1);(3,2);...,(n, n-1)}, D = {(1,1),...,(n,n)}. Then ac-priority vector $v^*=(v_1^*,v_2^*,\ldots,v_n^*)$ with respect to K is given as

$$v_1^*=\frac{1}{S},\ \text{and}\ v_{i+1}^*=a_{i,i+1}v_i^*,\ \text{for } i=1,2,..,n\text{-}1, \tag{17}$$

where $S=1+\sum_{i=1}^{n-1}a_{i,i+1}a_{i+1,i+2}\cdot\ldots\cdot a_{n-1,n}$ and $a_{ij}=\frac{1-b_{ij}}{b_{ij}}$ for all $(i,j)\in K$.

By (12) and Proposition 4 in [5] it follows that $\boldsymbol{B}^{ac}(K)=\{\,b_{ij}^{ac}\,\}_K$ is a-consistent.

Now, we investigate the at-extension $\boldsymbol{B}^{at}(K)$ of $\boldsymbol{B}(K)$. We obtain the following result.

Proposition 3. Let L={(1,2);(2,3);...,(n-1,n)}, $0 < b_{ij} < 1$ with $b_{ij} + b_{ji} = 1$ for all $(i,j)\in K$, $K = L\cup L'\cup D$. Then at-priority vector $u^*=(u_1^*,u_2^*,\ldots u_n^*)$ with respect to K is given as

$$u_i^*=\frac{2}{n^2}\sum_{j=1}^{n-1}\alpha_j-\frac{2}{n}\alpha_{i-1}-\frac{n-i-1}{n}\ \text{ for } i=1,2,...,n, \tag{18}$$

$$\text{where } \alpha_0=0,\ \alpha_j=\sum_{i=1}^{j}b_{i,i+1}\ \text{ for } j=1,2,...,n\text{-}1. \tag{19}$$

Remark. In general, the optimal solution $u^*=(u_1^*,u_2^*,\ldots,u_n^*)$ of (P_{at}) does not satisfy condition

$$0\le\tfrac{1}{2}(1+nu_i^*-nu_j^*)\le 1,\ \text{for all } i,j=1,2,...,n, \tag{20}$$

i.e. $\boldsymbol{B}=\{b_{ij}\}=\{\tfrac{1}{2}(1+nu_i-nu_j)\}$ is not a fuzzy preference matrix. We can easily prove the necessary and sufficient condition for satisfying (20) based on evaluations $b_{i,i+1}$.

Proposition 4. Let L={(1,2);(2,3);...,(n-1,n)}, $0\le b_{ij}\le 1$ with $b_{ij} + b_{ji} = 1$ for all $(i,j)\in K$, $K = L\cup L'\cup D$. Then the at-extension $\boldsymbol{B}^{at}(K)=\{\,b_{ij}^{at}\,\}_K$ is a-transitive if and only if

$$\left|\sum_{k=i}^{j-1}b_{k,k+1}-\frac{j-i}{2}\right|\le\frac{1}{2}\ \text{ for } i=1,2,...,n\text{-}1,\ j=i+1,...,n. \tag{21}$$

Example 1. Let L={(1,2);(2,3);(3,4)}, let the expert evaluations be b_{12}= 0.9, b_{23}= 0.8, b_{34}= 0.6, with $b_{ij} + b_{ji} = 1$ for all $(i,j)\in L$, let $K = L\cup L'\cup D$. Hence $\boldsymbol{B}(K)=\{b_{ij}\}_K$ is a fuzzy preference matrix with missing elements as follows

$$\boldsymbol{B}(K)=\begin{pmatrix}0.5 & 0.9 & - & -\\ 0.1 & 0.5 & 0.8 & -\\ - & 0.2 & 0.5 & 0.6\\ - & - & 0.4 & 0.5\end{pmatrix}.$$

Solving (P_{ac}) we obtain ac-priority vector $\boldsymbol{v}^*$ with respect to K, particularly, $v^* = (0.864; 0{,}096; 0{,}024; 0{,}016)$. By (18) we obtain $\boldsymbol{B}^{ac}(K)$ - ac-extension of $\boldsymbol{B}(K)$ as follows

$$\boldsymbol{B}^{ac}(K) = \begin{pmatrix} 0.5 & 0.9 & 0.97 & 0.98 \\ 0.1 & 0.5 & 0.8 & 0.86 \\ 0.03 & 0.2 & 0.5 & 0.6 \\ 0.02 & 0.14 & 0.4 & 0.5 \end{pmatrix},$$

where, $\boldsymbol{B}^{ac}(K)$ is a-consistent, as $d_{ac}(\boldsymbol{v}, \boldsymbol{B}(K)) = 0$, hence $I_{ac}(\boldsymbol{B}^{ac}(K)) = 0$.
Solving (P_{at}) we obtain at-priority vector u^* with respect to K, $u^* = (0{,}487; 0{,}287; 0{,}137; 0{,}088)$. Then $\boldsymbol{B}^{at}(K)$ is an at-extension of $\boldsymbol{B}(K)$ as

$$\boldsymbol{B}^{at}(K) = \begin{pmatrix} 0.5 & 0.9 & 1.0 & 1.0 \\ 0.1 & 0.5 & 0.8 & 0.9 \\ 0.0 & 0.2 & 0.5 & 0.6 \\ 0.0 & 0.1 & 0.4 & 0.5 \end{pmatrix},$$

where, $\boldsymbol{B}^{at}(K)$ is not a-transitive, as $d_{ac}(\boldsymbol{v}, \boldsymbol{B}(K)) > 0$. It can be easily verified as $I^9_{at}(\boldsymbol{B}^{at}(K)) = 0.057$.

5.2 Case $L=\{(1,2);(1,3);...;(1,n)\}$

Now, we assume that the expert evaluates a fixed element with the remaining n-1 elements, i.e. the fuzzy preference matrix $\boldsymbol{B}(K)$: $b_{12}, b_{13},...,b_{1n}$. We investigate the ac-extension of $\boldsymbol{B}(K)$ and obtain the following result.

Proposition 5. Let $L=\{(1,2);(1,3);...,(1,n)\}$, $0 < b_{ij} < 1$ with $b_{ij} + b_{ji} = 1$ for all $(i,j) \in K$, $K = L \cup L' \cup D$, and $L' = \{(2,1);(3,1);...;(n,1)\}$, $D = \{(1,1),...,(n,n)\}$. Then ac-priority vector $v^* = (v_1^*, v_2^*, \ldots, v_n^*)$ with resp. to K is given as

$$v_1^* = \frac{1}{V}, \text{ and } v_{i+1}^* = a_{1,i+1} v_i^*, \text{ for } i=1,2,..,n\text{-}1, \tag{22}$$

where $V = 1 + \sum_{i=1}^{n-1} a_{1,i+1}$, and $a_{ij} = \dfrac{1-b_{ij}}{b_{ij}}$ for all $(i,j) \in K$.

We conclude that the ac-extension of $\boldsymbol{B}(K)$, i.e. matrix $\boldsymbol{B}^{ac}(K) = \{ b_{ij}^{ac} \}_K$ is a-consistent.

Now, we investigate the at-extension matrix $\boldsymbol{B}^{at}(K)$ of $\boldsymbol{B}(K)$. We can prove the following result.

Proposition 6. Let $L=\{(1,2);(1,3);...;(1,n)\}$, $0 < b_{ij} < 1$ with $b_{ij} + b_{ji} = 1$ for all $(i,j) \in K$, $K = L \cup L' \cup D$. Then at-priority vector $u^* = (u_1^*, u_2^*, \ldots u_n^*)$ with respect to K is given as

$$u_1^* = \frac{2}{n^2} \sum_{j=1}^{n-1} b_{1,j+1} + \frac{1}{n^2}, \text{ and } u_{i+1}^* = u_1^* + \frac{1 - 2b_{1,i+1}}{n} \text{ for } i=1,2,...,n\text{-}1. \tag{23}$$

Remark. In general, the optimal solution $u^* = (u_1^*, u_2^*, \ldots u_n^*)$ of (P_{at}) does not satisfy condition (20), i.e. $\boldsymbol{B} = \{b_{ij}\} = \{\frac{1}{2}(1 + nu_i - nu_j)\}$ is not necessarily a fuzzy preference matrix. We can, however, easily prove the following necessary and sufficient condition for (20):

$$\left| b_{1j} - b_{1i} \right| \le \frac{1}{2} \text{ for } i,j=1,2,...,n. \tag{24}$$

Example 2. Let $L=\{(1,2);(1,3);(1,4)\}$, the expert evaluations be $b_{12}=0.9$, $b_{13}=0.8$, $b_{14}=0.3$, with $b_{ij} + b_{ji} = 1$ for all $(i,j) \in K$, $K = L \cup L' \cup D$. Hence $\boldsymbol{B}(K) = \{b_{ij}\}_K$ is a fuzzy preference matrix with missing elements

$$\boldsymbol{B}(K) = \begin{pmatrix} 0.5 & 0.9 & 0.8 & 0.3 \\ 0.1 & 0.5 & - & - \\ 0.2 & - & 0.5 & - \\ 0.7 & - & - & 0.5 \end{pmatrix}.$$

Solving (P_{ac}) we obtain ac-priority vector $\boldsymbol{v}^*$ with respect to K, particularly, $v^* = (0{,}271; 0{,}030; 0{,}068; 0{,}632)$. Then we obtain $\boldsymbol{B}^{ac}(K)$ - the ac-extension of $\boldsymbol{B}(K)$ as

$$\boldsymbol{B}^{ac}(K) = \begin{pmatrix} 0.5 & 0.90 & 0.80 & 0.30 \\ 0.10 & 0.5 & 0.30 & 0.04 \\ 0.20 & 0.70 & 0.5 & 0.10 \\ 0.70 & 0.96 & 0.90 & 0.5 \end{pmatrix},$$

where, by (12) $\boldsymbol{B}^{ac}(K)$ is a-consistent, as $d_{ac}(\boldsymbol{v}, \boldsymbol{B}(K)) = 0$, hence $I_{ac}(\boldsymbol{B}^{ac}(K)) = 0$. Solving (P_{at}) we obtain at-priority vector u^* with respect to K as $u^* = (0{,}312; 0{,}113; 0{,}162; 0{,}412)$. By (20) we obtain $\boldsymbol{B}^{at}(K)$ – the at-extension of $\boldsymbol{B}(K)$ as

$$\boldsymbol{B}^{at}(K) = \begin{pmatrix} 0.5 & 0.90 & 0.80 & 0.30 \\ 0.10 & 0.5 & 0.40 & 0.00 \\ 0.20 & 1.00 & 0.5 & 0.00 \\ 0.70 & 1.00 & 1.00 & 0.5 \end{pmatrix},$$

where $\boldsymbol{B}^{at}(K)$ is not a-transitive, as condition (24) fails for $|b_{12} - b_{14}| = 0.6 > \frac{1}{2}$.

6 Conclusion

In this paper we have dealt with some properties of fuzzy preference relations, particularly reciprocity, consistency and transitivity of relations given in the form of square matrices with the entries from the unit interval. We have shown how to measure the grade of a-consistency and/or a-transitivity, and also how to evaluate the pairs of elements by fuzzy values. Also, we have proposed a new method for measuring of inconsistency based on Saaty´s principal eigenvector method. Moreover, we have dealt with two particular cases of the incomplete fuzzy preference matrix, where some elements of pairwise comparison are missing. We have proposed some special methods for dealing with these cases. Two illustrating examples have been presented to clarify our approach.

Acknowledgements

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Modeling price of dwelling based on socioeconomic factors

Michael Rost[1], Ludvík Friebel[2]

Abstract: In our contribution we focused our attention to modelling relationship between average price of dwelling in the given districts on one side and socioeconomic variables as explanatory variables on the other side. More concretely our regression model considered several socioeconomic variables, resp. indexes: unemployment rate, criminality, average length of incapacity to work, index of ageing and proportion of economic active inhabitants in conjunction with their spatial coordinates. Dataset were combined from two sources. The first data set was obtained from real estate agency server http://www.realitymorava.cz, where the comparison of dwelling prices in all districts of the Czech Republic is periodically published. The second data set was selected through SQL commands from database provided by the Czech Statistical Office. During the modeling process we respected the spatial character of the observed data and for estimation of regression coefficients we used spatial simultaneous autoregressive lag model. Data preprocessing and model building with numeric evaluation was carried out in programming language R 3.0.1 and MS Access. For this purpose we used several specific libraries like "sp" or "maptools".

Keywords: dwelling, economic activity, unemployment, spatial regression, autoregressive lag model,

JEL Classification: C51, C38
AMS Classification: 62H11, 62P10, 93E24

1. Introduction

Evaluation of economic situation and life quality has been recently dealt with many authors. Economic situation and life quality at the level of administrative districts (LAU1) of South Moravia is investigated, for example, by Živělová and Jánský [7]. In their work, life quality is assessed on the base of analysis of the population and unemployment increase. Moreover, the authors involve indicators of medical care and transport and technical infrastructure.

The same authors, in another work [8], investigate the development of life quality in regions (NUTS 3) primarily from the social aspects' point of view. The quality of life in given regions is assessed in terms of the unemployment rate, the number of job applicants per one vacancy, the number of physicians per 1000 inhabitants, the number of completed flats and the length of road network. Data envelopment analysis model as the tool for assessment of life quality is used in work [2].

The assessment of regions resp. regional disparities is considered by the Ministry of Regional Development of the Czech Republic within the scope of the WD-05-07-3 – Regional disparities Program in the availability and affordability of housing, their socioeconomic consequences and tools directed to decrease of regional disparities, for details see [4].

The above mentioned works are focused upon the comparison of the regions, with respect to particular aspects. In the scope of multicriteria regional evaluation, the output of the MasterCard Czech Centres of Development Project (http://www.centrarozvoje.cz) are considered to be important. Life quality and economic potential of towns were assessed in this project, using 11 selected indicators (Master CARD WordlWIDE, 2010), whose weights were determined empirically, see [5]

Furthermore, Kuprová and Kamenický [3] in their work deal with evaluation of life quality in the NUTS 3 regions. The evaluation is based on 47 particular indicators.

In our contribution we focused our attention to modeling relationship between average price of dwelling in the given districts on one side and socioeconomic variables as explanatory variables on the other side. More concretely our regression model considered several socioeconomic variables, resp. indexes: unemployment rate

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(UR), criminality (CR), average length of incapacity to work (IW), index of ageing (IA) and proportion of economic active inhabitants (EA) in conjunction with their spatial coordinates.

In different multicriteria decision makings in above mentioned works is used as a level of life quality a theoretical integral assessing criterion. In our work we tried to establish a market criterion besides socio-economical criterions. There is questionable how much is possible the price of dwelling as a level attractiveness of a region to explain with these predictive variables. We use multinomial regression analysis method to study just this dependency.

Price of dwelling is dependent on many factors. That are technical and legal characteristics, for example number of rooms, surface, floor, age, condition of estate, character of ownership, rental etc. These objective parameters are the secondary considering differences among regions. Let's consider multiple differences between prices of dwelling between Prague and Ústí nad Labem. Thus the importance of the location and implied socio-economic factors is permanently rising. In our contribution we attempt to explain the price of dwelling with the aid macro-economic characteristics just arising from the location of flat. As the elementary unit of localization of dwelling we have chosen the districts (LAU1).

2. Methodology

The average price of dwelling in all districts of the Czech Republic was obtained as an average value from statistics which is periodically published on http://www.realitymorava.cz [10]. Respecting that the owner of this website publish average prices of flats within different categories, we have derived the average price as a scalar product of the vector representing share the particular category on flats sold and their price vector. The mentioned period covered 12 months – V/2011-VI/2012. Obtained data included 78 districts of the Czech Republic.

We have considered five predictive variables – indices: index of economic activity as the proportion of inhabitants aged from 15 to 64 to the whole population (EA), ageing index as the proportion of inhabitants aged from 15 to 64 to the whole population (AI), unemployment rate as the number of unemployed in % (UR), average length of incapacity to work as the average number of calendar days of incapacity to work per one registered event (IW), criminality as the number of crimes per 10 thousand inhabitants (CR).

The source data was obtained from website of Czech Statistical Office and are available at http://www.czso.cz [9]. On account of prediction we have chosen year 2010. We have adjusted six month time distance from the origin of the file with dwelling price.

At first we use classical linear regression methodology to construct a model in order to predict price of dwelling in particular districts of the Czech Republic in dependence on the socio-economic variables as the explanatory variables. Dependent variable Y was a natural logarithm of the average price of dwelling:

$$\ln(y) = a + b_1 x_1 + b_2 x_2 + \cdots + b_n x_n + \xi, \tag{1}$$

where ξ is a sum of impact of the all explanatory variables that are not included in model which follows N(0, σ). Regression coefficients was estimated with classical ordinary least square method (OLS).

With respect to hypothesis about spatial autocorrelation of the error term after fitting OLS we compute Moran's I measuring the extent to which is variable of interest, e.g. residuals resulting from OLS model, correlated with itself, considering the spatial distribution of this variable at the same time. The Moran's I could be characterized through following equation:

$$I = \frac{n}{S_0} \cdot \frac{\sum_{i=1}^{n}\sum_{i=1}^{n} w_{ij}(x_i - \mu)(x_j - \mu)}{\sum_{i=1}^{n} w_{ij}(x_i - \mu)^2} \tag{2}$$

Because the spatial dependence was proved, see vide infra, we consequently used for the modeling purpose so called spatial simultaneous autoregressive lag model [1] which incorporates spatial effects by including a spatially lagged dependent variable as an additional predictor. This regression model could be expressed as $\mathbf{y} = \rho\mathbf{W}\mathbf{y} + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}$, where $\mathbf{W}\mathbf{y}$ is the spatially lagged dependent variables for weight matrix $\mathbf{W}$, $\mathbf{X}$ is a design matrix containing observations on the explanatory variables included in model with one column of ones, $\boldsymbol{\beta}$ is vector of estimated regression coefficient, $\boldsymbol{\varepsilon}$ is vector of error terms following $N(0;\sigma)$ and ρ is the simultaneous autoregressive lag coefficient.

The matrix of weight **W** was determined on the basis of so called "neighborhood matrix" with elements d_{ij}. Values of these elements were determined in the following manner: "Neighbors are areas with main town within radius 50 km" or more formally:

$$d_{ij} = \begin{cases} 0 & \text{if the distance of main town of the area is greather than 50 km} \\ 1 & \text{if the distance of main town of the area is less or equal 50 km} \end{cases} \quad (3)$$

Before utilization of the weight matrix **W** we carried out her row standardization. For better clarity we provide graphical illustration of the "neighborhood matrix" see figure 1.

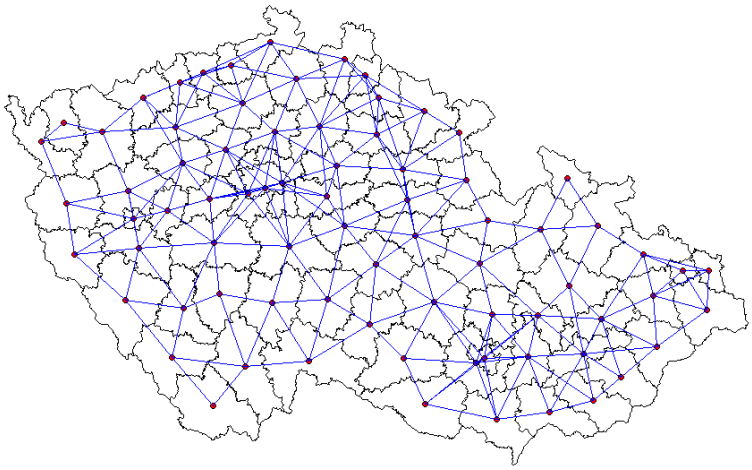

Figure 1: Non null elements of neighborhood matrix

With aim to identify if the spatial dependence still exists in residuals from our spatial model we finally tested hypothesis about existence of spatial clusters. For this purpose we used global Moran's I.

Data preprocessing and model building with numeric evaluation was carried out in programming language R 3.0.1 [6] and DBMS MS Access. For this purpose we used several specific libraries, for example "sp" or "maptools" and others.

3. Results and Discussion

At first we excluded Prague from the model, because of its exceptionality originating in capital status and in dotation policy of EU. Results of the OLS regression model are depicted in table 1.

Explanatory variable	Estimate	S.E. of Estimate	t	*p*-value
Intercept	19.27318	3.492642	5.518	5.42×10-7
EA	-0.08361	0.03985	-2.098	0.01968
AI	0.28865	0.279906	1.031	0.3060
UR	-0.05697	0.012211	-4.665	1.44×10^{-5}
IW	-0.11043	0.058782	-1.879	0.0645
CR	-0.00037	0.000373	-0.995	0.3230
Residual standard error	0.2578	70 df		
F-test	11.45	5 a 70 df		4.299×10^{-8}
R^2_{adj}	0.4106			
AIC	16.514			

Table 1 Results of OLS regression - excluding Prague

From the results of derived regression analysis is evident that value od adjusted index of determination reached 0.4106. We could see this value as relatively sufficient with respect to studied phenomenon. The same

statement could be expressed for whole model on the basis of F-test (p -value = 4.299×10^{-8}). The value for the Akaike information criterion reached the value 16.514 see Table 1.

In this model, there was identified as the most significant explanatory variable the rate of unemployment (UR) with p-value 1.44×10^{-5}. So it could be said that there is agreement with our expectation which could be expressed in following statement. With increasing rate of unemployment in the studied area the price level of dwelling is decreasing inside this area. As the second most significant explanatory variable was identified index of economic activity with p-value 0.0197. Estimate for this explanatory variable is negative too. In our case it means that with greater economic activity in studied area the price level of dwelling has decreasing tendency. The last statistically significant variable is the average length of incapacity to work. In this case the identified relationship is in the agreement with common theoretical expectation.

Consequently we can say that others explanatory variables, more concretely CR and AI are non-significant. In case of criminality it is logical because, for example the regions with the highest rate of criminality like Ostrava city and Prague has both the big discrepancies in average price of dwelling. These discrepancies are nearly three times higher.

After reducing the full linear model we obtained the following equation only with two explanatory variables (EA and NZ). More concretely: $\ln(y)=-0.1EA-0.528UR+\varepsilon$ with following statistics $R^2_{adj.}=0{,}4114$ and with p-value of F-test $3{,}966\times10^{-9}$. Cartogram of the residuals derived from the simplified version of the OLS regression model is provided in the Figure 2. From this cartogram is obvious that there is present spatial clustering of some higher, resp. lower residual values.

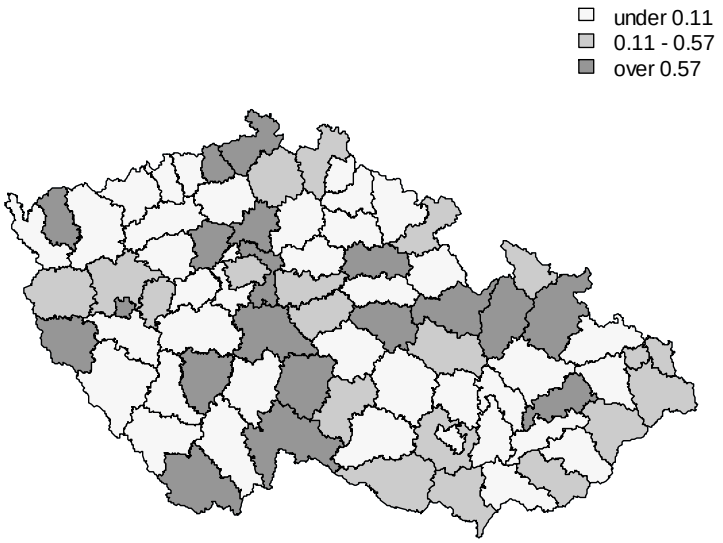

Figure 2: Cartogram with the residuals obtained with OLS regression

Spatial correlation hypothesis was verified using Moran's I. Test results about occurrence of spatial clusters for residual term obtained by OLS regression are depicted in Table 2.

Moran's *I*	Estimated mean value	Variance	Standard score	p-value
0.3121	-0.029466751	0.0042077	5.266	6.972×10^{-8}

Tab.2: Test results for spatial autocorelation test – OLS regression

From the results depicted in table 2 is obvious that Moran's I for the estimated residuals derived from OLS regression model is 0.312 with variance 0.004208. With these values is associated the value of standard score 5.266 which is considerably higher than estimated men value -0.0294 (p-value $=6.972\times10^{-8}$). So it is evident that assumption of independence among the particular observations in OLS regression used for estimates of regression coefficients does not hold and that there exist spatial clustering in our data.

With respect to these findings we decided to revise our linear regression model and change our approach. So we used regression model with spatially lagged dependent variable. This model takes into account the spatial character of our data. The results obtained by this model are provided in following Table 3.

Explanatory variable	Estimate	S.E. of Estimate	t	*p*-value
Intercept	1.3778	2.9892	4.6092	4.043×10^{-6}
EA	-0.061695	0.030557	-2.0191	0.04348
AI	0.31998	0.24237	1.3202	0.18677
UR	-0.045308	0.011271	-4.0197	5.826×10^{-5}
IW	-0.11780	5.2097×10^{-2}	-2.2611	0.02375
CR	-8.3355×10^{-5}	3.1220×10^{-4}	-0.2670	0.78947
Simultaneous autoregressive lag coefficient ρ	0.36879	Likelihood ratio 9.5626		0.0019858
Asymptotic residual standard error	0.1816			
AIC	9.1486			

Table 3: Results of spatial simultaneous autoregressive lag model - excluding Prague

After reducing the full spatial simultaneous autoregressive lag model we obtained the reduced model only with two explanatory variables: EA and UR, for estimates of regression coefficients see Table 4. Akaike information criterion increased to 11.617 in the comparison to non-reduced model.

Explanatory variable	Estimate	S.E. of Estimate	t	*p*-value
Intercept	15.7575	3.469717	4.5414	5.587×10-6
EA	-0.085450	0.038666	-2.2099	0.027109
UR	-0.045023	0.011831	-3.8056	0.00141
Simultaneous autoregressive lag coefficient ρ	0.33457	Likelihood ratio 6.8967		0.008635
Asymptotic residual standard error	0.13094			
AIC	11.617			

Table 4: Results of reduced spatial simultaneous autoregressive lag model - excluding Prague

By interpreting spatial simultaneous autoregressive lag model is necessary to remember that given regression coefficients have rather different interpretation, than at classical regression model based on OLS. In this case regression coefficient explain only the part of the variability of the dependent variable which remains after subtracting of the effect of the price change in adjacent districts. That is caused by autoregressive characteristic of the model. Estimated regression coefficients of explanatory variables (economic activity and unemployment rate) is possible to interpret as short time effect of those variables rather than classical net effects.

Regarding simultaneous autoregressive parameter ρ it represents average effect of the impact price of dwelling in adjacent areas on the price level in assessed area. In our case is this parameter positive and significant with the value 0.33457. In the other words, if the price of dwelling in adjacent areas is increasing, in the assessed value the price is changed in the same way. This dependence has double side effects until stability will be established.

Absence of spatial autocorrelation for the residuals in above mentioned model was verified using Lagrange Multiplier test (p-value=0.18435). Furthermore by the comparison Akaike information criterion (AIC in table 5) at reduced OLS regression model and reduced spatial autoregressive model we come to conclusion, that the preferred model will be the one with consideration of spatial autocorrelation.

We can state, based on the coefficients of used regression model after its reduction (see Table 4), that with the increase of economic activity by one percent the price of dwelling decreases by 8.19%, respectively we multiply this price 0.9181 (exp(-0.08545)= 0.9181). This negative dependency is rather surprising, because we expected that higher economic activity is accompanied by higher purchasing power of inhabitants and so higher price of dwelling on the supply side. The question is if observed negative trend is not results of explosion development activity in economically strong districts before economic crisis. Persisting oversupply of dwelling in past years causes decrease of price level of it in the mentioned districts. It has to be notified, that the dynamics of demand is very high in comparison to supply and that demand follows the economic situation in particular region with minimal time lag. In case of unemployment rate it means decrease by 4.4%, respectively we have to

multiply this price 0.956 (exp(-0.045023)=0.956). This dependence is quite logical, because it could be expected that with increasing unemployment rate the price of dwelling in selected district is decreasing.

It could be concluded, that price of dwelling is primarily predeterminated by change of the dwelling price level in the adjacent regions, which confirms Tobler's geography law. The correlation among the prices in this case is positive. From the set of our explanatory variables the significant influence on the price level has economic activity with unemployment rate. Influence of remaining explanatory variables was not proved by our models. It could be stressed that if we change the setting of our row standardized weight matrix we could obtain to different solution. It could be interesting to carry out the sensitivity analysis, or to consider possibility of inclusion of more relevant explanatory variables into our model.

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Quantification of Framing Effect in the Meat Distribution by ANP

Jan Rydval[1], Jan Bartoška[2]

Abstract. This paper deals with the form of semantic networks used in the creation of the producer's, distributor´s and consumer's information frames for meat purchases. It is based on the consideration that one side of the market (the producer, distributor or consumer) knows more than the other and takes the advantage of it. This leaves the other side in the disadvantage. The process of decision making in the market environment is affected by many factors. In the consumers' information frame consumers express their expectations and preferences for the product they demand. Producers usually work with a greater range of product information than the one available to the consumer. Distributors usually do not communicate complete information from producer to customer. The core motive of this paper is to show the distortion in information which occurs in the market. The essential idea is to capture the basic elements of the information frame and their mutual relationships to express the loss of information and its asymmetry in the market environment. Elements and their relationships are shown using a semantic network. The scales of identified elements in the network are then estimated using Analytic Network Process (ANP). The benefit of this paper is to make customers better informed about the information asymmetry in the market environment.

Keywords: Semantic network, framing effect, producer's frame, distributor´s frame, consumer's frame, asymmetric information, ANP.

JEL Classification: C44

AMS Classification: 90C29

1 Introduction

The integral part of any decision-making processes is the process of receiving information. As Fagley, Coleman and Simon [5] mention, decision-making is influenced by the quality of the information and by the effect of information distortion (framing effect). And as mentioned by Tversky and Kahneman [18] the framing effect included in some information can significantly influence decisions. There are various views on a particular issue in the decision process. Bishop [1] believes that if information is not sorted according to its relevance because we cannot properly decide who has the most important view, we may face the problem of being overloaded with too much information resulting in either poor information acquisition or the whole process is very time consuming and thus very ineffective. However, if we sort information preferring the certain point of view, we may lose the information needed for successful decision making. To reduce these negative frames we need firstly to define and understand them, secondly, as pointed out by Druckman [4], we need to evaluate them, and thirdly we need to use the appropriate method to limit them. For this kind of evaluation as Fagley, Coleman and Simon [6] write we need to know the importance of various frames and included points of view.

The typical example of the frame's impact on decision is the information written on the product's package. As Kozel [5] analyses in more details, the packaging inevitably influences our purchase decisions. According to Lindsey-Mullikin and Petty [7], the product packaging can attract our attention; affect our emotions, but on the other hand, its information value can contribute to the rational purchase decision. Each consumer has his unique view when buying a specific product (his unique way of perceiving the situation) based on his personality. His purchase decision is influenced by the preferences and expectations; we are talking about the framing effect of the decision situation, which discusses in detail Rydval [10], [11].

Semantic networks can be used to define the frames influencing consumer's behaviour. They were developed at the end of 1960s. In 1968 the American scientist used this term in his thesis to represent English words. Semantic networks were originally used to express meanings of various expressions in natural language. However,

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over time they evolved and in the form of directed graph consisting of nodes and edges they became more general graphical tool to represent knowledge, which consist of information about the particular fact.

According to Sowa [16] semantic networks are used for their ability to provide easily usable system to represent information focused mainly on organisation of large information sources, information integration in distribution and description of complex processes. Semantic networks offer the means to represent knowledge and they display their connections and relationships. Furthermore, semantic networks are used for database management. Semantic network can display individual elements influencing the decisions of producers, distributors and consumers. It provides us with information about relationships in the network between individual elements and how they can influence the decision. However, it does not give us the quantitative information about the importance of individual elements and how much they influence higher level elements.

We can convert semantic network to the network consisting of clusters and nodes for analytic network process to determine the cardinal quantitative information about the range in which the individual elements influence other elements of the same or higher level semantic network. Using this process we can calculate to what extent the network's elements influence other elements and how much influence they have in decision making process in the market environment. The Analytic Network Process is one of the multiple criteria decision making methods. It decomposes decision problems into a network of smaller parts (sub-problems) that can more easily be analysed and evaluated. It is specific for this method that the human judgment is involved. (Saaty [13])

The goal of this paper is to show the distortion in information which occurs in the market. The essential idea is to capture the basic elements of the information frame and their mutual relationships to express the loss of information and its asymmetry in the market environment. Elements and their relationships are shown using a semantic network. The scales of identified elements in the network are then estimated using Analytic Network Process (ANP). The benefit of this paper is to make customers better informed about the information asymmetry in the market environment.

2 Materials and Methods

2.1 Semantic Network

Semantic (associative) network can be seen, as Sowa wrote [16], as a directed graph consisting of nodes and edges. Nodes represent individual objects of described world and edges connecting these nodes represent relationships between these objects. Very often we can meet with following relations:

- IS-AN-INSTANCE-OF (ISA),
- A-KIND-OF (AKO),
- A-PART-OF (APO).

ISA relationship is used to express the fact that a particular object (an instance of a particular class) belongs to the specified class. AKO relationship can express that a class is a subclass of another class. And lastly APO relationship serves to express that a certain class of objects is composed of certain parts. With such a graph we can thus convey information. A simple example might be a semantic network of vegetation - nodes could be a single plant species, (oriented) edges between them could represent host-parasite relationship. The nodes could be of different types (for example, monocots, dicots, or self-pollinating plants), the edges could also be of different types (besides expressing who parasites on whom, completely different edges could be for example introduced in parallel representing an evolutionary tree).

Attributes of individual nodes could then have the names of the species, and any other necessary information about them. Attributes of the edges indicating the host-parasite relationship could be used to indicate whether in that particular relationship it represents the main or secondary source of nutrients for the parasite. The edges showing the evolutionary tree could indicate the time when the development branch separated from the original species. The basic advantage of such a semantic network is that (besides the fact that in principle it corresponds with the way information is stored in human memory) it is machine-understandable because of its attributes and relationships. In practice this means that it can be machine-processed - is it possible to analyse it and thus to acquire new knowledge about represented facts. In this simple example, we need to determine which plants have the most natural parasites and which the least, or other more complex relationships. Semantic networks have several basic characteristics:

- The network is arranged in a certain way - the individual nodes have only a limited number of edges leading to other nodes.
- The network allows quick navigation - from any part of the network we can relatively quickly browse through the edges to reach any other part.

- The network has the local subnet - a large part of the edge nodes in the cluster leads to another node of the same cluster.
- The network has a natural "entry points" - hubs (cores) act as a kind of signpost.

2.2 Framing Effect

Individual decisions are influenced by the presented information and by the formulation of problems. Each subject has his own preferences and expectations that create his own view (frame). The framing effect is made up of these frames. Tversky and Kahneman [18] pointed out the framing effect influences the way of information interpretation or misinterpretation, so it may influence decision-making significantly. As Rydval and Hornická [9] mentioned we can therefore define the framing effect as a set of preferences and expectations of involved subjects belonging to a particular decision-making problem. To quantify framing effect, as indicated Rydval [10] , [11], the methods for quantifying preferences of decision maker can be used. Quantification of the framing effect in education is discussed by Rydval and Brožová [8].

2.3 Saaty's method of pairwise comparison

It is a quantitative pairwise comparison method for the criteria, which analyzes in detail Saaty [12, 13]. A nine point scale (1, 3, 5, 7, 9) is provided to quantify pair wise importance of criteria and it is possible to use intermediate values (values 2, 4, 6, 8). Expert compares each pair of criteria and he records the size of the preferences of the *i* criterion to the *j* criterion in the Saaty matrix $S = (s_{ij})$:

$$S = \begin{pmatrix} 1 & s_{12} & \cdots & s_{1n} \\ 1/s_{12} & 1 & \cdots & s_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 1/s_{1k} & 1/s_{12} & \cdots & 1 \end{pmatrix} \qquad (1)$$

The most commonly used procedure calculates the weights v_i , $i = 1, 2, ..., n$ as normalized geometric mean of the rows' entries of the Saaty's matrix.

2.4 Analytic Network Process

The Analytic Network Process (ANP) is a generalization of the Analytic Hierarchy Process (AHP), by considering the dependence between the elements of the hierarchy. Many decision problems cannot be structured hierarchically because they involve the interaction and dependence of higher-level elements in a hierarchy on lower level elements. Therefore, ANP is represented by a network, rather than a hierarchy. (Saaty [12], [14]), [15])

The basic elements of the ANP method are following:

- The first step of ANP is based on the creation of a control network which describes dependency among decision elements. The ANP allows
  - inner dependence within a set (clusters) of elements, and
  - outer dependence among different sets (clusters).
- In the second step pairwise comparisons of the elements within the clusters and among the clusters are performed according to their influence on each element in another cluster or elements in their own cluster. So the ANP prioritizes not only decision elements but also their groups or clusters as it is often the case in the real world. The consistency of these comparisons has to be controlled.
- The third step consists of the supermatrix construction. The priorities derived from the pairwise comparisons are entered into the appropriate position in this supermatrix. This supermatrix has to be normalized using clusters weights.
- In the fourth step the limiting supermatrix is computed and global preferences of decision elements are obtained. These preferences serve as the best decision selection or for the purpose of analysis of preferences of decision-making elements. (Saaty [12], [13])

This method is carried out by the SuperDecisions software system (SuperDecisions [17]).

3 Results

A case study – Creation of distributor's frame using semantic network

In this case study, the essential elements were identified using structured interviews and analysis information on the product packaging. These elements are the basic elements of the producer's, distributor's and consumer's decision-making process in the pork meat distribution. The individual nodes of the semantic network represent the basic elements of the distributor's frame and individually oriented arrows between nodes represent connections and relationships between them. Three basic relationships are defined:

- Relationship ISA is used to express the fact that a specific element belongs to a class of elements.
- Relationship AKO expresses that a class is a subclass of another class.
- Relationship APO expresses that a certain class of objects is composed of certain parts.

Using these three basic relations and elements of the producer's, distributor's and consumer's frame allows us to display the semantic network description and graphic information about the preferences and expectations. The semantic network gives us the overview what aspects should be taken into account when buying meat from producers and distributing it to consumers. As shown in Figure 1, we can see that the distributor receives information about the quality of the meat from the producer. The distributor has information about consumer's preferences and expectations, and using marketing approaches he can affect the consumer's behavior. These steps may influence the decision-making process in consumer behaviour.

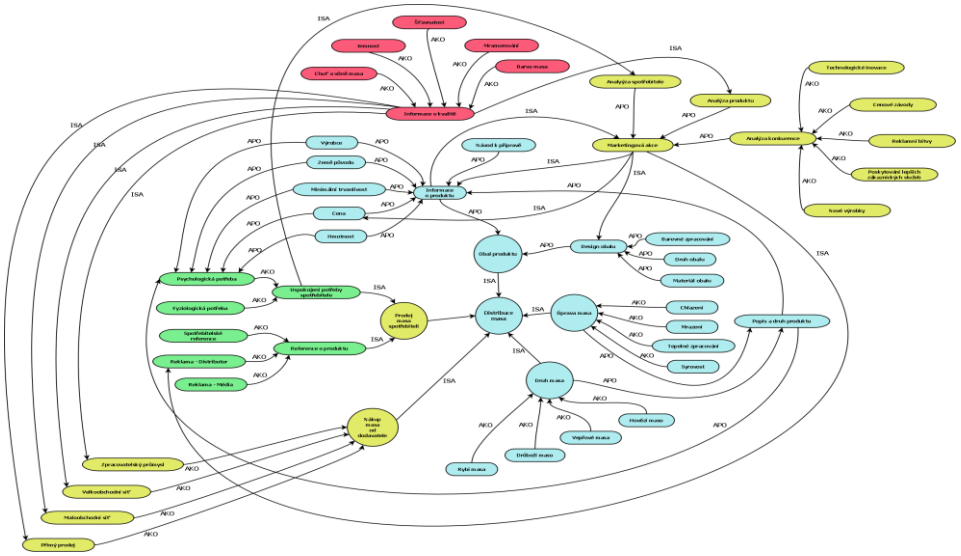

Figure 1 Semantic network of distributor

Multi-criteria approaches can be used to quantitatively measure the impact of each element of the semantic network. One of these approaches is the ANP. Using the SuperDecision software the semantic network can be transformed into ANP network. This new network consists of clusters and nodes within the cluster, as shown in Figure 2. Within this network the Saaty's pair wise comparison of the corresponding nodes and clusters is made. The pair wise comparisons correspond to the logical connections and relationships of the semantic network. Based on ANP the pair wise comparison tables of elements were created for all clusters. In these tables the survey participants expressed their preferences to the clusters' elements. Each participant filled out the questionnaire in the form of pair wise comparison table. In the table the participants expressed their preferences for each pair of elements by circling the value ranging from 1 to 9. Only consistent survey participants' responses were included in the final calculation. The values for each element were calculated like a simple average of all participants' answers, these values became the inputs to the ANP. Based on ANP the importance of role played by the colour, material, and type of packaging in design of product packaging was calculated, as shown in Table 2. Table 3 shows that the consumer analysis in a successful marketing strategy is the most important. In the last table it is calculated what kind of information on product packaging is the most important for the consumer.

Design of product packaging	color of packaging	75,31%
	material of packaging	6,29%
	type of packaging	18,40%

Table 2 Design of product packaging

Analysis for marketing events	competition analysis	10,47%
	consumer analysis	63,70%
	product analysis - quality information	25,83%

Table 3 Analysis for marketing events

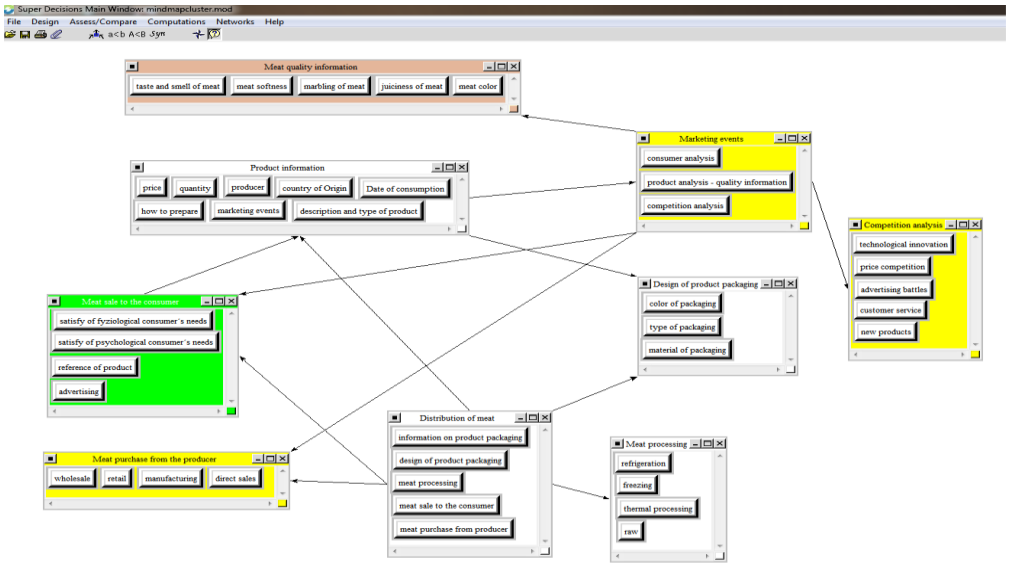

Figure 2 Network of ANP

Product information	country of origin	3,26%
	best before date	10,83%
	description and type of product	5,50%
	how to prepare	1,63%
	marketing events - quality information	24,75%
	price	39,32%
	producer	2,45%
	quantity	12,25%

Table 4 Product information

This case study shows how semantic networks and ANP help to model the consumer's preferences, and serves to quantify the criteria for his purchase. Thus it helps to decide what information to show on the product packaging and make the product more attractive for the consumers. Semantic networks and ANP may serve as supporting tools in the marketing strategy.

4 Discussion

Graphical representations of the mind processes affect human decision-making process, in this case the consumer's decision-making process. They have been used since the 70's by Tony Buzan. However, Buzan does not strictly use directed graph consisting of nodes and edges, he uses the mind map. T. Buzan and B. Buzan [2,3], define mind map as the graphic layout of text (accompanied by pictures) showing the connections. Mind maps are used as an aid to recap the ideas and opinions of consumers and mind maps are used to organize and facilitate the creation of decision. These approaches (resembling through their structure decision trees) of graphical representations of mind processes using structured texts supplemented by pictures do not work with oriented connections and nodes. But oriented connections and nodes can be used to express relationships of seemingly unrelated ideas and information in the semantic network. Unlike a mind map, the semantic network uses a network with evaluated and oriented arcs, what makes possible to see for example relationship between the ways of marketing communication and consumer's satisfaction, as shown in Figure 2. Using the mind map it is possible to show which elements affect the distributor's decision process. However it is not possible to show their mutual connections. Semantic network makes it possible to show both. For example we can see that the distributor receives information about the quality of the meat from the producer and at the same time we can see the value and the direction of the connection. This is important because unlike in the mind map we can spot the way the information asymmetry flow is made. Both approaches have their strengths and weaknesses. Mind map is more

graphically creative and user-friendly than the simple graph of a semantic network. A mind map does not clearly define the interaction of individual elements, but semantic network does. Danger of both approaches is an exponential increase in branching, which may increase the number of redundant elements.

5 Conclusion

This paper deals with the factors affecting our rational thinking, with our ability to make rational decisions, and in particular it serves to explain the framing effect in decision-making process and its representation in graphic form and quantification using semantic networks and the ANP method. The semantic network was used to define frames. Some quantitative characteristics of semantic networks were calculated after transferring individual clusters and nodes with their logic traceability to the network for ANP. Using ANP many other quantitative characteristics can be calculated from many other points of view. Transmission and loss of information in the producer-consumer chain can be defined by the comparison of semantic networks of individual actors of the market environment. Using this comparison we can also define the information asymmetry of market environment and the possible cause of externalities of the market process.

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Relationship between concentration and cost and profit efficiency in the Czech banking sector

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Abstract. The paper estimates the relationship between the banking concentration and cost and profit efficiency in the Czech banking industry in the period 2001–2011. To estimate the degree of banking concentration was used the concentration ratio and Herfindahl-Hirschman index and for estimation of the cost and profit efficiency of the Czech banks was applied the Stochastic Frontier Approach. In the Czech banking sector it was found a slight decrease in concentration during the analyzed period. The efficiency of the Czech banks increased in this period. Using a Johansen cointegration test the positive relationship between concentration and cost and profit efficiency was estimated in the Czech banking sector. This result confirms the efficiency hypothesis. These findings suggest that the increase of banking concentration will contribute to efficiency of the Czech banks.

Keywords: concentration, cost efficiency, profit efficiency, relationship, Johansen cointegration test, Czech banking sector.

JEL Classification: G21, C58
AMS Classification: 62P20

1 Introduction

The aim of this paper is to examine the relationship between concentration and cost and profit efficiency in the Czech banking industry during the period 2001–2011. For the estimation of the banking efficiency we applied the Stochastic Frontier Approach (SFA) on the data of the Czech commercial banks. First, the level of cost and profit efficiency and concentration was estimated in the Czech banking sector. In this paper the concentration ratio and Herfindahl-Hirschman index are applied to assess the degree of concentration. Next, the relationship between concentration and cost and profit efficiency was examined using the Johansen cointegration test.

In the empirical literature there has not been consensus in the relationship between concentration and efficiency in the banking sector. The existence of the relationship between market structure and efficiency was proposed by [10] who defined the quiet life hypothesis. Reference [10] argued that monopoly power allows managers a quiet life free from competition and thus increased concentration should bring about a decrease in efficiency. High market power enables efforts to reduce efficiency. Reference [12] stated the main determinant of the reduction of inefficiencies is the increase of competitive pressures for two reasons. First, competition provides incentives to managers to exert a higher effort. Second, a higher number of firms on the market improve the possibilities for owners to assess firm performance, relative to other firms. In contrast, the efficient structure hypothesis described by [7] and [14] who expected a positive relationship between cost efficiency and concentration. More efficient firms have lower costs, which in turn lead to higher profits. Therefore, the most efficient firms are able to increase their market share, resulting in higher concentration. Efficient structure hypothesis argued that it is mainly the efficiency of the largest banks that explains the bank market consolidation. The structure-conduct-performance hypothesis [2] argues that greater concentration causes less competitive bank conduct and leads to greater profitability.

The structure of the paper is follow. Next section presents methodology and data, the measure of concentration, the Stochastic Frontier approach and selection of variables are described. Section 3 reveals and discusses the estimated results and Section 4 concludes the paper with summary of key findings.

2 Methodology and data

To estimate the degree of banking concentration the concentration ratio (CR) and the Herfindahl-Hirschman index (HHI) was used. Simplicity and limited data requirements make the *k* bank concentration ratio one of the most frequently used measures of concentration in the empirical literature. Summing only over the market shares of the *k* largest banks in the market, it takes the form:

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$$CR_k = \sum_{i=1}^{k} s_i. \quad (1)$$

The HHI is the most widely treated summary measure of concentration in the theoretical literature. It is based on the total number and size distribution of firms in an industry. It is computed as the sum of the squares of the market shares of all firms in the industry. HHI ranges from zero in a market with many very small firms, to 10000 in a pure monopoly. It takes the form:

$$HHI = \sum_{k=1}^{n} \left(\frac{q_k}{Q}\right)^2 = \sum_{k=1}^{n} r_k^2, \quad (2)$$

The two general approaches used to assess efficiency of an entity, parametric and non-parametric methods, employ different techniques to envelop a data set with different assumptions for random noise and for the structure of the production technology. The nonparametric methods are Data Envelopment Analysis and Free Disposal Hull. The parametric methods are Stochastic Frontier Approach, Distribution Free Approach and Thick Frontier Approach, which assume specific functional form for the cost function or production technology.

For estimation of the cost and profit efficiency of the Czech banks was applied the Stochastic Frontier Approach. The SFA originated with papers [13], [1] and [4]. The SFA is based on the economics of cost minimization or profit maximization by banks, and starts with a standard cost or profit function with factors of input, output, and their prices. It estimates the minimal cost or maximum profit based on these functions, and generates distance of its cost or profit to the frontier value. The parametric approach has the advantage of allowing noise in the measurement of inefficiency. However, the approach needs to specify the functional form for cost or profit.

Cost efficiency

Cost efficiency measures the performance of banks relative to the best-practice banks that produces the same output under the same exogenous conditions. Cost efficiency function is based on a cost equation that relates a bank's cost to variables that incur those expenses, such as output levels and input prices. The cost equation contains a composite error structure that distinguishes random cost fluctuations from cost inefficiencies. To put it simply, the cost function describes the relationship between the cost with quantities of output and input variables plus the inefficiency and random error. The following cost equation:

$$C_{it} = f(y_{it}, w_{it}, z_{it}) + \varepsilon_{it}, \quad (3)$$

where C_{it} measures the total costs of a bank i incurs at time t, including operating and financial costs, y_{it} is a vector of outputs, w_{it} is a vector of input prices, z_{it} represents the quantities of fixed bank parameters, such as physical capital and equity and ε_{it} is the error term. The error term ε_{it} is composed of two parts:

$$\varepsilon_{it} = \mu_i + \upsilon_{it}, \quad (4)$$

where μ_i represents the inefficiency term that captures the difference between the efficient level of cost for given output levels and input prices and the actual level of cost and υ_{it} is the random error. More specifically $\mu_i$and υ_{it} are assumed to follow the following distributions:

$$\mu_i \sim N^+(0, \sigma_\mu^2), \quad (5)$$
$$\upsilon_{it} \sim N^+(0, \sigma_\upsilon^2). \quad (6)$$

We assume μ_i follows a half-normal distribution. Alternatively, μ_i can be modelled to follow a truncated normal distribution or exponential distribution so that it can only take non negative values. It measures the difference of bank's i cost compared with that of the frontier $f(y_{it}, w_{it}, z_{it})$. The cost efficiency of the bank can be written in a natural logarithm form as follows:

$$\ln TC = \ln f(y, w, z) + \ln u_t - \ln v_t, \quad (7)$$

where f denotes a functional form. After estimating a particular cost function, cost efficiency for bank i is measured as the ratio between the minimum cost ($C_{\min}$) necessary to produce bank's output and the actual cost (C_i):

$$CE_i = \frac{C_{\min}}{C_i} = \frac{\exp[f(y, w, z)]x\exp(lnu_{\min})}{\exp[f(y, w, z)]x\exp(lnu_i)} = \frac{u_{\min}}{u_i}, \quad (8)$$

where u_{min} is the minimum u_i across all banks in the sample. Under this formulation, an efficiency score of 0.95 for example, implies the bank would have incurred only 95% of its actual costs had it operated in the frontier.

Profit efficiency

Despite the wide agreement on the relevance of profit efficiency analysis, the technical difficulties with the measurement and decomposition of profit inefficiency were the main reasons for the small number of empirical studies on banking profit efficiency. Unlike the cost function, the profit function has an additive structure implying that the Shephard type distance functions, which are radial, are not the appropriate dual model of technology [8]. The profit frontier is derived as follows:

$$P = f(y, w, z) + u + v, \tag{9}$$

where P measures the profit of a bank, including both interest and fee income, less total costs of a bank, y is a vector of outputs, w is a vector of input prices, z represents the quantities of fixed bank parameters, u is the inefficiency term that captures the difference between the efficient level of cost for given output levels and input prices and the actual level of cost, and v is the random error term. The profit function of the bank can be written in a natural logarithm form as follows:

$$\ln P = f(y, w, z) + \ln u_t - \ln v_t. \tag{10}$$

where f denotes a functional form. Profit efficiency is measured by the ratio between the actual profit of a bank and the maximum possible profit that is achievable by the most efficient bank.

$$PE_i = \frac{P_i}{P_{\max}} = \frac{\exp[f(y, w, z)]x\exp(lnu_i)}{\exp[f(y, w, z)]x\exp(lnu_{\max})}, \tag{11}$$

where u_{max} is the maximum u_i across all banks in the sample. For example, if the profit efficiency score of a bank is 90%, it means that the bank is losing about 10% of its potential profits to managerial failure in choosing optimum output quantities and input prices.

2.1 Data and selection of variables

The data set used in this paper was obtained from the annual reports of banks for the period 2001–2011. All the data is reported on unconsolidated basis. The data set consists of data of banks that represent almost 80% of the assets of the national banking sector. We analyzed only commercial banks that are operating as independent legal entities due to the homogeneity of the data set. All foreign branches, building societies, specialized banks or credit unions were excluded from the estimation data set.

In order to conduct SFA estimation, inputs and outputs were defined. We adopt intermediation approach which assumes the banks' main aim is to transform deposits into loans. Consistently with this approach, banks use the two inputs and produce two outputs. The intermediation approach is considered relevant for the banking industry, where the largest share of activity consists of transforming the attracted funds into loans. Total costs are the sum of the interest cost and operation cost. Total profit is the sum of interest income and fee income. We employed two inputs (labour and deposits), and two outputs (loans and net interest income). We measure price of labour (w_j) as a ratio of personnel expenses to number of employees, and price of deposits (w_h) as a ratio of interest expenses to total deposits. Loans (y_l) are measured by the value of loans to customers and other financial institutions and net interest income (y_m) as the difference between interest incomes and interest expenses.

The functional form of the stochastic frontier was determined by testing the adequacy of the Cobb Douglas relative to the less restrictive translog. As in e.g. [5] or [11], we normalize dependent variable with all output quantities y by equity capital Z to account for heterogeneity. The normalization by equity capital has economic meaning [5]. The frontier models estimated are defined as:

$$\begin{aligned}\ln\left(\frac{C}{Z}\right)_{it} = \alpha_1 + \sum_{l=1}^{2}\beta_l \ln\frac{y_l}{Z} + \frac{1}{2}\sum_{l=1}^{2}\sum_{m=1}^{2}\beta_{lm}\ln\frac{y_l}{Z_l}\ln\frac{y_m}{Z} + \sum_{j=1}^{2}\gamma_j \ln w_j + \frac{1}{2}\sum_{j=1}^{2}\sum_{h=1}^{2}\gamma_{jh}\ln w_j \ln w_h \\ + \sum_{l=1}^{2}\sum_{j=1}^{2}\beta_{lj}\ln\frac{y_l}{Z_l}\ln w_j + \ln u_{it} + \ln v_{it},\end{aligned} \tag{12}$$

where C is total cost, y_l, y_m are the outputs l or m, w_j, w_h are the price of inputs, u_{it} is the random error, v_{it} is the inefficiency term, i denotes the bank (i = 1, ..., N) and t denotes time (t = 1, …, T).

$$\ln\left(\frac{P}{Z}\right)_{it} = \alpha_1 + \sum_{l=1}^{2}\beta_l \ln\frac{y_l}{Z} + \frac{1}{2}\sum_{l=1}^{2}\sum_{m=1}^{2}\beta_{lm}\ln\frac{y_l}{Z}\ln\frac{y_m}{Z} + \sum_{j=1}^{2}\gamma_j \ln w_j + \frac{1}{2}\sum_{j=1}^{2}\sum_{h=1}^{2}\gamma_{jh}\ln w_j \ln w_h + \sum_{l=1}^{2}\sum_{j=1}^{2}\beta_{lj}\ln\frac{y_l}{Z}\ln w_j + \ln u_{it} - \ln v_{it}, \tag{13}$$

where P is the total profit. The use of duality implies the necessity to impose the following homogeneity restrictions:

$$\sum_{l=1}^{2}\beta_l = 1\,, \sum_{j=1}^{2}\gamma_j = 0\,, \sum_{h=1}^{2}\sum_{k=1}^{2}\gamma_{hk} = 0. \tag{14}$$

3 Empirical analysis and results

Following section estimates the degree of concentration and cost and profit efficiency in the Czech banking sector during the period 2001–2011. Next, the relationship between concentration and cost and profit efficiency is assessed in the banking industry.

The common measures of concentration, especially the concentration ratio and the Herfindahl-Hirschman index, are computed. It is used the three largest bank concentration ratio (CR3), the five largest bank concentration ratio (CR5) and the ten largest bank concentration ratio (CR10), which defined as the ratio of the total assets of the three, five and ten largest banks to the total assets of all banks in a given year. Table 1 illustrates the results of the concentration ratio and HHI in the Czech banking sector from 2001 to 2011.

Year	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
CR3	59%	57%	57%	54%	56%	54%	55%	51%	51%	52%	51%
CR5	68%	66%	66%	64%	65%	64%	66%	62%	62%	62%	62%
CR10	81%	80%	79%	78%	79%	78%	80%	78%	79%	79%	78%
HHI	1018	1089	1137	1084	1110	1066	1112	947	974	989	960

Table 1 Concentration ratio and Herfindahl-Hirschman index in the Czech banking sector

The development of concentration in the Czech banking sector shows a slight decrease in the total assets. Despite the declining shares of the three largest banks, activities of the banking sector are mostly concentrated in the three leading banks, which reach values over 50% of the market. Market shares of the group of the largest banks in the banking sector have falling gradually despite the strengthening of competition and changes in the internal structure of the banking sector. Since 2001, the value of the Herfindahl-Hirschman index has been declining in the total assets, which means a slight decrease in concentration. The Czech banking sector would be classified as moderately concentrated industry. The concentration ration and HHI show a trend of a modest decrease, meaning that market concentration changed appreciably over the sample period. The Czech banking market could be described as a moderately concentrated market over the period of 2001–2011.

The cost and profit efficiency function is estimated using the maximum likelihood estimation of parameters in the Cobb-Douglas [3]. The computer programme FRONTIER 4.1 developed by [6] and R Project have been used to obtain the maximum likelihood estimates of parameters in estimating the technical efficiency. The programme can accommodate cross sectional and panel data; cost and production function; half-normal and truncated normal distributions; time-varying and invariant efficiency; and functional forms which have a dependent variable in logged or original units.

Year	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011
Cost efficiency	85	83	86	88	90	88	84	90	91	78	89
Profit efficiency	85	91	76	94	99	93	89	97	80	64	75

Table 2 Cost and profit efficiency of the Czech commercial banks (in %)

Table 2 presents the average value of the cost and profit efficiency in the period 2001–2011. The value of average cost efficiency was in the range 78-91%. The value of the profit efficiency was in the range 64-99%. This suggests that an average of about 64% to 99% of potential maximum profit was gained due to profit efficiency. The development of the average efficiency show that the efficiency score was increasing in the period

2001–2009. Large Czech banks were privatized in the period 1999–2001, it is probably that the new owners and managers learnt to adapt in the new environment. In 2009-2010 the efficiency decreased, we can suppose that this development was as a result of the financial crisis. The analyzed outputs decreased in the balance sheet of the individual banks. The decrease in the net profit was registered in the balance sheet of the most Czech banks (e.g. LBBW, J&T Bank) in this period. In 2011 the cost and profit efficiency increased.

3.1 Relationship between concentration and cost and profit efficiency

The Johansen cointegration test is used to determine the relationship between concentration and cost and profit efficiency in the Czech banking sector. First, time series have tested for stationarity. For testing stationarity augmented Dickey-Fuller test (include trend and intercept in the test equation) has been used. Unit root test has found that the time series are stationary at the first difference $I(1)$, and results can perform testing the cointegration relationship between variables. Next, to estimate the cointegration relationship VAR model has been estimated to determine the optimal lag to eliminate residual autocorrelation vector components. Based on the Akaike information criterion the optimal lag has been selected one year for the relationships. As the optimal model we have picked up the model assuming no linear deterministic trend in data with intercept and no trend in cointegration equation and no intercept in test VAR. The finding of long-term relationships between variables in the Johansen cointegration test it has been identified using the Trace test and Maximum Eigenvalue test. The result of estimating the relationship between concentration and cost efficiency is presented in Table 3.

	Trace Statistic	**Critical value 5%**	**Max-Eigen Statistic**	**Critical value 5%**
r = 0	23.99305[b]	20.26184	22.77714[a]	15.89210
r ≤ 0	1.215909	9.164546	1.215909	9.164546

[a], [b] denotes the existence of cointegration relationship at a significance 1%, 5% level

Table 3 Johansen cointegration test illustrating the relationship between the concentration and cost efficiency in the Czech banking industry

The Johansen cointegration test indicates one cointegration equation at the 0.05 level. The relationship between concentration and cost efficiency in the Czech banking industry shows cointegration eq. (15) which indicates the effect of concentration to cost efficiency:

$$\underset{\quad}{\text{Cost efficiency}} = \underset{(6.12101)}{72.12326} + \underset{(0.00576)}{0.013784}\ \mathbf{concentration}. \qquad (15)$$

Standard error is in parentheses. The cointegration equation (15) indicates that concentration has a positive influence to cost efficiency. Thus, the increased of the banks' market power will contribute to higher cost efficiency of the Czech banks. Eq. (16) indicates the effect of cost efficiency to concentration:

$$\underset{\quad}{\text{Concentration}} = \underset{(872.461)}{-5232.334} + \underset{(10.0347)}{72.54711}\ \text{cost efficeincy}. \qquad (16)$$

Eq. (16) shows that cost efficiency affect concentration positively in the Czech banking sector. The cointegration equations (15) and (16) show that concentration has a positive effect on cost efficiency and vice versa. The estimation of the relationship between profit efficiency and concentration in the Czech banking sector is presented in Table 4.

	Trace Statistic	**Critical value 5%**	**Max-Eigen Statistic**	**Critical value 5%**
r = 0	31.70382[a]	20.26184	30.21735[a]	15.89210
r ≤ 0	1.486462	9.164546	1.486462	9.164546

[a] denotes the existence of cointegration relationship at a significance 1% level

Table 4 Johansen cointegration test illustrating the relationship between the concentration and profit efficiency in the Czech banking sector

The Johansen cointegration test indicates one cointegration equation at the 0.05 level. The relationship between concentration and profit efficiency in the Czech banking industry shows cointegration Eq. (17) which indicates the effect of concentration to profit efficiency:

$$\underset{\quad}{\text{Profit efficiency}} = \underset{(15.4833)}{\mathbf{-78.81071}} + \underset{(0.01452)}{\mathbf{0.1549580}}\ \mathbf{concentration}. \qquad (17)$$

Standard error is also in parentheses. The cointegration eq. (17) indicates that concentration has a positive influence to profit efficiency in the Czech banking market. Thus, the increase of the concentration will contribute to higher profit efficient bank in the banking sector. Eq. (18) indicates the effect running from profit efficiency to concentration:

$$\underset{(56.4688)}{\text{Concentration} = 508.5228} + \underset{(0.63537)}{6.452458}\ \text{profit efficiency}. \qquad (18)$$

Eq. (18) shows that profit efficiency has a positive effect to concentration in the Czech banking sector. The equations (17) and (18) show that the relationship between profit efficiency and concentration is positive in the Czech banking sector. We found a positive causation running from concentration to cost and profit efficiency and also from cost and profit efficiency to concentration in the Czech banking sector. These results confirm the efficiency hypothesis suggesting that the increase of banking concentration will contribute to higher level of the cost and profit efficiency of the Czech banks.

4 Conclusion

The aim of this paper was to examine the relationship between concentration and cost and profit efficiency in the Czech banking industry during the period 2001–2011. Using the concentration ratio and Herfindahl-Hirschman index was found a slight decrease in concentration during the analyzed period. The efficiency of the Czech commercial banks slightly increased in this period. Using a Johansen cointegration test, the paper contributes to the empirical literature, testing the causality running from concentration to efficiency and also from cost and profit efficiency to concentration. The positive relationship between the concentration and cost and profit efficiency was estimated in the Czech banking sector. The results confirm the findings of [9] who found that a higher degree of concentration was associated with higher degree of efficiency. Our result is in line with the efficiency hypothesis. These findings suggest that the increase of banking concentration will contribute to higher cost and profit efficiency of the Czech banks.

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Modeling Probability of Corporate Defaults in Constructions

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Abstract: The paper is focused to examine condition in the Constructions of the Czech economy with final goal to build micro scoring model which would be able to predict probability of corporate defaults. The scoring model found main explanatory variables which led us to serious conclusions about firms in the Constructions. We proved importance of earnings and cash together with market share, productivity of a firm and its stability. We also revealed strong influence of employees to stability of a firm; the higher number of employees the lower probability of default, which led us to conclusion that employees incorporate stability and quality of inner processes of a firm. Macroeconomic environment seems insignificant in the Constructions, however, we assume that in the population would be significant.

Keywords: scoring, econometric modeling, default, prediction, bankruptcy model.

JEL Classification: C52, C53

1 Introduction

Actual discussion about modeling probability of corporate default offers many opportunities how to build an appropriate model in the context of corporate risk [1], [5]. However, what are the measurable factors which could be behind of firm's default? Moreover, is there any connection with the macroeconomic situation? These types of questions are with no doubt serious and it's actual to build a model which could be able predicts probability of default and provides us relevant answers. An useful application of a default model is related with today's economy which is based on quick and correct information. Correct information in right time about suppliers and consumers could divide firms between those which will survive in the economic competition and those which have to be liquidated as a result of their unsatisfying decision making, insufficient information. Scoring is a tool which could provide such information - a probability of default about customers and suppliers. In this way it could be prevented possible damage of a firm's stability which could come mainly from disruption of its cash flow.

The article starts with a theoretical modeling process, followed by empirical models of defaults of the Constructions of the Czech Republic and statistical accuracy with main economic findings.

2 Data and model building details

The listed below demographic, economic and financial data of the Constructions was obtained from Bisnode company in the year periodicity from 2007 to 2011. On the basis of the financial data we constructed 12 financial indicators up to second lag thus we obtained 32 indicators for our analysis. Further, for each firm we obtained category of employees (CoE) which is defined by the Czech Statistical Office[2], legal form of business (LF) and region of firm's place of business (REG). Finally, firms which were liquidated we had a concrete date of a liquidation.

Following part shows construction of financial indicators for an actual period, an indicator in the first lag has the same construction but it's based on previous year, analogously second lagged indicator.

Rentability of Assets provides one of the most important information about firm's condition to its owners and investors, it could be written as

$$ROA = \frac{Earnings}{Assets}. \tag{1}$$

The ratio between liabilities and assets is a key component to reveal if a firm is stable, especially if we examine this indicator in following periods, formally

$$LA = \frac{Liabilities}{Assets}. \tag{2}$$

Turnover assets is also very useful indicator which can say what is the productivity of the invested capital (assets) of a firm. It could be written as

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[2] i.e. 0 empl., 1-5 empl., 6-9., ..., 3000-3999, 4000-4999, 5000 and more, thus we have 18 categories

$$RA = \frac{Revenues}{Assets}. \tag{3}$$

The ratio between Cash Flows and Liabilities can provide risk level of creditors' loans to a company, thus

$$CFL = \frac{Cash\ Flow}{Liabilities}. \tag{4}$$

The EL financial indicator covers message for investors to a firm about stability and ability to pay its debts, it could be expressed as

$$EL = \frac{Earnings}{Liabilites}. \tag{5}$$

The ratio between current assets and short term liabilities provides information about firm's stability and ability to control its structure of capital, formally

$$CASL = \frac{Current\ Assets}{Short\ Term\ Liabilities}. \tag{6}$$

The Working Capital in the context of Assets is crucial to determine whether firm is able to optimize its production, capital needs., formally

$$WCA = \frac{Working\ Capital}{Assets}. \tag{7}$$

The indicator calls as Margin Rate provides information about firm's ability to optimize its costs in the context of revenues, thus it says mainly about efficiency and stability of a firm.

$$MR = \frac{Earnings}{Revenues}. \tag{8}$$

Finally, we included ratios between actual and previous year of the main characteristics of a firm, which are

$$AA = \frac{Assets_t}{Assets_{t-1}}. \tag{9}$$

$$EE = \frac{Equity_t}{Equity_{t-1}}. \tag{10}$$

$$LL = \frac{Liabilities_t}{Liabilities_{t-1}}. \tag{11}$$

$$TT = \frac{Revenues_t}{Revenues_{t-1}}. \tag{12}$$

We started our analysis with data mining. On the first place we divided firms into two groups. In the first group were firms with no sign of default, in the second group we had all firms which were liquidated. Then we examined an exact date of a liquidation thus we decided to which year will be a firm included, in other words, which year was previous one before liquidation process. In the next step we constructed 32 financial indicators on the basis of the equations 1-12, maximally to lag two. Together with demographic and economic data we created our date set for the model building purposes.

In the first step with the date set we applied correlation test to reveal if exist strong connections between financial indicators. To avoid multicolinearity in our estimations we excluded those variables which had correlation coefficient higher than 0.85 in absolute value, thus they were redundant in an estimated equation. After this corrections we used the Stepwise technique to determine whether exist variables with high influence to the binary endogenous variable Defaults, whereas Defaults have been set as follows: value 1 means that firm was liquidated in next period; and value 0 had firms without any signs of problems. This analysis gives us a good clue about relationships between the Defaults and the explanatory variables. However, standard OLS doesn't have optimal characteristics for endogenous binary variable, hence we used logit models [2], [3], [4] to find the best model for the Default in Construction. On the basis of the distribution of logit we was looking for the best solution which could provide us the best distinction between good and bad firms.

3 Empirical Modeling

For the Constructions we obtained data for 758 firms where 77 of them defaulted in the next period. We would like to present quality and robustness of our final model thus we will sequentially represent changes in the model when number of firms increasing.

The final model has 7 explanatory variables, whereas only 4 of them are financial indicators. The first one is an annual ratio between assets. We can see negative sign which indicates decreasing probability of default if firm increase level of assets. From the economic point of view the higher assets - or from the opposite side higher capital - mean higher stability of a firm. A firm disposes with higher capital, compared with previous year, thus it's more stable to the future period. We would like to emphasize importance of this variable - as number of firms increasing in the model, it could be expected stronger influence in the population (compare Table 1 and 2). The second found indicator is the annual ratio between liabilities. It reveals positive influence to the number of defaults in the Constructions, which means higher probability of default if a firm increases its liabilities. Finally, the last two indicators are rentability of assets in current and previous period, in both cases we can observe growing significance which assure us about their importance and adequacy. An important fact is a stronger influence of ROA from previous period than in current period. This result warn us about firm's rentability, it is crucial to examine it with high attention even several years back.

Variable	Coefficient	Std. Error	z-Statistic	Prob.
AA	-0.514013	0.279379	-1.839839	0.0658
LL	0.022232	0.013172	1.687775	0.0915
ROA	-0.595725	0.271773	-2.191995	0.0284
ROA_{t_1}	-1.078511	0.699862	-1.541033	0.1233
CoE	-0.803002	0.096325	-8.336373	0.0000
REG	0.039586	0.051050	0.775431	0.4381
LF	-0.377180	0.156222	-2.414388	0.0158
C	3.511245	0.737166	4.763166	0.0000
R-squared	0.594084			
Obs with Dep=0	323	Total obs		400
Obs with Dep=1	77			

Variable	Coefficient	Std. Error	z-Statistic	Prob.
AA	-0.529676	0.287651	-1.841382	0.0656
LL	0.023702	0.012529	1.891774	0.0585
ROA	-0.547364	0.255993	-2.138204	0.0325
ROA_{t_1}	-1.020231	0.617117	-1.653222	0.0983
CoE	-0.793084	0.088565	-8.954860	0.0000
REG	0.079748	0.046098	1.729962	0.0836
LF	-0.329084	0.142878	-2.303257	0.0213
C	2.711176	0.645116	4.202614	0.0000
R-squared	0.574516			
Obs with Dep=0	488	Total obs		565
Obs with Dep=1	77			

Table 1 Robustness of the Constructions model on the sample 400 - 565 obs.
Source: The Author.

In the Constructions are very important demographic and economic variables. We tested macroeconomic environment but coefficient was insignificant thus we have removed it from the final model. However, we assume that in the population the coefficient is significant as its probability grew up with the number of firms. Category of Employees (CoE) significantly decreases probability of default thus we can assume that CoE incorporates many factors, like a position of a firm on the market, its stability, quality of inner processes etc.

Variable	Coefficient	Std. Error	z-Statistic	Prob.
AA	-0.599574	0.296333	-2.023312	0.0430
LL	0.026647	0.012667	2.103677	0.0354
ROA	-0.530839	0.253977	-2.090103	0.0366
ROA_{t_1}	-1.067954	0.615288	-1.735698	0.0826
CoE	-0.795613	0.086754	-9.170939	0.0000
REG	0.081568	0.043273	1.884986	0.0594
LF	-0.377440	0.135081	-2.794178	0.0052
C	2.656053	0.617540	4.301022	0.0000
R-squared	0.567133			
Obs with Dep=0	583	Total obs		660
Obs with Dep=1	77			

Variable	Coefficient	Std. Error	z-Statistic	Prob.
AA	-0.683703	0.303555	-2.252317	0.0243
LL	0.029479	0.012107	2.434967	0.0149
ROA	-0.528843	0.253390	-2.087073	0.0369
ROA_{t_1}	-1.367297	0.623012	-2.194656	0.0282
CoE	-0.650759	0.072852	-8.932604	0.0000
REG	0.124958	0.039451	3.167419	0.0015
LF	-0.404722	0.128771	-3.142973	0.0017
C	1.489114	0.519211	2.868032	0.0041
R-squared	0.493278			
Obs with Dep=0	681	Total obs		758
Obs with Dep=1	77			

Table 2 Robustness of the Constructions model on the sample 660 - 758 obs.
Source: The Author.

Next variable, region of firm's place of business, has a positive sign which implies that firms located far from Prague and the Central region have higher probability of default. This could be impressive result for firms operating in the Constructions, especially for investors which planning their new activities in this field. Causes behind this fact are probably (1) lower stability of demand and (2) prices of final products could be lower in these regions, thus firm's cash flow has lower border to default. Together with a fact that these regions are more prone to worse performance if the economic condition goes down, we definitively confirm our empirical result

of our model. Finally, a legal form with negative sign implies, that, if a firm has different form than Ltd. or Inc. we can expect higher probability of default. This is likely correct conclusion which is based on a fact that an individual or another legal form has less stable corporate background and corporate processes.

Table 3 assure us about quality and robustness of our model, we can see significant differences between means and standard deviations in the case of all of the explanatory variables, mainly ROA and CoE have very strong distinctions between defaulted and healthy firms. Interesting differences are also in the case of assets and liabilities, in average healthy firms increase their assets more than 20% annually instead of defaulted firms which decrease their assets more than 10%. Liabilities are similar but with inverse tendency, thus healthy firms decrease their liabilities and defaulted increase.

	Mean			Standard Deviation		
Variable	Dep=0	Dep=1	All	Dep=0	Dep=1	All
AA	1.205474	0.894064	1.173840	2.280536	1.200751	2.196689
LL	1.814927	2.095315	1.843409	16.65854	9.028345	16.04588
ROA	0.072272	-2.206108	-0.159173	0.089744	12.62243	4.059227
$ROA_{t\ 1}$	0.030910	-0.222645	0.005153	0.106802	0.755240	0.270899
CoE	7.487518	2.519481	6.982850	2.852796	1.651239	3.136856
REG	5.700441	7.350649	5.868074	4.016307	4.695368	4.117304
LF	2.494860	1.389610	2.382586	1.551201	1.125726	1.549300
C	1.000000	1.000000	1.000000	0.000000	0.000000	0.000000

Table 3 Differences between defaulted and healthy firms in Mean and SD
Source: The Author.

Very radical difference can be also found in CoE, here it is 3 times higher in the case of healthy firms which leads us again to emphasize how important is number of employees which reflects stability of a firm and its ability to survive in the future.

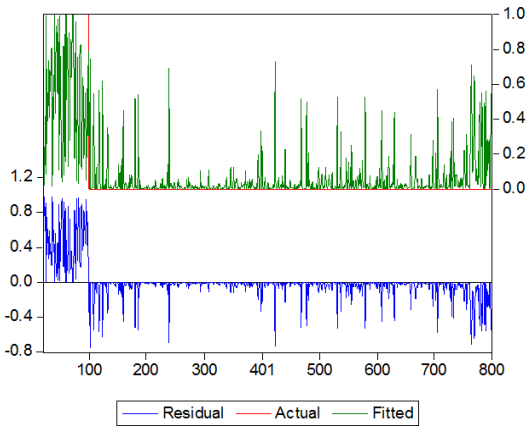

Figure 1 Prediction quality of the model
Source: The Author.

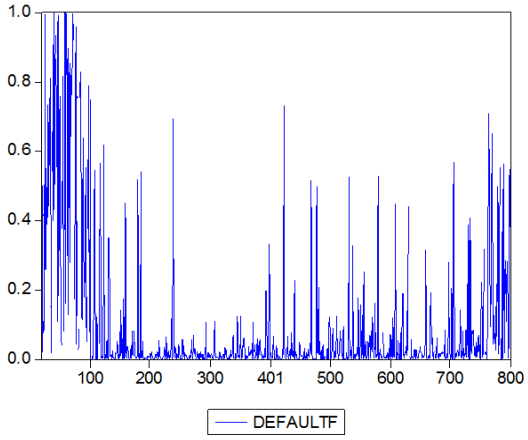

Figure 2 Forecast of the Defaults
Source: The Author.

Figure 1 and 2 represent prediction quality of the final model, it could be seen that residuals are relatively small, thus forecast value have good coherency with the empirical data set.

Table 4 depicts final output of the Constructions model, its ability to distinguish between healthy and defaulted firms. In the case of healthy firms we can observe that the prediction ability is more than 96% which is fine result. Lower, but still fine performance has distinction for defaulted firms where we can observe more than 57% success. Total performance is 92.08% which give us 2.24% better performance than constant model and we were able to explain more than 22.8% of incorrect prediction.

	Estimated Equation			Constant Probability		
	Dep=0	Dep=1	Total	Dep=0	Dep=1	Total
P(Dep=1)<=C	654	33	687	681	77	758
P(Dep=1)>C	27	44	71	0	0	0
Total	681	77	758	681	77	758
Correct	654	44	698	681	0	681
% Correct	96.04	57.14	92.08	100.00	0.00	89.84
% Incorrect	3.96	42.86	7.92	0.00	100.00	10.16
Total Gain*	-3.96	57.14	2.24			
Percent Gain**	NA	57.14	22.8			

*Change in "% Correct" from default (constant probability) specification
**Percent of incorrect (default) prediction corrected by equation

Table 4 Prediction quality result
Source: The Author.

4 Conclusion

The article has revealed the probability model of defaults in the Constructions of the Czech Republic. We used standard Logit techniques together with Stepwise analysis. Our data set has been obtained from Bisnode company, thus we were able verified stability and robustness of our model. We have found high importance of cash flow which is reflected to rentability indicators and the macroeconomic condition which in the Constructions doesn't have significant influence, however, we assume that in the population the influence is significant with no doubt. We've also revealed high importance in the variable category (i.e. number) of employees which incorporates stability and quality of a firm. Furthermore, total level of assets and their evolution in time are very important, as an indicator of firm's development and investments. Finally, we've found important factors in demographic and economic indicators. Hence, firms in the Constructions are more prone to their location of business and legal form, which is also important information for investors - at least they should require a higher rate of their investments to compensate a higher risk.

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On Structural Breaks and Volatility Persistence: Evidence from Central European Stock Market

Petr Seďa[1]

Abstract. In recent years, an extensive literature has been developed on studying the volatility of financial markets. One of the most challenging econometric problems is to understand and model the behavior of the volatility through time. This paper deals with an impact of structural breaks on long memory of volatility using iterated cumulative sum of squares (ICSS) algorithm. Using this algorithm one can identify sudden changes in volatility of stock markets corresponding to major economic and political events that have been affecting stock markets worldwide. In this paper we considered daily data of PX and BUX indexes in period from 2004 till 2012 which includes recent global financial crisis of 2008-2009 years. For a comprehensive analysis, several sophisticated models of quantitative analysis have been adopted. We especially worked with GARCH and FIGARCH models with multiple sudden change dummies. When incorporating these sudden changes into conditional volatility models, volatility persistence or long memory property has significantly disappear. This finding means that ignoring an influence of sudden changes leads to overestimation of volatility persistence and potential errors by risk managers when interpreting Value-at-Risk. Therefore, inclusion of information on sudden shocks in conditional variance may improve the precision of volatility dynamics and forecasting.

Keywords: emerging stock market, GARCH, ICSS algorithm, structural breaks, stock market, volatility persistence.

JEL Classification: C32, C52, G10, E37
AMS Classification: 60G99

1 Introduction

Financial markets, due to their key role in the economics, have been studied from different points of view. In this regard, one key aspect of the stock markets that has attracted much attention in financial literature is the analysis of the stock returns and its volatility. Volatility is usually a symptom of a highly liquid stock market. An increase in stock market volatility brings a large stock price change of advances or declines. Investors interpret a raise in stock market volatility as an increase in the risk of investment and consequently they usually shift their funds to less risky assets.

Volatility modeling and especially volatility dynamics are central to many issues in financial markets including leverage ratios, credit spreads, and portfolio decisions. In times of low market volatility it is relatively easy to measure volatility and understand volatility dynamics. At other times, financial markets are affected by severe disruptions which may be largely isolated events like IT dot com bubbles, U.S. terror attack of 2001 year or the global financial crisis of 2008-2009 years. During such periods, apparent spikes in volatility and large movements in asset prices complicate estimation and forecasting of volatility and understanding its dynamics. These crises or political events dramatically influenced market volatility and diversification opportunities for investors.

In the financial econometrics literature many different techniques have been utilized to model volatility. The autoregressive conditional heteroskedasticity (ARCH) model derivate by Engle [5], and later generalized by Bollerslev [4] is one of the most popular methods used for modeling high-frequency return series in financial engineering. Generalized autoregressive conditional heteroskedasticity (GARCH) models have been used to model the spillover effects among different markets. It have been empirically confirmed that higher levels of conditional volatility in the past are followed by higher conditional volatility in the current period. In other words, it is associated with a volatility persistence value that is normally very close to one. A persistence value close to one indicates an integrated GARCH (IGARCH) process. The estimates of volatility persistence for return series in fact provide information on a size to which past shocks and volatility influence the estimation of forecast of future conditional variance. The greater value of volatility persistence means that more weight should be given to recent observations of volatility when explaining future behavior of conditional variance.

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The volatility of financial time series is affected by infrequent market structural breaks, usually corresponding to economic turmoil. Innumerable expert studies have been provided to identify the volatility of stock prices, and the structural characteristics of risk. Inclán and Tiao [7] introduced an Iterated Cumulative Sum of Squares algorithm in order to identify break points in volatility. Aggarwal et al. [1] analyzed break points in volatility of some Asian and South American emerging stock markets. They successfully included the influence of structural breaks on volatility persistence by incorporating structural shifts dummies directly into the GARCH model. Wang and Moore [10] investigated structural changes in volatility of new European Union stock markets using weekly data over the period 1994-2006. They found that structural breaks are caused mainly by economic and political factors.

The aim of this paper is to examine the impact of structural breaks on volatility persistence of Central European stock markets represented by Czech Republic and Hungary using weekly data over the period of 2004-2012 years. The studies conducted in this paper can be summarized as follows: First, the methodology that is used to detect potential changes in volatility is discussed. In particular, it includes the ICSS algorithm and GARCH, respectively FIGARCH models. Second, the descriptive statistics of Czech and Hungarian stock markets and time series features will be computed. Section 4 reports the main results of analysis we achieved and Section 5 concludes the analysis.

2 Methodology

The aim of this section is to describe methods that will be used for analysis of structural shifts on long memory of volatility of respective time series. In this paper the sudden changes in volatility will be identified using the ICSS algorithm, and the GARCH and FIGARCH models with and without sudden shifts will be described.

2.1 ICSS Algorithm

In the paper of Inclan and Tiao [7] there was derivate the Iterative cumulative Sums of Squares (ICSS) algorithm. Using this algorithm one can detect multiple break points in variance of respective time series. However, Fenandéz [6] has shown that using the ICSS algorithms tend to overestimate the number of breaks in variance. In addition, Bacmann and Dubois [3] stressed that the ICSS algorithm express patterns that are questionable assuming the presence of conditional heteroskedasticity. They found that this problem can be solved by filtering the returns applying a GARCH (p, q) model in the first step, and then using the ICSS algorithm on the standardized residuals. In this paper we will test for volatility breaks at first and as a next step we will filter the data for conditional heteroskedascity and serial correlation.

In general, the ICSS algorithm was utilized in order to identify time periods of significant exogenous changes in volatility of series returns. It is assumed that the conditional volatility of returns is stationary over an initial period of time until an exogenous change appears as a result of an unexpected event. The conditional variance is then predicted to become stationary again until another sudden change appears. This process is repeating over the time. Let assume that ε_t is an independent time series with zero mean and variance s_j^2. The unconditional variance s_j^2 for each particular period is given by s_j^2 for $j=0,1,...,N_T$, where N_T means the total number of structural breaks in variance in T observations, and assuming $1<K_1<...<K_{N_T}<T$ is the set of break point. To sum up, the unconditional variance s_t^2 over N_T intervals can be written as follows:

$$s_t^2 = \begin{cases} s_0^2, & 1<t<K_1 \\ s_1^2, & K_1<t<K_2 \\ \cdot & \\ \cdot & \\ \cdot & \\ s_{Nt}^2, & K_{Nt}<t<T. \end{cases} \tag{1}$$

A cumulative sum of squares (CSS) was counted in order to determine the total number of structural shifts in volatility or variance and time point when variance breaks occurs. Therefore, the cumulative sum of square residuals from the first observation to the K-th time point can be defined as follows:

$$S_K = \sum_{t=1}^{K} e_t^2, \tag{2}$$

where K=1,…,T.

Let´s define statistic D_K as follows:

$$D_K = \left(\frac{S_K}{S_T} - \frac{K}{T} \right), \tag{3}$$

where $D_0 = D_T = 0$, and S_T is the sum of squared residuals of the whole sample period. In case of no change in unconditional variance, the D_K statistic will oscillate around zero. On the other hand, if there are one or more changes in unconditional variance, the statistical values will considerably drift up, or down, from zero. To summarize, structural changes in unconditional variance are estimated applying the critical values obtained from the distribution of D_K assuming the null hypothesis of constant variance. In the case when the maximum absolute value of D_K is greater than the critical value, the null hypothesis of homogeneity in variance is rejected. Let´s define K^* as a value when $max_K |D_K|$ is achieved. If $max_K \sqrt{(T/2)} |D_K|$ exceeds the critical value, then parameter K^* means the time point when a volatility suddenly changed. Inclan and Tiao [7] computed the critical value of 1,358 as the 95th percentile of $max_K \sqrt{(T/2)} |S_K|$ assuming asymptotic distribution. Thus, the upper and lower boundaries can be established at ± 1.358. Therefore, a change point in volatility can be identified when the value of D_K exceeds these boundaries. The ICSS algorithm works on the basis of evaluating the D_K function over different time periods.

2.2 (FI)GARCH Models

The seminal papers of Engle [5] and Bollerslev [4] generalized autoregressive conditional heteroskedasticity or GARCH models have become a standard tool in modeling the conditional variances of the returns from financial time series data. The popularity of these models lies mainly in their compatibility with major stylized facts for asset returns, the existence of efficient statistical methods for estimating model parameters, and the availability of useful volatility forecasts. Normally, the AR(1)-GARCH(1,1) model taking the form of discrete data and considering heteroskedasticity may be according to [4] defined as follows:

$$\sigma_t^2 = \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \beta_2 \sigma_{t-1}^2, \tag{4}$$

$$\varepsilon_t = \xi_t \sigma_t, \tag{5}$$

$$\xi_t \sim N(0,1), \tag{6}$$

where σ_t^2 is conditional variance estimated here. The GARCH heteroskedasticity is defined by (6). β_0, β_1, β_2 are constants satisfying the conditions of $\beta_0 > 0$, $\beta_1 + \beta_2 < 1$. For instance, in the GARCH(1,1) model, there is a stringent trade-off between the possibility of having sharp changes in the short term volatility represented by high value of the parameters $\beta_1 + \beta_2$ and the ability to capture the long memory behavior of volatility through high values of $\beta_1 + \beta_2$. Moreover, even with high value of $\beta_1 + \beta_2 < 1$, GARCH models are subject to exponential decline in the autocorrelation, which is at odds with the observed hyperbolic decline observed in the data. Hence the recent interest in long memory process.

Another modification of ARCH type model which provides greater flexibility for modeling the conditional variance is the FIGARCH model. According Baillie et al. [2] the FIGARCH(1,*d*,1) model can be specified as follows:

$$\sigma_t^2 = \beta_0 (1-\beta_2)^{-1} + \left[1 - (1-\beta_2 L)^{-1} (1-\beta_1 L)^{-1} (1-L)^d \right] \varepsilon_t^2, \tag{7}$$

where $0 \le d \le 1$ is fractional difference parameter and L means the lag operator. In the case of FIGARCH model the persistence of shocks to either the conditional variance or the degree of long memory is measured by the fractional differencing parameter d and estimated by quasi-maximum likelihood estimation technique.

Volatility persistence estimated by the GARCH and FIGARCH models usually tend to be overestimated when sudden shift or changes in regime occurs in volatility. Once the change points in variance of respective

series have been identified using ICSS algorithm, the GARCH and FIGARCH models might be estimated with and without changes in variance. In order to calculate estimation of GARCH and FIGARCH models parameters more accurately sudden shifts in volatility using ICSS algorithm should be incorporated.

Therefore, the GARCH(1,1) and FIGARCH(1,*d*,1) models with sudden changes can be defined as follows:

$$\sigma_t^2 = \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \beta_2 \sigma_{t-1}^2 + d_1 D_1 + ... + d_n D_n, \tag{8}$$

$$\sigma_t^2 = \beta_0 (1-\beta_2)^{-1} + \left[1-(1-\beta_2 L)^{-1}(1-\beta_1 L)^{-1}(1-L)^d\right]\varepsilon_t^2 + d_1 D_1 + ... + d_n D_n, \tag{9}$$

where $D_1 + ... + D_n$ are dummy variables which can be equal value of 1 when sudden shifts of volatility occurs, elsewhere take a value of 0.

3 Description of Data Set

The original data set used in this paper consists of PX and BUX indexes. Empirical analysis is performed on weekly data in period from 2004 till 2012, which includes total of 426 weekly observations. This period was chosen purposely, to investigate changes of the Central European equity markets volatility during time with a special emphasis on the behaviour in the time before, during and after the global financial crisis of 2008-2009 years, and also taking into consideration local economic and political events. As it has been already empirically confirmed, crises are not devoted to developed markets only. Emerging markets including Czech Republic and Hungary aren't excluded from this rule and may face such instability sometimes, see Nevima and Melecký [8] or Seďa [9]. We have more than 8 years long time series of the closing prices of PX and BUX indexes. The sample period was limited by the availability of the data on emerging Central European stock markets. The returns r_t at time *t* were defined as the logarithm of respective indices *p*, that is, $r_t = \log(p_t - p_{t-1})$.

Table 1 shows several descriptive statistics and the results of the unit root test for both Central European stock returns. Symbol * means a rejection of relevant null hypothesis at the 5% significance level.

Index	Mean	Stand. dev.	Skewness	Kurtosis	*J-B* test	*L-B Q*(12)
PX	0,0009	0,0376	-0,3399	10,9765	1137,543*	188,950*
BUX	0,0016	0,0413	0,0112	9,2602	695,633*	177,142*

Table 1 Descriptive statistics of PX and BUX indexes

The means of both sample returns are quite small, while the standard deviations are significantly higher. Based on the values of skewness, increased values of kurtosis and *J-B* test of normality, the weekly PX and BUX return series show mostly leptokurtic distribution which has a higher peak and heavy tail, instead of normal distribution. The Ljung-Box statistics Q_n for the squared return series are high which indicate rejection of the null hypothesis of no correlation. The sample statistics of analysed time series data indicate that it is desirable to consider heteroscedasticity and jump risk when estimating the volatility of the Czech and Hungarian markets.

4 Empirical Results

In this chapter estimation results will be presented. The studies will be summarized as described in the Introduction section. The aim of this section is to detect large shifts in volatility of PX and BUX returns. Once the time points of changes in variance are identified, the standard GARCH and FIGARCH models are estimated. In the next step of our analysis the GARCH family models are reestimated with dummy variables corresponding to the shift points as specified by the ICSS algorithm.

4.1 Sudden Changes in Volatility

With a help of the ICSS algorithm we calculated the standard deviations in percent between the change points in order to identify the number of sudden breaks in volatility of respective return series. Table 2 shows time periods of sudden breaks in volatility of both return series as indicated by the ICSS algorithm. According Table 2 we can identify similar time points of sudden breaks in volatility of both stock markets, with more sudden changes occurring in Hungarian stock market. Based on the results shown in Table 2, most changes in volatility in both

Central European stock markets were linked rather with global economic and political events than specific local events. Both investigated return series exhibit similar changes in volatility because of volatility spillovers across stock markets. In other words, a single political or economic event can influence different markets simultaneously and cause analogous breaks in volatility.

Index	Standard deviation	Time period	Economic or political events
PX	2,824	2.1.2004 – 5.8.2008	Entry to European Union in 2004, economic growth
	7,536	14.8.2008 – 16.6.2009	Global financial crisis
	3,302	25.6.2009 – 5.3.2012	Debt crisis, economic stagnation
BUX	3,108	2.1.2004 – 7.6.2006	Entry to European Union in 2004, economic growth
	3,873	15.6.2006 – 27.9.2006	Austerity measures in order to reduce the budget deficit
	3,026	2.10.2006 – 11.8.2008	Economic stagnation
	8,458	19.8.2008 – 22.6.2009	Global financial crisis
	3,832	30.6.2009 - 5.3.2012	Debt crisis, economic stagnation

Table 2 Sudden changes in volatility of PX and BUX indexes detected by the ICSS algorithm

4.2 GARCH and FIGARCH Estimations

Once we identified sudden changes in volatility, the next step of our analysis is to include these sudden shifts into the GARCH and FIGARCH models, and investigate the influence of sudden shifts on volatility persistence or long memory. Table 3 and Table 4 show the results of the GARCH(1,1) and FIGARCH(1,*d*,1) models with and without dummies. The Ljung-Box test statistic $Q(12)$ verifies the serial correlation of squared residuals of respective returns. The $LM(4)$ test statistic verifies the ARCH effect in the residuals, and AIC means the Akaike information criterion.

The GARCH(1,1) model without dummy variables shows highly significant parameters b_1 and b_2, and moreover, the sums of these parameters are close to one. In other words, sudden shocks have a persistence impact on the volatility of both respective returns. On the other hand, when including dummy variables for sudden shifts, it reduced the persistence of conditional volatility in both cases significantly. The PX index shows decline in volatility persistence with 0,1826, while the BUX index shows even larger decline in volatility persistence with 0,1974. It seems that the standard GARCH(1,1) model tends to overestimate volatility persistence by ignoring sudden shifts in conditional volatility.

GARCH(1,1) model without dummies							
Index	b_1	b_2	$b_1 + b_2$		$Q(12)$	$LM(4)$	AIC
PX	0,3534*	0,5598*	0,9131		8,5683	0,3396	-4,1043
BUX	0,1763*	0,6810*	0,8573		10,838	0,7327	-3,7536
GARCH(1,1) model with dummies							
Index	b_1	b_2	$b_1 + b_2$	Persistence decline	$Q(12)$	$LM(4)$	AIC
PX	0,2564*	0,4741*	0,7305	0,1826	10,5881	0,4065	-4,0465
BUX	0,0727*	0,5872*	0,6599	0,1974	13,9729	0,5927	-3,4972

Table 3 GARCH(1,1) model parameters with and without dummy variables for sudden breaks in volatility

The FIGARCH(1,*d*,1) models without dummy variables detect that fractional difference parameter *d* significantly from those cases when *d*=0 (GARCH model) or *d*=1 (IGARCH model). It means that volatility of both series reveals long memory patterns. Nevertheless, when including sudden breaks into the FIGARCH(1,*d*,1), the values of estimated parameter *d* significantly decreased. In addition, it becomes statistically insignificant, and the volatility persistence disappears. To sum up, controlling structural breaks in volatility sufficiently reduce volatility persistence of both Central European stock markets. In the final step we verified the model specification using

selected diagnostic tests as shown in Table 3 and 4. The insignificance of Ljung-Box test statistic $Q(12)$ and LM ARCH(4) tests imply that all the models were correctly specified.

FIGARCH(1,*d*,1) model without dummies						
Index	b_1	b_2	d	$Q(12)$	$LM(4)$	AIC
PX	0,2482*	0,4295*	0,4628*	8,5683	0,3396	-4,0985
BUX	0,1046	0,5839*	0,3201*	10,838	0,7327	-3,7392
FIGARCH(1,*d*,1) model with dummies						
Index	b_1	b_2	d	$Q(12)$	$LM(4)$	AIC
PX	0,4280*	0,3716*	0,0916	14,7542	0,2851	-4,0219
BUX	0,3037	0,4850*	0,1389	22,8410	0,4072	-3,4680

Table 4 FIGARCH(1,*d*,1) model parameters with and without dummy variables for sudden breaks in volatility

5 Summary and Conclusions

This paper investigates an impact of structural breaks caused by exogenous factors on volatility using the ICSS algorithm and analyses the effect of these changes on volatility persistence with a help of GARCH and FIGARCH models. For this purpose we used two Central European stock indexes, namely PX and BUX return series. Using the ICSS algorithm, we found that identification of sudden shocks in both markets is rather caused by global political or economic event like the global financial crisis of 2008-2009 years. When incorporating sudden breaks into GARCH and FIGARCH models, the long memory of volatility significantly declines. In other words, when ignoring sudden shifts in conditional variance it causes the presence of long memory in volatility of return series.

Our findings suggested that major political or economic events usually lead to overestimation of the persistence or long memory in volatility. It may bring about potential errors by risk managers for instance when interpreting Value-at-Risk (VaR) or in option pricing. Excessive volatility can lead to inefficient resource allocation, and upward pressure on interest rates is due to the excessive uncertainty in the equity markets. Reasonable level of volatility is certainly a natural on stock markets, while from some level of volatility it can be harmful.

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University timetable construction – computational approach

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Abstract. Problem of university timetable construction is problem that each university deals each term with. Some of them use time-tested scheduling board, the other use more or less sophisticated computer-based system. The University of Economics, Prague, belongs to the first group. The problem of this process is that the timetable from the previous year is only modified a little bit. Therefore the timetable does not react on specific changes between the two academic years. In this paper a computational approach to timetable construction at the department level is presented. The timetabling problem is divided into four interrelated stages, that each is solved via integer goal programming model. Then there is a part focused on difficulties in constructing timetabling models and a part describing the specificity of timetable of University of Economics, Prague. Also a short summary of current state-of-the-art in the field of university timetabling is presented.

Keywords: timetabling, integer goal programming, assignment problem.

JEL Classification: C61
AMS Classification: 90C29

1 Introduction

Timetable construction is a yearly problem of each university. In the age of computers, there are still many schools and universities that use scheduling board instead of a sophisticated software. As the latter research shows (eg. [2], [12], [15]), university timetabling attracts many researchers.

The idea of constructing university timetables via mathematical model came out in middle 70's (Harwood and Lawless [9]; Van Dusseldorp et al. [16]). There are two main principles of timetable construction. One of them consists in decomposition of the problem into several stages (usually two (Badri [4]; Mirrazavi, Mardle and Tamiz [11]) or three stages (Al-Husain, Hasan and Al-Qaheri [1])). These stages are interrelated. This means outputs of one stage are inputs in the next stage. The other principle is solving the problem as one complex model. In the complex model the integer programming is usually used (Daskalaki, Birbas and Housos [7]; Bakir and Aksop [5]). In general, solving integer programming models is NP-hard problem [6], so it leads to utilizing various heuristic or metaheuristic methods (Aladag, Hocaoglu and Basaran [3]; De Causmaecker, Demeester and Berghe [8]). Heuristic and metaheuristic methods give us solutions that are relatively close to optimal solution in relatively reasonable time.

In the next part the current way of timetable construction at the University of Economics, Prague is described in brief. Then the four-stage mathematical model for the specific process of timetable construction at the university at the level of a department is presented. The mathematical model that is improved all the time utilizes integer goal programming.

2 Timetabling at the University of Economics, Prague

The process of a timetable construction starts on the beginning of term previous to the term for which the timetable is prepared. At first, all the departments of the university have to provide their requirements for classrooms and lecture halls to the educational department. This means to provide information which subjects are expected to be taught in that term, how many courses and lectures of each subject might be opened for students and what should be the capacity of the rooms. Also requirements for special equipment of the room such as computers or specialized software must be provided. Each department has to estimate the interest of students in each subject, because students at the University of Economics, Prague do not have a fixed timetable except their first term at the bachelor's level. For the other terms there is only a recommended curriculum of obligatory subjects. Nevertheless, students do not have to follow it. The estimation is based on previous experiences, on the number of students at each grade and the estimation is always a little bit overestimated.

According to the requirements of the departments the educational department provides a timetable for each department. In these timetables there are assigned only classrooms to time slots; there are no faculties assign-

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ments. These timetables are usually just a little bit modified timetables from the previous terms. The modifications are done manually, because at the university there is not used any automatic system. These timetables are the first version of the department timetable and the department uses it usually as list of disposable classrooms. Next step is that the department assigns subjects and faculties to given classrooms in given time slots with respect to faculties' preferences. This timetable is proposed to the students. The students via university information system construct their own timetable combined of mandatory and optional subjects. They register their interest in courses and lectures in given times. After the registration the department can modify the timetable according to student's preferences. This means that courses or subjects with low number of registered students can be cancelled. On the other hand, if there is any subject, which for registered more students than expected, a course or lecture can be added. Sometimes the students give a notice during the registration that there is some collision of subjects that are planned for the same term in the recommended curriculum. In cooperation with the educational department the department can modify its timetable.

3 Four-stage integer goal programming model

In this part a four-stage mathematical model for timetable construction at the department level is described. The model utilizes integer goal programming approach. The presented model is based on the three-stage model introduced by Al-Husain, Hasan and Al-Qaheri [1]. It is an extension of the model presented by Skocdopolova [13]. In the model the timetable construction problem is divided into four interrelated stages. Output of each stage serves as input in the next stages. Using of goal programming enables formulation of soft constraints. Soft constraints (in opposite to hard constraints that have to be fulfilled) might be fulfilled, but some deviations are allowed. In each stage a sum of weighted deviations is minimized.

The model is solved via LINGO optimization system. The input and output data are stored in Excel sheet. Modification of data between the stages is done by short macros written in Visual Basic for Application (VBA).

3.1 Stage I

In the first stage the timetable for lectures is constructed. Due to small number of lectures at the department, the timetable of lectures can be done in one stage. Faculty *i* is assigned to lecture *j* at the time slot *k* in the classroom *l*. Mathematical model of the stage I can be formulated as follows.
Minimize

$$z = p_1\delta_1^- + p_2\sum_{i=1}^{32}\left(\delta_{2i}^- + \delta_{2i}^+\right) + p_3\sum_{l=1}^{84}\left(\delta_{3i}^- + \delta_{3i}^+\right) \tag{1}$$

subject to

$$\sum_{i=1}^{32}\sum_{j=1}^{18} SP_{ij}x_{ij} + \delta_1^- - \delta_1^+ = 90, \tag{2}$$

$$x_{ij} \le SP_{ij}, \quad i = 1,2,...,32,\ j = 1,2,...,18, \tag{3}$$

$$\sum_{j=1}^{18} x_{ij} + \delta_{2i}^- - \delta_{2i}^+ = L_i, \quad i = 1,2,...,32, \tag{4}$$

$$\sum_{i=1}^{32} x_{ij} = 1, \quad j = 1,2,...,18, \tag{5}$$

$$\sum_{k=1}^{35} y_{ijk} \le x_{ij}, \quad i = 1,2,...,32,\ j = 1,2,...,18, \tag{6}$$

$$\sum_{j=1}^{18} y_{ijk} \le P_{jk}, \quad i = 1,2,...,32,\ k = 1,2,...,35, \tag{7}$$

$$\sum_{i=1}^{32} y_{ijk} = a_{jk}, \quad j = 1,2,...,18,\ k = 1,2,...,35, \tag{8}$$

$$\sum_{j=1}^{18} y_{ijk} \le 1, \quad i = 1,2,...,32,\ k = 1,2,...,35, \tag{9}$$

$$\sum_{k=1}^{35} a_{jk} = 1, \quad j = 1,2,...,18, \tag{10}$$

$$\sum_{l=1}^{84} b_{jl} = 1, \quad j = 1,2,...,18, \tag{11}$$

$$\sum_{j=1}^{18} b_{jl} \le 1, \quad l = 1,2,...,84, \tag{12}$$

$$\sum_{j=1}^{18} C_j b_{jl} + \delta_{3l}^{-} - \delta_{3l}^{+} = \sum_{j=1}^{18} K_l b_{jl}, \quad l = 1,2,...,84, \tag{13}$$

$$\sum_{k=1}^{35} T_k a_{jk} = \sum_{l=1}^{84} TW_l b_{jl}, \quad j = 1,2,...,18, \tag{14}$$

$$\sum_{l=1}^{84} PC_l b_{jl} = COMP_j, \quad j = 1,2,...,18, \tag{15}$$

$$\delta_{3l}^{+} \le 0{,}05K_l, \quad l = 1,2,...,84, \tag{16}$$

$$\begin{gathered} x_{ij} \in \{0,1\}, \quad i = 1,2,...,32, \; j = 1,2,...,18, \\ y_{ijk} \in \{0,1\}, \quad i = 1,2,...,32, \; j = 1,2,...,18, \; k = 1,2,...,35, \\ a_{jk} \in \{0,1\}, \quad j = 1,2,...,18, \; k = 1,2,...,35, \\ b_{jl} \in \{0,1\}, \quad j = 1,2,...,18, \; l = 1,2,...,84, \\ \delta_1^{-}, \delta_1^{+}, \delta_{2i}^{-}, \delta_{2i}^{+} \ge 0, \quad i = 1,2,...,32, \\ \delta_{3l}^{-}, \delta_{3l}^{+} \ge 0, \quad l = 1,2,...,84, \end{gathered} \tag{17}$$

where the binary decision variable x_{ij} equals 1, if the teacher i is assigned to the lecture j, and 0 otherwise; the binary decision variable y_{ijk} equals 1, if the lecture j assigned to the teacher i is assigned to time slot k, and 0 otherwise; the binary decision variable a_{jk} (or b_{jl}) equals 1, if the lecture j (assigned to the teacher i) is assigned to time slot k (or classroom l), and 0 otherwise. SP_{ij} represents faculty's preferences of the lectures. It gathers values from 0 to 5, where 0 means that the faculty i cannot teach lecture j and 5 points means the faculty i prefer the most to teach lecture j. L_i is the maximum number lecture loads for faculty i; P_{jk} equals 1, if faculty assigned to lecture j is available to teach the course in time slot k, and 0 otherwise; C_j is the required capacity of lecture j, K_l is the capacity of classroom l, T_k is number of the time slot k and TW_l is the time slot in which is the classroom l available. PC_l equals 1, if the classroom l is equipped with computers, and 0 otherwise; $COMP_j$ equals 1, if students computers are necessary for the lecture j. δ are the deviation variables and p_1, p_2, p_3,are the priorities of the minimal deviation.

After this stage data in the Excel sheet are modified via VBA macro. The modification consists in adaptation of faculty's presence (if the faculty have a lecture in time slot k, he/she cannot have seminar in the same time slot) and disposable classrooms. Next stages deal with timetabling of seminars.

3.2 Stage II

The second stage assigns faculty i to seminar j. The entire notation has the same meaning as in the previous stage with seminar instead of lecture. S_i is the maximum number seminar loads for faculty i. Mathematical model of the stage II is formulated as follows.
Minimize

$$z = p_1\delta_1^{-} + p_2 \sum_{i=1}^{32} \left(\delta_{2i}^{-} + \delta_{2i}^{+}\right) \tag{18}$$

subject to

$$\sum_{i=1}^{32} \sum_{j=1}^{62} SP_{ij} x_{ij} + \delta_1^{-} - \delta_1^{+} = 310, \tag{19}$$

$$x_{ij} \le SP_{ij}, \quad i = 1,2,...,32, \; j = 1,2,...,62, \tag{20}$$

$$\sum_{j=1}^{62} x_{ij} + \delta_{2i}^{-} - \delta_{2i}^{+} = S_i, \quad i = 1,2,...,32, \tag{21}$$

$$\sum_{i=1}^{32} x_{ij} = 1, \quad j = 1,2,...,62, \tag{22}$$

$$\begin{aligned} \delta_{2i}^{-} &\le 1, \quad i = 1,2,...,32, \\ \delta_{2i}^{+} &\le 1, \quad i = 1,2,...,32, \end{aligned} \tag{23}$$

$$\begin{aligned} x_{ij} &\in \{0,1\}, \quad i = 1,2,...,32,\ j = 1,2,...,62, \\ \delta_1^{-}, \delta_1^{+}, \delta_{2i}^{-}, \delta_{2i}^{+} &\ge 0, \quad i = 1,2,...,32. \end{aligned} \tag{24}$$

3.3 Stage III

The third stage assigns seminar *j* (assigned to faculty *i*) to the time slot *k*. Mathematical model of the stage III is formulated as follows.
Minimize

$$z = \sum_{k=1}^{35} \delta_{1k}^{+} \tag{25}$$

subject to

$$\sum_{k=1}^{35} y_{ijk} \le x_{ij}, \quad i = 1,2,...,32,\ j = 1,2,...,62, \tag{26}$$

$$\sum_{j=1}^{62} y_{ijk} \le P_{jk}, \quad i = 1,2,...,32,\ k = 1,2,...,35, \tag{27}$$

$$\sum_{i=1}^{32} y_{ijk} = a_{jk}, \quad j = 1,2,...,62,\ k = 1,2,...,35, \tag{28}$$

$$\sum_{j=1}^{62} a_{jk} + \delta_{1k}^{-} - \delta_{1k}^{+} = R_k, \quad k = 1,2,...,35, \tag{29}$$

$$\sum_{j=1}^{62} COMP_j a_{jk} \le PCR_k, \quad k = 1,2,...,35, \tag{30}$$

$$\sum_{j=1}^{62} y_{ijk} \le 1, \quad i = 1,2,...,32,\ k = 1,2,...,35, \tag{31}$$

$$\sum_{k=1}^{35} a_{jk} = 1, \quad j = 1,2,...,62, \tag{32}$$

$$\begin{aligned} y_{ijk} &\in \{0,1\}, \quad i = 1,2,...,32,\ j = 1,2,...,62,\ k = 1,2,...,35, \\ a_{jk} &\in \{0,1\}, \quad j = 1,2,...,62,\ k = 1,2,...,35, \\ \delta_{1k}^{-}, \delta_{1k}^{+} &\ge 0, \quad k = 1,2,...,35, \end{aligned} \tag{33}$$

where the entire notation has the same meaning as in the stage I with seminar instead of lecture. R_k is the number of classrooms available in time slot *k*, and PCR_k is the number of classrooms equipped with computers available in time slot *k*.

3.4 Stage IV

In the last stage seminar *j* (assigned to faculty *i* and to time slot *k*) is assigned to classroom *l*. The binary decision variable z_{ikl} equals 1, if the course *j* assigned to the time slot *k* is assigned to classroom *l*, and 0 otherwise. Mathematical model of the stage IV can be formulated as follows.

Minimize

$$z = p_1 \sum_{l=1}^{84} \delta_{1l}^{+} + p_2 \sum_{l=1}^{84} \delta_{1l}^{-} \tag{34}$$

subject to

$$\sum_{l=1}^{84} z_{jkl} = a_{jk}, \quad j = 1,2,...,62,\ k = 1,2,...,35 \tag{35}$$

$$\sum_{k=1}^{35} z_{jkl} = b_{jl}, \quad j = 1,2,...,62,\ l = 1,2,...,84 \tag{36}$$

$$\sum_{k=1}^{35} \sum_{l=1}^{84} z_{jkl} = 1, \quad j = 1,2,...,62, \tag{37}$$

$$\sum_{j=1}^{62} \sum_{k=1}^{35} z_{jkl} \le 1, \quad l = 1,2,...,84, \tag{38}$$

$$\sum_{j=1}^{62} C_j b_{jl} + \delta_{1l}^{-} - \delta_{1l}^{+} = K_l, \quad l = 1,2,...,84, \tag{39}$$

$$\sum_{j=1}^{62} T_j b_{jl} = TW_l \sum_{j=1}^{62} b_{jl}, \quad l = 1,2,...,84, \tag{40}$$

$$\sum_{j=1}^{62} COMP_j b_{jl} \le PC_l, \quad l = 1,2,...,84, \tag{41}$$

$$\begin{aligned} &z_{jkl} \in \{0,1\}, \quad j = 1,2,...,62,\ k = 1,2,...,35,\ l = 1,2,...,84, \\ &b_{jl} \in \{0,1\}, \quad j = 1,2,...,62,\ l = 1,2,...,84, \\ &\delta_{1l}^{-}, \delta_{1l}^{+} \ge 0, \quad l = 1,2,...,84, \end{aligned} \tag{42}$$

where T_j is the time slot assigned to course j and the rest of the notation has the same meaning as in the stage I with seminar instead of lecture.

4 Conclusion

The presented approach deals with timetable construction at the department of econometrics at the University of Economics, Prague. The result of the mathematical model is a timetable for the department that meets all the hard constraints. The described automated approach of timetable construction has not been implemented yet. As the future research we want to add to the model constraints that enable providing time preferences in the second above mentioned way. For this improvement the approach of Zouhar and Havlova (2012) might be utilized. Also the presented model cannot ensure parallel courses of chosen subjects and the problem of computer classrooms is not yet solved. The first implementation of this model might be possible in October 2013. This means that the timetable for summer term 2013/2014 might be constructed via the presented model.

As the mentioned parts will be solved, the next step will be to extend the model for the whole university. The first part of this step would be a unification of the form on which the departments' room requirements are provided to the educational department.

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Comparison of V4 economies using CGE model

Vaclav Skoluda[1]

Abstract. The presented paper deals with comparative analysis of V4 countries – Czech Republic, Hungary, Poland and Slovakia. This goal is achieved using computable general equilibrium model (CGE). The analysis is based on simulation of selected government policies and comparison of their impacts on country economies. Changes in tax policy or in external trade are examples of government policies used in simulation. Computable general equilibrium model is based on specific set of mostly nonlinear equations and it's basically problem of nonlinear optimization. The input data are in special form of social accounting matrix which is based on input-output tables and other sources. Matrixes are composed for year 2005 because of the data availability. The goal of this analysis is comparison of impacts of selected government policies using the same framework for different input data.

Keywords: general equilibrium, CGE model, SAM matrix, policy modeling

JEL Classification: C68, D58
AMS Classification: 91B5

1 Introduction

The aim of the presented article is to compare the V4 countries, for which will be used a specific computable general equilibrium model which allows us to follow the impacts of selected shocks to economy. Given the comparative nature of the analysis, we will focus on single period, year 2005, whereas is the latest year for which relevant data are available for all observed countries. The actual model used to comparison of selected countries is work of author of the article and is described in more detail in [17]. Compared with the original model is the biggest difference in extension of the model to 10 production sectors. CGE model is based on the assumption of equilibrium in the base period and compares the data from this period with data obtained from a new equilibrium after the introduction of an external shock. Such shocks are represented in the analysis as government action in the field of taxation, wages and foreign trade. Computable general equilibrium model compares data from two runs and the result is the percentage difference between two equilibrium states.

In its essence it is a non-linear optimization. Part of the microeconomic equations describing the events in the economy and entering into the model is in fact non-linear. Initial equilibrium is calculated by non-linear optimization algorithms. Part of the model construction is also calibration of parameters and closure of the model. CGE models are discussed in more detail in [3], [4] or [9], which were used by the creation of the present model.

The entire analysis emphasizes the principle of ceteris paribus. According to this rule, a model was constructed and as well results obtained, which are comparable to each other. To obtain the results the same model was used with the same calibration set and the same solver. Different were just the input data, which had of course for all countries the same form and the results obtained. The procedure of obtaining results was for all analyzed countries identical.

2 The Model

Means of achieving the results is to construct a model. This model is adapted to the conditions of Central Europe and has been explained in detail in the paper [17], with slight changes mainly in the field of foreign trade and calibration. Part of the calibrated values came from econometric estimation, [11], [15] and [18]. The actual equations are based on microeconomic theory and describe the demand of companies, households, government and investment, abroad, the balance of payments, expressed zero earnings, household income, government and investments and represents a balance in the markets. Together model consists of 29 blocks of equations.

The model is using Leontieff production function, Cobb-Douglas production function, CES and CET production function. For all these functions it is necessary to derive the conditional demand (1), (2), (3), which will be used in the model specification. Model assumes behavior of firms according to Cobb-Douglas production function, the conditional demand for labor (4), capital (5) and j-th commodity (6). The model is using Cobb-Douglas function for Demand of households (7), government (8) and investment (9). Model assumes a different nature of

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the products from domestic production and other from abroad so the rate of substitution is low (10) to (13). Equation (14) expresses the deficit or surplus from account balance. Equations for zero profit and budget limitations are (15) to (19). Household income (20) consists of wages, capital and transfers. Government revenues (21) consist of tax, capital and transfers. Income of investment (22) is formed by transfers and resources, which households and government misses for consumption. Balance of the markets is represented by (23) and (24).

$$\hat{X}_j = \frac{\alpha_j}{\gamma} Y \tag{1}$$

$$\hat{X}_j = \frac{Y \alpha_j}{P_j \gamma} \prod_{i=1}^{n} \left(\frac{P_i}{\alpha_i} \right)^{\alpha_i} \tag{2}$$

$$\hat{X}_j = \frac{Y}{\gamma} \left(\frac{P_j}{\alpha_j} \right)^{\frac{1}{\rho-1}} \left(\sum_{i=1}^{n} \alpha_i \left(\frac{P_i}{\alpha_i} \right)^{\frac{\rho}{\rho-1}} \right)^{-\frac{1}{\rho}} \tag{3}$$

$$L_i = L_i\left(Y_i, w^E, r, P^{DS}\right) \tag{4}$$

$$K_i = K_i\left(Y_i, w^E, r, P^{DS}\right) \tag{5}$$

$$X_j^i = X_j^i\left(Y_i, w^E, r, P^{DS}\right) \tag{6}$$

$$H_j = H_j\left(TH, P^{DS}\right) \tag{7}$$

$$G_j = G_j\left(TG, P^{DS}\right) \tag{8}$$

$$INV_j = INV_j\left(TINV, P^{DS}\right) \tag{9}$$

$$DP_j = DP_j\left(DS_j, P_j, P_j^{IM}\right) \tag{10}$$

$$IM_j = IM_j\left(DS_j, P_j, P_j^{IM}\right) \tag{11}$$

$$DP_i = DP_i\left(Y_i, P_i, P_i^{EX}\right) \tag{12}$$

$$EX_i = EX_i\left(Y_i, P_i, P_i^{EX}\right) \tag{13}$$

$$BP = \sum_i P_i^{IM} IM_i - \sum_j P_j^{EX} IM_j - ER\left(t_{ROW}^L - t_L^{ROW} + t_{ROW}^H - t_H^{ROW} + t_{ROW}^G - t_G^{ROW}\right) \tag{14}$$

$$P_i DP_i + P_i^{EX} EX_i = w^E L_i + rK_i + \sum_j P_j^{DS} X_j^i \tag{15}$$

$$P_j^{DS} DS_j = P_j^{IM} IM_j + P_j DP_j \tag{16}$$

$$P^{TH} TH = \sum_j P_j^{DS} H_j \qquad con. \quad P^{TH} TH = b_C^H M^H \tag{17}$$

$$P^{TG} TG = \sum_j P_j^{DS} G_j \qquad con. \quad P^{TG} TG = b_C^G M^G \tag{18}$$

$$P^{TINV} TINV = \sum_J P_j^{DS} INV_j \qquad con. \quad P^{TINV} TINV = M^{INV} \tag{19}$$

$$M^H = \sum_i wL_i + b_K^H \sum_i rK_i + t_H^G - t_G^H + ER\left(t_L^{ROW} - t_{ROW}^L + t_H^{ROW} - t_{ROW}^H\right) \tag{20}$$

$$M^G = \sum_i t_i w L_i + b_K^G \sum_i r K_i - t_H^G + t_G^H + ER\left(t_G^{ROW} - t_{ROW}^G\right) \tag{21}$$

$$M^{INV} = \left(1 - b_C^H\right) M^H + \left(1 - b_C^G\right) M^G + ER\left(t_{INV}^{ROW} - t_{ROW}^{INV}\right) \tag{22}$$

$$DS_j = \sum_j X_j^i + H_j + G_j + INV_j \tag{23}$$

$$TK = \sum_i K_i \tag{24}$$

3 Data

The input data for computable general equilibrium model are in a specific table, the so-called Social Accounting Matrix (SAM). It is a symmetric table in which rows represent revenue and columns expenditure, input-output principle. The amounts for lines must be equal to amounts in the corresponding columns, the principle of national accounts. Specific Social Accounting Matrixes are not available and therefore they had to be constructed during the preparation of analysis. It was also necessary to decide on the degree of aggregation in the model. It has shown as a reasonable and simple to combine the production sectors to 10 subgroups, where the allocation key was NACE code. Main activities in each production sector are shown Table 1.

Sector	**sec0**	**sec1**	**sec2**	**sec3**	**sec4**	**sec5**	**sec6**	**sec7**	**sec8**	**sec9**
NACE	**01-09**	**10-19**	**20-29**	**30-39**	**40-49**	**50-59**	**60-69**	**70-79**	**80-89**	**90-99**
Main activities	Agriculture	Mining manufacture of food, textile	Chemicals and machinery	Electro and automotive	Energy supply and construction	Retail and hotels	Transport and insurance	Real estate and social security	Education and health	Cultural activities and other

Table 1 Aggregation according to NACE

The data sources for the creation of a Social Accounting Matrix were primarily national statistical offices and the statistical office of the European Union - Eurostat. The aim of the author was to work with the most current data, but this was not possible and therefore the input data correspond to the year 2005. This year was in fact the last in which they can ensure the same input data for all the countries compared. Main data source are the symmetrical Input-Output tables, which are produced for most European countries every five years. For 2010 data will be available in December 2013. Given limited scope and dimensions of the input data the SAMs are not part of the contribution. Further details on the development of specific SAM may be found in [5], [9] and [16].

4 Computation

Computing phase precedes obtaining the results. In this step are theoretically derived equations combined with pre-built data in the Social Accounting Matrix and are entered as a set of commands in GAMS. This program is designed, among others, for non-linear optimization and this capability was used to solve the model. The model itself consists of about 200 individual equations and the program code is around 450 lines, without subprograms importing data and performing sensitivity analysis. A sensitivity analysis was performed as to change of selected parameters so the change in calibrated values.

Computable general equilibrium models, as well as model used for the analysis are inherently nonlinear optimization. It follows that the obtained solution is a local optimum, but not certainly a global optimum. In the environment of GAMS the tasks are solved through nonlinear optimization algorithms, solvers, and it is appropriate to define starting values with respect to the speed of convergence to solution. These starting values were specified in the vicinity of the baseline equilibrium, so it can be expected that the results are relevant. In other words, the global optimum may be too far away and give irrelevant results considering the economic nature of the problem. The sensitivity of solution to used solver was also tested (CONOP, MINOS, SNOPT), while individual solutions showed no significant differences.

The results obtained by the analysis are expressed as a percentage. Input data are indeed in monetary units, but given the system solvability one variable is chosen as unity, the so-called numeraire and others are then

calculated in relation to it. This option has no effect when using percentage changes. In the model is for the numeraire chosen the cost of labor because of the equation set and actually no impact on obtained results.

5 Results

The final step in the analysis based on computable general equilibrium models is the analysis of the impacts of various changes in the model parameters through scenarios. Changed may be different entries, all depends on the focus of the model and its possibilities. Model, on which is based the presented analysis, is focused on the impacts of selected government policies. The analysis consists of comparing the two equilibria. The first is the initial steady state and the second is the steady state after the introduction of an external shock to the model. These shocks are in our case, government policy actions. Differences between the equilibrium values are expressed as a percentage, due to the fact that the model is not internally expressed in monetary units. Comparison of the individual countries is based on following the impacts of selected four scenarios, which summarizes results Table 2. Monitored are the percentage changes in total household, government and investment consumption, changes in import and export and the impact on production of production sectors.

	Czech republic				**Hungary**				**Poland**				**Slovakia**			
	S1	**S2**	**S3**	**S4**	**S1**	**S2**	**S3**	**S4**	**S1**	**S2**	**S3**	**S4**	**S1**	**S2**	**S3**	**S4**
% change in total consumption of Households	0	1.11	*1.44*	1.11	0.03	0.71	*1.65*	0.74	-0.03	0.42	*1.28*	0.71	-0.01	0.84	*1.36*	0.82
% change in total consumption of Government	*0.52*	0.34	1.57	0.86	*0.37*	0.36	1.87	0.74	*0.92*	0.13	1.64	1.19	*0.98*	0.5	1.73	1.48
% change in total consumption of Investment	-0.28	9.72	3.61	*9.43*	*3.97*	3.48	1.81	*7.51*	-0.36	0.58	-1.36	*9.68*	-0.07	-0.74	-0.37	*-0.77*
% change in total Import	0.01	2.05	0.88	2.06	-0.09	1.45	1.21	1.37	-0.02	0.67	0.25	3.83	-0.02	1.04	0.82	1.03
% change in total Export	0.03	*-0.79*	0.09	-0.76	-1.59	*0.53*	1.08	-1.09	-0.01	*0.68*	1.53	-1.01	-0.1	*1.72*	1.36	1.61
% change in production in sec0	0.17	0.35	1.88	0.53	0.79	0.56	1.26	1.35	0	0.33	1.35	-0.2	0.13	0.62	1.73	0.75
% change in production in sec1	0.03	0.06	1.93	0.09	0.11	0.27	1.94	0.37	-0.04	0.39	2.13	-0.53	0.06	0.93	1.97	0.99
% change in production in sec2	-0.09	-0.34	1.61	-0.43	-0.37	0.44	1.6	0.08	-0.05	0.67	1.84	-0.29	0.01	2.13	1.35	2.13
% change in production in sec3	0	0	*0*	0	-2.72	0.62	1.51	-2.12	-0.15	0.92	1.66	1.86	-0.53	1.73	2.52	1.18
% change in production in sec4	-0.15	5.04	2.77	4.88	1.84	2	2.16	3.88	-0.11	0.39	0.67	3.61	-0.03	-0.21	0.54	-0.22
% change in production in sec5	-0.02	0.35	1.84	0.33	0.27	0.44	2.45	0.72	-0.1	0.11	1.49	-0.22	-0.1	0.01	1.95	-0.09
% change in production in sec6	0.06	0.15	1.61	0.21	-0.11	0.13	2.32	0.02	-0.11	0.17	1.79	-0.18	-0.06	0.24	1.67	0.17
% change in production in sec7	0.09	0.83	1.96	0.92	0.31	0.41	2.2	0.72	0.13	0.14	1.55	0.77	0.24	0.16	1.81	0.4
% change in production in sec8	0.41	0.31	2.98	0.71	0.2	0.27	3.5	0.47	0.66	0.1	3.29	0.86	0.77	0.33	*3.43*	1.1
% change in production in sec9	-0.04	0.63	2.04	0.59	0.07	0.28	2.14	0.35	0	0.17	1.72	0.64	-0.02	0.03	2.19	0.01

Table 2 Results of various policy scenarios (S1 to S4)

The **first** scenario (S1) represents an increase in the overall tax burden by 3%. The impact of this scenario by looking at the change in the total consumption of national governments is similar. In all cases brings higher tax burden higher government consumption. The smallest impact is in the case of Hungary (0.37%), followed by the Czech Republic (0.52%) and almost 1% change for Poland and Slovakia. Noteworthy is also the change in total

consumption of investments (3.97%) in Hungary, which has the opposite sign then in the case of the other three countries.

The **second** scenario (S2) represents a reduction in exports by 5%. Given the fresh country membership in the European Union and the characteristics of equations of the model it is a volume limit of the share of exports in total output of individual sectors, namely the reduction of 5%. The impact of such foreign trade policy is questionable, though only in the Czech Republic this restriction reduced the total exports about 0.79%. In three other countries has such a restriction opposite effect, with the greatest impact is in total exports of Slovakia, which in response to this scenario increased by 1.72%. Of course you can follow by the model similar effects of reducing the share of exports in one or more sectors, depending on the scope of analysis.

The **third** scenario (S3) corresponds to the general wage increase of 5%. Wages in the model are understood in a broader sense and correspond in statistical terms to compensation of employees. The impact of wage increases has an expected impact of increasing the total household consumption by 1.28% for Poland to 1.65% in the case of Hungary. Worth noting is also an increase in production in all production sectors and all countries. The lowest is the plus zero for sec4 (energy and construction) in Czech Republic and at the highest for sec9 (culture and other) in Slovakia with a value of 3.43%.

The **fourth** scenario (S4) is a combination of the first two scenarios and shows the current impact of increasing the overall tax burden by 3% and reduction in the share of exports in total output of each sector by 5%. The impact of this scenario on each country is different. On the one hand are Czech Republic, Hungary and Poland, where the impact of the shock increases the consumption of investment by up to 9.68% for Poland. Changes in the production of various sectors of these countries are similar. On the other hand is Slovakia, where the overall consumption of investment is slightly reduced and also production in several sectors has a different trend than in the other three countries.

Attention also draws the sec3 (electro and automotive) in the Czech Republic, where all values are zero. More precisely zero when rounded to two decimal places. The results seem unusual but even after checking and performing sensitivity analysis are confirmed. Similar results were obtained and discussed in more detail in [18]. For other countries the results of this production sector meet the expectations and since the only difference is in the input data, it indicates the specific position of that sector in the Czech economy.

Scenario 5	Czech republic	Hungary	Poland	Slovakia
% change in total consumption of Households	1.4	1.62	1.41	1.35
% change in total consumption of Government	0.67	1.19	0.11	0.05
% change in total consumption of Investment	1.58	-4.99	1.83	-1.19
% change in total Import	1.19	1.47	1.1	0.71
% change in total Export	1.18	3.9	1.1	1.72
% change in production in sec0	1.33	-0.09	1.19	1.55
% change in production in sec1	1.75	1.7	1.94	1.98
% change in production in sec2	1.58	2.15	1.66	1.56
% change in production in sec3	2.44	6.46	2.19	3.53
% change in production in sec4	1.71	-1.01	1.76	0.15
% change in production in sec5	1.94	1.97	1.6	2.11
% change in production in sec6	1.47	2.5	1.91	1.81
% change in production in sec7	1.64	1.66	1.45	1.36
% change in production in sec8	2.25	3.11	2.17	2.08
% change in production in sec9	1.94	2.02	1.85	2.26

Table 3 Results of policy scenario 5

The **fifth** scenario is again the current change in the form of compensation of employees increased by 5% and reducing the tax burden as well by 5%. Results are shown for convenience in the following Table 3. This shock acts in the Czech Republic positively where all changes are positive. The same case is also in Poland, although the changes are not so distinctive. In the case of Hungary, there was a negative change in consumption of investment and in the production of two sectors (sec1 and sec5). Negative change also occurred in the consumption of investment in Slovakia. Based on the analysis, Czech Republic and Poland, as well as the Slovak Republic and Hungary exhibit certain common features. Impact of the shock in the form of the fifth scenario this time did not cause nearly zero change in sec3 (electro and automotive) in the Czech Republic. Change between the baseline equilibrium and the equilibrium after the introduction of shock was at 2.44% in sec3 production.

6 Conclusion

The paper focuses on examining and comparison of four different but similar countries - Czech Republic, Hungary, Poland and Slovakia. The results have limited information value for policy makers, as they relate to relatively remote period, to 2005, because of data issuing schedule. However, this does not change the fact that the economy of the Czech Republic and Poland on the one hand and the Hungarian and Slovak economies on the other had similar structure according to the model and its results. However, in some impacts Slovakia differs from other countries. An interesting result is also a low sensitivity to changes in the electro equipment manufacturing and automotive sector in the economy of the Czech Republic. Here it would be interesting to see if a similar trend is reflected in other countries in the next years.

The presented work discusses in addition to the performance of the model and the concept of using CGE modeling, mainly the great possibilities of application of such models. Well-constructed model can provide answers to a wide range of questions about economic policy. The reader himself can see the effects of selected scenarios on sectors or the economy as a whole. Commented results are just partial and only demonstration of the potential impacts and use of CGE models in practice.

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Cumulative Optimality in Risk-Sensitive and Risk-Neutral Markov Reward Chains

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Abstract.

This contribution is devoted to risk-sensitive and risk-neutral optimality in Markov decision chains. Since the traditional optimality criteria (e.g. discounted or average rewards) cannot reflect the variability-risk features of the problem, and using the mean variance selection rules that stem from the classical work of Markowitz present some technical difficulties, we are interested in expectation of the stream of rewards generated by the Markov chain that is evaluated by an exponential utility function with a given risk sensitivity coefficient. Recall that for the risk sensitivity coefficient equal zero we arrive at traditional optimality criteria. In this note we present necessary and sufficient risk-sensitivity and risk-neutral optimality conditions; in detail for unichain models and indicate their generalizations to multichain Markov reward chains.

Keywords: dynamic programming, stochastic models, risk analysis and management.

JEL classification: C44, C61, C63
AMS classification: 90C40, 60J10, 93E20

1 Notation and Preliminaries

The usual optimization criteria examined in the literature on stochastic dynamic programming, such as a total discounted or mean (average) reward structures, may be quite insufficient to characterize the problem from the point of a decision maker. To this end it may be preferable if not necessary to select more sophisticated criteria that also reflect the variability-risk features of the problem. Perhaps the best known approaches stem from the classical work of Markowitz on mean variance selection rules. On the other hand risky decisions can be also eliminated when expectation of the stream of one stage rewards (or costs) is evaluated by an exponential utility function. Recall that exponential utility functions are separable and hence suitable for sequential decisions.

In what follows, we consider Markov decision chain $X = \{X_n,\, n = 0, 1, \ldots\}$ with finite state space $\mathcal{I} = \{1, 2, \ldots, N\}$ and an infinite (compact) set $\mathcal{A}_i \equiv [0, K_i] \subset \mathbb{R}$ of possible decisions (actions) in state $i \in \mathcal{I}$. Supposing that in state $i \in \mathcal{I}$ action $a \in \mathcal{A}_i$ is selected, then state j is reached in the next transition with a given probability $p_{ij}(a)$ and one-stage transition reward $r_{ij}(a) > 0$ will be accrued to such transition. We assume that each $p_{ij}(a)$, $r_{ij}(a)$ is a continuous function of $a \in \mathcal{A}_i$.

A (Markovian) policy controlling the chain, $\pi = (f^0, f^1, \ldots)$, is identified by a sequence of decision vectors $\{f^n, n = 0, 1, \ldots\}$ where $f^n \in \mathcal{F} \equiv \mathcal{A}_1 \times \ldots \times \mathcal{A}_N$ for every $n = 0, 1, 2, \ldots$, and $f^n_i \in \mathcal{A}_i$ is the decision (or action) taken at the nth transition if the chain X is in state i. Let π^k be a sequence of decision vectors starting at the k-th transition, hence $\pi = (f^0, f^1, \ldots f^{k-1}, \pi^k)$. Policy which selects at all times the same decision rule, i.e. $\pi \sim (f)$, is called stationary; $P(f)$ is transition probability matrix with elements $p_{ij}(f_i)$. Stationary policy $\tilde{\pi}$ is randomized if there exist decision vectors $f^{(1)}$, $f^{(2)}, \ldots, f^{(m)} \in \mathcal{F}$ and on following policy $\tilde{\pi}$ we select in state i action $f_i^{(j)}$ with a given probability $\kappa_i^{(j)}$ (of course, $\kappa_i^{(j)} \geq 0$ with $\sum_{j=1}^{N} \kappa_i^{(j)} = 1$ for all $i \in \mathcal{I}$.)

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Let ξ_n be the cumulative reward obtained in the n first transition of the considered Markov chain X. Since the process starts in state X_0, $\xi_n = \sum_{k=0}^{n-1} r_{X_k,X_{k+1}}$. Similarly let $\xi_{(m,n)}$ be reserved for the cumulative (random) reward, obtained from the mth up to the nth transition (obviously, $\xi_n = r_{X_0,X_1} + \xi_{(1,n)}$, we tacitly assume that $\xi_{(1,n)}$ starts in state X_1).

In this note, we assume that the stream of rewards generated by the Markov processes is evaluated by an exponential utility function (so-called risk-sensitive models) with a given risk sensitivity coefficient.

To this end, let us consider an exponential utility function, say $\bar{u}^\gamma(\cdot)$, i.e. a separable utility function with constant risk sensitivity $\gamma \in \mathbb{R}$. Then the utility assigned to the (random) outcome ξ is given by

$$\bar{u}^\gamma(\xi) := \begin{cases} (\text{sign } \gamma)\, \exp(\gamma\xi), & \text{if } \gamma \neq 0, \quad \text{risk-sensitive case,} \\ \xi \qquad \text{for } \gamma = 0 & \text{risk-neutral case.} \end{cases} \tag{1}$$

Obviously $\bar{u}^\gamma(\cdot)$ is continuous and strictly increasing. For $\gamma > 0$ (risk averse case) $\bar{u}^\gamma(\cdot)$ is convex, if $\gamma < 0$ (risk seeking case) $\bar{u}^\gamma(\cdot)$ is concave. Finally if $\gamma = 0$ (risk neutral case) $\bar{u}^\gamma(\cdot)$ is linear. Observe that exponential utility function $\bar{u}^\gamma(\cdot)$ is separable and multiplicative if the risk sensitivity $\gamma \neq 0$ and additive for $\gamma = 0$. In particular, we have $u^\gamma(\xi_1 + \xi_2) = u^\gamma(\xi_1) \cdot u^\gamma(\xi_2)$ if $\gamma \neq 0$ and $u^\gamma(\xi_1 + \xi_2) \equiv \xi_1 + \xi_2$ for $\gamma = 0$.

Moreover, recall that the certainty equivalent corresponding to ξ, say $Z^\gamma(\xi)$, is given by

$$\bar{u}^\gamma(Z^\gamma(\xi)) = \mathsf{E}\,[\bar{u}^\gamma(\xi)] \qquad \text{(the symbol } \mathsf{E} \text{ is reserved for expectation).} \tag{2}$$

From (1), (2) we can immediately conclude that

$$Z^\gamma(\xi) = \begin{cases} \gamma^{-1} \ln\{\mathsf{E}\; u^\gamma(\xi)\}, & \text{if } \gamma \neq 0 \\ \mathsf{E}\,[\xi] & \text{for } \gamma = 0. \end{cases} \tag{3}$$

Considering Markov decision process X, then if the process starts in state i, i.e. $X_0 = i$ and policy $\pi = (f^n)$ is followed, for the expectation of utility assigned to (cumulative) random reward ξ_n obtained in the n first transitions we get by (1)

$$\mathsf{E}_i^\pi\, \bar{u}^\gamma(\xi_n) := \begin{cases} (\text{sign } \gamma)\, \mathsf{E}_i^\pi\, \exp(\gamma\xi_n), & \text{if } \gamma \neq 0, \quad \text{risk-sensitive case} \\ \mathsf{E}_i^\pi\, \xi_n \quad \text{for } \gamma = 0 & \text{risk-neutral case.} \end{cases} \tag{4}$$

In what follows let

$$\bar{U}_i^\pi(\gamma, n) := \mathsf{E}_i^\pi\, \bar{u}^\gamma(\xi_n), \qquad U_i^\pi(\gamma, n) := \mathsf{E}_i^\pi\, \exp(\gamma\xi_n), \qquad V_i^\pi(n) := \mathsf{E}_i^\pi\, \xi_n. \tag{5}$$

2 Risk-Neutral Optimality in Markov Processes

In this section we focus attention primarily on so called unichain models, i.e. when the underlying Markov chain contains a single class of recurrent states. Then on introducing for arbitrary $g, w_j \in \mathbb{R}$ $(i, j \in \mathcal{I})$ the discrepancy function (cf. [8])

$$\tilde{\varphi}_{i,j}(w, g) := r_{ij} - w_i + w_j - g \tag{6}$$

we can easily verify the following identity:

$$\xi_n \;=\; ng + w_{X_0} - w_{X_n} + \sum_{k=0}^{n-1} \tilde{\varphi}_{X_k,X_{k+1}}(w, g). \tag{7}$$

For the risk-neutral models (i.e. if the risk sensitivity coefficient $\gamma = 0$, and $u^\gamma(\xi) = \xi$) we can conclude:

If the process starts in state i and policy $\pi = (f^n)$ is followed then for the expected (undiscounted) total reward $V_i^\pi(n) := \mathsf{E}_i^\pi\, \xi_n$ we immediately get by (7)

$$V_i^\pi(n) \;=\; ng + w_i + \mathsf{E}_i^\pi\{\sum_{k=0}^{n-1} \tilde{\varphi}_{X_k,X_{k+1}}(w, g) - w_{X_n}\}, \qquad \text{where} \tag{8}$$

$$\mathsf{E}_i^\pi \sum_{k=0}^{n-1} \tilde{\varphi}_{X_k,X_{k+1}}(w, g) \;=\; \sum_{j\in\mathcal{I}} p_{ij}(f_i)\{\tilde{\varphi}_{i,j}(w, g) + \mathsf{E}_j^{\pi^1} \sum_{k=1}^{n-1} \tilde{\varphi}_{X_k,X_{k+1}}(w, g)\} \tag{9}$$

It is well-known from the dynamic programming literature (cf. e.g. [1, 6, 9, 10]) that

If there exists state $i_0 \in \mathcal{I}$ that is accessible from any state $i \in \mathcal{I}$ for every $f \in \mathcal{F}$ then $(*)$

(i) For every $f \in \mathcal{F}$ the resulting transition probability matrix $P(f)$ is *unichain* (i.e. $P(f)$ has no two disjoint closed sets),

(ii) For every $f \in \mathcal{F}$ there exist numbers $g(f)$, and $w_i(f), i \in \mathcal{I}$ (unique up to additive constant) such that

$$w_i(f) + g(f) = \sum_{j\in\mathcal{I}} p_{ij}(f_i)[r_{ij} + w_j(f)], \qquad (i \in \mathcal{I}) \tag{10}$$

$$\text{i.e.} \quad \sum_{j\in\mathcal{I}} p_{ij}(f_i)\,\tilde{\varphi}_{i,j}(w,g) = 0 \quad \text{if} \quad \tilde{\varphi}_{i,j}(w,g) := r_{ij} - w_i(f) + w_j(f) - g(f).$$

(iii) There exists decision $\hat{f} \in \mathcal{F}$ (resp. $f^* \in \mathcal{F}$) along with numbers $\hat{g}$, (resp. g^*), $\hat{w}_i, i \in \mathcal{I}$ (resp. $w_i^*, i \in \mathcal{I}$) (unique up to additive constant) such that

$$\hat{w}_i + \hat{g} = \min_{a\in\mathcal{A}_i} \sum_{j\in\mathcal{I}} p_{ij}(a)[r_{ij} + \hat{w}_j] = \sum_{j\in\mathcal{I}} p_{ij}(\hat{f}_i)[r_{ij} + \hat{w}_j], \tag{11}$$

$$\varphi_i(f,\hat{f}) := \sum_{j\in\mathcal{I}} p_{ij}(f)[r_{ij} + \hat{w}_j] - \hat{w}_i - \hat{g} \geq 0 \ \text{ with } \ \varphi_i(\hat{f},\hat{f}) = 0, \tag{12}$$

resp.

$$w_i^* + g^* = \max_{a\in\mathcal{A}_i} \sum_{j\in\mathcal{I}} p_{ij}(a)[r_{ij} + w_j^*] = \sum_{j\in\mathcal{I}} p_{ij}(f_i^*)[r_{ij} + w_j^*], \tag{13}$$

$$\varphi_i(f,f^*) := \sum_{j\in\mathcal{I}} p_{ij}(f)[r_{ij} + w_j^*] - w_i^* - g^* \leq 0 \ \text{ with } \ \varphi_i(f^*,f^*) = 0. \tag{14}$$

From (8),(10),(12),(14) we immediately get that $\hat{g} \leq g(f) \leq g^*$, and

$$V_i^{\hat{\pi}}(n) = n\hat{g} + \hat{w}_i - \mathsf{E}_i^{\hat{\pi}}\hat{w}_n, \qquad V_i^{\pi^*}(n) = ng^* + w_i^* - \mathsf{E}_i^{\pi^*} w_n^*. \tag{15}$$

Hence for stationary policy $\pi \sim (\hat{f})$ and arbitrary policy $\pi = (f^n)$

$$\lim_{n\to\infty} \frac{1}{n} V_i^{\hat{\pi}}(n) = \lim_{n\to\infty} \frac{1}{n} V_i^{\pi}(n) = \hat{g} \quad \text{if and only if} \quad \lim_{n\to\infty} \frac{1}{n} \mathsf{E}_i^{\pi} \sum_{k=0}^{n-1} \varphi_{X_k}(f^n,\hat{f}) = 0. \tag{16}$$

$$\lim_{n\to\infty} \frac{1}{n} V_i^{\pi^*}(n) = \lim_{n\to\infty} \frac{1}{n} V_i^{\pi}(n) = g^* \quad \text{if and only if} \quad \lim_{n\to\infty} \frac{1}{n} \mathsf{E}_i^{\pi} \sum_{k=0}^{n-1} \varphi_{X_k}(f^n,f^*) = 0. \tag{17}$$

Remark. For the general multichain models it is necessary to modify $\tilde{\varphi}_{i,j}(w,g)$ such that $\tilde{\varphi}_{i,j}(w,g) := r_{ij} - w_i + w_j - g_i$ and introduce $\tilde{\psi}_{i,j}(g) := g_j - g_i$. Then (7) is replaced by

$$\xi_n = ng_{X_0} + w_{X_0} - w_{X_n} + \sum_{k=0}^{n-1} \Big[(n-1-k)\tilde{\psi}_{X_k,X_{k+1}}(g) + \tilde{\varphi}_{X_k,X_{k+1}}(w,g)\Big] \tag{18}$$

and (8) reads (see [12], [13])

$$V_i^{\pi}(n) = ng_i + w_i + \mathsf{E}_i^{\pi}\{\sum_{k=0}^{n-1} \Big[(n-1-k)\,\tilde{\psi}_{X_k,X_{k+1}}(g) + \tilde{\varphi}_{X_k,X_{k+1}}(w,g)\Big] - w_{X_n}\}. \tag{19}$$

Then (10) should be completed with $\sum_{j\in\mathcal{I}} p_{ij}(f_i)[g_j(f) - g_i(f)] = 0$ and considered in the form $w_i(f) + g_i(f) = \sum_{j\in\mathcal{I}} p_{ij}(f_i)[r_{ij} + w_j(f)]$. Similarly optimal policy $\hat{\pi} \sim (\hat{f})$, $\pi^* \sim (f^*)$ must fulfil

$$\min_{a\in\mathcal{A}_i} \sum_{j\in\mathcal{I}} p_{ij}(a)[\hat{g}_j - \hat{g}_i] = \sum_{j\in\mathcal{I}} p_{ij}(\hat{f}_i)[\hat{g}_j - \hat{g}_i], \qquad \max_{a\in\mathcal{A}_i} \sum_{j\in\mathcal{I}} p_{ij}(a)[g_j^* - g_i^*] = \sum_{j\in\mathcal{I}} p_{ij}(f_i^*)[g_j^* - g_i^*] \tag{20}$$

and in (11) (resp. in (13)) minimization (resp. maximization) of $a \in \mathcal{A}_i$ should be considered only for $a \in \bar{\mathcal{A}}_i \subset \mathcal{A}_i$ fulfilling (20). Then, if *the action set is finite* it is guaranteed that for sufficiently large n $(n-1-k)\tilde{\psi}_{X_k,X_{k+1}}(\hat{g}) + \tilde{\varphi}_{X_k,X_{k+1}}(\hat{w},\hat{g}) \leq 0$ (resp. $(n-1-k)\tilde{\psi}_{X_k,X_{k+1}}(g^*) + \tilde{\varphi}_{X_k,X_{k+1}}(w^*,g^*) \geq 0$).

3 Risk-Sensitive Optimality in Unichain Markov Processes

Similarly to risk-neutral models we get by (5), (6), (7) for the risk-sensitive case

$$U_i^{\pi}(\gamma, n) = \mathrm{e}^{\gamma[ng+w_i]} \times \mathsf{E}_i^{\pi} \mathrm{e}^{\gamma[\sum_{k=0}^{n-1} \tilde{\varphi}_{X_k, X_{k+1}}(w,g) - w_{X_n}]}. \tag{21}$$

Now observe that

$$\mathsf{E}_i^{\pi}\, \mathrm{e}^{\gamma \sum_{k=0}^{n-1} \tilde{\varphi}_{X_k, X_{k+1}}(w,g)} = \sum_{j \in \mathcal{I}} p_{ij}(f_i^0)\, \mathrm{e}^{\gamma[r_{ij} - w_i + w_j - g]} \times \mathsf{E}_j^{\pi^1}\, \mathrm{e}^{\gamma \sum_{k=1}^{n-1} \tilde{\varphi}_{X_k, X_{k+1}}(w,g)} \tag{22}$$

$$\mathsf{E}_j^{\pi} \{\mathrm{e}^{\gamma \sum_{k=m}^{n-1} \tilde{\varphi}_{X_k, X_{k+1}}(w,g)} | X_m = j\} = \sum_{\ell \in \mathcal{I}} p_{j,\ell}(f_j^m)\, \mathrm{e}^{\gamma[r_{j,\ell} - w_j + w_\ell - g]} \times \mathsf{E}_\ell^{\pi^{m+1}}\, \mathrm{e}^{\gamma \sum_{k=m+1}^{n-1} \tilde{\varphi}_{X_k, X_{k+1}}(w,g)}. \tag{23}$$

In analogy with the risk-neutral case if stationary policy $\pi \sim (f)$ is followed, we are looking for numbers g, w_j's such that $\sum_{j \in \mathcal{I}} p_{ij}(f_i)\, \mathrm{e}^{\gamma \tilde{\varphi}_{ij}(g,w)} = 1$ and for stationary policy with maximal/minimal value of $g(f)$. To this end we consider the following sets of linear and nonlinear equations

$$\mathrm{e}^{\gamma[g(f) + w_i(f)]} = \sum_{j \in \mathcal{I}} p_{ij}(f_i)\ \mathrm{e}^{\gamma[r_{ij} + w_j(f)]} \quad (i \in \mathcal{I}) \tag{24}$$

$$\mathrm{e}^{\gamma[g^* + w_i^*]} = \max_{f \in \mathcal{F}} \sum_{j \in \mathcal{I}} p_{ij}(f_i)\ \mathrm{e}^{\gamma[r_{ij} + w_j^*]}, \qquad \mathrm{e}^{\gamma[\hat{g} + \hat{w}_i]} = \min_{f \in \mathcal{F}} \sum_{j \in \mathcal{I}} p_{ij}(f_i)\ \mathrm{e}^{\gamma[r_{ij} + \hat{w}_j]} \quad (i \in \mathcal{I}) \tag{25}$$

for the values $g(f)$, $\hat{g}$, g^*, $w_i(f)$, w_i^*, $\hat{w}_i$ $(i = 1, \ldots, N)$; obviously, these values depend on the selected risk sensitivity γ. Eqs. (25) can be called the *γ-average reward/cost optimality equation*. In particular, if $\gamma \downarrow 0$ using the Taylor expansion by (24), resp. (25), we have

$$g(f) + w_i(f) = \sum_{j \in \mathcal{I}} p_{ij}(f_i)\, [c_{i,j} + w_j(f)], \qquad \text{resp.} \quad \hat{g} + \hat{w}_i = \min_{f \in \mathcal{F}} \sum_{j \in \mathcal{I}} p_{ij}(f_i)\, [c_{i,j} + \hat{w}_j]$$

that well corresponds to (11).

On introducing the new variables $v_i(f) := \mathrm{e}^{\gamma w_i(f)}$, $\rho(f) := \mathrm{e}^{\gamma g(f)}$, and on replacing transition probabilities $p_{ij}(f_i)$'s by general nonnegative numbers defined by $q_{ij}(f_i) := p_{ij}(f_i) \cdot \mathrm{e}^{\gamma r_{ij}}$ (24) can be alternatively written as the following set of equations

$$\rho(f) v_i(f) = \sum_{j \in \mathcal{I}} q_{ij}(f_i)\, v_j(f) \quad (i \in \mathcal{I}) \tag{26}$$

and (25) can be rewritten as the following sets of nonlinear equations (here $\hat{v}_i := \mathrm{e}^{\gamma \hat{w}_i}$, $v_i^* := \mathrm{e}^{\gamma w_i^*}$, $\hat{\rho} := \mathrm{e}^{\gamma \hat{g}}$, $\rho^* := \mathrm{e}^{\gamma g^*}$)

$$\rho^* v_i^* = \max_{f \in \mathcal{F}} \sum_{j \in \mathcal{I}} q_{ij}(f_i)\, v_j^*, \quad \hat{\rho}\, \hat{v}_i = \min_{f \in \mathcal{F}} \sum_{j \in \mathcal{I}} q_{ij}(f_i)\, \hat{v}_j \quad (i \in \mathcal{I}) \tag{27}$$

called *γ-average reward/cost optimality equation in multiplicative form.*

For what follows it is convenient to consider (26), (27) in matrix form. To this end we introduce (cf. [5]) $N \times N$ matrix $Q(f) = [q_{ij}(f_i)]$ with spectral radius (Perron eigenvalue) $\rho(f)$ along with its right Perron eigenvector $v(f) = [v_i(f)]$, and right Perron eigenvectors $v(f^*) = v^* = [v_i^*]$, $v(\hat{f}) = \hat{v} = [\hat{v}_i]$. Then (26),(27) can be written in matrix form as

$$\rho(f) v(f) = Q(f) v(f), \quad \rho^* v^* = \max_{f \in \mathcal{F}} Q(f) v^*, \quad \hat{\rho}\, \hat{v} = \min_{f \in \mathcal{F}} Q(f) \hat{v}. \tag{28}$$

Recall that vectorial maximum and minimum in (28) should be considered componentwise and $\hat{v}$, v^* are unique up to multiplicative constant. Furthermore, if the transition probability matrix $P(f)$ is irreducible then also $Q(f)$ is irreducible and the right Perron eigenvector $v(f)$ can be selected strictly positive. Unfortunately, if $P(f)$ is unichain in contrast to condition $(*)$ to guarantee that $v(f)$ can be selected strictly positive it is necessary to assume existence of state

$i_0 \in \mathcal{I}$ accessible from any state $i \in \mathcal{I}$ for every $f \in \mathcal{F}$ that belongs to the *basic* class[1] of $Q(f)$. $(**)$

If condition $(**)$ is fulfilled it can be shown (cf. [15], [16]) that

(i) In (28) eigenvectors $v(f)$, $\hat{v}$, v^* can be selected strictly positive and ρ^*, resp. $\hat{\rho}$, is the maximum, resp. minimum, Perron eigenvalue of the matrix family $\{Q(f), f \in \mathcal{F}\}$.

(ii) From (3), (21), (22), (24) we immediately get for stationary policy $\pi \sim (f)$ that

$$U_i^{\pi}(\gamma, n) = \mathrm{e}^{\gamma[ng(f)+w_i(f)]} \times \mathsf{E}_i^{\pi} \mathrm{e}^{\gamma w_{X_n}(f)}, \qquad Z_i^{\pi}(\gamma, n) = \frac{1}{\gamma} \ln U_i^{\pi}(\gamma, n).$$

Similarly, for the mean value of the certainty equivalent for stationary policies $\hat{\pi} \sim (\hat{f})$, $\pi^* \sim (f^*)$, and an arbitrary policy $\pi = (f^n)$ we get

$$\lim_{n\to\infty} \frac{1}{n} Z_i^{\pi}(\gamma, n) = g^*, \quad \text{resp.} \quad \lim_{n\to\infty} \frac{1}{n} Z_i^{\pi}(\gamma, n) = \hat{g} \quad \text{if and only if}$$

$$\lim_{n\to\infty} \frac{1}{n} \ln[\mathsf{E}_i^{\pi} \mathrm{e}^{\gamma \sum_{k=0}^{n-1} \bar{\varphi}_{x_k, x_{k+1}}(w^*, g^*)}] = 0, \quad \text{resp.} \quad \lim_{n\to\infty} \frac{1}{n} \ln[\mathsf{E}_i^{\pi} \mathrm{e}^{\gamma \sum_{k=0}^{n-1} \bar{\varphi}_{x_k, x_{k+1}}(\hat{w}, \hat{g})}] = 0. \quad (29)$$

In particular, for unichain models condition $(**)$ is fulfilled if this risk sensitive coefficient γ is sufficiently close to zero (cf. [3, 4, 15]). Finding solution of (28) can be performed by policy or value iteration. Details can be found e.g. in [2, 3, 7, 14, 15, 16].

4 Risk-Sensitive Optimality in Multichain Markov Processes

To begin with, recall that a (reducible) nonnegative matrix $Q(f), f \in \mathcal{F}$ has only nonnegative (not necessary positive) right Perron eigenvectors. Then (cf. [11, 14, 15, 17, 18])

i) On suitably permuting rows and corresponding columns each $Q(f), f \in \mathcal{F}$ can be written in block-triangular form such that its (possible reducible) diagonal blocks of $Q(f)$, say $Q_{ii}(f)$, are the biggest submatrices of $Q(f)$ with strictly positive right Perron eigenvectors, i.e. for $Q_{ii}(f)$, the ith diagonal block of $Q(f)$ and for the corresponding (strictly positive) right Perron eigenvector $\bar{v}_i(f)$ it holds

$$\rho_i(f)\, \bar{v}_i(f) = Q_{ii}(f)\, \bar{v}_i(f), \quad \text{where} \quad \bar{v}_i(f) > 0 \ \text{ and } \ \rho_{i-1}(f) \le \rho_i(f) \le \rho_{i+1}(f). \quad (30)$$

Considering diagonal blocks $Q_{i-1,i-1}(f)$ and $Q_{ii}(f)$ where $\rho_{i-1}(f) = \rho_i(f)$ accessibility of basic classes of $Q_{i-1,i-1}(f)$ and $Q_{ii}(f)$ is of great importance.

ii) Considering the set of nonnegative matrices $Q(f), f \in \mathcal{F}$ (i.e. is a family of nonnegative matrices fulfilling the "product property") it is possible to construct (using policy iteration algorithms) the matrix $Q(\hat{f})$ (resp. $Q(f^*)$ whose diagonal blocks are the biggest submatrices with positive right Perron eigenvectors and minimum (resp. maximum) possible spectral radii of the set $Q(f), f \in \mathcal{F}$. In particular:

There exist $f^{\odot} = \hat{f}, f^* \in \mathcal{F}$ such that the matrix

$$Q(f^{\odot}) = \begin{bmatrix} Q_{11}(f^{\odot}) & Q_{12}(f^{\odot}) & \dots & Q_{1s}(f^{\odot}) \\ 0 & Q_{22}(f^{\odot}) & \dots & Q_{2s}(f^{\odot}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & Q_{ss}(f^{\odot}) \end{bmatrix} \quad (31)$$

induces the *basic partition* of the state space $\mathcal{I}$, such that for $f^{\odot} = \hat{f}, f^* \in \mathcal{F}$

$$\mathcal{I} = \mathcal{I}_1(f^{\odot}) \cup \mathcal{I}_2(f^{\odot}) \cup \ldots \cup \mathcal{I}_s(f^{\odot}), \quad \text{where} \quad \mathcal{I}_i(f^{\odot}) \cap \mathcal{I}_j(f^{\odot}) = \emptyset \quad \text{for} \quad i \neq j,$$

and in (31) elements of the $Q_{ii}(f^{\odot})$ are labelled from $\mathcal{I}_i(f^{\odot})$.

Furthermore, on keeping the basic partition given by $Q(f^*)$ then $Q_{ji}(f^*) \equiv 0$ for all $j < i$ and

$\rho_i(f^*)\, v_i(f^*) = Q_{ii}(f^*)\, \bar{v}_i(f^*) \ge Q_{ii}(f)\, \bar{v}_i(f^*)$, where $\bar{v}_i(f^*) > 0$ and $\rho_{i-1}(f^*) \le \rho_i(f^*) \le \rho_{i+1}(f^*)_i$ with $\rho_i(f^*) = \rho_{i+1}(f^*)$ if and only if each basic class of $Q_{ii}(f^*)$ has access to some basic class of $Q_{i+1,i+1}(f^*)$. On considering submatrices $Q_{ii}(f)$ with elements from $\mathcal{I}_i(f^{\odot})$ we can apply results of Section 3 to $Q_{ii}(f)$.

[1] (i.e. irreducible class with spectral radius equal to the Perron eigenvalue of $Q(f)$)

5 Conclusions

In this note necessary and sufficient optimality conditions for discrete time Markov decision chains are obtained along with equations for average optimal policies both for risk-neutral and risk-sensitive models. Our analysis is mostly restricted to unichain models, and for the risk-sensitive case some additional assumptions are made. If no such assumptions are made, it is indicated how to handle this problem by partition of the state space into suitable classes that inherit from properties of unichain models. Some further results in this direction can be found in [11, 14, 15, 17, 18].

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Analysis of Structural Differences and Asymmetry of Shocks using Posterior Distributions

Martin Slanicay[1]

Abstract. The goal of this paper is to examine asymmetry of shocks and structural differences between the Czech economy and the euro area. For this purpose I use New Keynesian DSGE model of two economies which I estimate on the data of the Czech economy and the euro area, using Bayesian techniques. The applied method examines differences in values of the parameters by comparison of their posterior distributions. Results suggest that the main differences are in volatility, persistence, and timing of macroeconomic shocks. There are also several structural differences. It seems that the ECB smooths less the interest rate and reacts more to the development in output and inflation than the CNB. It also seems that prices in the Czech economy are more sticky than prices in the euro area, especially in the non-tradable sector. Results also suggest that domestic households are less willing to smooth consumption over the time then their foreign counterparts.

Keywords: DSGE model, Bayesian estimation, structural differences, asymmetry of shocks, posterior distributions.

JEL classification: C51, C68, E32
AMS classification: 91B51

1 Motivation

Asymmetric shocks and structural differences are regarded to be the main causes of a potential suboptimality of common monetary policy. Asymmetry of shocks is understood as differences in timing, magnitude or persistence of macroeconomic shocks among economies. Structural differences are, on the other hand, perceived as differences in propagation mechanisms of macroeconomic shocks among economies.

The reason why asymmetric shocks and structural differences can cause suboptimality of common monetary policy is as follows. Business cycle fluctuations of the main macroeconomic variables are driven by macroeconomic shocks. Asymmetric shocks and/or structural differences between economies cause differences in development of their macroeconomic variables over the business cycle. Optimal monetary policy should react to the business cycle fluctuations of the macroeconomic variables (driven by shocks) in a way that minimizes existing distortions caused by these shocks. In the case of asymmetric shocks and/or structural differences in a monetary union, these distortions differ across countries, and applied common monetary policy is likely to be suboptimal for some countries. Therefore, an analysis of asymmetric shocks and structural differences plays an important role in evaluating costs and benefits of common currency.

Much economic research is devoted to the issues of asymmetric shocks and structural differences between economies because of their important role in evaluating costs and benefits of common currency. Using model comparison based on Bayes factor, several papers examine presence and relative importance of different sources of heterogeneity among economies. Jondeau and Sahuc [4] distinguish three main sources of heterogeneity: (i) structural heterogeneity which corresponds to differences in preferences, technology, etc.; (ii) policy heterogeneity which corresponds to differences in the conduct of economic policy; and (iii) stochastic heterogeneity which corresponds to differences in shocks hitting respective economies.[1] Pytlarczyk [7] estimates a two-country DSGE model of Germany and the rest of the euro area.

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[1]Note, that both structural and policy heterogeneity influence propagation mechanisms of macroeconomic shocks among

He finds that the model with structural heterogeneity between countries fits the data much better than the model with imposed structural homogeneity. Jondeau and Sahuc [4] examine sources of heterogeneity within the euro area and come to the conclusion that asymmetric shocks are the main sources of a different behavior of countries in the euro area, while structural differences play almost no role. Similar results are provided by Kolasa [5] who investigates sources of heterogeneity between Poland and the euro area, and finds out that volatility and synchronization of shocks hitting both economies are the main sources of the heterogeneity between Poland and the euro area. Similarly, Slanicay [8] examines sources of heterogeneity between the Czech economy and the euro area. I do not find substantial evidence in favor of heterogeneity in household preferences. I find slight differences in price and wage formation and substantial difference in interest rate smoothing. However, the main differences are in timing, persistence and volatility of structural shocks. Herber and Němec [3] provide similar results. They find out that price and wage rigidities and the asymmetry of shocks are the main sources of heterogeneity between the Czech economy and the euro area. On the other hand, they find a strong evidence in favour of homogeneity in parameters describing preferences of households.

The goal of this paper is to examine asymmetry of shocks and structural differences between the Czech economy and the euro area. For this purpose I use New Keynesian DSGE model of two economies which I estimate on the data of the Czech economy and the euro area, using Bayesian techniques. The applied method examines differences in values of the parameters by comparison of their posterior distributions.

2 Model

I use a New Keynesian DSGE model of two economies, originally presented in Kolasa [5].[2] In this section I restrict my description of this model to a brief non-technical overview of its structure.

The model assumes that there are only two economies in the world: a domestic economy (represented by the Czech economy) and a foreign economy (represented by the euro area). The problematic fact that one economy is much smaller than the other is solved by parameter , which governs the relative size of the two economies.

The model assumes five types of representative agents in each economy. Households consume tradable and non-tradable goods produced by firms. There is an assumption of habit formation in consumption and an assumption that consumption of a final tradable good requires consumption of ω units of non-tradable distribution services. Households trade bonds, too, and their intertemporal choice about consumption is influenced by preference shocks. Households supply labor and set wages on a monopolistically competitive labor market. Their labor supply is influenced by labor supply shocks and their wage-setting is subject to a set of labor demand constraints and to the Calvo constraint on the frequency of wage adjustment, see Calvo [1]. According to the Calvo constraint, in every period each household resets its wage with probability $1-\theta_w$ and keeps its wage unchanged with probability θ_w. Households also accumulate capital, which they rent to firms. Capital accumulation is subject to investment-specific technological shocks and to adjustment costs.

There are two types of firms in each economy: producers of tradable goods and producers of non-tradable goods. Both of them employ a Cobb-Douglas production function with constant returns to scale. Productivity in both sectors is influenced by productivity shocks. Firms hire labor on the labor market and sell their goods on monopolistically competitive goods markets. They set prices on the goods market subject to a set of demand constraints and to the Calvo constraint on the frequency of price adjustment, see Calvo [1]. According to the Calvo constraint, in every period each firm resets its price with probability $1-\theta_p$ and keeps its price unchanged with probability θ_p.

The fiscal authority collects lump-sum taxes, which it uses for government expenditures and transfers to households so that the state budget is balanced each period. Government expenditures consist only of domestic non-tradable goods and are modeled as a stochastic process - a government expenditures shock. Given our assumptions about households, Ricardian equivalence holds in this model. The monetary authority follows a Taylor–type rule, and deviations from this rule are explained as monetary shocks.

economies and thus can be included in the term "structural differences" while stochastic heterogeneity is an alternative label for asymmetric shocks.

[2] I depart from the original specification of the model in several aspects, however, these modifications are generally minor. While I used the model in my previous paper, see Slanicay [9], derivation of the modified version of the model from microfoundations, as well as its log-linear form can be found in http://journal.fsv.cuni.cz/storage/1272_slanicay_upr1_-_nově_comments_-_appendix_-_upr_pro_web.pdf.

The model is closed with an assumption of a complete bond market and with an assumption of goods and labor markets clearing.

The behavior of the model is driven by seven structural shocks in both economies: a productivity shock in the tradable sector and the non-tradable sector, a labor supply shock, an investment efficiency shock, a consumption preference shock, a government spending shock, and a monetary policy shock. Except for the monetary policy shock, which is modeled as an IID process, all shocks are represented by an AR1 process. I allow for correlations between innovations of corresponding shocks in both economies.

3 Estimation

For the estimation of the model I used the quarterly data of the Czech economy and the euro area-12 economy from the 1st quarter of 2000 to the 3rd quarter of 2011. The employed data series were downloaded from the web database of Eurostat. I use following 14 time series (seven for each economy): real GDP, consumption, investment, HICP, real wage, short-term interest rate and internal exchange rate defined as prices of non-tradable goods relative to prices of tradable goods. Except for the nominal interest rates, all observed variables are seasonally adjusted and expressed as demeaned 100*log differences. Nominal interest rates are demeaned and expressed as quarterly rates per cent.

The model is estimated with Random Walk Chain Metropolis-Hastings algorithm, using Dynare toolbox for Matlab. I generated four independent chains, each with 2,000,000 draws. From each chain I used only 25% percents of last draws, i. e. 1,500,000 initial draws from each chain were discarded. Average acceptance rate in each chain is about 29%, which is in line with the informal recommendation about ideal acceptance rate, see for example Koop [6]. According to the MCMC convergence diagnostics, smoothed shocks and variables, and the prior and posterior distributions of the parameters, the model was estimated successfully.[3]

4 Analysis based on Posterior Distributions

Asymmetry of shocks and structural differences manifest itself as significant differences in values of some parameters. The estimates of the parameters are given by their estimated posterior distributions. Therefore, it seems intuitive to compare posterior distributions of corresponding parameters in both economies and evaluate how much they overlap. Note that this method can be applied only on the structural parameters and on the parameters governing the volatility and persistence of macroeconomic shocks. It can not be applied on the correlations of macroeconomic shocks. The correlations between shocks are therefore analysed differently via comparing their point estimates given by the estimated posterior means. Because of using identical priors for both economies, it is correct to construct a natural measure of the overlap of corresponding parameters such as the intersection of the posterior distributions of the domestic and foreign parameter.[4] The obtained number must lie between zero and one, where zero represents absolute asymmetry in the parameter values while unity represents absolute symmetry in the parameter values.

4.1 Structural Parameters

Table 4.1 displays calculated intersections of posterior distributions, ordered from the lowest to the highest. As regards the structural parameters, the biggest structural difference is in the way how central banks react to the development of output (ψ_y and ψ_y^*, overlap 0.11). It seems that unlike the CNB, the ECB reacts more to the development of output. Point estimates[5] imply that one percent deviation of output from the steady state (trend) brings about change in the interest rate of 0.012% in the Czech economy, and 0.032% in the euro area.

[3] For the sake of space I do not present these things in the paper. For the same reasons I also do not provide detailed justification for the calibration of several structural parameters and for the prior setting of the estimated parameters, as well as interpretation of the parameters estimates. However, all these things are available on the request from the author.

[4] One would imagine a lot of different measures of the overlap of two posterior distributions. For detailed discussion of this topis see Čapek [2]

[5] Point estimates are represented by the estimated posterior mean.

The second biggest difference within the structural parameters is in the degree of price stickiness in the non-tradable sector (θ_N and θ_N^*, overlap 0.26), such that domestic prices of the non-tradable goods are more sticky than foreign prices of the non-tradable goods. Point estimates of the degree of price stickiness in the non-tradable sector imply that the average duration of domestic prices of the non-tradable goods is 12.24 months, while the average duration of foreign prices of the non-tradable goods is 9.35 months. Both results are in line with the empirical evidence on price stickiness surveyed in the section **??**.

The third biggest structural difference is in the inverse elasticity of intertemporal substitution (σ and σ^*, overlap 0.43). It implies that the domestic households are less willing to smooth consumption over the time then their foreign counterparts.

parameter	intersection
structural parameters	
ψ_y, ψ_y^* - weight on output in MP rule	0.11
θ_N, θ_N^* - price stickiness in non-tradables	0.26
σ, σ^* - inverse elasticity of intertemporal subs.	0.43
ρ, ρ^* - interest rate smoothing	0.48
ψ_π, ψ_π^* - weight on inflation in MP rule	0.48
ϕ, ϕ^* - inverse elasticity of labor supply	0.64
θ_W, θ_W^* - wage stickiness	0.71
$S^{''}, S^{''*}$ - adjustment costs	0.75
h, h^* - habit formation in consumption	0.85
θ_H, θ_F^* - price stickiness in tradables	0.87
parameters related to shocks	
σ_g, σ_g* - volatility of shocks in government expenditures	0.001
$\rho_{a^H}, \rho_{a^F}^*$ - persistence of productivity shocks in tradables	0.04
$\sigma_{a^N}, \sigma_{a^N}^*$ - volatility of productivity shocks in non-tradables	0.09
$\rho_{a^N}, \rho_{a^N}^*$ - persistence of productivity shocks in non-tradables	0.18
ρ_g, ρ_g* - persistence of shocks in government expenditures	0.27
ρ_d, ρ_d^* - persistence of preference shocks	0.37
$\sigma_{a^H}, \sigma_{a^F}^*$ - volatility of productivity shocks in tradables	0.65
ρ_i, ρ_i^* - persistence of shocks in investment efficiency	0.76
σ_m, σ_m^* - volatility of monetary shocks	0.77
σ_l, σ_l^* - volatility of labor supply shocks	0.83
σ_i, σ_i^* - volatility of shocks in investment efficiency	0.95
σ_d, σ_d^* - volatility of preference shocks	0.96
ρ_l, ρ_l^* - persistence of labor supply shocks	0.98

Table 1 Overlap of Posterior Distributions

Other structural differences are related to the parameters in the monetary policy rule, namely the degree of interest rate smoothing (ρ and ρ^*, overlap 0.48) and the way how central banks react to the development of inflation (ψ_π and ψ_π^*, overlap 0.48). It seems that the ECB smooths its interest rate less than the CNB does and that the ECB reacts more to the development of inflation than the CNB does. Point estimates imply that one percent deviation of inflation from the steady state (trend) induce a change in the interest rate of 0.12% in the Czech economy and 0.17% in the euro area.

The overlap of remaining parameters is higher than 0.6, which suggests that differences in these parameters are less significant. However, there are still some interesting differences which are worth interpreting. It seems that labor supply in the Czech economy is more elastic than labor supply in the euro area. Point estimates imply that the 1% increase of the real wage induces 1.93% increase of the labor supplied in the Czech economy, while in the euro area the 1% increase of the real wage induces only 1.1% increase of the labor supplied. Results also suggest that wages in the Czech economy are less sticky than in the euro area. Point estimates imply that the average duration of wage is 12.1 months in the

Czech economy, and 13.6 months in the euro area. It seems that there is almost no difference regarding price stickiness in the tradable sector. Point estimates imply that the average duration of prices in the tradable sector is 10.5 in the Czech economy months, and 10 months in the euro area.

4.2 Parameters Related to Shocks

Results suggest that the main differences regarding volatility and persistence of shocks are in volatility and persistence of shocks in government expenditures (σ_g and σ_g*, overlap 0.001, ρ_g, and ρ_g*, overlap 0.27), volatility and persistence of productivity shocks in the non-tradable sector (σ_{a^N} and $\sigma^*_{a^N}$, overlap 0.09, ρ_{a^N} and $\rho^*_{a^N}$, overlap 0.18), persistence of productivity shocks in the tradable sector (ρ_{a^H} and $\rho^*_{a^F}$, overlap 0.04), and persistence of preference shocks (ρ_d and ρ^*_d, overlap 0.37). Except for monetary policy shocks, the shocks in the Czech economy are more volatile than the same kind of shocks in the euro area. Volatility of monetary policy shocks is a bit larger in the Euro Area than in the Czech economy, nevertheless, the difference in the estimated posterior means is almost negligible. Except for productivity shocks in non-tradables, the shocks in the Czech economy are more persistent than the same kind of shocks in the euro area.

As regards the correlations between corresponding shocks in both economies, for obvious reasons it can not be analysed via the method based on overlap of posterior distributions. Therefore, I decided to analyse these correlations via their point estimates given by the estimated posterior means. Table 4.2 displays correlations between corresponding innovations in the shocks and, except for the monetary policy shocks which are modeled as IID processes, correlations between the corresponding whole shocks represented by AR1 processes. We can see that the results for the innovations and for the whole shocks differ dramatically. For example, the innovations of the consumption preference shocks and the innovations of the shocks in government expenditures are much more correlated than the innovations of the investment efficiency shocks. However, in case of the whole shocks (represented by AR1 processes) the consumption preference shocks and the shocks in government expenditures are much less correlated than the investment efficiency shocks. Another example, the innovations of productivity shocks in the tradable sector are negatively correlated (cor=-0.26) while the whole productivity shocks in the tradable sector are positively correlated (cor=0.18).

	correlation	
shock	innovation	AR1
productivity shocks in tradables	-0.26**	0.18
productivity shocks in non-tradables	0.27**	0.59***
consumption preference shocks	0.45***	0.67***
labor supply shocks	-0.17	0.03
shocks in government expenditures	0.35***	0.20*
shocks in investment efficiency	0.33**	0.81***
monetary policy shocks	0.57***	

Table 2 Estimated Correlations between Shocks
Note: */**/*** - significance level 0.1/0.05/0.01

If we are interested in the asymmetry of shocks actually hitting the economies, then we should pay more attention to the correlations between the corresponding whole shocks, presented in the last column of the Table 4.2.[6] We can see that the most correlated shocks are (ordered from the highest correlation to the lowest) investment efficiency shocks (cor=0.81), consumption preference shocks (cor=0.67), productivity shocks in the non-tradable sector (cor=0.59), and monetary policy shocks (cor=0.57). Correlations of these shocks are quite high, all of them are statistically significant on the significance level $\alpha = 0.01$. We can say that these shocks are very symmetric between the Czech economy and the euro area. On the other hand, the shocks with the lowest correlations are (ordered from the lowest correlation to the highest) labor supply shocks (cor=0.03), productivity shocks in the tradable sector (cor=0.18), and shocks in

[6] In Kolasa [5], which is the reference paper for the model employed in this paper, the asymmetry of shocks is analysed using correlations between corresponding innovations in the shocks. From my point of view, these results might be misleading because these correlations are only between innovations and not between shocks actually hitting the economies. I view the results based on the correlations between the whole shocks (in most cases represented by AR1 process) as more meaningful.

government expenditures (cor=0.20). These shocks can be regarded as very asymmetric between the Czech economy and the euro area.

5 Conclusion

In this paper I examined asymmetry of shocks and structural differences between the Czech economy and the euro area. For this purpose I used New Keynesian DSGE model of two economies, originally presented in Kolasa [5]. The model is estimated on the data of the Czech economy and the euro area, using Bayesian techniques. The applied method examines differences in values of the parameters by comparison of their posterior distributions.

Results suggest that the main differences are in volatility, persistence, and timing of macroeconomic shocks. As regards volatility of shocks, the main differences are in volatility of shocks in government expenditures and productivity shocks in non-tradables, with domestic shocks being more volatile than the foreign shocks. As regards persistence of shocks, the main differences are in persistence of productivity shocks in tradables and non-tradables, shocks in government expenditures, and consumption preference shocks. Except for productivity shocks in non-tradables, domestic shocks are more persistent than foreign shocks. As regards timing of shocks, the main differences are in timing of labor supply shocks, productivity shocks in tradables and shocks in government expenditures.

I also found several structural differences which are worthy to mention. It seems that the ECB smooths less the interest rate and reacts more to the development in output and inflation than the CNB. It also seems that prices in the Czech economy are more sticky than prices in the euro area, especially in the non-tradable sector. Results also suggest that domestic households are less willing to smooth consumption over the time then their foreign counterparts.

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Asymmetric Price Transmission within the Atlantic Steam Coal Market

Sławomir Śmiech[1]

Abstract. The purpose of this paper is to investigate long and short term relationships between prices on the Atlantic steam coal market. The analysis considers coal prices set by the largest exporter (fob RB) and importer (cif ARA ports) on the Atlantic market. A diverse character of ARA and RB prices and their subsequent different behaviour results from several reasons. A presence of financial institutions on the Euromarket is the first reason. Therefore, ARA prices are strongly connected with the financial markets. Next, the specificity of RB prices results from the fact that coal from the Republic of South Africa is sent both to the Atlantic and to the Pacific regions. Therefore, RB prices depend directly on the situation on the market in the Atlantic and the Pacific Ocean. The data used in the analysis consisted of weekly prices from the period from October 2006 to February 2013. The analysis was based on two models: two-regime threshold cointegration approach (TAR and M-TAR) proposed by Enders and Siklos [7] and asymmetric error correction model with threshold cointegration proposed by Sun [14]. The results obtained revealed that, in the long run, there is asymmetry in price transmission. Prices come back to long run equilibrium sooner after the decrease then after the increase.

Keywords: international coal market, asymmetric cointegration, price transmission

JEL Classification: C22, F49, Q37
AMS Classification: 62M10, 91B52, 91B76

1 Introduction

According to International Energy Outlook [9], global energy production will increase by 49% by the year 2035. Coal (together with crude oil and natural gas) is expected to remain the main source of energy with 32% share in 2035. Because of the share of transport costs in the price of coal (coal takes up a lot of space and is heavy), the global market of coal is divided into two zones: the Atlantic and the Pacific Ocean regions. On the Atlantic market Northwest Europe is the biggest importing region with the postage Amsterdam-Rotterdam-Antverp (cif ARA), whereas South Africa remains the largest exporter with the Richards Bay postage (fob RB). Japan and South Korea are the biggest importing countries on the Pacific Ocean market, while Australia and Indonesia export the highest amounts of coal.

Over the last few years a geographical structure of the demand for coal has changed. In Europe, on account of the limits of the CO2 emission, participation of coal in energy production is decreasing. The situation is different in developing countries, especially in China, India and Brazil, where coal consumption is steadily growing. On the other hand, the Euromarket is a region where the volume of coal trade increases. The participants of this trade include various financial institutions, banks and funds, which make the turnover several times higher than the turnover of physical goods. In 2002 the trading volume of world steam coal amounted to 463.7 Mt, and the trading volume of steam coal derivatives amounted to 190 Mt. In 2010 world steam coal trade increased to 777.7 Mt, and the volume of paper trade increased to about 3000 Mt, that is about 4.4 times of total physical steam coal trade. Most paper trade is based on the supplies from the Atlantic region.

The division of the international coal trade poses several questions connected with economic functioning of the international market. The first one refers to the integration of the market and the hypothesis of one price (accurate within fright prices). Another refers to the price adjustment mechanisms and the roles of the participants of the international coal market that is the price setters, and the links between different regions.

Research carried out so far is not conclusive. Ellermann[4], analysing the data from the period 1970 to 1990, argued that there was a unified global coal market. The USA had a special dominating role as a prices setter. Warell [18] used cointegration analysis for European and Japanese prices for different periods and showed that

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the global market was more regional in scope than before. Australia was a region linking the marketplace of the Pacific and the Atlantic Oceans. Ekawan et al. [3] pointed at the integration of international coal trade at the beginning of the 21st century. Zaklan et al. [19] conduced a detailed analysis which took into account the links between importers and exporters. They showed that price adjustment differed on different routes; the system achieved equilibrium sooner on some routes than on others.

Assuming the existence of one market, Śmiech et al.[16] analysed areas most influential in shaping coal prices. The results obtained from weekly data from the first decade of the 21st century revealed that price setters were located on the Pacific market. Using Cheung and Ng and Hong tests, Papież and Śmiech [11] analysed the Granger causality-in-mean and causality-in-variance. The results for weekly return rates indicated that the Atlantic market, especially ARA prices, Granger caused other prices.

Li et al. [10] analysed the convergence price for coal exporters and concluded that the international steam coal market was on the whole integrated. Their methodology was based on Kalman filter techniques which complemented cointegration analysis. Other important area of research of prices of coal and primary fuels includes analysis of connections of prices with financial markets (Śmiech and Papież [15][17], Papież and Śmiech[12]).

The main purpose of this paper is the analysis of the relations between prices of the main importer and exporter on the Atlantic market that is cif ARA and fob RB prices. The methodology used should provide the answers to two questions. Firstly, whether ARA and RB coal prices are in a long-term equilibrium. Secondly, whether adjustments of prices are symmetrical or whether the hypothesis of asymmetric price transmission is confirmed. The analysis conducted allowed to assess mutual price relationships on the Atlantic market more precisely than before.

The article consists of 5 parts. Apart from the introduction, it contains the introduction of threshold models tar and mtar, as well as of cointegration and asymmetric price transmission tests, which will be used in the analysis. It also presents the characteristics of time series, the results of the application of the models and final conclusions.

2 Methodology

Long run equilibrium is usually assessed by two linear methods. The first one is the two step Engle-Granger procedure, and the other one is the Johansen approach. In both cases the adjustment process to maintain the equilibrium is symmetric and does not depend on the direction of the deviation from long run equilibrium. Balke and Fomby [1] proposed a two-step approach for threshold cointegration, which was a generalization of Engle-Granger procedure. Engle and Granger [6] and Enders and Siklos [7] developed the standard Dickey-Fuller test to assess asymmetric movements in time-series data. This approach was used many times especially for the analysis of prices of agricultural products (Ghoshray[8]), stock market indexes (Shen et al. [13]), and the prices of wooden bed market in the United States (Sun [14]). Engel-Granger two-stage procedures were applied in order to confirm whether pairs of variables are asymmetrically cointegrated. Let us follow two-stages Enders and Siklos [7] procedures[2]. During the first stage, for two I(1) time series we estimate the long-run equilibrium as:

$$y_t = c_0 + c_1 x_t + \varepsilon_t \tag{1}$$

where c_0, c_1 are parameters and ε_t is a disturbance term. If variables y_t, x_t are cointegreted, then ε_t should be stationary. In the second stage we estimate two-regime threshold cointegration approach:

$$\Delta\hat{\varepsilon}_t = \rho_1 I_t \hat{\varepsilon}_{t-1} + \rho_2 (1 - I_t)\hat{\varepsilon}_{t-1} + \sum_{i=1}^{p} \varphi_i \Delta\hat{\varepsilon}_{t-i} + \mu_t \tag{2}$$

$$I_t = 1 \text{ if } \hat{\varepsilon}_{t-1} \geq \tau \text{, 0 otherwise or} \tag{2a}$$

$$I_t = 1 \text{ if } \Delta\hat{\varepsilon}_{t-1} \geq \tau \text{, 0 otherwise or} \tag{2b}$$

where I_t is the Heaviside indicator, P the number of lags, ρ_1, ρ_2 and φ_i model coefficients, τ the threshold value. The number of lags is specified using AIC or BIC, but should be large enough to avoid serially correlated

[2]Frey and Manera[5] present a large scope of econometric models of asymmetric price transmission.

residuals. The Heaviside indicator specified with definition (a). Equation (2) and (2b) have been named as the Threshold Autoregression model (TAR). Eq. 2 and 2b give so called the Momentum Threshold Autororegression model (MTAR).

The variables are nonlinear cointegrated, if the hypothesis of no cointegretion and the hypothesis of symmetric adjustment are rejected at the same time. Test statistics for the null hypothesis of no cointegration has a nonstandard distribution (Enders and Siklos[7]).

The TAR model is useful to capture asymmetric deep movements in the residuals. MTAR should be used when the adjustment is believed to exhibit more momentum in one direction than the other (Sun[14]). The threshold value should be specified (usually as zero) or estimated (consistent estimate was proposed by Chan [2]. There is no presumption on which specification should be used, thus in this study selection criteria methods AIC and BIC help to decide on the proper one. If, as a result of estimation, the parameters obtained for the residuals in model (2) fulfil the condition $|\rho 1| \leq |\rho 2|$), it means that increases tend to revert towards equilibrium much more slowly than decreases.

3 Data

The analysis of cointegration and asymmetric price transmission was conducted using weekly data from the period January 2006 – February 2013. The analysis covered the period of activity of investment funds on the European steam coal market. In Figure 1 the logarithms of ARA and RB prices are compared. They exhibit very similar dynamics. In the initial period RB prices were lower than ARA prices. Descriptive statistics compared in Table 1 confirm it: the mean, median, and quartiles are lower for RB. However, standard deviation is higher for RB prices. Since the end of 2008 there were periods in which RB prices were higher than ARA prices. The prices reached the maximum point in July 2008, and then dropped rapidly. Increases appeared at the beginning of 2009 and lasted till the middle of 2011. Since then a gradual decrease of prices has been observed.

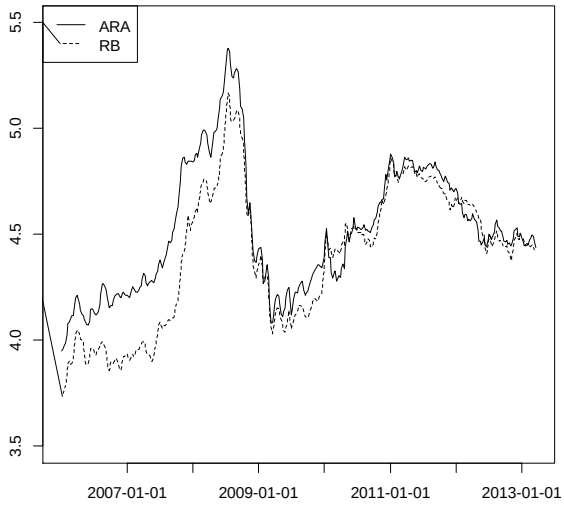

Figure 1 Logarithms of steam coal prices from the period January 2006 – February 2013

	ARA	**RB**
Min	3.951	3.720
1st Qu.	4.267	4.092
Median	4.482	4.448
Mean	4.519	4.395
3rd Qu.	4.775	4.658
Max	5.381	5.170
sd	0.306	3.332

Table 1 Summary statistics for logarithms of ARA and RB weekly prices

Table 2 presents the results of augmented Dickey-Fuller (ADF) and Philips-Perron (PP) unit root tests. The number of lags in the tests was established using AIC criterion. The tests were conducted for models with a constant and trend specification. The values of test statistics are similar and show that both series are 1 order integrated. The condition necessary to examine cointegration between prices is met.

	ARA	**RB**	**5%**	**Diff ARA**	**Diff RB**	**5%**
ADF	-1.732	-1.616	-3.41	-9.134	-9.094	-2.86
PP	-4.099	-4.571	-18.1	- 10.857	-11.272	-3.02

Table 2 ADF and PP tests of unit roots in weekly data (January 2006 – February 2011)

4 Results

Within the analysis four types of threshold models were built. For two of them, c.tar and c.mtar the value of threshold was estimated. At the same time, the number of lag lengths of the models were estimated (AIC and BIC criterion were established and the analysis of SSE residuals was carried out). In all cases the value of lags equal 3 was obtained (it should be mentioned that after shortening or lengthening the windows, a different number of lags and different values of thresholds were obtained).

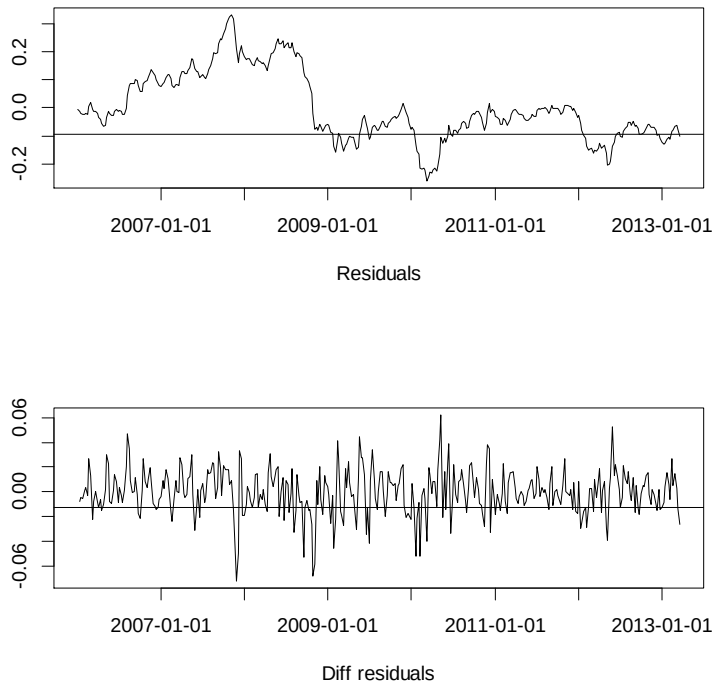

Figure 2 Thresholds values for c.tar and c.mtar models

As a result of searching for the optimal value of threshold, the following results were obtained (Table 3) for residuals the threshold was -0.094, and for differences (diff) of residuals the threshold was -0.013. In the first case, 302 observations out of 377 were above the threshold (in the first regime), while in the second case 299 observations were above the threshold. Both approaches (see Figure.2) indicate the changes of regimes in different periods. In c.tar model in the second regime (below the threshold) there are no observations from before 2009 and between the middle of 2010 and the middle of 2012, while in c.mtar model observations in the second regime can be found in every consecutive year.

	tar	**c.tar**	**mtar**	**c.mtar**
lag	3	3	3	3
thresh	0	-0.094	0	-0.013
ρ_1	-0.008	-0.006	-0.009	-0.003
ρ_2	-0.021	-0.031	-0.018	-0.047
total obs	377	377	377	377
sse	0.105	0.105	0.106	0.104
aic	-1977.16	-1979	-1976.94	-1983.46
bic	-1953.63	-1955.47	-1953.41	-1959.93
H1: no CI	1.94	2.859	1.829	5.107
H1: p.value	0.145	0.059	0.162	0.006
H2: no APT	0.679	2.501	0.458	6.958
H2: p.value	0.411	0.115	0.499	0.009
LB test(4)	0.945	0.945	0.937	0.814
LB test(8)	0.753	0.743	0.762	0.676
LB test(12)	0.74	0.731	0.747	0.736

Table 3 The results obtained for threshold models

It should be noted that the results of cointegration tests and APT (asymmetric price transmission) depend on the fact whether the model used an arbitrarily chosen threshold or whether it was estimated. In the latter case, that is for c.tar and c.mtar models, the tests indicate the possibility of asymmetric price transmission. For c.mtar model tests results are significant for standard test levels (*p*-value equals 0.006 and 0.009), and for c.tar model p-value slightly exceeds 5% of confidence level for the cointegration test, and 10% for the APT test. Different results were obtained for tar and mtar models with threshold equals 0. In both cases there is no evidence allowing to reject the hypothesis of the lack of cointegration between prices.

Taking into consideration the values of information criteria AIC and BIC, c.mtar and c.tar models proved the best.

Conditional models reveal a better adjustment than models with a default value of threshold. Ljung-Box test for 4, 8 and 12 lags indicate that residual of all models are not autocorrelated.

No matter which model will be finally chosen, in each of them the absolute value of parameter is smaller than value. This means that independent of model specification, there are differences in the rate of return to equilibrium. Faster return can be observed for all negative deviations (below threshold or below zero), which means that equilibrium is obtained sooner after a drop in prices than in case of their growth.

5 Conclusion

The results of research presented in this paper are not conclusive. Cointegration and asymmetric prices transmission depend on the type of the econometric model used. Conditional models, such as, c.tar and c.mtar (models with estimated threshold) reject the hypothesis of no cointegration and no asymmetric price transmission. These models were preferred taking into account the information criteria. The opposite results were obtained with the use of the remaining models. Regardless of the type of model used, the return to the long run equilibrium (assuming that equilibrium exists) had a different rate in the positive and negative regime. A sooner return to the balance took place after declines in prices than after their rise.

Acknowledgements

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Simulation of Socio-Economic Processes on the Municipal Level

Martin Sobotka[1], Milan Půček[2], Lucie Sobotková[3], Radka Kněžáčková[4]

The monograph elaborates on strategic management in selected priorities of municipal development in which system dynamics is being applied. The method enables quantitative capturing of relations between individual development actors, subjects or processes themselves that may occur during development of a municipality. It is then possible to explore causalities between individual elements of the whole municipal system. The main benefit of the system dynamics, however, is not only capturing of these simple causalities, but also understanding the feedbacks between the system elements. This means that one single element through its behaviour affects another one within the same system. But response of this second element has a loop influence on the initial one. System dynamics introduces a dynamic view, in this case applied on the topic of municipal development. The approach is applicable also in modern management systems. In that case it would be an application of the Balanced Scorecard method which is based on system dynamics principles described above. In addition this method recommends capturing the development through both financial and also non-financial indicators. This generates further possibilities of application of quantitative approach of system dynamics to processes of rather a qualitative character.

Keywords: System dynamics, Balanced Scorecard method, municipal development.

JEL Classification: R11, R58
AMS Classification: 68U20

1 Introduction

Management of any organization needs an approach that creates the order in its organization and determines the direction in which the organization is going to follow directions through the applicable instruments for reaching the desired objectives. This thesis should be valid for both of organizations – private and public sector as well. This fact appears to be needed to have a strategy and its corresponding methods for management.

One of the methods that allow incorporate the strategic direction into the life of the organization is the method of Balanced Scorecard (BSC). This is currently a very popular method which diffuses into organizations of the private sector and to institutions of the public sphere. BSC method can be used at municipal level, where it has the potential to promote the development of the municipality when it is used in its proper and consistent application. Its basic characteristic is the use of even non-financial characteristics to support governance and promoting development. The further rated aspect is the involvement of system dynamics into this method. Benefits of system dynamics is the overcoming of the simplified causality.

Practical use of dynamic BSC method that is applicable at the municipal level will be presented in this article on the example of Vsetín municipality. Vsetín is the county seat with around 28,000 inhabitants, which simultaneously performs delegated state administration role for 32 surrounding villages (about 68,000 inhabitants). The city began to develop a strategy and methodology to apply the BSC in 2004, as one of the first municipalities in the country. The strategic map of Vsetín and a set of scales were published by Půček and Ochrana [7]. Other towns in the Czech Republic which apply this method are Děčín and Uherské Hradiště.

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2 Theoretical outlines

2.1 Approach of system dynamics

The discipline of system dynamics was created in the late fifties the 20th century by J.W. Forrenster from Sloan School of Management at the Massachusetts Institute of Technology. It was called "industrial dynamics" at the time. And this method dealt with the construction of a mathematical model of a dynamic system. This discipline has been widened since its inception.

System dynamics is a methodology used to understand how systems change over time. The way in which the elements or variables composing a system vary over time is referred to as the behavior of the system. System dynamics can also be used to analyze how structural changes in one part of a system might affect the behavior of the system as a whole. Perturbing a system allows one to test how the system will respond under varying sets of conditions (see e.g. Martin [6]).

According to Sterman [9] the system dynamics is a method to enhance learning in complex systems. But learning about complex dynamic systems requires more than technical tools to create mathematical models. System dynamics is fundamentally interdisciplinary.

The access of the system dynamics allows us to create models. The models can be used to solve a lot of problems, not only simple, but very complex and extensive problems too. To better understand the system structures we introduce a notation for representing system structures (cf. e.g. Kirkwood [5]). System dynamics uses graphical notations to display the mutual interrelations among the examined elements. We use feedbacks.

As noted by Sterman [9] feedback is one of the core concepts of system dynamics. We can define feedback like a closed sequence of causes and effects, that is, a closed path of action and information (cf. e.g. Kirkwood [5]).

We use several diagramming tools to capture the structure of systems, including causal loop diagrams and stock and flow diagrams. Causal loop diagrams are an important tool for representing the feedback structure of systems.

The casual loop diagram shows the links among elements of the system. It uses arrows (called causal links). These causal links are complemented by signs "+" to express a positive dependence (reinforcing manner) and "-" to express a negative dependence (balancing manner). These signs have the following meanings (cf. e.g. Kirkwood [5], Sterman [9]):

- a causal link from one element A to another element B is positive if either A adds to B or a change in A produces a change in B in the same direction,
- a causal link from one element A to another element B is negative if either A subtracts from B or a change in A produces a change in B in the opposite direction.

Mathematical expression for positive dependence:

$$\frac{\partial Y}{\partial X} > 0, \tag{1}$$

In the case of accumulations:

$$Y_t = \int_{t_0}^{t} (X + \cdots) ds + Y_{t_0}. \tag{2}$$

Mathematical expression for negative dependence:

$$\frac{\partial Y}{\partial X} < 0, \tag{3}$$

In the case of accumulations:

$$Y_t = \int_{t_0}^{-t} (-X + \cdots) ds + Y_{t_0}. \tag{4}$$

The stock and flow diagram is the second way for expressing the system and links among elements of the system. This method helps to overcome some limitations of the causal loop diagram. One of the most important limitations of causal diagrams is their inability to capture just the stock and flow structure of systems.

Stocks characterize the state of the system and generate the information upon which decisions and actions are based. Stocks create delays by accumulation the differences between the inflow to a process and its outflow. Stocks are the source of disequilibrium dynamics in systems. Stocks integrate their flows. The net flow into the stock is the rate of change of the stock.

For expression we can use the following integral equation:

$$\text{Stock}(t) = \int_{t_0}^{t} [\text{Inflow}(s) - \text{Outflow}(s)]ds + \text{Stock}(t_0). \quad (5)$$

Inflow represents the value of the inflow at any times between the initial time t_0 and the current time t.

This dependence can be also written by differential equation:

$$\frac{d(Stock)}{dt} = Inflow(t) - Outflow(t). \quad (6)$$

The flows will be functions of the stock and other state variables and parameters.

2.2 Methods and goals of the article

Modeling is based on a combination of positive and normative methodologies. When creating a dynamic model, it is next to impossible to disregard one aspect or the other. Positive methodology has been used particularly in the problem analysis and description. It was also used for the study of theoretical sources and the analysis of the BSC method application experience at the municipal level. The result of a positive methodology is represented by the recognition of the actual state and situation. Normative approach is related mainly to the basic revision of assembled models which were compared to the newly created municipal strategy and utilized scales. The result is a certain degree of optimization of the model as a whole.

The mix of positive and normative methodology has been used in many scientific research methods. The analysis method related to the study of literature and other information sources was used in the first row. The analysis method was used particularly for the creation of partial models (subsystems) where it was necessary to decompose the problem into individual elements. Besides that, the synthesis method was used as well, which is typical for dynamics models. Linking sub-models into a single unit enables us to examine mutual causalities and the behavior of the model as a whole.

Goals of the article: The analysis provided in this paper is focused on examining the potential of system dynamics in the form of the dynamic BSC method, related to the municipal development. In this context, authors aimed to (1) show using the system dynamics in the strategic management of town Vsetín, (2) present two models, including the simulation related to the municipal development.

3 Application of system dynamics

System dynamics can be used for modeling various processes within the various disciplines (cf. Vojtko a Mildeová [11]). Approach of system dynamics will be used for municipality strategic planning in the article. Town which was chosen for it is Vsetín municipality. There was applied Balanced Scorecard in practice.

3.1 The Balanced scorecard and dynamics

Management of the development of any organization requires inclusion in the various aspects of the decisions that have different impacts on the organization. Financial indicators are often generally preferred in the management of institutions. However, their use is often associated with many deficiencies. Namely the authors of Kaplan and Norton [4] highlighted the need to perceive and to apply also non-financial factors into decision-making processes. The authors created currently very popular method of Balanced Scorecard (BSC) for this purpose. BSC is a concept that uses traditional financial measures. These, however, reveal a history of development. For it for the determination of the future strategy, therefore, are not very suitable (cf. Kaplan and Norton, [3]. For this reason the BSC method completes the financial measures by the new measures which evaluate the new dimension of non-financial nature.

The BSC method is criticized because of static approach (cf. for example Bianchi and Montemaggiore [2], Sloper et al. [8]. The relationships among the four perspectives do not express their mutual internal dynamic relationship. This method is not considered directly with the possible delay between the cause and the resulting effect. It is stressed, in particular in the case of the linear approach to the BSC method to the evaluation of the string causes and the resulting effects, although of the authors themselves, Kaplan and Norton [4] recommend that the BSC method can be captured in a systems dynamics model that provides a comprehensive, quantified model of a business´s value creation process.

3.2 The Balanced Scorecard on the municipality level

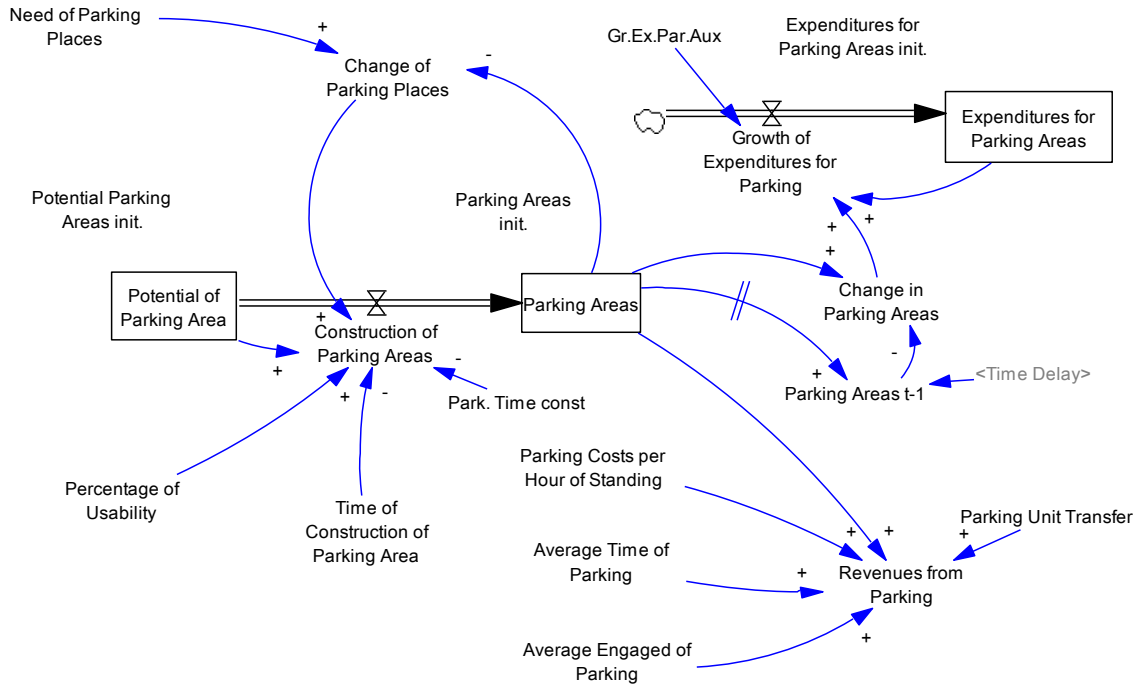
The BSC method can be applied in any organization. Therefore, it may not always be the private sector organization. Public sector institutions can apply ideas the BSC method in the same manner. Only their content will be modified in certain cases.

Using the Balanced Scorecard method in its dynamic form is shown on the town Vsetín in the Czech Republic.

For creation of this model there were applied models which have been drawn up under the professional management team Šusta – Půček. The BSC Model had been subsequently generalized by the authors of this article, and edited for the need of the simulation. Using synthesis is typical for dynamic modeling. The model is divided into several sub-models but it is simulated as a whole. So it is possible to examine causalities among elements of the system and simultaneously behavior of the whole system can be study. When creating case studies, the authors had access to all the necessary data and information from the point of view of executive management and strategic documents and data from a variety of databases which are available through the Internet. It can be mentioned the data from different base, in particular, the Czech Statistical Office, the ARIS System which concludes information of economies of municipalities in the Czech Republic, the system TIMUR is used, and the Dataplan database of Healthy Cities of the Czech Republic is used as well. By reason of data´s availability and consistency of the model on applied method of BSC in the town Vsetín simulations were carried out for period 2009 – 2019 in the Vensim PLE Plus 5.

Dynamic model of town Vsetín which were treated by authors based on materials of the town and others dates, includes the following models and simulations: the model of population of the town, the model of quality of life (the attractiveness of the area, quality of life), the model of sporting opportunities, the model of town culture, the transport models, the models of environment, the models for education sector, the labor market model, the model of housing sector, the model of budget of the town.

In view of the scope of this article we included only some models. Using of the system dynamics in balanced scorecard will be explained on these models. The first model which was chosen is the model of parking places in the town (see figure No. 1). For this model there is typical inclusion of basic limits like limitation of place. Creation of new places for parking reduces open space. Also there is evident the model is connection to the strategic plan. The plan determines the number of necessary parking places in the town. This sub-model is incorporated into the overall model for development of the town. It is through budget. Parking places need finances for service. But they bring income from parking fee. Linking the number of parking places on the budgeted expenditures is by relative changes in parking places in the time. Size relative annual changes in the number of parking places is find through the using the time delay. Another important element of the system dynamics is the ability to work with time delay.

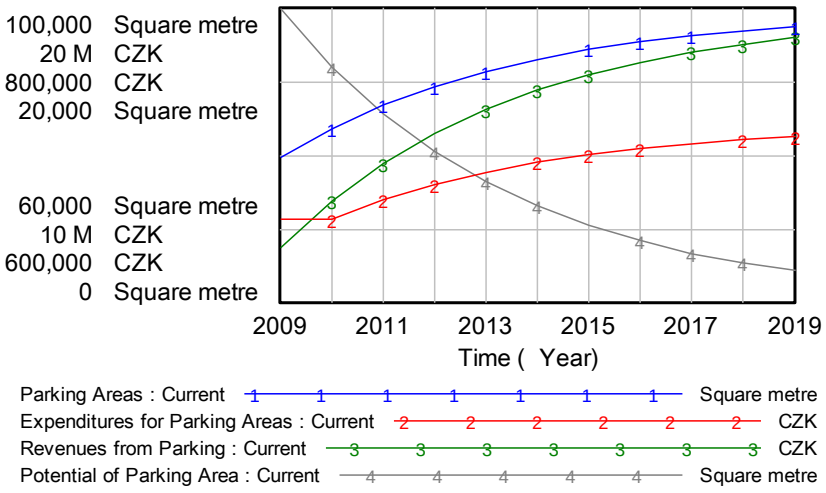


Source: Authors according to internal documents of the Town of Vsetín

Figure 1 Model of parking places in Town of Vsetín

Through simulation of model BSC was achieved the following scenarios of expected development. In expenditures for Parking Areas can be expected grow of parking places. Gradually it will be achieved the

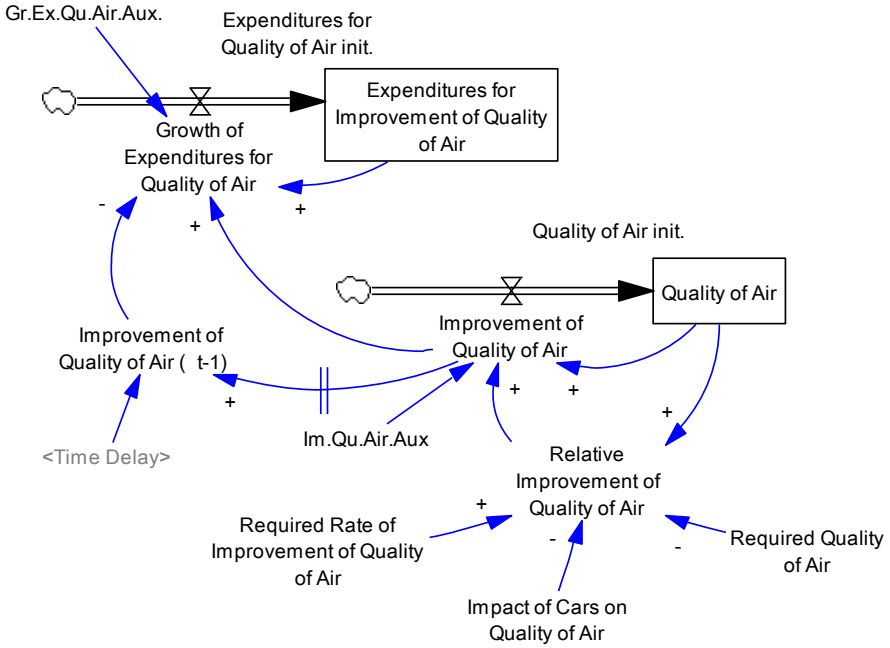
expected state of parking spaces as is set in strategic development. Also similar development is expected in the case of budget management .



Source: Authors

Figure 2 The simulation of development for the model of parking areas in the town Vsetín

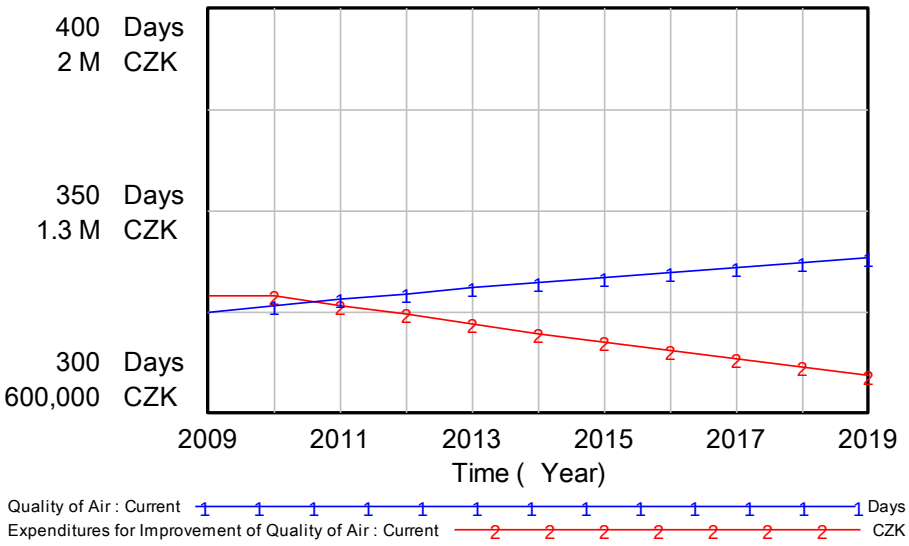
The model of air quality is the other illustrative model. This model describes the values from qualitative aspect and for the reason this model was added to this article. In order to simulate quality of air, it had to be expressed through quantitative indicators. The town Vsetín chose as an indicator the number of days in one year with good air. It is the same in the article. The connection between the assessment of quality of air and municipal budget expenditures is through annual examined the relative improvement of quality.



Source: Authors according to internal documents of the Town of Vsetín

Figure 3 Model of air quality of the Town of Vsetín

Quality of air should improve according to simulations. In this time Vsetin isn´t exposed to industrial emissions. For the reason expenditures for improvement of air can gradually drop.



Source: Authors

Figure 4 The simulation of model of air quality of the Town of Vsetín

These models are sub-models of comprehensive model of development of town. They are based on balanced scorecard. The outputs of them influence the quality of life in the town and they have impact on the attractiveness of the town. We can see the limits in the fact that these sub-models are mainly dependent on development of the town. So the model cannot make provision for actual development of quality in town. On the other side strategy of municipality is a stabilizing element and it is not subject to frequent changes.

4 Conclusion

The aim of the article was describing of system dynamics in municipality management. The dynamic model for development of the town Vsetín is a model example for this article. The second goal was to present two already processed city models (see Figure 1–4). This model is based on principles of balanced scorecard. The advantage of this approach is the fact that it points out on using non-monetary indicators. They can be represented by various factors of development and their character can be purely qualitative. But here is a question: is it possible use the factors for modeling and can we simulate their supposed development? Using balanced scorecard and the system dynamics in the article demonstrates their suitability for the strategic management in municipalities. Both methods leave the static view of development, but they allow see connections and interactions of elements. Additionally this approach allows examining qualitative factors and we can evaluate their impacts.

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Access price regulation and investment incentives

Rostislav Staněk [1]

Abstract. The paper provides a model of an industry with a vertically integrated infrastructure provider that has a possibility to invest into quality-upgrading infrastructure technology. The investment increases consumer's willingness to pay. The paper examines investment incentives and welfare implication of access price regulation without the possibility of ex-ante commitment. Regulation with imperfect information is considered as well as regulation with perfect information. The model shows that regulation causes underinvestment in both cases. The imperfect information model has only pooling equilibria which makes the underinvestment problem more severe. The regulation increases welfare only if the marginal investment costs are high enough and the degree of information asymmetry is low enough.

Keywords: access price, imperfect information, investment

JEL classification: L43, L51
AMS classification: 91B24, 91B26

1 Introduction

One of the main goal of economic regulation in the last years has been to increase competition in industries that has been traditionaly less competitive. Many industries such as telecommunications, railways, electricity, broadband interconnection etc. are characterized by monopolistic supply of essential infrastructure which prevents efficient competition on the retail market. In order to admit competitive supply of retail services regulators require the incumbent to make parts of its infrustructure available to entrants for regulated access price. For example, some national regulators have allowed mobile operators which do not possess their own frequency spectrum and infrastructure (virtual mobile network operator) to lease the network facilities of mobile network operators. With the emergence of advanced infrastructure technologies, such as next generation networks, there has been an increasing focus of regulators on the infrastructure investment incentives (see [2], [5]). The paper provides a model which studies welfare implications of acces price regulation in cases when the infrastructure owner is able to invest into the upgrade of the quality of services supplied through its infrastructure.

The paper follows literature on the access price regulation and investment incentives, which emphasizes regulator's limited ability to make credible ex-ante commitment to the pre-specified regulation scheme. The regulators inability to make credible commitment influences firms' investment decisions. If the regulator is free to change the access price ex-post, he faces a dynamic inconsistency problem. Investment costs are sunk after the investment is made and the regulator ignores them when setting an acess price. Beacuse the access price does not reflect investment costs, the infrastructure provider has lack of incentives to invest. This situation is studied by [1], [3], [6], [4]. Foros [3] analyzes competition between verticaly integrated incumbent and other retailers. Foros supposes that investment not only increases quality of the final good but creates also vertical differentiation on the retail market. Foros shows that access price regulation can decrease consumer surplus and total welfare. Kotakorpi [6] supposes that competitive fringe operates on the retail market and product is horizontally diferentiated. Kotakorpi then finds that the incumbent's investment is negatively affected by investment spillovers. Vareda [4] considers two types of investment: cost-reduction investment and quality-upgrading investment. Vareda shows that if the costs of investment are sufficiently high, then higher access price increases the incentives to invest into quality-upgrading technology and reduces the investment into cost-reducing technology. Brito et al. [1] analyze if two-part access tariffs solve the dynamic consistency problem of the regulator. They

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assume Hotelling competition on the retail market and show that investment can be induced only for low investment cost or for high product differentiation.

Above mentioned models are based on the perfect information assumption. Hence, the literature ignores possible asymmetry of information between infrastructure provider (incumbent) and the regulator. Above mentioned papers (with the exception of [3]) compare also only investment levels under different regulatory regimes. Welfare comparison, which is probably the main issue for policy makers, is not provided. This paper extends existing literature by incorporating asymmetry of information between regulator and infrastrucutre owner into the model. Concretely, I assume that the regulator has imperfect information about costs of providing access. Moreover, I provide comparison the access price regulation regime with the outcome of unregulated industry in terms of welfare.

2 Model

Model describes an industry with two markets: infrastructure market and retail market. The infrastructure is an essential input for production of the final good on the retail market. There are two kinds of firms in the model: one vertically integrated incumbent and its $n-1$ competitors. The incumbent owns an essential infrastructure and competes with $n-1$ retail competitiors on the retail market. The retail market competition is characterized by Cournot oligopoly. The incumbent sells the access to the infrastructure to its competitors on the retail market for the access price a for each unit of final good produced on the retail market. The costs of providing access per unit of final output is denoted as c. The access price can be set by the incumbent or by the regulator. To implement the assumption that the regulator has imperfect information, suppose that the incumbent can be of two different types depending on the costs of providing acces. Both types occur with equal probabilities. Low cost incumbent has zero cost of providing access $c = 0$. High cost incumbent has positive cost of providing access $c = \delta$, where $\delta < 1$. If regulator has imperfect information, parameter δ measures the degree of information asymmetry.

The incumbent has a possibility to invest into a quality-upgrading infrastructure technology. The investment causes parallel shift in market demand for final good. Concretely, suppose that the inverse market demand is $P(Q, I) = 1 + I - Q$, where I is the level of quality-upgrading investment and Q is the quantity produced on the retail market. As the competitors on the retail market use the incumbent's infrastructure, infrastructure investment have a postive effect on them, because the whole market demand increases. Suppose that investment I brings costs $\frac{\Psi}{2}I^2$. Hence, investment costs are convex and the slope of invesment marginal costs is Ψ.

The model is an extensive form game with three stages. The timing of the model is following. The incumbent decides about investment into the infrastructure in the first stage. In the second stage, the incumbent sets the access price a if the access price is unregulated. Otherwise the access price is set by the regulator. The regulator can have perfect or imperfect information when setting an access price. If the regulator has imperfect information, he does not know whether the access costs are δ or 0. In the third stage, retail market competition takes place. Incumbent chooses quantities q_1, and $n-1$ rivals choose quantities $q_2, \ldots, q_n$. Incumbent's profit is $\Pi_1 = q_1(P(Q, I) - c) + (a - c)(q_2 + \cdots + q_n) - \frac{\Psi}{2}I^2$. The first term represents the profit from the retail market, the second term is the profit obtained from providing access to the infrastructure, the third term is the cost of infrastructure investment. Profit of other retailers is given as $\Pi_i = q_i(P(Q, I) - a)$.

3 Equilibrium

This section derives equilibrium outcome of an industry under three different situations: unregulated access price, regulated access price with perfect information and regulated access price with imperfect information.

3.1 Unregulated access price

Consider the case when the access price is set freely by the incumbent. The model is a game with perfect information which can be solved by backward induction. In the third stage, Nash equilibrium condition

determines equilibrium quantities for a given access price and infrastructure investment.

$$q_1^* = \frac{1 + I - Nc + (N-1)a}{N+1} \tag{1}$$

$$q_2^* = \cdots = q_n^* = \frac{1 + I + c - 2a}{N+1} \tag{2}$$

In the second stage, the incument sets the access price in order to maximize its profit. The profit maximizing level of access price is given by the following expression

$$a^* = \frac{1 + I + c}{2} \tag{3}$$

By substituting 3 into 2 it can bee seen that the incumbent sets an access price such that $q_2 = \cdots = q_n = 0$. Access price is so high such that incumbent's rivals cannot operate on the retail market without making a loss. This eliminates competition on the retail market and market foreclosure occurs. The equilibrium outcome in unregulated industry is therefore the same as monopoly outcome. In the third stage, the incumbent chooses investment in order to maximize its profit. Taking into account the monopoly outcome on the retail market, the incumbent's profit function is $\Pi_1 = (\frac{1+I-c}{2})^2 - \frac{\Psi}{2}I^2$. In order to ensure that the profit function is concave and the maximum exists, assume that $\Psi > \frac{1}{2}$. The profit-maximizing level of investment in unregulated case I_U^* is given as

$$I_U^* = \frac{1 - c}{2\Psi - 1} \tag{4}$$

3.2 Regulation with perfect information

Let me next consider the case of regulation with perfect information. As in the previous case, the equilibrium quantities produced on the retail market are given by expressions (1) and (2). The motivation behind regulation is to prevent foreclosure and increase welfare by decreasing deadweight loss. The regulator thus maximizes welfare measured as a sum of expected consumer surplus and expected firms' profits. The regulator is also constrained by the incumbent's participation constraint which states that the incumbent cannot obtain negative profit from providing access to its infrastructure. The only instrument available for the regulator is the access price. The regulator has perfect information, so he can set different access prices for a given realization of the access costs. The regulator solves the following problem where Q^* denotes equilibrium quantity produced.

$$\max \int_0^{Q^*(c)} (1 + I - Q - c)dQ - \frac{\Psi}{2}I^2 \quad \text{such that} \quad (q_2 + \cdots + q_n)(a - c) \geq 0 \tag{5}$$

Inserting the equilibrium conditions (1) and (2) into the the regulator's problem (5) and taking derivative with respect to a it can be shown that the socially optimal access price is equal to the costs of providing access, that is $a^* = c$. Note that the participation constraint is binding at the optimal access price. The reason is that the downstream market is characterized by Cournot competition which results in a positive mark-up over marginal costs. This implies that the first-best solution, which is given by the condition that the price of final good is equal to the marginal costs, is not consistent with the participation constraint. In the first stage, the incumbent chooses the investment level in order to maximize its profit $\Pi_1 = (\frac{1+I-c}{N+1})^2 - \frac{\Psi}{2}I^2$. Profit maximizing level of investment in case of perfect information regulation is denoted as I_P^*.

$$I_P^* = \frac{1 - c}{\frac{\Psi(N+1)^2}{2} - 1} \tag{6}$$

3.3 Regulation with imperfect information

Finally, consider the case of regulation with imperfect information. The regulator cannot distinguish whether the costs of providing access are zero or δ in this case. However, the regulator observes the investment into the infrastructure. The model is a signaling game where the incumbent can send a signal about its access costs by its choice of investment. Assume that the solution of the model is given by sequential equilibrium requirement. Next proposition shows that there is no equilibrium where regulator can infer incumbent's costs from its choice of investment.

Proposition 1. *There is no separating equilibrium if $N \geq 3$ or $N = 2$ and $\delta < \frac{45}{49}$*

Proof. Suppose there is some separating equilibrium, then $I^*(\delta) \neq I^*(0)$ where $I^*(c)$ is the optimal investment level at a given access costs. Because of consistency of beliefs requirement, the regulator believes that $c = 0$ when he observes $I^*(0)$. In this case, he sets the access price $a = 0$. On the other hand if the regulator observes $I^*(\delta)$, he believes that $c = \delta$ and sets the access price $a = \delta$. Hence, the optimal investment levels $I^*(\delta)$ and $I^*(0)$ are given in the same way as in the case of perfect information, i.e. by (6). Now fix beliefs and strategies of other players and consider the investment decision of the first type incumbent. Separating equilibrium exists if the first type incumbent does not want to deviate by investing the amount $I^*(\delta)$ instead of $I^*(0)$. This condition translates into the following inequality

$$\left(\frac{1 + I^*(0)}{N+1}\right)^2 - \frac{\Psi}{2} I^*(0)^2 \geq \left(\frac{1 + I^*(\delta) + (N-1)\delta}{N+1}\right)^2 + \frac{(N-1)\delta(1 + I^*(\delta) - 2\delta)}{N+1} - \frac{\Psi}{2} I^*(\delta) \qquad (7)$$

After substituting equation (6) for $I^*(\delta)$ and $I^*(0)$ and after some algebraical manipulations it can be shown that this condition is not satisfied if $N \geq 3$ or if $N = 2$ and $\delta < \frac{45}{49}$ □

Taking into consideration that $\delta \in (0, 1)$ proposition 1 shows that the parameter space for which the separating equilibrium exists is negligible. Therefore, we have to look for some pooling equilibrium. There is a multiplicity of pooling equilibria in the model. The equlibria differ by the regulator's reaction on out-of-equilibrium investment, so we have to specify regulator's out-of-equilibrium conjectures before deriving equilibrium strategies. Assume that the regulator believes that the incumbent has zero access costs whenever the observed investment level is higher than the optimal investment level of the second type incumbent, i.e. whenever $I > I^*(\delta)$. This assumption is adequate because high cost incumbent has no incentives to invest more than $I^*(\delta)$ for any access price $a \in [0, \delta]$. Next propostition describes pooling equilibrium given by this assumption.

Proposition 2. *There is a pooling equilibrium characterized by the following conditions: $I_I^* = \frac{1-\delta}{\Psi(N+1)^2/2-1}$, $a^* = \delta$, equilibrium quantities $q_1^*(0)$, $q_2^*(0)$ $q_1^*(\delta)$ and $q_2^*(\delta)$ are given by substituting I_I^* and a^* into equations (1) and (2). If $I > I_I^*$, then the regulator believes that $c = 0$ and sets $a = 0$.*

Proof. The regulator is not able to infer incumbent's cost from the investment decision. The regulator therefore solves the following problem.

$$\max \frac{1}{2}\int_0^{Q^*(0)} (1 + I_I^* - Q - c)dQ + \frac{1}{2}\int_0^{Q^*(\delta)} (1 + I_I^* - Q - c)dQ - \frac{\Psi}{2}(I_I^*)^2 \quad \text{such that} \quad a \geq \delta \qquad (8)$$

The optimal access price is set such that the participation constraint is binding. Note that the participation constraint says that neither type of incumbent can obtain negative profit from providing access to the infrastructure. The optimal access charge is thus $a^* = \delta$. It is obvious that the investment of the high cost incumbent is optimal. If the low cost incumbent invests $I > I_I^*$, then the regulator sets the access price $a = 0$. The most profitable deviation of the low cost incumbent is $I^*(0) = \frac{1}{\Psi(N+1)^2/2-1}$. The equilibrium exists if and only if the incumbent cannot increase its profit by investing $I^*(0)$. This condition is satisfied if the opposite inequality to the inequality (7) holds. Therefore the pooling equilibrium exists for any $\delta \leq \frac{45}{49}$ or $N \geq 3$. □

4 Welfare and investment comparison

This section presents investment and welafare comparison of the three cases analyzed in the previous section. First, I compare equilibrium investment levels. This comparison is summarized in the following proposition.

Proposition 3. *Consider the three cases stated above. By comparison of expected equilibrium investment levels, it holds that $I_U^* > I_P^* > I_I^*$*

Proof. It holds that $\frac{1-\delta/2}{2\Psi-1} > \frac{1-\delta/2}{\frac{\Psi(N+1)^2}{2}-1} > \frac{1-\delta}{\frac{\Psi(N+1)^2}{2}-1}$ for any $N \geq 2$, $\Psi > \frac{1}{2}$ and $\delta \in (0, 1)$ □

Proposition 3 shows that incumbent has greater incentives to invest when regulator has perfect information compared to the case of imperfect information. This could be quite surprising because imperfect information regulation leads to higher access price than perfect information regulation. Yet higher access price does not imply higher investment in this case. The reason is that the low cost incumbent does not want to signal its type and therefore he has to invest the same amount as the high cost incumbent. Proposition 3 also implies that there is a trade-off between static effect and dynamic effect of regulation. The static effect is given by the fact that the regulator sets lower access price to counteract the foreclosure tendecy of the incumbent. This increases market output as well as total welfare given the infrastructure investment. On the other hand, there is also dynamic effect. The dynamic effect is caused by the fact that regulation decreases investment incentives and as a consequence also total welfare. Because of these opposite effects the welfare consequences of the access price regulation are ambiguous.

Now, I compare welfare of unregulated industry with welfare obtained under perfect information and imperfect information regulation. Whether access price regulation is welfare improving depends on whether static effect or dynamic effect prevails. Analytical comparison of welfare is not very tractable and does not provide easy survey. But we can get the main insight into the welfare comparison by showing the space of parameter values for which the outcome of regulated industry is welfare superior to the outcome of unregulated industry. This comparison is presented in figures 1 and 2.

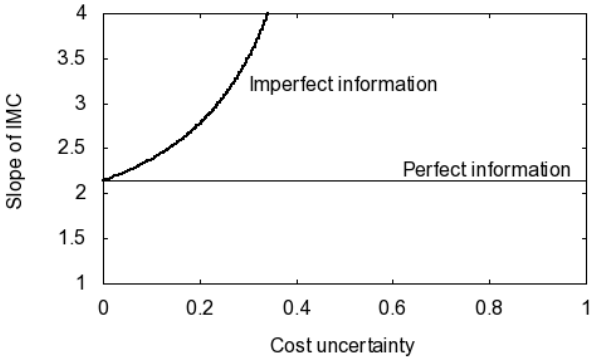

Figure 1 Welfare comparison, N=2

The model contains three parameters: number of firms N, slope of invesment marginal costs Ψ and the measure of regulator's uncertainty about access costs δ. Figure 1 ilustrates how welfare comparison depends on parameters Ψ and δ when there are two firms operating on the retail market. There are two curves in the figure. Perfect information curve ilustrates welfare comparison between unregulated industry and perfect information regulation. The interpretation of imperfect information curve is analogous. Each curve divides the parameter space into two areas. Regulation improves welfare in the area above the curve and decreases welfare in the area below the curve. We can see that it is preferable to regulate the industry if the regulator has perfect information and the slope of investment marginal costs is high enough, namely $\Psi > 2,14$. The figure also shows that regulation becomes less preferable aas the regulator's uncertainty about access costs increases.

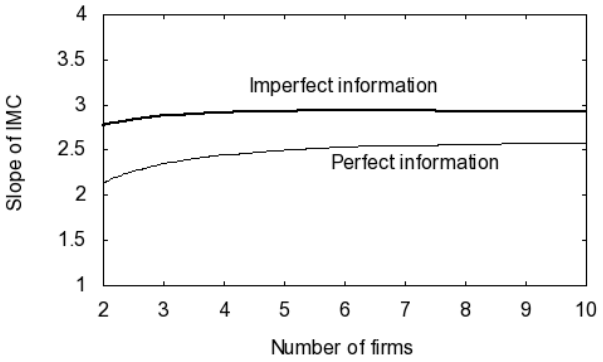

Figure 2 Welfare comparison, c=0,2

Figure 2 shows how welfare comparison depends on parameters N and Ψ in the case when $c = 0, 2$. Again, regulation is welfare superior in the area above the curves and welfare inferior in the area below the curves. Figure 2 confirms the conclusion that regulation is welfare improving if investment marginal costs are high enough. The figure also shows that the parameter space where regulation improves welfare becomes smaller as the number of firms increases. In both figures we can see that the imperfect information curve lies above the perfect information curve. Therefore, both figures reveal that regulation is less likely welfare improving if the regulator has imperfect information. This conclusion is quite intuitive because regulation under imperfect information leads to higher access price and lower investment compared to regulation with perfect information. Hence, regulation under perfect information is better in terms of static efficiency as well as dynamic efficiency.

5 Conclusion

The paper analyzes an industry with verticaly integrated incumbent that invests into quality-upgrading technology. The paper extends existing literature on the subject in two ways. First, it compares different regulatory regimes directly in terms of welfare. Second, the paper explicitly considers the problem of information asymmetry between regulator and incumbent. The model assumes that the regulator cannot commit to an access pricing regime prior to the incumbent's investment decision. Therefore, regulation brings underinvestment problem. Insufficient investment implies losses to welfare. For this reason, welfare implications of regulation are ambiguous. The welfare analyzes of the model shows that regulation is welfare improving if the marginal costs of investment are high enough. This means that the access price regulation is not advantageous in industries with the possibility of cheap quality-upgrading investment. The welfare analysis also shows that higher competition on the retail market makes regulation less profitable.

The underinvestment problem is more severe if the regulator has imperfect information about incumbent's cost of providing access. This occurs because more efficient incumbet wants to profit from higher access price. In order to do so, the more efficient incumbent has to imitate the strategy of the less efficient incumbent and decrease the investment into the infrastructure. If regulator has imperfect information, then regulation is welfare improving if and only if marginal costs of investment are high enough and the degree of information asymmetry is not very high.

Acknowledgements

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Measuring of efficiency of EU27 countries by Malmquist index

Michaela Staníčková[1]

Abstract. The paper deals with an application of Data Envelopment Analysis (DEA) method in an efficiency analysis of the European Union (EU) Member States during reference period 2000-2011. DEA method measures numerical grades of efficiency of economical processes within evaluated countries, and therefore it becomes a suitable tool for setting an effective/ineffective position of each country. The main aim of the paper is to measure efficiency changes over the reference period and to analyse a level of productivity in individual countries based on the Malmquist Index, and then to classify the EU Member States to homogeneous units (clusters) according to efficiency results. The theoretical part of the paper is devoted to the fundamental basis of efficiency theory and DEA method – especially the Malmquist Index. The empirical part is aimed at measuring the degree of productivity and level of efficiency changes of evaluated countries by the Malmquist Index, measuring the change of technical efficiency and the movement of the production possibility frontier. The final part of the paper offers a comprehensive comparison of results obtained by calculating the Malmquist Index.

Keywords: Cluster analysis, competitiveness, correlation, DEA method, efficiency, EU, Malmquist index.

JEL Classification: C67, C82, O11, O33, Y10
AMS Classification: 62H99, 90C05, 93B15, 93D25

1 Introduction

In the European Union (EU), the process of achieving an increasing trend of performance and a higher level of competitiveness is significantly difficult by the heterogeneity of countries and regions in many areas. Although the EU is one of the most developed parts of the world with high living standards, there exist significant and huge economic, social and territorial disparities having a negative impact on the balanced development across Member States and their regions, and thus weaken EU's performance and competitiveness in a global context. The European integration process is thus guided by striving for two different objectives: *to foster economic competitiveness* and *to reduce differences* [7]. The support of cohesion and balanced development together with increasing level of competitiveness belong to the temporary *EU's key development objectives*. In relation to competitiveness, performance and efficiency are *complementary objectives*, which determine the long-term development of countries in a globalized economy [5].

1.1 Background of efficiency analysis

In recent years, the topics about assessment of efficiency have enjoyed economic interest. Although there is no *uniform definition* and understanding of this term, *no mainstream approach* for measuring of efficiency, this multidimensional concept remains one of the basic standards of performance evaluation (besides the concepts of competitiveness and productivity) and it is also seen as a reflection of success of area in a wider comparison. *Performance management* is one of the major sources of sustainable national efficiency and effectiveness. Efficiency and effectiveness analysis is based on the *relationship between the inputs* (entries), *the outputs* (results) and *the outcomes* (effects). In general sense, *the efficiency* can be achieved under the conditions of maximizing the results of an action in relation to the resources used, and it is calculated by comparing the effects obtained in their efforts. As it can be seen in Figure 1, the *efficiency is given by the ratio of inputs to outputs*, but there is difference between *the technical efficiency* and *the allocative efficiency*. The technical efficiency implies a relation between inputs and outputs on the frontier production curve, but not any form of technical efficiency makes sense in economic terms, and this deficiency is captured through the allocative efficiency that requires a cost/benefit ratio. *The effectiveness implies a relationship between outputs and outcomes*.

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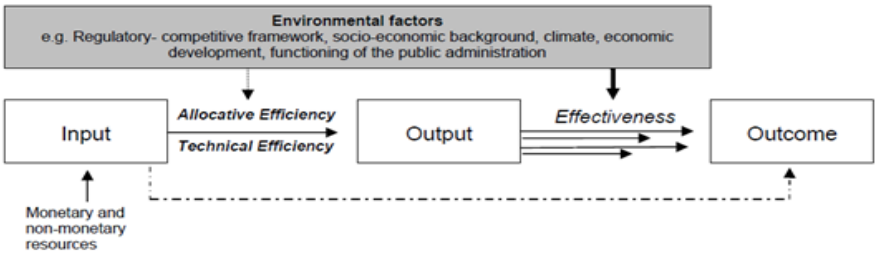

Figure 1 The relationship between the efficiency and the effectiveness
Source: [4], p. 3

1.2 Evaluation of efficiency

The analysis of efficiency is about the relationships between inputs and outputs. Techniques to measure efficiency are improved and investigations of efficiency become more frequent. The early research study focused on separate measures for productivity and there was a failure to combine the measurements of multiple inputs into any satisfactory measure of efficiency. These inadequate approaches included forming an average productivity for a single input (ignoring all other inputs), and constructing an efficiency index in which a weighted average of inputs is compared with output. Responding to these inadequacies of separate indices of labor productivity, capital productivity, etc., Farrell [3] proposed an activity analysis approach that could more adequately deal with the problem. His measures were intended to be applicable to any productive organization. Farrell confined his numerical examples and discussion to single output situations, although he was able to formulate a multiple output case. Twenty years after Farrell's model, and building on those ideas, *A. Charnes, W. W. Cooper* and *E. Rhodes* in 1978 [2], responding to the need for satisfactory procedures to assess the relative efficiencies of multi-input/multi-output production units, introduced a powerful methodology – **Data Envelopment Analysis** (DEA) in the form of CCR model assuming constant returns to scale (CRS).

2 Methodological background of efficiency analysis

Measurement of efficiency is an important issue for at least *two reasons*. One is that in a group of units where only limited number of candidates can be selected, the efficiency of each must be evaluated in a fair and consistent manner. The other is that as time progresses, better efficiency is expected. Units with declining efficiency must be identified in order to make the necessary improvements. The efficiency of countries can be evaluated in either a cross-sectional or a time-series manner; *DEA is useful method for both types of efficiency evaluation* [7].

2.1 Theoretical framework of DEA method

DEA is data oriented approach for providing a relative efficiency assessment and evaluating the efficiency of a set of peer entities called *Decision Making Units* (DMUs) which convert multiple inputs into multiple outputs. The aim of DEA method is to examine DMU if they are efficient or inefficient by the size and quantity of consumed resources by the produced outputs. Efficient DMUs have equivalent efficiency score. However, they don't have necessarily the same performance. DMU is efficient if the observed data correspond to testing whether the DMU is on the imaginary *'production possibility frontier'*. All other DMU are simply inefficient, and DEA identifies a set of corresponding efficient units that can be utilized as benchmarks for improvement of inefficient units. Efficiency score of DMUs is defined as follows (1):

$$Efficiency\ of\ \ DMU = \frac{\text{weighted sum of outputs}}{\text{weighted sum of inputs}}. \tag{1}$$

2.2 Data base characteristics for empirical analysis

The efficiency analysis, based on application of DEA approach, is used for evaluating national development quality and potential (with respect to the national factors endowment). Based on the above facts, it is possible to determine **the initial hypothesis of the analysis**, which is based on the assumption that more advanced EU countries achieving best results in efficiency (old EU Member States (EU15)) are countries best at converting inputs into outputs and therefore having greater performance and productive potential than new EU Member States (EU12); see e.g. [6]. Database of indicators is part of a common approach of WEF and EU in the form of **Country Competitiveness Index** (CCI) [1]. Eleven pillars of CCI are grouped according to the different dimensions (input versus output aspects) of national competitiveness they describe. The terms '*inputs*' and '*outputs*' are meant to classify pillars into those which describe driving forces of competitiveness, also in terms of long-term potentiality, and those which are direct or indirect outcomes of a competitive society and economy. *Meth-*

odology of CCI is suitable for measuring national competitiveness by DEA method. Set of CCI data file consists of 66 CCI indicators – 38 of them are inputs and 28 outputs. In this paper, all CCI indicators are not used because all indicators were not available for the whole period for each country, but for some indicators were found comparable indicators. The pillars and 62 used indicators are listed in Appendix 1. Empirical analysis is based on a frontier non-parametric approach and aims to study productivity growth and performance effectiveness. This is based on **Malmquist index** (MI) for measuring the change of technical efficiency and the movement of the frontier in terms of individual countries, in during reference period 2000-2011, in years of growth period 2000-2007, and in years of crisis and post-crisis period 2008-2011. For solution of DEA method software tool based on solving linear programming problems is used in the paper – Solver in MS Excel 2010, such as *the DEA Frontier*.

Suppose we have a production function in time period *t* as well as period *t+1*. Malmquist index calculation requires two single period and two mixed period measures. The two single period measures can be obtained by using the *CCR model with Constant Returns to Scale* (CRS). Suppose each DMU_j *(j=1, 2… n)* produces a vector of output $y_j^t = \left(y_{1j}^t, \ldots, y_{sj}^t\right)$ by using a vector of inputs $x_j^t = \left(x_{1j}^t, \ldots, x_{mj}^t\right)$ at each time period *t, t=1... T*. MI measuring the efficiency change of production units between successive periods *t* and *t+1*, is formulated via (2):

$$M_0\,(x^{t+1}, y^{t+1}, x^t, y^t) = E_0 \cdot P_0 \tag{2}$$

where E_0 is change in the relative efficiency of DMU_0 in relation to other units (i.e. due to the production possibility frontier) between time periods *t* and *t+1;* P_0 describes the change in the production possibility frontier as a result of the technology development between time periods *t* and *t+1*. Modification of M_0 (3) makes it possible to measure the change of technical efficiency and the movement of the frontier in terms of a specific DMU_0:

$$M_0 = \frac{\theta_0^t\left(x_0^t, y_0^t\right)}{\theta_0^{t+1}\left(x_0^{t+1}, y_0^{t+1}\right)}\left[\frac{\theta_0^{t+1}\left(x_0^{t+1}, y_0^{t+1}\right)}{\theta_0^t\left(x_0^{t+1}, y_0^{t+1}\right)} \cdot \frac{\theta_0^{t+1}\left(x_0^t, y_0^t\right)}{\theta_0^t\left(x_0^t, y_0^t\right)}\right]^{1/2}. \tag{3}$$

where $\theta_0^t\left(x_0^t, y_0^t\right)$ is a function that represents the production technology S^t in the time period *t* and assigns to evaluated DMU_0 the efficiency rate. Function $\theta_0^{t+1}\left(x_0^t, y_0^t\right)$ gives the relationship of inputs and outputs of the time period *t* with production technology S^t in the time period *(t +1)* and function $\theta_0^t\left(x_0^{t+1}, y_0^{t+1}\right)$ present inputs and outputs of the time period *(t +1)* with production technology S^t in the time period *t*. Function $\theta_0^{t+1}\left(x_0^{t+1}, y_0^{t+1}\right)$ gives the relationship of inputs and outputs of the time period *t+1* with production technology S^t in the time period *(t +1)*. The first component E_0 measures the magnitude of *technical efficiency change* (TEC) between time periods *t* and *t+1*. Obviously, $E_0 <\,=\,> 1$ indicating that technical efficiency improves remains or declines. The second terms P_0 measures the shift in the possibility frontier, i.e. *technology frontier shift* (FS), between time period's *t* and *t+1*. Productivity declines if $P_0 > 1$, remains unchanged if $P_0 = 1$ and improves if $P_0 < 1$. In Table 1, characteristics and trends of MI are shown.

Malmquist Index	Productivity	Efficiency Change	Technical Efficiency
MI > 1	Declining ↓	Change < 1 Improving	Change < 1 Improving
MI = 1	Unchanging —	Change = 1 Unchanging	Change = 1 Unchanging
MI < 1	Improving ↑	Change > 1 Declining	Change > 1 Declining

Table 1 Characteristics and trends of Malmquist index
Source: Own elaboration, 2013

3 Results of efficiency analysis in EU27 countries

The *initial hypothesis* was partly confirmed through analysis by *Malmquist index* (based on *IO CCR CRS model)*, as it is illustrated in following evaluation. Some of countries have reached the best results and recorded predominantly total efficiency increase through the time period and other countries have reached predominantly total efficiency decrease during reference years. Most of evaluated countries have recorded both increasing and decreasing trend in efficiency development during reference years of period 2000-2011 and 2000-2007, but in years 2007-2008, most of countries have recognized considerable deterioration in efficiency (due to economic crisis). It is recognized gradually improving in economic development in years 2008-2011, but it is very slow. Apparently the best results are traditionally achieved by economically powerful countries (some EU12 and EU15 countries) which were 'highly efficient' during the reference period. In Table 2, results of 'efficient' countries are recorded and also development´ trend in efficiency of individual countries in the context of their effec-

tive/ineffective position based on efficiency results is recorded. The best results (of all evaluated countries and in compared years 2000-2011) have recognized from EU15 countries Germany, Ireland, Spain, France, Italy, Portugal and Sweden; from EU12 countries Bulgaria, Estonia, Slovenia, Latvia, Lithuania, Poland, Romania and Slovenia. These countries have recorded clear increasing trend of MI in referred years. The worst results (of all evaluated countries and in compared years 2000-2011) have recognized from EU15 countries Belgium, Denmark, Greece, Luxembourg, Netherlands and Austria; from EU12 countries Czech Republic, Cyprus, Hungary and Malta. These countries have recorded clear decreasing trend of MI in referred years.

In Table 2 **efficient' countries** are coloured by bold and dark grey colour. These are *Ireland*, *Spain* and *Slovenia*, which were efficient in all comparing periods and recognized increasing performance trend in all periods. In the frame of paper hypothesis, could be countries with the best competitive potential and perspective to further development. The efficient countries are followed by a group of countries which are also **highly efficient**. These countries do not achieved increasing performance trend in all periods, but in most of periods. These countries are highlighted by bold in Table 2. These countries are *Germany*, *France*, *Italy*, *Latvia*, *Poland* and *Sweden.* These countries show high level of competitive potential. Countries with worse and the worst levels of efficiency' scores and trends are classified as **slightly inefficient** countries or **inefficient** i.e. these countries are considered as countries with lower competitive potential in compared reference years. From the group of evaluated countries belong to the group of slightly efficient countries *Belgium*, *Denmark*, *Cyprus*, *Luxembourg* and *United Kingdom*. These countries are highlighted by italics in Table 2. In Table 2, the most inefficient countries are highlighted by light grey colour and italics; these countries are *Netherlands* and *Finland* in compared years. It is necessary to note that some of EU15 countries have recognized worse results in efficiency analysis than EU12 countries. Most of EU15 countries are among the most competitive EU countries, but other countries, especially EU12 countries, have recorded the convergence of their economic performance towards these advanced countries, and thus is confirmed the convergence process of new EU Member States to old ones.

Code		Country/Time	IO CCR CRS MI*							
			2000-2011		2000-2007		2007-2008		2008-2011	
1	*BE*	*Belgium*	*1,7048*	↓	*1,5728*	↓	*1,2179*	↓	*0,7325*	↑
2	BG	Bulgaria	0,9153	↑	1,3847	↓	1,7665	↓	0,6643	↑
3	CZ	Czech Republic	1,1321	↓	0,9617	↑	1,0884	↓	1,0102	↓
4	*DK*	*Denmark*	*2,1959*	↓	*1,0056*	↓	*3,3186*	↓	*0,9584*	↑
5	**DE**	**Germany**	**0,5214**	↑	**0,4281**	↑	**0,8808**	↑	**1,1811**	↓
6	EE	Estonia	0,8407	↑	1,0823	↓	1,2268	↓	0,7206	↑
7	**IE**	**Ireland**	**0,8374**	↑	**0,9340**	↑	**0,9202**	↑	**0,7807**	↑
8	EL	Greece	1,5674	↓	2,3659	↓	0,8398	↑	1,3062	↓
9	**ES**	**Spain**	**0,6093**	↑	**0,6302**	↑	**0,8893**	↑	**0,7188**	↑
10	**FR**	**France**	**0,8579**	↑	**0,7385**	↑	**0,5342**	↑	**1,2365**	↓
11	**IT**	**Italy**	**0,8798**	↑	**0,8833**	↑	**0,6513**	↑	**1,4473**	↓
12	*CY*	*Cyprus*	*1,7085*	↓	*1,4294*	↓	*1,1640*	↓	*0,7153*	↑
13	**LV**	**Latvia**	**0,1107**	↑	**0,1173**	↑	**1,4020**	↓	**0,8170**	↑
14	LT	Lithuania	0,7309	↑	1,1112	↓	1,1167	↓	0,6396	↑
15	*LU*	*Luxembourg*	*4,5500*	↓	*1,6689*	↓	*0,9806*	↑	*1,0032*	↓
16	HU	Hungary	1,2908	↓	1,3510	↓	0,9979	↑	0,8083	↑
17	MT	Malta	1,7850	↓	1,8909	↓	0,7373	↑	0,7734	↑
18	*NL*	*Netherlands*	*1,5885*	↓	*1,3730*	↓	*1,0240*	↓	*1,1321*	↓
19	AT	Austria	1,2576	↓	0,7836	↑	1,0769	↓	1,7336	↓
20	**PL**	**Poland**	**0,8955**	↑	**1,2124**	↓	**0,9053**	↑	**0,9317**	↑
21	PT	Portugal	0,9240	↑	1,0004	↓	0,8792	↑	0,8267	↑
22	RO	Romania	0,7994	↑	0,8148	↑	1,1680	↓	0,7517	↑
23	**SI**	**Slovenia**	**0,5000**	↑	**0,6363**	↑	**0,8026**	↑	**0,8316**	↑
24	SK	Slovakia	1,0599	↓	0,4801	↑	1,7704	↓	1,0027	↓
25	*FI*	*Finland*	*1,3637*	↓	*1,1770*	↓	*1,0605*	↓	*1,2941*	↓
26	**SE**	**Sweden**	**0,7968**	↑	**0,8533**	↑	**0,9802**	↑	**1,0844**	↓
27	*UK*	*United Kingdom*	*2,1699*	↓	*1,0661*	↓	*1,5256*	↓	*0,9070*	↑

Table 2 Application of MI for EU27 countries
Source: Own calculation and elaboration, 2013

Cluster analysis (CA) is used for comparing of efficiency results based on presented DEA models. The first step of cluster analysis is to select the criterion of similarity (dissimilarity) of the objects. As a measure of dissimilarity, the *Squared Euclidean Distance* was selected. This measure is based on *the Ward's method*. The column "*Coefficients*" helps to decide how many clusters are optimal for representation of the data. The cluster formation should be stop when the increase in the Coefficients is large. In this case, the best interpretation of data ensures **five-cluster solution in Malmquist index** for overview of development trend across referred period, thus 2000-2011. The graphical representation of distance between which clusters are combined is presented by *Dendogram*. The gradual clustering of countries and the final optimal number of the determined clusters in DEA approach is shown in example of Dendogram in Figure 2. CA is solved by software tool *IBM SPSS Statistics 20*.

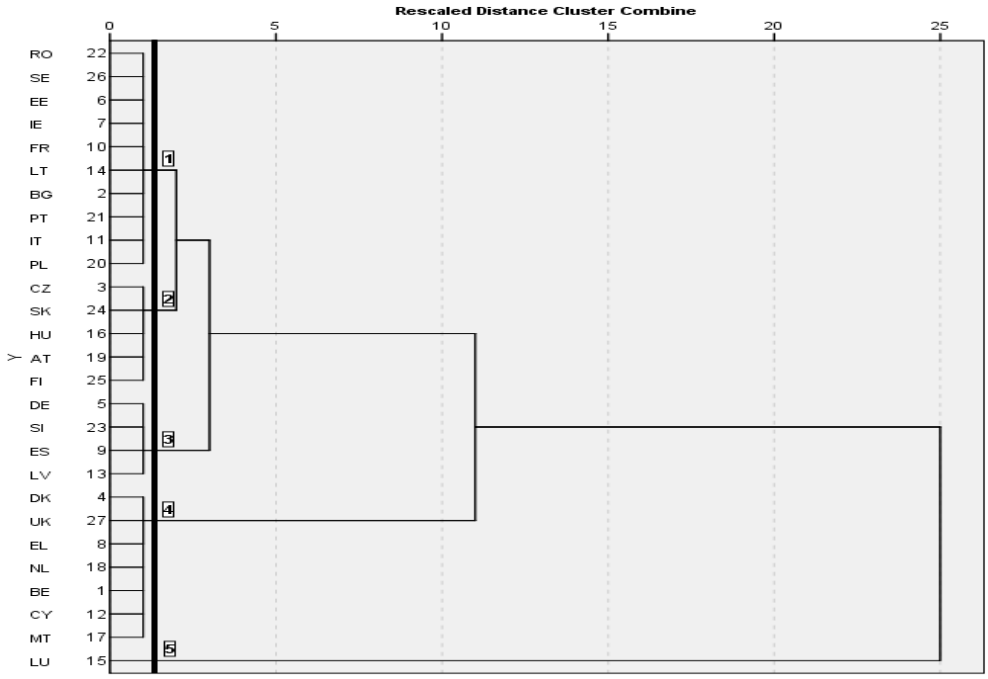

Figure 2 Dendogram of EU27 countries´ clusters using Ward linkage
Source: Own calculation and elaboration, 2013

Cluster I is created by *Romania, Sweden, Estonia, Ireland, France, Lithuania, Bulgaria, Portugal, Italy and Poland*, which have very similar level of MI and have recognized increasing trend of efficiency development. **Cluster II** is characterized by countries as *Czech Republic, Slovakia, Hungary, Austria* and *Finland*, which have recognized worse levels of efficiency scores and have recorded deteriorating trend in efficiency. **Cluster III** represents *Germany, Slovenia, Spain* and *Latvia*, which have recognized efficiency increasing, they have best levels of MI. **Cluster IV** is created by *Denmark, United Kingdom, Greece, Netherlands, Belgium, Cyprus* and *Malta*. **Cluster V** represents also only one country – *Luxemburg*; this is caused by impacts of the economic crisis and in particular the not favourable ratio of inputs and outputs in year 2011 compared to year 2000. Most of countries, especially DK, UK, NL and BE, belonging to Cluster IV and Cluster V are among the most competitive EU countries, but other countries recorded the convergence of their economic performance towards these advanced countries.

4 Conclusion

Based on DEA approach has been found out that in evaluated countries is a *distinct gap between economic and social standards*, so differences still remain. Measuring the Malmquist index on the basis of DEA is an important method which has many applications. This index has been used in this paper to analyse and evaluate performance of EU25 countries across selected years of reference period 2000-2011. Regarding the findings and the analysis of each country can decide whether it had an efficiency increase during the time period, or not. By having this information and dividing productivity into its elements, the basic trend in performance whether it be increase or decrease is observed. According to MI results, countries noticeable productivity decreases and increases were mostly achieved (both in EU12 and EU15 countries); more or less balanced performance and efficiency trend were recognized during reference years. Most countries experienced decline in their performance as a result of economic crisis. The economic crisis has threatened the achievement of sustainable development in the field of competitiveness. The crisis has underscored importance of competitiveness-supporting economic environment to enable economies better absorb shocks and ensure solid economic performance going in future.

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Annex

Dimension	Pillar	Indicator of input*
Inputs	**Institution**	Political Stability, Voice and Accountability, Government Effectiveness, Regulatory Quality, Rule of Law, Control of Corruption
	Macroeconomic Stability	Harmonized Index of Consumer Prices, Gross Fixed Capital Formation; Income, Saving and Net Lending/Net Borrowing, Total Intramural Research & Development Expenditure, Labour Productivity per Person Employed; ~~General Government Gross Debt~~
	Infrastructure	Railway transport - Length of Tracks, Air Transport of Passengers, Volume of Passenger Transport, Volume of Freight Transport; Motorway Transport -Length of Motorways, Air Transport of Freight
	Health	Healthy Life Expectancy, Infant Mortality Rate, Cancer Disease Death Rate, Heart Disease Death Rate, Suicide Death Rate; Hospital Beds, Road Fatalities
	Primary, Secondary and Tertiary Education; Training and Lifelong Learning	Mathematics-Science-Technology Enrolments and Graduates, Pupils to Teachers Ratio, Financial Aid to Students, Total Public Expenditure at Primary Level of Education, Total Public Expenditure at Secondary Level of Education, Total Public Expenditure at Tertiary Level of Education, Participants in Early Education, Participation in Higher Education, Early Leavers from Education and Training, Accessibility to Universities; Lifelong Learning
	Indicators for Technological Readiness	Level of Internet Access; E-government Availability
Dimension	**Pillar**	**Indicator of output***
Outputs	**Labour Market Efficiency**	Labour productivity, Male employment, Female employment, Male unemployment, Female unemployment, Public expenditure on Labour Market Policies; Employment rate, Long-term unemployment, Unemployment rate
	Market Size	Gross Domestic Product; Compensation of employees, Disposable income
	Business Sophistication	Gross Value Added in sophisticated sectors, Employment in sophisticated sectors, ~~Venture capital (investments early stage), Venture capital (expansion- replacement)~~
	Innovation	Human resources in Science and Technology, Total patent applications, Employment in technology and knowledge-intensive sectors, Employment in technology and knowledge-intensive sectors-by gender, Employment in technology and knowledge-intensive sectors-by type of occupation, Human resources in Science and Technology – Core, Patent applications to the EPO, Total intramural R&D expenditure, High-tech patent applications to the EPO, ICT patent applications to the EPO, Biotechnology patent applications to the EPO; ~~Employment in technology and knowledge-intensive sectors-by level of education~~

Table 1 Indicators of Inputs/Outputs in Period 2000-2011 Relevant to DEA Analysis
Note: * Number of indicators for inputs was decreased from 38 to 37; Number of indicators for outputs was decreased from 28 to 25
Source: [1]; own elaboration, 2013

On permutation tests with several test statistics to compare two populations

Jacek Stelmach[1]

Abstract. One of the practical problems during estimation real processes with regression models seems to be inevitable obsolescence such models as a result of changes of these processes. Updating the model, however make sense when such changes would be statistically significant. Indication of it can be carried out with comparing the populations – today and when the model was created.
Usually parametric tests are carried out but it needs to require the set of assumptions related to the knowledge of distribution. Permutation tests let to pass over typical limitation and don't require knowledge of the distribution of examined populations. In the classic form – proposed by R. A. Fisher, single test statistics must be used. This paper proposes the extension, using at the same time two test statistics to verify the hypothesis of equality of two multi-dimensional populations. The critical region is determined used original division based on cells. Theoretical considerations were supplemented with the results of simulations carried out with R 2.15.0.

Keywords: permutation tests, multi-dimensional populations, computer simulations.

JEL Classification: C14
AMS Classification: 62H15

1 Problem description

The testing of differences between populations is one of a crucial point in multivariate statistical inference. It enables explore how independent predictors influence a patterning of response on the dependent variable or to monitor the change of the investigated phenomenon. Taking into considerations two multivariate populations (with *p* dimensions), that are observed as multivariate samples with *n* number of observations, the null hypothesis can be written as:

$$H_0 : F_1(x) = F_2(x) \tag{1}$$

where: *Fi(x)* is a distribution of i-th population.

The test that compares two populations may base (see [1]) on conformity of:

- densities;
- moments;
- measures of positions.

Often comparing of two populations bases on the study of certain parameters like variance or average values. The most known tests (MANOVA, T^2-Hotelling test) need to fulfill the assumptions, see [7]:

- the distribution of multivariate data set is Gauss distribution (normal distribution);
- both sets have common variance/covariance (homogeneity);
- the data are independently sampled (independence).

Although Ito [6] says that MANOVA is relatively robust to violations of the assumptions (except the case where the number of samples is limited), from the formal point of view, no parametric method shall be carried out where any assumption is not fulfilled. In such cases nonparametric tests should be performed. Very often however, the tables of empirical critical values are known only for the limited number of variables – see [1]. Then permutation tests could be the best solution.

1.1 Permutation tests in classic form

The idea of permutation test was described by R. A. Fisher. This test doesn't need any knowledge of the distribution of test statistics because instead of using any theoretical distribution, ASL (Achieved Significance Level) is estimated by Monte Carlo sampling from permutation distribution. The power of permutation test is similar to parametric tests, see [3]. The permutation test sequence used is presented below:

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1. calculate the value of chosen statistics for tested sample – *T*;*
2. proceed a permutation (*N* cases, it is recommended *N*>1000 – see [4]) of data – it destroys existing dependencies in sets;
3. calculate values of test statistics for the permutations: T_i*; i=1, 2, …, N*;
4. estimate *p-value* as *ASL* according to the formula (for one-side tests), see [2].

$$ASL \approx \frac{card\{T_i : T^* \le T_i\}}{N} \tag{2}$$

5. if *ASL* value is greater than α value (one-sided rejected region), H_0 hypothesis cannot be rejected.

In certain cases, a power of permutation tests could be increased with other information than used in test statistics. However, classic permutation test sequence doesn't allow using another statistics simultaneously because rejected region is to be fixed only for one dimension (one statistics).

1.2 Permutation tests with two statistics

Using two statistics requires different approach to determining critical region which becomes a two-dimensional figure because the empirical distribution of statistics takes the form of three-dimensional instead of planar figure – see Figure 1**.**

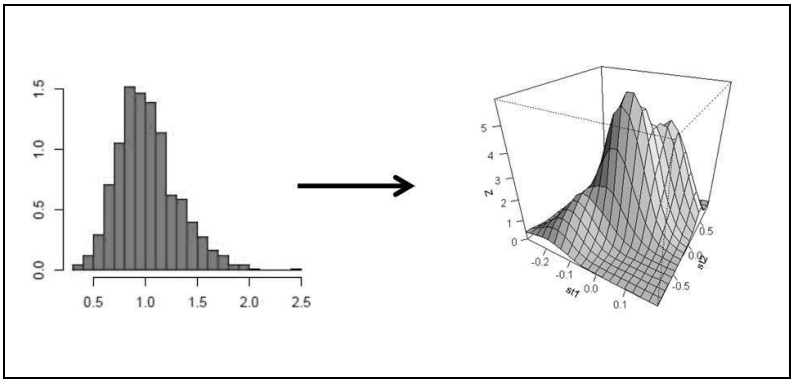

Figure 1 Transformation two-dimensional figure into three-dimensional solid for two test statistics.

Then the estimation of critical region as *ASL* value for any statistics cannot be used. And because of usually irregular shape of real critical region, the estimation for separate both dimensions would create improper empirical critical region. No known literature gave an advice so it was proposed the estimation of planar critical region implementing division into cells and ranking them.

The ranking is often used in statistics especially in non-parametric statistical inference. Let's assume that two test statistics: ST_1 and ST_2 were chosen to verify null hypothesis. In this case to determine critical region following sequence is proposed:

1. calculate the values of both statistics: ST_1* and ST_2*,
2. proceed a permutation *N* times – like for permutation test with one statistics only, in such situation *N* value should be much bigger, in this experiment *N*=10000 was set;
3. calculate values of test statistics for the permutations: T_i*; i=1, 2, …, N* that can be interpreted as the points [ST_{1i}, ST_{2i}], these points trace a rectangle containing all the points;
4. make a transformation that minimize a number of empty cells and transform the rectangle into square:
   a) carry out Principal Component Analysis for a population of calculated points and transform coordinate system – new axis shall be both eigenvectors,
   b) carry out a normalization - to achieve that all the values of main diagonal of covariance matrix will be ones,
5. divide transformed square into *k* x *k* cells – a number of points inside the cell describes each cell, a number of cells is very important – more cells increase the accuracy of critical region but needs bigger number of permutations (it can make the method inefficient), less cells decrease time of calculation but decrease the accuracy – it is even possible that inside confidence area empty cells can occur;
6. rank the cells starting from the cells with bigger number of points;
7. empirical critical region is set as the cells that are empty (no points inside) or the cells that are in (1-α) percentage of ranked cells - see Figure 2. Light gray and white cells show the critical region, so if a point [ST_1*, ST_2*] is placed as **A** point – null hypothesis cannot be rejected while it can be rejected for **B** or **C** points.

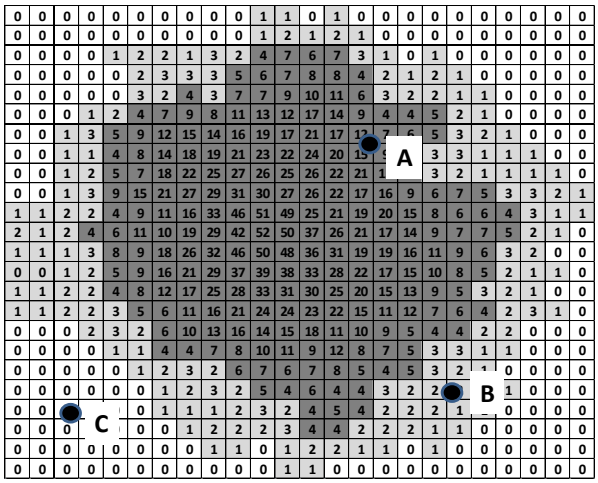

Figure 2 An example of critical region based on cells division.

2 Experiment description

An experiment was carried out in order to investigate the properties and advantages of permutation test with two statistics to compare two populations. Therefore the results of this test were compared with other tests results. Additionally permutation test was carried out with different number of cells to estimate its optimum number and with different sample sizes to estimate the sensitivity test power. All the tests were performed using simulated data sets with Monte Carlo sampling to compare the average power of these tests.

2.1 Simulated data sets

It was decided to simulate pairs of data sets (5 variables, 200 samples) with well-known distributions as four cases:

1. both data sets ~ N(0,1);
2. first data set ~ N(0,1), second - multivariate normal with zero vector of mean values and covariance matrix as in Table 1;
3. first data set ~ N(0,1), second - multivariate normal with zero vector of mean values and covariance matrix as in Table 1;
4. both data sets – uniform distribution with unitary covariance matrix and zero vector of mean values.

Second case					**Third case**				
0	0	0	0	0	0	0	0	0	0
1	0.2	0	0	0	1	0.4	0	0	0
0.2	1	0.2	0	0	0.4	1	0.4	0	0
0	0.2	1	0.2	0	0	0.4	1	0.4	0
0	0	0.2	1	0.2	0	0	0.4	1	0.4
0	0	0	0.2	1	0	0	0	0.4	1

Table 1 Covariance matrix for second data set of second and third cases.

During the experiment second data set was simultaneously transformed changing both: covariance matrix and mean vector creating 60 data cases:

1. all the samples were moved with a vector $\boldsymbol{\mu}=[x, x, x, x, x]$, where $x=0.0, 0.1, \ldots, 0.6$;
2. a variance of each variable (main diagonal of covariance matrix) was increased with a vector $\boldsymbol{\sigma}^2=[y, y, y, y, y]$, where $y=1.0, 1.1, \ldots, 1.9$;

2.2 Selected tests

Although last data set (uniform distribution) from the formal point of view does not allow carrying out parametric tests, then both: parametric and nonparametric tests were chosen to make the investigations as deep as possible. It was a reason to extend last five tests (permutation tests) with two statistics, using different number of cells .The tests and their test statistics are presented in Table 2.

Test no	Test description	Test statistics
1	T^2-Hotelling parametric test.	$T^2 = \frac{n_1 n_2}{n_1 + n_2}(\bar{x}_1 - \bar{x}_2)^T S_{pl}^{-1}(\bar{x}_1 - \bar{x}_2)$
2	Permutation test with T^2-Hotelling statistics.	
3	Permutation test with M test statistics (indicates equality of covariance matrixes).	$M = 1 - \frac{\lvert S_1\rvert^{\frac{n_1-1}{2}} \bullet \lvert S_2\rvert^{\frac{n_2-1}{2}}}{\lvert S_{pl}\rvert^{\frac{n_1+n_2-2}{2}}}; S_{pl} = \frac{(n_1-1)S_1 + (n_2-1)S_2}{n_1+n_2-2}$
4	Permutation test with test statistics as weighted sum of T^2 and M statistics.	$ST = \left(\frac{T^2}{F_{\alpha,k,n_1+n_2-k-1}}\right)^2 + M^2$
5	Permutation test (two test statistics) with: ST1: weighted sum of T^2 and M squares; ST2: a quotient of maximum and minimum of confidence ellipsoid volumes (described deeper later) for number of cells 7 x 7.	$ST_1 = \left(\frac{T^2}{F_{\alpha,k,n_1+n_2-k-1}}\right)^2 + M^2$ $ST_2 = \left(\frac{\max(v(1), v(2), v(1\cup 2))}{\min(v(1), v(2), v(1\cup 2))}\right) - 1$
6	As test no 5, number of cells 10 x 10.	
7	As test no 5, number of cells 15 x 15.	
8	As test no 5, number of cells 20 x 20.	
9	As test no 5, number of cells 30 x 30.	
10	As test no 5, number of cells 50 x 50.	

Table 2 Description of the tests used in the experiment.

Volumes of confidence ellipsoids mentioned for test cases 5-10 (contained in ST_2 statistics) were chosen as a consequence of following inference: if there are no significant differences between first and second data set, a volume of first set *v(1)*, second set *v(2)* and a sum of both sets *v(1u2)* are very similar and there are no significant differences between minimum and maximum of these volumes. Then ST_2 value should be close to zero. Otherwise value of this statistics is significantly greater than zero – that allows to distinguish the differences between both data sets.

2.3 Monte Carlo sequence

The experiment was carried out with Monte Carlo method: from simulated four pairs of data sets, which were transformed according to the description presented above, the subsets of different sizes (25, 50 and 100 observations) were randomly samples (500 times). Calculated percentage of cases for which the null hypothesis should be rejected was the indicator of the performance of examined tests. A value of α level was set to 0.05.

2.4 Experiment results

The results are presented in Tables 3-8. Tables 3-5 present cases with first data set (25, 50 and 100 observations in subsets) to show the sensitivity the tests on the number of observations. Tables 6-8 present cases with second, third and fourth data sets, only for subsets with 50 observations. Rows represents cases for which second set was moved with $\boldsymbol{\mu}$ vector while columns – the cases for which the variance was increased.

3 Conclusions

Permutation tests allow verify the hypothesis with no additional calculations concerning critical values and don't need any knowledge of the distribution of analyzed populations. The tests with pair of statistics (Test 5 – Test 10) have significantly bigger power than with one statistics (Test 4) and are even more efficient than parametric tests (Test 1, Test 2), especially for the samples with lower number of observations because let to identify the

differences of both: mean vector and covariance matrix. Parametric test sensitive to differences of covariance matrix (Test 3) has the biggest power only for data sets with differed covariance matrix (second and third data set). Its advantage decreases for smaller number of observations. Number of cells affects the permutation test power: bigger number increases the power, too big number of cells however significantly reduced the power (Test 10). In this case 30 x 30 cells with 10000 permutations are optimal test parameters (Test 9).

Acknowledgements

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Test power for subsets with 25 observations. First data set.																									
σ^2	1	1.2	1.4	1.6	1.8	1	1.2	1.4	1.6	1.8	1	1.2	1.4	1.6	1.8	1	1.2	1.4	1.6	1.8	1	1.2	1.4	1.6	1.8
μ	Test 1					Test 2					Test 3					Test 4					Test 5				
0	0.02	0.03	0.04	0.08	0.04	0.02	0.04	0.04	0.08	0.04	0.06	0.01	0.09	0.16	0.35	0.02	0.04	0.04	0.08	0.04	0.01	0.06	0.18	0.36	0.52
0.1	0.07	0.05	0.12	0.05	0.10	0.08	0.05	0.11	0.06	0.11	0.05	0.05	0.06	0.14	0.28	0.08	0.05	0.11	0.06	0.11	0.08	0.07	0.19	0.27	0.46
0.2	0.14	0.14	0.17	0.16	0.12	0.14	0.16	0.18	0.16	0.13	0.05	0.06	0.13	0.18	0.30	0.14	0.16	0.18	0.16	0.13	0.12	0.14	0.16	0.41	0.62
0.3	0.38	0.42	0.43	0.31	0.26	0.40	0.42	0.42	0.29	0.23	0.10	0.08	0.15	0.15	0.27	0.40	0.42	0.42	0.29	0.24	0.22	0.27	0.36	0.40	0.60
0.4	0.71	0.71	0.59	0.60	0.63	0.70	0.72	0.61	0.60	0.61	0.05	0.10	0.10	0.19	0.35	0.69	0.73	0.61	0.60	0.61	0.55	0.55	0.52	0.61	0.71
0.5	0.94	0.88	0.90	0.80	0.74	0.95	0.87	0.90	0.81	0.74	0.03	0.09	0.15	0.30	0.38	0.94	0.88	0.90	0.81	0.74	0.85	0.71	0.82	0.74	0.91
0.6	0.98	1.00	0.96	0.96	0.92	0.98	1.00	0.96	0.95	0.92	0.06	0.04	0.09	0.19	0.24	0.98	1.00	0.96	0.95	0.92	0.95	0.95	0.87	0.92	0.95
μ	Test 6					Test 7					Test 8					Test 9					Test 10				
0	0.05	0.09	0.23	0.43	0.56	0.05	0.09	0.26	0.47	0.64	0.05	0.12	0.32	0.51	0.67	0.07	0.13	0.30	0.51	0.71	0.01	0.01	0.01	0.01	0.02
0.1	0.09	0.06	0.23	0.36	0.53	0.08	0.09	0.28	0.39	0.56	0.10	0.11	0.35	0.44	0.64	0.12	0.12	0.37	0.43	0.65	0.01	0.00	0.02	0.02	0.01
0.2	0.12	0.14	0.23	0.48	0.59	0.13	0.22	0.33	0.49	0.66	0.16	0.24	0.36	0.56	0.69	0.16	0.34	0.39	0.62	0.68	0.01	0.01	0.05	0.04	0.01
0.3	0.28	0.37	0.44	0.51	0.66	0.33	0.42	0.50	0.57	0.69	0.40	0.48	0.56	0.56	0.71	0.44	0.55	0.58	0.67	0.74	0.05	0.03	0.05	0.06	0.02
0.4	0.56	0.59	0.59	0.67	0.76	0.68	0.68	0.67	0.72	0.79	0.70	0.72	0.70	0.76	0.80	0.79	0.77	0.71	0.83	0.82	0.08	0.12	0.12	0.09	0.09
0.5	0.90	0.78	0.88	0.81	0.90	0.91	0.83	0.91	0.89	0.95	0.94	0.87	0.93	0.91	0.94	0.94	0.90	0.95	0.92	0.94	0.14	0.13	0.14	0.12	0.12
0.6	0.95	0.96	0.91	0.92	0.96	0.97	0.97	0.96	0.97	0.97	0.96	0.98	0.97	0.98	0.98	0.97	0.98	0.97	0.99	0.98	0.13	0.14	0.13	0.14	0.12

Table 3 Test power for subsets with 25 observations and first data set.

Test power for subsets with 50 observations. First data set.																									
σ^2	1	1.2	1.4	1.6	1.8	1	1.2	1.4	1.6	1.8	1	1.2	1.4	1.6	1.8	1	1.2	1.4	1.6	1.8	1	1.2	1.4	1.6	1.8
μ	Test 1					Test 2					Test 3					Test 4					Test 5				
0	0.02	0.03	0.02	0.00	0.01	0.02	0.03	0.01	0.00	0.01	0.05	0.11	0.22	0.43	0.77	0.02	0.03	0.01	0.00	0.01	0.03	0.09	0.26	0.51	0.68
0.1	0.09	0.09	0.10	0.07	0.09	0.10	0.08	0.09	0.07	0.08	0.02	0.07	0.28	0.54	0.81	0.10	0.08	0.09	0.07	0.09	0.06	0.14	0.28	0.47	0.78
0.2	0.32	0.31	0.23	0.21	0.21	0.33	0.29	0.22	0.21	0.22	0.03	0.09	0.29	0.44	0.80	0.33	0.29	0.22	0.21	0.22	0.27	0.26	0.30	0.53	0.69
0.3	0.82	0.80	0.77	0.61	0.63	0.82	0.81	0.74	0.64	0.63	0.06	0.11	0.29	0.55	0.83	0.82	0.80	0.74	0.64	0.63	0.71	0.63	0.70	0.76	0.87
0.4	0.98	0.97	0.99	0.93	0.94	0.98	0.97	0.98	0.93	0.94	0.03	0.11	0.35	0.60	0.85	0.98	0.97	0.98	0.93	0.94	0.97	0.93	0.97	0.97	0.97
0.5	1.00	1.00	1.00	1.00	0.99	1.00	1.00	1.00	1.00	1.00	0.01	0.03	0.21	0.57	0.80	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.99
0.6	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.03	0.12	0.30	0.47	0.78	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
μ	Test 6					Test 7					Test 8					Test 9					Test 10				
0	0.02	0.16	0.27	0.55	0.78	0.01	0.16	0.30	0.59	0.80	0.02	0.18	0.35	0.64	0.81	0.03	0.20	0.35	0.67	0.84	0.00	0.00	0.00	0.00	0.00
0.1	0.07	0.12	0.41	0.52	0.83	0.09	0.17	0.45	0.57	0.84	0.10	0.21	0.46	0.62	0.89	0.13	0.26	0.50	0.66	0.90	0.00	0.00	0.02	0.02	0.02
0.2	0.24	0.33	0.38	0.58	0.77	0.30	0.40	0.46	0.66	0.82	0.39	0.44	0.50	0.73	0.85	0.37	0.44	0.50	0.76	0.89	0.06	0.06	0.03	0.03	0.02
0.3	0.76	0.73	0.77	0.86	0.94	0.84	0.84	0.82	0.91	0.98	0.90	0.87	0.88	0.95	0.95	0.89	0.85	0.88	0.92	0.96	0.09	0.13	0.13	0.08	0.11
0.4	0.98	0.92	0.97	0.99	0.98	0.97	0.96	0.98	1.00	0.99	0.98	0.99	1.00	1.00	0.99	0.99	0.99	1.00	1.00	0.99	0.14	0.13	0.13	0.13	0.13
0.5	1.00	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.14	0.14	0.14	0.14	0.14
0.6	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.14	0.14	0.14	0.14	0.14

Table 4 Test power for subsets with 50 observations and first data set.

Test power for subsets with 100 observations. First data set.																									
σ^2	1	1.2	1.4	1.6	1.8	1	1.2	1.4	1.6	1.8	1	1.2	1.4	1.6	1.8	1	1.2	1.4	1.6	1.8	1	1.2	1.4	1.6	1.8
μ	Test 1					Test 2					Test 3					Test 4					Test 5				
0	0.02	0.02	0.00	0.01	0.01	0.04	0.02	0.00	0.01	0.01	0.02	0.12	0.64	0.96	1.00	0.04	0.02	0.00	0.01	0.01	0.03	0.09	0.22	0.71	0.92
0.1	0.13	0.13	0.10	0.05	0.12	0.12	0.12	0.10	0.05	0.12	0.05	0.09	0.66	1.00	1.00	0.12	0.12	0.10	0.05	0.12	0.12	0.11	0.43	0.71	0.94
0.2	0.84	0.76	0.73	0.67	0.49	0.83	0.76	0.74	0.68	0.52	0.03	0.15	0.64	0.97	0.99	0.84	0.77	0.74	0.68	0.52	0.76	0.72	0.67	0.95	0.94
0.3	1.00	1.00	0.98	1.00	0.95	1.00	1.00	0.98	1.00	0.96	0.04	0.16	0.67	0.98	1.00	1.00	1.00	0.99	1.00	0.96	0.99	1.00	0.97	0.98	1.00
0.4	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.01	0.17	0.62	0.96	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.5	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.00	0.14	0.73	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.6	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.04	0.11	0.64	0.98	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
μ	Test 6					Test 7					Test 8					Test 9					Test 10				
0	0.01	0.04	0.24	0.68	0.95	0.08	0.07	0.26	0.77	0.97	0.06	0.07	0.33	0.77	0.99	0.07	0.16	0.35	0.83	0.99	0.00	0.00	0.00	0.00	0.00
0.1	0.11	0.09	0.39	0.71	0.93	0.12	0.15	0.44	0.82	1.00	0.13	0.19	0.46	0.81	1.00	0.13	0.19	0.49	0.82	0.99	0.02	0.01	0.02	0.02	0.02
0.2	0.78	0.72	0.78	0.95	0.98	0.88	0.79	0.82	0.96	0.99	0.87	0.87	0.89	0.97	1.00	0.89	0.83	0.88	0.99	1.00	0.12	0.08	0.08	0.08	0.07
0.3	0.99	1.00	0.98	1.00	1.00	1.00	1.00	0.99	1.00	1.00	1.00	1.00	0.99	1.00	1.00	0.99	1.00	1.00	1.00	1.00	0.14	0.14	0.14	0.14	0.14
0.4	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.14	0.14	0.14	0.14	0.14
0.5	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.14	0.14	0.14	0.14	0.14
0.6	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.14	0.14	0.14	0.14	0.14

Table 5 Test power for subsets with 100 observations and first data set.

Test power for subsets with 50 observations. Second data set.																									
σ^2	1	1.2	1.4	1.6	1.8	1	1.2	1.4	1.6	1.8	1	1.2	1.4	1.6	1.8	1	1.2	1.4	1.6	1.8	1	1.2	1.4	1.6	1.8
μ	Test 1					Test 2					Test 3					Test 4					Test 5				
0	0.04	0.02	0.03	0.03	0.06	0.04	0.02	0.02	0.04	0.06	0.19	0.29	0.57	0.82	0.96	0.04	0.02	0.02	0.04	0.06	0.05	0.04	0.34	0.54	0.79
0.1	0.10	0.03	0.08	0.05	0.02	0.10	0.03	0.08	0.05	0.02	0.22	0.29	0.58	0.81	0.92	0.10	0.03	0.08	0.05	0.02	0.07	0.02	0.36	0.42	0.71
0.2	0.24	0.17	0.19	0.11	0.12	0.24	0.14	0.17	0.11	0.11	0.21	0.36	0.54	0.77	0.91	0.24	0.14	0.18	0.12	0.11	0.15	0.19	0.26	0.52	0.80
0.3	0.51	0.45	0.45	0.46	0.33	0.52	0.43	0.42	0.44	0.32	0.22	0.36	0.46	0.77	0.95	0.52	0.43	0.43	0.44	0.32	0.46	0.42	0.44	0.67	0.82
0.4	0.92	0.91	0.79	0.73	0.66	0.92	0.92	0.79	0.74	0.66	0.12	0.28	0.54	0.71	0.92	0.92	0.92	0.79	0.74	0.67	0.83	0.83	0.72	0.82	0.90
0.5	0.98	0.98	0.99	0.94	0.91	0.98	0.98	0.99	0.94	0.91	0.19	0.27	0.56	0.79	0.93	0.98	0.98	0.99	0.94	0.91	0.98	0.97	0.97	0.95	0.98
0.6	1.00	1.00	0.99	0.99	0.98	1.00	1.00	0.99	0.98	0.98	0.20	0.34	0.56	0.81	0.93	1.00	1.00	0.99	0.98	0.98	1.00	1.00	0.99	1.00	1.00
μ	Test 6					Test 7					Test 8					Test 9					Test 10				
0	0.06	0.04	0.34	0.64	0.82	0.08	0.06	0.36	0.67	0.86	0.12	0.07	0.42	0.68	0.89	0.11	0.10	0.45	0.78	0.88	0.00	0.00	0.00	0.01	0.01
0.1	0.06	0.06	0.37	0.45	0.77	0.13	0.07	0.50	0.58	0.81	0.14	0.10	0.51	0.54	0.84	0.17	0.11	0.50	0.57	0.87	0.01	0.00	0.02	0.01	0.00
0.2	0.19	0.22	0.34	0.59	0.83	0.26	0.28	0.34	0.65	0.85	0.30	0.34	0.40	0.68	0.83	0.33	0.38	0.44	0.66	0.90	0.04	0.04	0.05	0.01	0.04
0.3	0.46	0.38	0.54	0.72	0.82	0.54	0.46	0.62	0.79	0.85	0.59	0.49	0.65	0.81	0.88	0.60	0.54	0.69	0.82	0.88	0.08	0.07	0.06	0.06	0.04
0.4	0.85	0.86	0.81	0.86	0.94	0.91	0.93	0.85	0.90	0.95	0.94	0.93	0.89	0.92	0.95	0.96	0.94	0.90	0.92	0.97	0.13	0.13	0.11	0.11	0.06
0.5	0.98	0.97	0.97	0.98	0.98	0.98	0.99	0.98	0.98	0.98	0.98	1.00	0.99	0.98	0.99	0.98	0.99	0.99	0.98	0.99	0.14	0.14	0.14	0.13	0.14
0.6	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.14	0.14	0.14	0.14	0.14

Table 6 Test power for subsets with 50 observations and second data set.

Test power for subsets with 50 observations. Third data set.																									
σ^2	1	1.2	1.4	1.6	1.8	1	1.2	1.4	1.6	1.8	1	1.2	1.4	1.6	1.8	1	1.2	1.4	1.6	1.8	1	1.2	1.4	1.6	1.8
μ	Test 1					Test 2					Test 3					Test 4					Test 5				
0	0.04	0.05	0.02	0.06	0.06	0.04	0.05	0.02	0.07	0.06	0.95	0.98	0.98	1.00	1.00	0.04	0.05	0.02	0.07	0.06	0.09	0.13	0.27	0.57	0.80
0.1	0.08	0.06	0.02	0.06	0.04	0.08	0.06	0.03	0.07	0.04	0.98	0.91	0.99	0.99	1.00	0.08	0.06	0.03	0.07	0.04	0.06	0.13	0.28	0.52	0.79
0.2	0.21	0.14	0.19	0.09	0.11	0.22	0.15	0.16	0.07	0.12	0.96	0.99	0.98	0.98	1.00	0.23	0.16	0.16	0.07	0.12	0.12	0.20	0.42	0.64	0.74
0.3	0.46	0.47	0.39	0.30	0.35	0.43	0.48	0.38	0.33	0.35	0.94	0.98	0.99	0.99	1.00	0.43	0.48	0.38	0.33	0.35	0.32	0.39	0.44	0.69	0.82
0.4	0.85	0.83	0.68	0.68	0.61	0.87	0.83	0.68	0.67	0.56	0.98	0.98	0.97	0.99	1.00	0.87	0.83	0.68	0.67	0.58	0.69	0.73	0.69	0.76	0.86
0.5	0.99	0.89	0.90	0.90	0.93	0.99	0.89	0.89	0.91	0.93	0.94	0.94	1.00	1.00	0.99	0.99	0.89	0.89	0.91	0.93	0.95	0.88	0.86	0.92	0.98
0.6	0.99	1.00	1.00	0.98	0.95	0.99	1.00	1.00	0.98	0.95	0.94	0.99	0.98	1.00	0.99	0.99	1.00	1.00	0.98	0.95	0.99	0.99	0.99	0.96	0.98
μ	Test 6					Test 7					Test 8					Test 9					Test 10				
0	0.14	0.14	0.32	0.68	0.89	0.14	0.17	0.36	0.67	0.91	0.16	0.17	0.36	0.73	0.92	0.18	0.23	0.38	0.73	0.93	0.00	0.00	0.00	0.02	0.02
0.1	0.11	0.15	0.34	0.58	0.89	0.12	0.18	0.43	0.67	0.90	0.15	0.24	0.51	0.77	0.93	0.14	0.19	0.54	0.77	0.92	0.02	0.02	0.01	0.01	0.01
0.2	0.15	0.21	0.48	0.66	0.77	0.23	0.28	0.53	0.72	0.82	0.26	0.33	0.55	0.74	0.82	0.30	0.40	0.57	0.78	0.86	0.02	0.01	0.02	0.00	0.01
0.3	0.37	0.49	0.55	0.74	0.87	0.49	0.60	0.61	0.73	0.93	0.52	0.64	0.62	0.79	0.96	0.57	0.66	0.65	0.82	0.96	0.04	0.05	0.05	0.04	0.04
0.4	0.78	0.80	0.71	0.84	0.92	0.87	0.85	0.82	0.87	0.94	0.92	0.88	0.84	0.90	0.95	0.91	0.90	0.86	0.91	0.95	0.09	0.12	0.10	0.08	0.08
0.5	0.95	0.92	0.92	0.95	1.00	0.98	0.93	0.96	0.95	0.99	0.99	0.95	0.97	0.96	0.99	0.99	0.95	0.97	0.99	0.99	0.14	0.14	0.14	0.14	0.14
0.6	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	0.99	1.00	0.99	0.99	0.99	1.00	1.00	0.99	0.99	1.00	0.99	0.99	0.14	0.14	0.14	0.14	0.14

Table 7 Test power for subsets with 50 observations and third data set.

Test power for subsets with 50 observations. Fourth data set.																									
σ^2	1	1.2	1.4	1.6	1.8	1	1.2	1.4	1.6	1.8	1	1.2	1.4	1.6	1.8	1	1.2	1.4	1.6	1.8	1	1.2	1.4	1.6	1.8
μ	Test 1					Test 2					Test 3					Test 4					Test 5				
0	0.07	0.04	0.13	0.08	0.05	0.07	0.04	0.13	0.06	0.05	0.11	0.13	0.57	0.90	0.99	0.08	0.04	0.14	0.07	0.06	0.01	0.17	0.66	1.00	1.00
0.1	0.63	0.47	0.41	0.46	0.33	0.62	0.46	0.41	0.46	0.31	0.14	0.21	0.58	0.84	1.00	0.58	0.50	0.46	0.49	0.33	0.18	0.52	0.85	0.99	1.00
0.2	0.98	0.96	0.95	0.95	0.90	0.98	0.96	0.95	0.95	0.91	0.05	0.17	0.64	0.85	0.99	0.98	0.96	0.95	0.97	0.92	0.77	0.94	0.98	1.00	1.00
0.3	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.08	0.20	0.53	0.85	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.4	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.09	0.21	0.45	0.87	0.98	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.5	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.06	0.17	0.51	0.83	0.96	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.6	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.03	0.12	0.44	0.90	0.96	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
μ	Test 6					Test 7					Test 8					Test 9					Test 10				
0	0.01	0.12	0.69	0.98	1.00	0.06	0.23	0.78	0.98	1.00	0.03	0.31	0.90	0.99	1.00	0.10	0.39	0.93	1.00	1.00	0.01	0.00	0.02	0.00	0.00
0.1	0.30	0.49	0.88	0.99	0.99	0.34	0.52	0.92	0.99	0.99	0.43	0.71	0.97	1.00	1.00	0.45	0.73	1.00	1.00	1.00	0.09	0.07	0.06	0.07	0.05
0.2	0.79	0.94	0.97	1.00	1.00	0.88	0.93	0.97	1.00	1.00	0.90	0.97	1.00	1.00	1.00	0.93	0.98	0.99	1.00	1.00	0.14	0.12	0.13	0.14	0.12
0.3	0.99	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.14	0.14	0.14	0.14	0.14
0.4	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.14	0.14	0.14	0.14	0.14
0.5	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.14	0.14	0.14	0.14	0.14
0.6	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.14	0.14	0.14	0.14	0.14

Table 8 Test power for subsets with 50 observations and fourth data set.

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Fuzzified AHP in evaluation of R&D outputs - a case from Palacky University in Olomouc

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Abstract. In this paper we present a method developed at the Faculty of Science, Palacky University in Olomouc for the assessment of scientific quality of books (for the purposes of funds distribution among departments). The method is based on the fuzzified AHP used to determine fuzzy weights of predefined categories of publishers prior to the evaluation process. The weights reflect the reputation of the publishers from the respective category. The following evaluation process then combines expert assessment (peer review) with the less subjective criterion of publishers reputation. In the first stage of the two-stage evaluation process, each book is evaluated by a fuzzy number which corresponds with the reputation of the publisher and is interpreted as an interval of possible evaluations - scores (the support) with a most typical score (default evaluation - the only element in the kernel of the fuzzy number). In the second stage, the default evaluation can be altered during the peer review process within the proposed interval of scores and the reasons for changes (if made) are reported. The whole development process of the method will be briefly summarized to stress the possibility of using AHP (and its fuzzified version) for visualisation and interpretation of expert preferences expressed in terms of intuitively set intervals of scores.

Keywords: AHP, fuzzy, R&D, evaluation, MCDM, management.

JEL classification: C44
AMS classification: 90C15

1 Research and development evaluation in the Czech Republic

The evaluation of R&D outputs is a complex task. In the Czech Republic, the evaluation of R&D outputs for the purposes of funds distribution among research institutions (including universities) is regulated by the Methodology for evaluation of R&D outputs [8], that is under constant development as the issues of quality assessment are becoming more and more important (see also [1]). Although some outputs are assessed according to criteria that can be assumed to correlate with scientific quality (e.g. the evaluation of papers in journals with nonzero impact factor), others are assigned a fixed score (this is the case of patents, books etc.).

On the task of the evaluation of books we will show our proposal how to combine expert assessment (peer review) with other criteria that are easier to assess - in this case the reputation of the publisher of the book. The first stage of the evaluation is based on a classification of publishers into 4 categories according to their reputation. The classification was provided by a board of experts prior to the development of the presented mathematical model. Based on pairwise comparison the categories were assigned intervals of scores (represented by triangular fuzzy numbers) with "default" evaluation for typical books published by the publishers from the current category. In the second stage, the peer review process is used to adjust the evaluation of the book according to its scientific quality within predefined limits for each category of publishers. The underlying mathematical apparatus used to reflect the preferences of the board of evaluators and to derive default evaluations for books from a given category of publishers is described

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in Sections 2 and 3. The mathematical model is based on Saaty's AHP method (see [9, 10]) and its fuzzification that was introduced in [4]. Section 4 summarizes the computation of scores for categories of publishers at Palacky University in Olomouc (UP), discussess the results and presents the consecutive two stage evaluation procedure for books, that is currently being used at the Faculty of Science of UP [5].

2 Pairwise comparison matrices

Let us consider a set of objects $K_1, K_2, \ldots, K_n$ for which we need to find evaluations $h_1, h_2, \ldots, h_n$. If these evaluations are known, we can construct the multiplicative matrix of relative preferences $H = \{h_{ij}\}_{i,j=1}^n$ such that $h_{ij} = \frac{h_i}{h_j}$. The elements of such a matrix describe the relative preference of K_i over K_j. The evaluations $h_1, h_2, \ldots, h_n$ are however usually not known. To determine the values of $h_1, h_2, \ldots, h_n$ we can construct the Saaty's matrix $S = \{s_{ij}\}_{i,j=1}^n$, where the elements s_{ij} (provided by experts) describe the estimated ratio of the evaluation of K_i to the evaluation of K_j. As such, s_{ij} are expert estimations of the actual elements h_{ij} of the matrix H, hence we require the matrix S to be reciprocal, i.e. $s_{ij} = \frac{1}{s_{ji}}$ for all $i, j = 1, 2, \ldots, n$. It is easy to see that we need to set only $\frac{n(n-1)}{2}$ elements of the matrix to provide all the information necessary to complete it. For the purpose of expressing expert's intensities of preferences between pairs of objects, Saaty [9, 10] proposes to use the scale shown in Table 1. The decision maker can also use the intermediate values $2, 4, 6$ and 8 with the respective intermediate linguistic meanings (e.g. "between moderately and strongly" for 4).

numerical value of s_{ij}	linguistic description
1	decision maker is indifferent between K_i and K_j
3	K_i is moderately preferred to K_j
5	K_i is strongly preferred to K_j
7	K_i is very strongly preferred to K_j
9	K_i is absolutely preferred to K_j

Table 1 Saaty's scale.

Saaty defines the full consistency of the matrix S by $s_{ik} = s_{ij} \cdot s_{jk}$, for all $i, j, k = 1, 2, \ldots, n$. Full consistency is, however, unachievable for larger matrices. Saaty therefore introduces an inconsistency index $CI = (\lambda_{\max} - n)/(n-1)$, where n is the order of S (i.e. the number of objects that are being compared). A Saaty's matrix S is considered to be consistent enough, when it's inconsistency ratio $CR = CI/RI < 0.1$, where RI is the so called random inconsistency index representing the inconsistency of a randomly generated reciprocal pairwise comparison matrix of the order n. Other approaches to the assessment of consistency of Saaty's matrices can be found in the literature (see [2] for a comparison). Stoklasa et al. suggest the concept of weak consistency in [11, 12] as a minimum requirement on the consistency of the expertly defined Saaty's matrix S. Saaty's matrix of preference intensities is weakly consistent, if and only if for all $i, j, k \in \{1, 2, \ldots, n\}$ the following holds:

$$s_{ij} > 1 \wedge s_{jk} > 1 \quad \Longrightarrow \quad s_{ik} \geq \max\{s_{ij}, s_{jk}\}; \tag{1}$$

$$(s_{ij} = 1 \wedge s_{jk} \geq 1) \vee (s_{ij} \geq 1 \wedge s_{jk} = 1) \quad \Longrightarrow \quad s_{ik} = \max\{s_{ij}, s_{jk}\}. \tag{2}$$

This weak consistency can be checked during the input of the preference intensities by experts and represents a weakening of Saaty's notion of full consistency (for more details see [11, 12]). As follows from the Perron-Frobenius theorem, Saaty' matrix always has a maximum real eigenvalue (spectral radius - see [7]). A fully consistent Saaty's matrix has a single nonzero eigenvalue $\lambda_{\max}$, which is equal to the order of the matrix. The eigenvector corresponding to this maximum eigenvalue represents the evaluations $h_1, h_2, \ldots, h_n$ (see [9, 10]).

Alternatively to find $h_1, h_2, \ldots, h_n$ based on the expertly defined matrix S, we need to find the arguments of the minimum of the following expression:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left(s_{ij} - \frac{h_i}{h_j} \right)^2. \tag{3}$$

The minimisation problem (3) can be rewritten into:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\ln(s_{ij})-\ln(h_i)-\ln(h_j)\right)^2, \tag{4}$$

thus obtaining the logarithmic least square problem, for which the solution can be found in the form of (5), see [7] for more details.

$$h_i = \sqrt[n]{\prod_{j=1}^{n} s_{ij}} \tag{5}$$

3 Fuzzy pairwise comparison matrices

A fuzzy set A on a nonempty universal set U is defined by a mapping $A : U \to [0, 1]$. For each $x \in U$ the value $A(x)$ is called a membership degree of the element x in the fuzzy set A; $A(.)$ is called a membership function of the fuzzy set A. A fuzzy number B is a fuzzy set on $\mathbb{R}$ satisfying three conditions: 1) the kernel of B, $\mathrm{Ker}(B) = \{x \in \mathbb{R}|B(x) = 1\}$ is a nonempty set, 2) the α-cuts of B, $B_\alpha = \{x \in \mathbb{R}|B(x) \geq \alpha\}$ are closed intervals for all $\alpha \in (0, 1]$, and 3) the support of the fuzzy set B, $\mathrm{Supp}(B) = \{x \in \mathbb{R}|B(x) > 0\}$ is bounded. A triangular fuzzy number B has a membership function in the form:

$$B(x) = \begin{cases} 0, & x < b_1 \\ \frac{x-b_1}{b_2-b_1}, & b_1 \leq x \leq b_2 \\ 1, & x = b_2 \\ \frac{b_3-x}{b_3-b_2}, & b_2 \leq x \leq b_3 \\ 0, & x > b_3, \end{cases} \tag{6}$$

and can be represented by the triplet of its significant values $B = (b_1, b_2, b_3)$.

fuzzy value of s_{ij}	linguistic description
$(\frac{1}{3}, 1, 3)$	decision maker is indifferent between K_i and K_j
$(1, 3, 5)$	K_i is moderately preferred to K_j
$(3, 5, 7)$	K_i is strongly preferred to K_j
$(5, 7, 9)$	K_i is very strongly preferred to K_j
$(7, 9, 9)$	K_i is absolutely preferred to K_j

Table 2 Properly fuzzified Saaty's scale [4].

Saaty's method allows the experts to express their preferences using linguistic terms Table 1. It is surely more appropriate to represent the uncertainty of the linguistic terms not by real numbers (that is by elements from the set $\{1, 3, 5, 7, 9\}$), but by triangular fuzzy numbers. The support of each of the fuzzy numbers used is an interval defined by the single real values from the kernels of the neighboring fuzzy numbers (see Table 2). The meaning of "is indifferent between" is defined in a way such that the reciprocity of the Saaty's matrix is preserved. The fuzzification of a Saaty's scale with intermediate values (numerical and linguistic) is analogical.

There are also many ways how to assess the consistency of fuzzy Saaty's matrices (see e.g. [3, 6]). For the purposes of this application, we can approach consistency in the fuzzy case in the same way as in the crisp case - using just the real values from the kernels of the respective fuzzy number (middle significant values). Again, it is reasonable to require the matrix S to be weakly consistent - that is to require (1) and (2) to hold for the middle significant values of the fuzzy numbers.

The evaluations of the objects computed from this fuzzy Saaty's matrix will be fuzzy numbers $\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n$. For simplicity and easier interpretability we can approximate them by triangular fuzzy

numbers, that is $\tilde{h}_i = (h_{i1}, h_{i2}, h_{i3})$ for all $i = 1, 2, \ldots, n$. The significant values of these triangular fuzzy numbers can then be computed using the formulas (7), (8) and (9) proposed by Krejčí in [4].

$$h_{i1} = \min \left\{ \sqrt[n]{\prod_{j=1}^{n} s_{ij}^*} \Big/ \sum_{k=1}^{n} \sqrt[n]{\prod_{j=1}^{n} s_{kj}^*} \; ; \; s_{kj}^* \in \langle s_{kj1}, s_{kj3} \rangle, k = 1, \ldots, n, \; j = 1, \ldots, n, \; s_{kj}^* = \frac{1}{s_{jk}^*} \right\} \quad (7)$$

$$h_{i2} = \sqrt[n]{\prod_{j=1}^{n} s_{ij2}} \Big/ \sum_{k=1}^{n} \sqrt[n]{\prod_{j=1}^{n} s_{kj2}} \quad (8)$$

$$h_{i3} = \max \left\{ \sqrt[n]{\prod_{j=1}^{n} s_{ij}^*} \Big/ \sum_{k=1}^{n} \sqrt[n]{\prod_{j=1}^{n} s_{kj}^*} \; ; \; s_{kj}^* \in \langle s_{kj1}, s_{kj3} \rangle, k = 1, \ldots, n, \; j = 1, \ldots, n, \; s_{kj}^* = \frac{1}{s_{jk}^*} \right\} \quad (9)$$

This method of computing fuzzy evaluations of objects provides as a result fuzzy numbers, which are less uncertain that the outputs of other approaches available in the literature. The condition $s_{kj}^* = \frac{1}{s_{jk}^*}$ in (7) and (9) ensures, that no unnecessary uncertainty is added by the computation.

4 Evaluation of books at Palacky University

In this section we will show how the fuzzified AHP can be used in the evaluation of R&D outcomes (books) and what benefits there might be in using fuzzy evaluation method for these purposes. A practical example from UP will be presented. We will stress how the outputs of the fuzzified AHP can then be used in a peer-review process to reflect the quality of the current R&D output.

4.1 Definition of the problem

This paper strives to suggest an evaluation methodology for R&D outcomes that would substitute the current assignment of fixed number of points to each book (see [8]) regardless of its quality. We propose a more objective approach to the evaluation. One that assesses the quality of the publisher (represented by its reputation) and combines it with the assessment of the quality of the book itself through peer review process. The former is achieved by categorizing publishers into 4 categories according to their reputation and by expressing preferences between the categories. As the publication process involves a detailed review of the book, this criterion is taken as a basis for the evaluation. Each book is assigned an initial evaluation according to its publisher's reputation. The fact that the scientific quality may vary for books from the same publisher category can subsequently be reflected in the peer review process.

Four categories of publishers are defined according to their reputation (*Category 1* being of highest reputation, ..., *Category 4* having the lowest reputation, but still fulfilling all the requirements for scientific publishers). For each category, a comprehensive description and a few "typical" members are provided for the purpose of mutual comparison of categories. In the case of UP, Faculty of Science, the board of experts responsible for the evaluation of R&D outcomes agreed on initial intervals of scores, that books from each category of publishers might be assigned. These intervals are presented as (10). Although the intervals of scores (10) are a result of consensus of experts, it is not obvious what preference structure underlies them. As such these were not considered satisfactory for the purposes of evaluation, as the evaluation using such intervals is in fact a "black box" impossible to be understood, so the correctness of such approach is difficult to assess.

$$\begin{aligned} &\text{Category 1:} && 50 - 75 \text{ points}, \\ &\text{Category 2:} && 30 - 40 \text{ points}, \\ &\text{Category 3:} && 15 - 20 \text{ points}, \\ &\text{Category 4:} && 5 - 10 \text{ points}. \end{aligned} \quad (10)$$

4.2 Preference structure representation

The consensus expressed by (10) is however a useful input for analysis. To understand this initial idea better we will still require at least confirmation of the results (if not additional information) from group

of experts at some stages. First we can use Saaty's AHP to reconstruct the preference structure and then use it as a starting point for the development of mathematical model. If we take the centers of the intervals as the most typical scores (evaluations) of each category of publishers, calculate the values of the elements of the matrix of relative preferences and round them up to be able to describe them linguistically, we obtain (11). The matrices describe the experts' preferences on the set of categories of publishers thus expressing the desirability of publication in each category of publishers from the point of view of the Faculty.

$$\text{rounded } S = \begin{pmatrix} 1 & 2 & 4 & 8 \\ \frac{1}{2} & 1 & 2 & 5 \\ \frac{1}{4} & \frac{1}{2} & 1 & 2 \\ \frac{1}{8} & \frac{1}{5} & \frac{1}{2} & 1 \end{pmatrix} \rightarrow \text{fuzzified } \tilde{S} = \begin{pmatrix} 1 & (1,2,3) & (3,4,5) & (7,8,9) \\ \left(\frac{1}{3},\frac{1}{2},1\right) & 1 & (1,2,3) & (4,5,6) \\ \left(\frac{1}{5},\frac{1}{4},\frac{1}{3}\right) & \left(\frac{1}{3},\frac{1}{2},1\right) & 1 & (1,2,3) \\ \left(\frac{1}{9},\frac{1}{8},\frac{1}{7}\right) & \left(\frac{1}{6},\frac{1}{5},\frac{1}{4}\right) & \left(\frac{1}{3},\frac{1}{2},1\right) & 1 \end{pmatrix}. \quad (11)$$

Let us note here that in the matrices S and $\tilde{S}$ the columns (and rows) are numbered in accordance with the ordering of the categories, hence ordered according to their preferences already. As we can see from (11) the relative importances of neighboring categories of publishers are the same for all categories and are described by the lowest value associated with preference. These values can be interpreted as "one book in the higher category can be compensated by two books from the neighboring lower category". The experts were provided this interpretation and subsequently were asked to input the pairwise comparison of the neighboring categories using the linguistic terms from Table 2 (we assisted them in completing the matrix to remain weakly consistent and to achieve CR as low as possible). This way we have obtained (12) which shows that the neighboring categories were in the eyes of the experts "further from each other", than was described by (10). The interpretation that "one book from higher category can be compensated by 3 books from the neighboring lower category" was closer to the intention of the expert evaluators (as was confirmed during the discussion with the evaluators concerning the results).

$$\begin{pmatrix} 1 & (1,3,5) & ? & ? \\ \left(\frac{1}{5},\frac{1}{3},1\right) & 1 & (1,3,5) & ? \\ ? & \left(\frac{1}{5},\frac{1}{3},1\right) & 1 & (1,3,5) \\ ? & ? & \left(\frac{1}{5},\frac{1}{3},1\right) & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & (1,3,5) & (3,5,7) & (7,9,9) \\ \left(\frac{1}{5},\frac{1}{3},1\right) & 1 & (1,3,5) & (3,5,7) \\ \left(\frac{1}{7},\frac{1}{5},\frac{1}{3}\right) & \left(\frac{1}{5},\frac{1}{3},1\right) & 1 & (1,3,5) \\ \left(\frac{1}{9},\frac{1}{9},\frac{1}{7}\right) & \left(\frac{1}{7},\frac{1}{5},\frac{1}{3}\right) & \left(\frac{1}{5},\frac{1}{3},1\right) & 1 \end{pmatrix} \quad (12)$$

4.3 Resulting evaluation according to the reputation of publisher - first stage evaluation

Both matrices in (11) are weakly consistent and the Saaty's inconsistency ratio $CR = 0.0023$. We can easily see that by changing s_{24} from 5 to 4, we would obtain an absolutely consistent matrix in Saaty's sense. The consistency ratio for (12), that is in the most important aspects similar to (11), but differs by the use of linguistic labels in the input phase, is 0.0286. If we compute the evaluations using the fuzzified AHP (see Section 3), normalize the evaluations and multiply them by 100 to avoid decimals, we get:

$$\begin{array}{lllll} & \tilde{h}_1 = (41,53,61) & & \tilde{h}_1 = (41,53,62) & \tilde{h}_1 = (38,58,69) \\ \text{from (11)}: & \tilde{h}_2 = (19,28,40) & \text{from (11)}' & \tilde{h}_2 = (18,27,39) & \text{from (12)}: \quad \tilde{h}_2 = (14,25,45) \\ & \tilde{h}_3 = (9,13,20) & \text{abs. consistent:} & \tilde{h}_3 = (9,13,21) & \tilde{h}_3 = (6,11,23) \\ & \tilde{h}_4 = (5,6,9)\,. & & \tilde{h}_4 = (5,7,10) & \tilde{h}_4 = (4,5,10)\,. \end{array}$$

Clearly the evaluations of categories do not differ very much between the slightly inconsistent and absolutely consistent version of (11). As the evaluators agreed that (12) captures their preferences best, we will restrict further interpretation to (12). The initial evaluation of each book (according to the category of its publisher) is now available as a triangular fuzzy number - $\tilde{h}_1, \tilde{h}_2, \tilde{h}_3$ or $\tilde{h}_4$.

4.4 Peer review integration - second stage of the evaluation process

For the purposes of the peer-review process, the support of each of the evaluations of categories of publishers is interpreted as an interval, from which scores for books published by a publisher from the respective category can be chosen. The value corresponding to the element in the kernel of the fuzzy number is then seen as a "default" evaluation - that is evaluation of a book typical as for scientific quality for the publishers from the given category. The peer review process following the first stage is intended

to expertly assess the quality of each book. If during the peer review the book is assessed as better or worse than "typical" for the current category of publishers, the score can be adjusted accordingly within the predefined interval. This way quality of the R&D outcome itself can be reflected.

5 Conclusions

We have presented a method for R&D outputs evaluation, that combines expert assessment of the reputation of the publisher (by expressing preferences between categories of publishers) with the peer review process for quality assessment of the actual R&D output. The fuzzified AHP is used to determine the evaluations based on the reputation of publishers. The quality of the outcome is subsequently assessed by a group of experts. The model is being used at the Faculty of Science of UP. As the described method integrates the evaluation of the medium with the evaluation of the outcome itself, it may also serve as inspiration for the further development of the R&D evaluation methodology in the Czech Republic.

Acknowledgements

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Branching process model in epidemiology

Radmila Stoklasová[1]

Abstract. This paper deals with mathematical models suitable to describe spread of influenza epidemic and validates the application of these models to the data of the Health Institute in Ostrava obtained under System of health and environment monitoring in the Czech Republic for the period 2003 – 2012. The aim is to design a suitable mathematical model of the spread of flu which is valid, i.e. such that the theoretical value, which are generated by the model are sufficiently close to actual observed values and to be able to accurately predict the future value using the model.

Key words: Epidemiology, modeling, classical epidemiological model, time series analysis, seasonal decomposition, Box-Jenkinson time series model.

JEL Classification: C13, C22, C63
AMS Classification: 60G20, 37M10

1 Introduction

After finding a suitable model for the simulation it is necessary to confront the simulated values with the actual values. There could be a problem with the availability of data. This article studied influenza epidemic modeling and data were provided by the Institute of Medicine, based in Ostrava.

National Influenza Center monitors the activity of flu and flu-like illness and performs laboratory isolation of the virus. Some laboratories also perform rapid and express diagnostics. The results are reported regularly to one of 4 international centers of the WHO, which also sends isolates for further antigen and genetic comparisons.

2 Basic characteristics of epidemiological data

Data that are used in this work come from a database Monaro for the years 2003 – 2012. The database contains the code district, day of the examination, year of birth, gender, diagnosis at resolution of acute respiratory diseases and others. The data source is the Institute of Public Health in Ostrava and data are obtained under the monitoring of health and environment of the Czech Republic for the period 2003 – 2012. [4]

It is the daily data of cooperating doctors with surgeries in districts Silesian Ostrava, Moravian Ostrava and Přívoz, Mariánské Hory and Hulváky, Poruba, Ostrava-South. The flu epidemic is examined from the 30th week one year to the 29th week following year.

Daily data on the number of patients were adjusted for a better interpretation of the weekly numbers of patients with acute respiratory diseases. In the period from 30th week 2003 to 53rd week 2012, the average weekly morbidity of acute respiratory illness including influenza was 116 patients per 100000 inhabitants.

3 Kermack – Mc Kendrik's model

This model is in its simplicity generally accepted approximation for the epidemic of infectious disease. Furthermore, following works were used: [5], [7].
We assume the following conditions:

- the disease is transmitted by contact between infected and healthy individual who is not immune against this disease. This one is called susceptible individual;
- all infected individuals may be as likely to transfer disease, as well as all susceptible individuals may get the disease with equal probability;
- for each disease it is important time period that is needed to make the susceptible individual contagious after meeting with infection. It is assumed that the latent period is so small that it can be considered zero.

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Therefore, the susceptible individual becomes infected individual immediately after contact with the disease;

- studied population is closed, i.e. there is no birth of new individuals, without considering any possible movement of people, dead individuals are not excluded from the population. The total number of individuals in the population is constant.

In addition, the following notation is used:
V … Size of the population susceptible to infection (i.e. those individuals who don't suffer from the disease, but they can get sick or become infectious),
E… size of the population in the latent period,
I… size of the population that is infected and capable of transmitting the infection to an individual susceptible to infection,
R… size of the population which become ill, but infection does not spread further, either because they are from other parts of the isolated population (quarantine), or after recovery has become permanently immune.
V, E, I, R are functions of time *t*. It is assumed that they are non-negative. Corresponding part of the population are marked (*V*), (*E*), (*I*), (*R*). The size of the entire population is assumed to be constant and is equal $N > 0$. Therefore $V(t) + E(t) + I(t) + R(t) \equiv N$.

(*V*)	(*E*)	(*I*)		(*R*)
period of individual's susceptibility to disease	latent period	contagious period		isolation period
	incubation period		disease with symptoms	

An important concept is the epidemic curve $W(t)$, which describes changes in the number of infected individuals. This curve is given by the equation $W(t) = \frac{dI(t)}{dt} = \alpha I(t)(N - I(t))$.

From which we get

$$W(t) = \frac{I_0 \alpha (N - I_0) e^{\alpha N t}}{\left(1 - \frac{I_0}{N}\left(1 - e^{\alpha N t}\right)\right)^2}. \qquad (1)$$

Epidemic curve $W(t)$ reaches its maximum for $I(t) = \frac{N}{2}$.

The calculations show that if $I_0 < \frac{N}{2}$, then the epidemic curve reaches its maximum at point $t^* = \frac{1}{\alpha N} \ln \frac{N - I_0}{I_0}$, so in this time the epidemic is spreading quickly and $W(t^*) = \frac{\alpha N^2}{4}$. If $I_0 > \frac{N}{2}$, then $t^* < 0$, then function $W(t)$ in time $t \in (0, \infty)$ does not reach its maximum.

Furthermore it can be shown that the epidemic curve $W(t)$ is symmetric at its maximum.

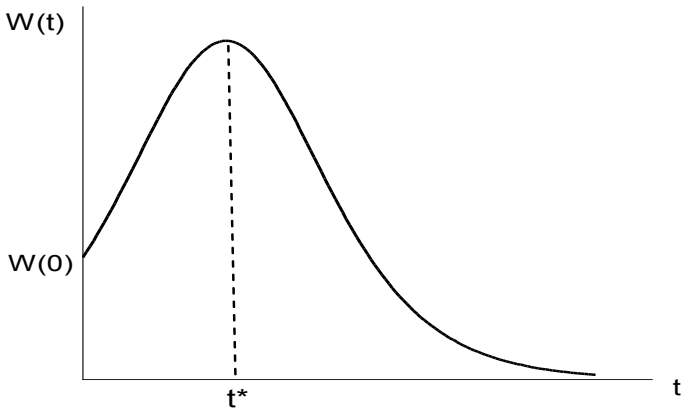

Figure 1 Epidemic Curve

For $t = 0$ the value of the epidemic curve is $W(0) = \alpha(N - I_0)I_0$ and reaches its maximum at the point t^*, where $W(t^*) = \frac{\alpha N^2}{4}$.

The system of three differential equations (2) – (4) is known as Kermack - Mc Kendrik's Model:

$$I'(t) = \alpha I(t)V(t) - \beta I(t), \tag{2}$$
$$R'(t) = \beta I(t), \tag{3}$$
$$V'(t) = -\alpha I(t)V(t). \tag{4}$$

Due to the closed condition of the population we can reduce the number of differential equations. For example $V(t) = N - I(t) - R(t)$. Substituting $V(t)$ in equation (2) leads to a system of two differential equations

$$I'(t) = I(t)(\alpha N - \beta - \alpha I(t) - \alpha R(t)), \tag{5}$$
$$R'(t) = \beta I(t). \tag{6}$$

The coefficients α and β are positive. It is assumed that $V(0) = V_0 > 0$, $I(0) = I_0 > 0$, $R(0) = 0$, $V_0 + I_0 = N$. Each initial problem for (5), (6) has the only complete solution.

Expression dependence *I* on *R* shows a linear differential equation

$$\frac{dI}{dR} = \frac{\alpha}{\beta}N - 1 - \frac{\alpha}{\beta}R - \frac{\alpha}{\beta}I. \tag{7}$$

The general solution is

$$I = h(R) = ke^{-\frac{\alpha}{\beta}R} - R + N. \tag{8}$$

Constant *k* can be determined by substituting the initial conditions: $I(0) = I_0, V(0) = V_0, R(0) = 0$, into (8) and using the conditions of closure and finality of the population at time $t = 0$; so $I_0 = k + N = k + V_0 + I_0$. From this relationship can be concluded: $k = -V_0$. Solution (7) is

$$I = -V_0 e^{-\frac{\alpha}{\beta}R} - R + N. \tag{9}$$

There is the following equivalence:

$$\frac{\beta}{\alpha}\ln\frac{\alpha}{\beta}V_0 > 0 \quad \Leftrightarrow \quad \ln\frac{\alpha}{\beta}V_0 > 0 \quad \Leftrightarrow \quad \frac{\alpha}{\beta}V_0 > 1 \quad \Leftrightarrow \quad V_0 > \frac{\beta}{\alpha}.$$

Epidemiological threshold law: an epidemic can occur only if the number of susceptible individuals is large enough at the beginning, i.e. it holds $V_0 > \frac{\beta}{\alpha}$. [7]

3.1 Application of the classical epidemiological model

For this epidemiological model was created a program written in C language. The same model was used to model the spread of epidemic influenza season of 2011– 2012 among children aged 6 to 14 years in selected parts of the city of Ostrava, namely Mariánské Hory and Hulváky. In these parts of Ostrava there live 12982 inhabitants, including children aged 6 to 14 years constitute about 7.8%. Therefore, the initial value of the number of susceptible individuals was chosen to be 1000 by iteration method.

The largest number of reported cases of influenza infection in the season 2011– 2012 was in the 7th week – 282. The simulation program in this week showed value 283.3 infected individuals.

The most complex problem was to determine the coefficients α, β. Here we use publications [2], [3], which shows the intervals for the estimates of these coefficients. These coefficients were computed from actual values and then experimentally adjusted by bisection method. For the simulation experiment the following values were set:

$$N = 1053, V(0) = 1000, R(0) = 0, I(0) = 53, \alpha = 6{,}5.10^{-4}\ den^{-1}, \beta = 0{,}262\ den^{-1}.$$

4 Application of time series analysis

In this part the data file is treated as a time series. First seasonal decomposition of time series is performed and on the basis of estimated development time series is predicted until the end of 2013. The second part is then modeled using ARIMA time series models.

For these purposes, the data had to be adjusted to monthly values of infected acute respiratory illness including influenza, because the program SPSS does not offer the option in its data variable "year-week".

4.1 The seasonal decomposition of time series

The program SPSS offers two variants of the seasonal decomposition time series: additive or multiplicative. For this time series was chosen multiplicative model because the amplitude time series decreases with increasing time. The following table shows the seasonal factors that say that the highest number of infections such as acute respiratory illness including influenza is usually in February (43.6% more than the average number of infected individuals). The lowest value we get for the month August (37.7%).

Month	Seasonal factor	Month	Seasonal factor
January	131.5	July	41.5
February	143.6	August	37.7
March	104.2	September	98.1
April	94.4	October	128.1
May	75.8	November	134.8
June	76.9	December	133.4

Table 1 Seasonal factors - the output from SPSS

The equation estimated by regression analysis has the form $y_t = 1618 - 6.3t \quad t = 1,2,\ldots$

The values of the original time series (real) values and the estimated model are graphically shown in the following graph.

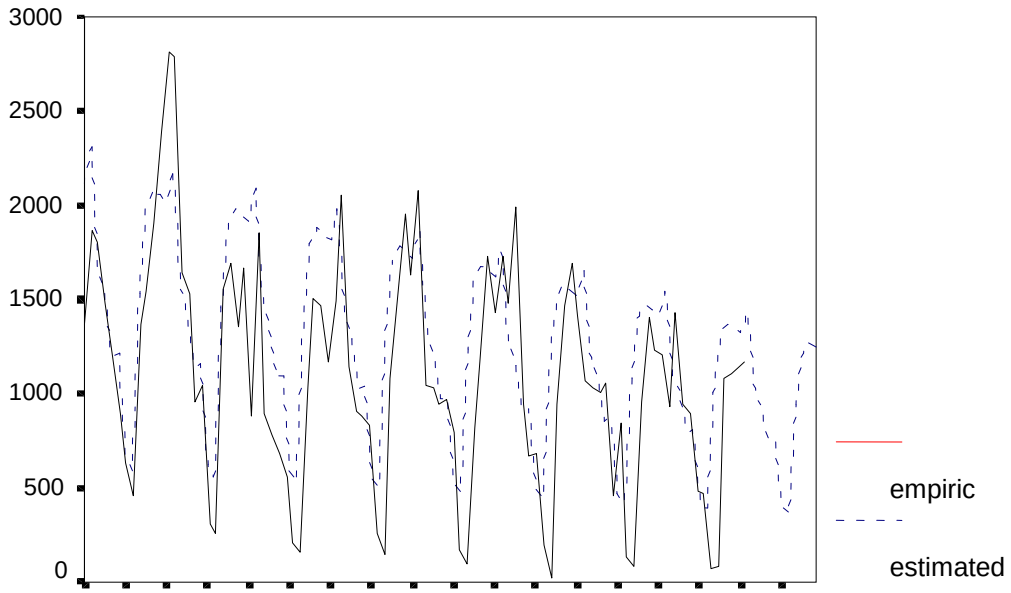

Figure 2 Comparison of actual and theoretical values of the number of infected individuals per 100 000 population from January 2003 to December 2013

4.2 Box – Jenkins time series model

In this method there is no emphasis on the systematic component of time series, but the unsystematic component, a residual component. The application of this method is that the modeled time series must be stationary. If it is not stationary, time series must be differentiated. When analyzing time series of the number of individuals infected with acute respiratory disease common difference of time series was performed and seasonal differences as well, in the model there were also included component of moving averages (ordinary and seasonal). The best model in terms of quality based was model SARIMA$(0,1,1)(0,1,1)_{12}$. [1]

	B	SEB	T-RATIO	PROB.
MA1	0.7378	0.0687	10.739	0.0000
SMA1	0.7873	0.1528	5.152	0.00000134
CONSTANT	– 3.2	3.6	– 0.88	0.381

Table 2 Values of coefficients in SARIMA model - the output from SPSS

Thus we get the following equation that describes this model:

$$y_t = -3.2 + y_{t-1} + y_{t-12} + y_{t-13} + a_t - 0.737a_{t-1} - 0.787a_{t-12} + 0.58a_{t-13}.$$

The following graph illustrates again the real value of the number of infected individuals, acute respiratory disease and the value estimated by SARIMA model. The graph shows the values predicted until December 2013.

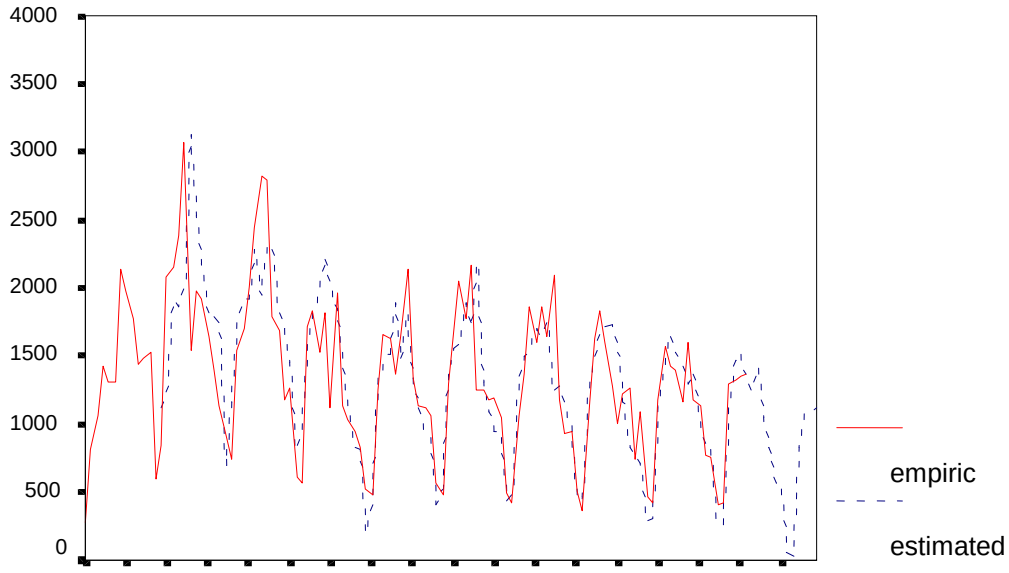

Figure 3 Comparison of actual and theoretical values of the number of infected individuals per 100000 inhabitants from January 2003 to December 2013

Period of 2013	Decomposition model	Box - Jenkins model	Period of 2013	Decomposition model	Box - Jenkins model
January	1321.69	1246.8	July	402.54	64.3
February	**1435.96**	**1420.6**	August	362.47	33.2
March	1034.42	1049.4	September	937.40	829.4
April	931.57	830.5	October	1216.29	1084
May	743.41	563.2	November	127.51	1085.6
June	749.80	569.4	December	1247.83	1121.8

Table 3 Comparison of predicted values using the two models

During influenza epidemics, i.e. 4^{th} – 12^{th} calendar week two time series reach their maximum values. Predicted values of the number of infected individuals vary greatly in summer months. In this period, the flu epidemic is not assumed, so it is not considered significant.

5 Conclusions

In this work mathematical models for describing the spread of epidemic infectious diseases are presented. Epidemiology is a science, which by definition of WHO studies the distribution and causes of illness and events related to health of human populations and application of knowledge to solve health problems.

Furthermore, in this work there are characterized epidemiological data and then there are carried out some mathematical models. There are two approaches applied to time series (seasonal decomposition time series and Box-Jenkins methodology). It also shows the classical model of the epidemic, and simulation experiments, depending on initial conditions.

In this work, which is certainly not a breakthrough in the epidemiology of influenza, there is outlined that simulation models can provide estimates of the number of infected individuals, to build the forecast of the epidemic and therefore to plan the financial, material and personnel resources that will be necessary in case of influenza epidemics.

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A Model of Risk Distribution in Czech Bonus-Malus Systems

Radek Stolín[1]

Abstract. By adopting a bonus-malus system in insurance industry the premium is no more a fixed value, but becomes a random variable. It implies that the standard deviation of annual premium paid by a policyholder is greater than zero. As risk is usually measured by means of the standard deviation of the correspondent random variable, the insurance company applying a bonus-malus system does not take the entire risk of the total coverage of all the damages the policyholder caused. A part of the risk carries a policyholder him or herself. So the risk is divided between the policyholder and the insurance company. It is possible to divide the risk according to the Schnieper covariance model of risk distribution.
Czech insurance companies have started using bonus-malus system in automobile insurance systematically since 2000. In this paper the Schnieper covariance model of risk distribution is used to investigate the ability of bonus-malus systems currently applied on Czech automobile insurance market to carry the risk to the policyholder. The insurance companies are ranked by the value of the risk that is being carried to the policyholder through their present bonus-malus system. The paper furthermore addresses potential development of the risk distribution in the future.

Keywords: bonus-malus system, risk distribution, Schnieper covariance model, transition matrix

JEL Classification: C46
AMS Classification: 91B30

1 Introduction

Bonus-malus system (BMS) in general is a system that adjusts the premium paid by a policyholder according to his or her individual claim history. Each BMS consists of several classes in which different premiums are charged. There is a basic class with the basic premium, furthermore other classes with discounts (the so called bonus classes) and some classes with premium surcharges (the so called malus classes). Drivers are divided into the classes on the history of the number of their individual claims during the given time period (usually the so called policy year). So the outlays of the policyholder in these systems are of the stochastic nature and the policyholder carries a part of the risk of the total costs associated with his or her insurance policy. The main goal of this paper is to calculate the proportion of the policyholder's risk to the entire risk for the BMSs applied on Czech automobile insurance market nowadays using the so called Schnieper covariance model of the risk distribution between the policyholder and the insurer.

Firstly features of the BMS currently applied by Czech insurance companies are introduced. After that I briefly present distribution models of some random variables which are needed for the calculation, namely the number of claims, the cost of claims, the total insurer's costs and the duration of insurance policy. Next Schnieper covariance model of the risk distribution is brought out and a formula for the calculation of the rate of policyholder's risk is derived. Finally the BMSs of the insurance companies are ranked by the rate of their policyholder's risk followed by a graph of the possible future development of the rate for some insurance companies. The text is accompanied by several illustrational examples.

2 Czech BMSs in automobile insurance

The first Czech automobile insurers have started applying BMS as late as at the end of the last century, systematically then since the 1st January 2000. A new policyholder is placed into the basic class and all their next placements depend on the so called determining period at the end of each policy year. The determining period is the duration of the policy diminished on any claim by given number of months.

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Example 1. For example AXA insurance company takes away 24 months for each claim, see [1]. Table 1 shows the dependency of positioning in its BMS on the determining period and the corresponding discounts or surcharges.

class	determining period (month)	discount/ surcharge
B8	96 and more	50 %
B7	84 up to 95	45 %
B6	72 up to 83	40 %
B5	60 up to 71	35 %
B4	48 up to 59	30 %
B3	36 up to 47	25 %
B2	24 up to 35	20 %
B1	12 up to 23	10 %
B0	0 up to 11	0 %
M1	-12 up to -1	30 %
M2	-36 up to -13	90 %
M3	-37 and less	150 %

Table 1 AXA (see [1])

Assuming that the maximum and minimum of the determining period of AXA insurance company makes 108 and -48 months respectively and dividing class M2 into two ones with the determining period in range of 12 months for both of them, the BMS has the following property. The class of a policyholder for a policy year is determined unequivocally by the class in the preceding policy year and the number of claims reported by the policyholder in that period. Besides, the yearly numbers of claims are independent. In that case the BMS makes a Markov chain and for the policyholder with the claim frequency ϑ there is a transition matrix $\mathbf{P}(\vartheta) = \left\| p_{ij}(\vartheta) \right\|_{i,j=1}^{s}$ consisting of transition probabilities of going from class *i* to class *j* at a policy anniversary.

3 Distribution models of the number and cost of claims

The number of claims *N* of the policyholder with the claim frequency ϑ is as usually modeled by a Poisson distribution

$$P(N = n) = \frac{\vartheta^n}{n!} e^{-\vartheta}, \quad n = 0, 1, \ldots, \vartheta > 0. \tag{1}$$

As each driver is differently prone to causing some accident, thus the parameter ϑ in (1) should be considered as a realization of a random variable Θ with the Gamma distribution, see [2]. Its probability density function can be written as

$$u(\vartheta) = \frac{\tau^h}{\Gamma(h)} \vartheta^{h-1} e^{-\tau\vartheta}, \ \vartheta \geq 0, \tau > 0, h > 0, \ \mathrm{E}(\Theta) = \frac{h}{\tau}, \ \mathrm{Var}(\Theta) = \frac{h}{\tau^2}. \tag{2}$$

The function $\Gamma(h)$ is the so called Gamma function

$$\Gamma(h) = \int_0^\infty z^{h-1} e^{-z} \mathrm{d}z.$$

The parameter τ is the so called rate parameter (it indicates the rate of the heterogeneousness of the portfolio, the greater value of τ the smaller heterogeneousness), and the parameter h is the so called shape parameter (it indicates the shape of the distribution of the random variable Θ).

Thus the number of claims N of a randomly picked driver is distributed as

$$P(N=n)=\int_0^\infty \frac{\vartheta^n}{n!}e^{-\vartheta}u(\vartheta)\,\mathrm{d}\vartheta, \quad n=0,\,1,\,\ldots, \quad \mathrm{E}(N)=\frac{h}{\tau}, \ \mathrm{Var}(N)=\frac{h(1+\tau)}{\tau^2}. \tag{3}$$

The total cost of claims S_0 of a random picked policyholder during a policy year is

$$S_0=\sum_{k=1}^{N} X_k, \tag{4}$$

where N denotes the number of claims of a driver during his or her policy year, $X_1, X_2, \ldots, X_N$ denote the costs of claims for reported accidents by the policyholder caused during the policy year.

Let us assume the cost of any claim X has the exponential distribution described by the probability density function

$$u(x)=\alpha e^{-\alpha x}, \ x\geq 0, \alpha>0, \quad \mathrm{E}(x)=\frac{1}{\alpha}, \ \mathrm{Var}(X)=\frac{1}{\alpha^2}, \tag{5}$$

which is in fact the Gamma distribution with $h=1$.

On the acceptable assumptions that the random variables $N, X_1, X_2, \ldots, X_N$ are mutually independent and all $X_1, X_2, \ldots, X_N$ have the same distribution as X, it holds (see [2]) that

$$\mathrm{E}(S_0)=\mathrm{E}(N)\mathrm{E}(X)=\frac{h}{\tau\alpha}, \ \mathrm{Var}(S_0)=\mathrm{E}(N)\mathrm{Var}(X)+\mathrm{Var}(N)\mathrm{E}^2(X)=\frac{h(1+2\tau)}{(\tau\alpha)^2}. \tag{6}$$

The equivalency principle for the calculation of premium π without BMS can be written in form

$$\pi=\frac{\mathrm{E}(S_0)}{1-\eta}=\frac{\mathrm{E}(N)\mathrm{E}(X)}{1-\eta}=\frac{h}{\tau\alpha(1-\eta)}, \tag{7}$$

where η denotes the expense and risk ratio in the premium.

4 A distribution model of insurer's costs

The total cost of insurance company S per a policyholder for his or her insurance year consists of the coverage of the claims and all kind of other expenses associated with the policy. Let us assume that the expense ratio in premium is $\frac{1}{2}\eta$ and the costs could be modeled as

$$S=S_0+\frac{1}{2}\eta\frac{1}{\alpha(1-\eta)}\Theta=S_0+a\Theta. \tag{8}$$

It is possible to derive that

$$\mathrm{E}(S)=\frac{h}{\tau}\left(\frac{1}{\alpha}+a\right), \quad \mathrm{Var}(S)=\frac{2h}{\tau a^2}+\left(\frac{1}{\alpha}+a\right)^2\frac{h}{\tau^2}. \tag{9}$$

The risk R is usually expressed as the square root of the variance, thus as the standard deviation. Mathematically

$$R=\sqrt{\mathrm{Var}(S)}. \tag{10}$$

If the premium charged was a fixed amount, the insurer would take the entire risk. Using BMS leads to variable premium, so a part of the risk carries to the policyholder.

5 Average premium level in BMS

It is obvious that premium paid by a policyholder within an applied BMS is a random variable, say $\bar{S}$, and it holds

$$\bar{S} = \pi_b \, c_I , \tag{11}$$

where c_I is the random variable whose realizations are the components of $\mathbf{c} = (c_1, c_2, \ldots, c_s)'$ - the column vector of premiums in single classes expressed relatively to the basic class premium π_b. For calculating this premium it is necessary to know the so called average premium level $\mathrm{E}(c_I)$.

Let us denote by $\mathbf{p}^r(\vartheta) = \left(p_1^r(\vartheta), p_2^r(\vartheta), \ldots, p_s^r(\vartheta)\right)$ the row vector, where $p_i^r(\vartheta)$ is representing the probability that the policyholder with the claim frequency ϑ will be in the class *i* during the year *r* of his or her insurance career. We can write that

$$\mathbf{p}^r(\vartheta) = \mathbf{p}^1 \mathbf{P}^{r-1}(\vartheta), \tag{12}$$

where $\mathbf{P}^{r-1}(\vartheta)$ is the $(r-1)th$ power of the transition matrix of the policyholder with the claim frequency of ϑ.

It is obvious that for the policyholder with the claim frequency ϑ in year *r* of his or her insurance career we have

$$\mathrm{E}(c_I) = \mathbf{p}^r(\vartheta)\, \mathbf{c}. \tag{13}$$

Considering that the random variable duration of policy *R* in the BMS has the geometric distribution

$$P(R = r) = q^{r-1}(1-q), \;\; r = 1, 2, \ldots, \;\; q \in (0, 1), \;\; \mathrm{E}(R) = \frac{1}{1-q}, \;\; \mathrm{Var}(R) = \frac{q}{(1-q)^2}, \tag{14}$$

it holds for a randomly picked policyholder that

$$\begin{aligned} \mathrm{E}(c_I) &= \int_0^\infty u(\vartheta) \sum_{r=1}^{\infty} q^{r-1}(1-q)\, \mathbf{p}^r(\vartheta)\, \mathbf{c}\, \mathrm{d}\vartheta = (1-q) \int_0^\infty u(\vartheta)\, \mathbf{p}^1 \sum_{r=1}^{\infty} \left(q\, \mathbf{P}(\vartheta)\right)^{r-1} \mathbf{c}\, \mathrm{d}\vartheta = \\ &= (1-q) \int_0^\infty u(\vartheta)\, \mathbf{p}^1 \left(\mathbf{I} - q\mathbf{P}(\vartheta)\right)^{-1} \mathbf{c}\, \mathrm{d}\vartheta = \bar{c}. \end{aligned} \tag{15}$$

Hence with (7) and (11) we have

$$\pi_b = \frac{h}{\tau \alpha (1-\eta) \bar{c}}. \tag{16}$$

Example 2. We will construct the transition matrix of the AXA insurance company BMS for the policyholder with the claim frequency ϑ and compute the corresponding average premium level and the basic premium considering that (according to [2]) $\mathrm{E}(\Theta) = 0.1$, $\mathrm{Var}(\Theta) = 0.0064$, and furthermore that $\mathrm{E}(R) = 10$ and $\alpha = 1$ (the average cost of one claim is a currency unit).

It holds

$$\frac{h}{\tau} = 0.1, \;\; \frac{h}{\tau^2} = 0.0064 \Rightarrow h = 1.5625, \;\; \tau = 15.625, \;\; \frac{1}{1-q} = 10 \Rightarrow q = 0.9.$$

In accordance with Table 1, formula (1) and with the assumptions stated at the end of the first chapter we have

$$\mathbf{P}(\vartheta)=\begin{pmatrix}
1-P(0) & P(0) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1-P(0) & 0 & P(0) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1-\sum_{n=0}^{1}P(n) & P(1) & 0 & P(0) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1-\sum_{n=0}^{1}P(n) & 0 & P(1) & 0 & P(0) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1-\sum_{n=0}^{2}P(n) & P(2) & 0 & P(1) & 0 & P(0) & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1-\sum_{n=0}^{2}P(n) & 0 & P(2) & 0 & P(1) & 0 & P(0) & 0 & 0 & 0 & 0 & 0 & 0\\
1-\sum_{n=0}^{3}P(n) & P(3) & 0 & P(2) & 0 & P(1) & 0 & P(0) & 0 & 0 & 0 & 0 & 0\\
1-\sum_{n=0}^{3}P(n) & 0 & P(3) & 0 & P(2) & 0 & P(1) & 0 & P(0) & 0 & 0 & 0 & 0\\
1-\sum_{n=0}^{4}P(n) & P(4) & 0 & P(3) & 0 & P(2) & 0 & P(1) & 0 & P(0) & 0 & 0 & 0\\
1-\sum_{n=0}^{4}P(n) & 0 & P(4) & 0 & P(3) & 0 & P(2) & 0 & P(1) & & P(0) & 0 & 0\\
1-\sum_{n=0}^{5}P(n) & P(5) & 0 & P(4) & 0 & P(3) & 0 & P(2) & 0 & P(1) & 0 & P(0) & 0\\
1-\sum_{n=0}^{5}P(n) & 0 & P(5) & 0 & P(4) & 0 & P(3) & 0 & P(2) & 0 & P(1) & 0 & P(0)\\
1-\sum_{n=0}^{6}P(n) & P(6) & 0 & P(5) & 0 & P(4) & 0 & P(3) & 0 & P(2) & 0 & P(1) & P(0)
\end{pmatrix}.$$

Hence with $\mathbf{c}=(2.5,\ 1.9,\ 1.9,\ 1.3,\ 1.0,\ 0.9,\ 0.8,\ 0.75,\ 0.6,\ 0.65,\ 0.6,\ 0.55,\ 0.5)'$ and using Maple we can get

$$\bar{c}=0.7046=70.46\%,\quad \pi_b=0.1774.$$

6 A risk distribution model

The random variable *S* introduced in chapter 3 can be written as

$$S=\bar{S}+\bar{\bar{S}}, \tag{17}$$

where $\bar{S}$ represents the part of costs paid by the policyholder and $\bar{\bar{S}}$ is the remainder paid by the insurer. The total risk *R* is possible to divide according to [3], [4] by means of the covariances between $S,\bar{S}$ and between $S,\bar{\bar{S}}$. It holds

$$\begin{aligned}
R=\sqrt{\mathrm{Var}(S)}&=\frac{\mathrm{Var}(S)}{\sqrt{\mathrm{Var}(S)}}=\frac{\mathrm{E}\left[(S-\mathrm{E}(S))^2\right]}{\sqrt{\mathrm{Var}(S)}}=\frac{\mathrm{E}\left[(S-\mathrm{E}(S))(\bar{S}+\bar{\bar{S}}-\mathrm{E}(\bar{S})-\mathrm{E}(\bar{\bar{S}})\right]}{\sqrt{\mathrm{Var}(S)}}=\\
&=\frac{\mathrm{E}\left[(S-\mathrm{E}(S))(\bar{S}-\mathrm{E}(\bar{S}))\right]+\mathrm{E}\left[(S-\mathrm{E}(S))(\bar{\bar{S}}-\mathrm{E}(\bar{\bar{S}})\right]}{\sqrt{\mathrm{Var}(S)}}=\frac{\mathrm{Cov}(S,\bar{S})}{\sqrt{\mathrm{Var}(S)}}+\frac{\mathrm{Cov}(S,\bar{\bar{S}})}{\sqrt{\mathrm{Var}(S)}}=\bar{R}+\bar{\bar{R}}.
\end{aligned} \tag{18}$$

The total risk R is divided into two parts. The part $\overline{R}$ carries through BMS the policyholder and the other part $\overline{\overline{R}}$ remains to the insurer. We will be interested in answering the following question: "How many per cent of entire risk the policyholder carries through the certain BMS?" Mathematically expressed, the value of

$$p = \frac{\overline{R}}{R} = \frac{\mathrm{Cov}(S, \overline{S})}{\mathrm{Var}(S)} \tag{19}$$

is the value of our interest. Let us call this quantity as the rate of policyholder's risk. It is obvious, that

$$p' = \frac{\overline{\overline{R}}}{R} = \frac{R - \overline{R}}{R} = 1 - p \tag{20}$$

could be called the rate of insurer's risk.

On condition that $\Theta = \vartheta$ and $I = i$ we have

$$\mathrm{E}(S\,\overline{S}) = \mathrm{E}\left[\left(\frac{\vartheta}{\alpha} + a\vartheta\right)\pi_b c_i\right] = \vartheta\left(\frac{1}{\alpha} + a\right)\pi_b c_i. \tag{21}$$

For the unconditioned covariance we get

$$\mathrm{Cov}(S, \overline{S}) = \mathrm{E}(S\,\overline{S}) - \mathrm{E}(S)\mathrm{E}(\overline{S}) = \pi_b\left(\frac{1}{\alpha} + a\right)\left[(1-q)\int_0^\infty \vartheta u(\vartheta)\mathbf{p}^1\left(\mathbf{I} - q\mathbf{P}(\vartheta)\right)^{-1}\mathbf{c}\,\mathrm{d}\vartheta - \frac{h}{\tau\,\bar{c}}\right]. \tag{22}$$

Example 3. We will compute the total risk R and the per cent parts of it p, p' for the AXA insurance company BMS.

With the values from Example 2 we gain (see (9), (19), (20) and (22))

$$R = 0.4562, \quad p = 0.4289\ \%, \quad p' = 99.5711\%.$$

7 Results

On Czech automobile insurance market twelve insurance companies work nowadays and each of them applies different BMS, see [5]. Table 2 shows the division of the risk through BMSs between them and their policyholders. It is obvious that the policyholders take only very small part of the risk, less than 1 % generally. This portion is not going to become significantly greater in the future. Figure 1 shows the future development of the rate of the policyholder's risk for some BMSs.

insurance company	p (%)	p' (%)
Wüstenrot	0.9715	99.0285
Česká pojišťovna	0.5703	99.4297
Uniqa	0.5698	99.4302
ČSOB	0.4973	99.5027
Hasičská vzájemná pojišťovna	0.4442	99.5558
Triglav	0.4389	99.5611
AXA	0.4289	99.5711
Generali	0.4020	99.5980
Allianz	0.3758	99.6242
Česká podnikatelská pojišťovna	0.3420	99.6580
Kooperativa	0.3156	99.6844
Slavia	0.1479	99.8521

Table 2 Ranking of Czech BMSs by the rate policyholder's risk

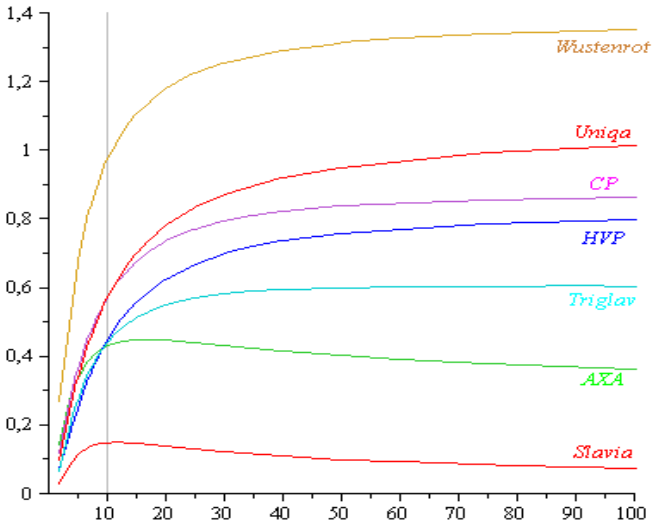

Figure 1 The dependency of the rate of the policyholder's risk on the average duration of policy

8 Conclusion

The paper deals with a feature of BMSs, namely with an extent of its ability to take a part of risk from the insurer. A model of the risk distribution between insurer and policyholder is presented. The model in question is the so called Schnieper covariance model using to distribution the relation between the covariance of the total costs and the insurer's costs on the one hand and the covariance of the total costs and policyholder's costs (premium) on the other hand. To express mentioned random variables the prevailing distributions and the values of their parameters (see [2]) are used. Further in the paper there is a derivation of the formulas for calculation of the proportion of the policyholder's risk to the entire risk of a particular BMS. In the end the values of the rates of the policyholder's risk for all twelve BMSs applied on Czech automobile insurance market nowadays are quantified, which was the main goal of the paper and moreover, some illustrations of their future development are plotted.

The calculations show that in spite of using BMS the insurer keeps on carrying nearly all the risk. Numerically expressed, the insurer takes more than 99 % of the total risk, see Table 2. This result is quite understandable, because the amount of premium depends directly only on the number of accidents caused by the policyholder and the associated costs of claims have no influence on this amount.

It is shown, however, that the rate of policyholder's risk is very various for different BMSs. In the Czech Republic the BMS applied by the Wüstenrot insurance company (one claim decreases the determining period by 36 months and there are rather high average differences between the premiums charged in adjacent classes) has the highest rate, namely of 0.97 %. On the contrary the Slavia insurance company (one claim decreases the determining period only by 12 months) has the lowest rate of policyholder's risk, namely of 0.15 %.

The future development of the rate of policyholder's risk in various Czech BMSs is obvious from Figure 1. There are BMSs where a point of inflection exists and where it does not. Nevertheless, all the investigated BMSs have a property in common. With increasing average duration of policy E(R) the rate of policyholder's risk is approaching the rate corresponding to the steady-state distribution of the vector $\mathbf{p}^r(\vartheta)$ of the BMSs in question.

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Modeling Customer Response by the Questionnaire Evaluation Method

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Abstract. Nowadays the trend is to obtain information from all range of available sources with the use of various methods. However, even today it is impossible to avoid standard methods which include questionnaire research. It can be carried out in a classic way, by means of a paper questionnaire or with implementing the Internet-based questionnaire. Questionnaire research must be equipped with the adequate answering possibility in order to meet the assumed information goal. Results of questionnaire research often influence the company's productivity in the short-, middle- and long-term time horizon. This is the main reason for paying attention to creating questionnaires and their subsequent evaluation. Evaluating questionnaires is always a specific issue in connection to its relation with a defined goal. The mathematical basis of it makes defining generalized models possible. These models are implemented directly or they require only small corrections. The aim of this paper is to highlight the way of creating the model of evaluating questionnaires which are filled in by customers and its relation to supporting a decision making process in a market-oriented company.

Keywords: questionnaire evaluation method, information response model, factor, engineering industry.

JEL Classification: C42
AMS Classification: 94A50

1 Introduction

The contemporary world which is highly competitive puts pressure on companies obliged to improve their performance continually [2]. It is extremely important to evaluate their effectiveness from the point of view of rapidly changing data not only for production companies [4] but also for entrepreneurs operating in the retail sector where the appropriate logistic approach ought to be implemented. Moreover, there is a need to acquire and evaluate information continually and react to changing conditions in a dynamic way. One of the possible ways of obtaining information are questionnaires which are a common and popular tool to gather data from a large number of people. A good questionnaire can be a powerful tool to help management in their decision-making, and a poorly designed questionnaire can make life difficult for both those that have to complete it and those that have to analyse the data. Brehop [1] defines a questionnaire to be a form that people fill out in order to obtain demographic information as well as views and interests of those questioned. The problem of questionnaires and interviews is also emphasized in [6]. Kirakowski [5] defines a questionnaire in a more structural way as a method for the elicitation, recording and collecting information. Researchers use questionnaires as tools (methods) to capture what is in users' minds (elicitation). The data collected from a group of respondents (users) is recorded onto a permanent medium to be analyzed and referenced later. Questionnaires surveys are frequently used in marketing and economics. By using questionnaire surveys we are able to identify and measure a large number of indicators. The problem of finding new sources is described in detail in [8], evaluation of quality of products and services are taken into account in [12], evaluation of quality of information and marketing communications productivity are the subject of [7], and so on. Moreover, methods of the development, evaluation and testing the survey questionnaires have undergone radical change in the recent years [9]. The paper is orientated towards presenting one of the possible models devoted to working out the data obtained by means of the questionnaire. The proposed model is applied to verify the research hypothesis concerning close sources of financing on the measurement basis of their usage in the area of the machine building industry in Slovakia.

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2 Information response model

Taking into account the contents of the article we come to the conclusion that there is a possibility to put forward the information response model presented hereby. This model is an original proposal of the authors, and has been validated on two examples in practice. One of them is presented in section 3.

Let us assume that there are *N* questions to be asked. Let us assume that there are *M* respondents. Let *k*, $k = 0,1,\ldots,K$ be a stage at which the survey is carried out. Let $R = \left[r_{m,n}^{k}\right]$, $k = 0,1,\ldots,K$, $m = 1,\ldots,M$, $n = 1,\ldots,N$ be the matrix of adjustment where $r_{m,n}^{k}$ - adjustment of the *m*-th respondent to the *n*-th question at the *k*-th stage. The element $r_{m,n}^{k}$ takes the following values:

$$r_{m,n}^{k} = \begin{cases} 1 & \text{if the } m\text{-th respondent is allowed to answer the } n\text{-th question at the } k\text{-th stage,} \\ 0 & \text{otherwise} \end{cases}$$

Each answer to the *n*-th question of the *m*-th respondent corresponds to the elements of the response matrix at the *k*-th stage: $A^{k} = \left[\alpha_{m,n}^{k}\right]$, $k = 0,1,\ldots,K$, $m = 1,\ldots,M$, $n = 1,\ldots,N$.

$$\alpha_{m,n}^{k} = \begin{cases} 1 & \text{if the } m\text{-th respondent answered the } n\text{-th question at the } k\text{-th stage positively,} \\ 0 & \text{otherwise} \end{cases}$$

The sum $\sum_{m=1}^{M} \alpha_{m,n}^{k}$ is decisive in terms of making a decision for the *n*-th question. The above sum equals β, $\beta = 0,1,\ldots,B$. Let us assume that the results are distributed. However, each *m*-th respondent is specific so responses to the questions are expected to be different at the *k*-th stage, $k = 1,\ldots,K$.

Let $D_n = \left[d_{\lambda}\right]$, $1 \le \lambda \le M$ be the decision vector where d_{λ} - the λ-th decision to be made after answering the *n*-th question by *M* potential representatives of companies (we assume that d_{λ} is adequate to the λ-th company strategy). Then:

$$\gamma^{k} = \begin{cases} \sum\limits_{m=1}^{M} \alpha_{m,n}^{k} = 0 & \rightarrow \quad d_{0} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \sum\limits_{m=1}^{M} \alpha_{m,n}^{k} = \beta & \rightarrow \quad d_{\lambda} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \sum\limits_{m=1}^{M} \alpha_{m,n}^{k} = M & \rightarrow \quad d_{M} \end{cases}$$

The state of responses changes at every stage as follows:

$$A^{0} \rightarrow A^{1} \rightarrow \ldots \rightarrow A^{k} \rightarrow \ldots \rightarrow A^{K}$$

The state of the element of the response matrix at the *k*-th stage changes as follows:

$$a_{m,n}^{k} \begin{cases} = a_{m,n}^{k-1} & \text{if the } m\text{-th respondent answered the } n\text{-th question at the } k\text{-th stage identically as at the stage } k\text{-1,} \\ \neq a_{m,n}^{k-1} & \text{otherwise} \end{cases}$$

As seen above, the model represents an online support tool to enable organizations to improve their performance by means of answering the *n*-th question, $n = 1,\ldots,N$.

Additionally, there is also a possibility to prepare a common response to *N* questions at the *k*-th stage. The sum $\sum_{m=1}^{M}\sum_{n=1}^{N} \alpha_{m,n}^{k}$ is decisive in terms of making a decision for the group of *N* questions. The sum takes the value *c*, $c = 0,1,\ldots,C$ where $C = M \cdot N$.

Let $D^M = \left[d_v^M\right]$, $1 \le v \le C$ be the strategy decision vector for the *N* asked questions where d_v^N - the *v*-th decision to be made for *N* answers to the questions (we assume that d_v^N is adequate to the *v*-th marketing price strategy for the group of *M* responses to *N* questions). Then:

$$\gamma_M^k = \begin{cases} \sum_{m=1}^{M}\sum_{n=1}^{N} \alpha_{m,n}^k = 0 \quad \rightarrow \quad d_0^M \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{m=1}^{M}\sum_{n=1}^{N} \alpha_{m,n}^k = c \quad \rightarrow \quad d_v^M \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \sum_{m=1}^{M}\sum_{n=1}^{N} \alpha_{m,n}^k = C \quad \rightarrow \quad d_V^M \end{cases}$$

The value γ_M^k is assumed to characterize the behaviour of *M* companies and it triggers the *v*-th decision, $v = 1,\ldots,V$. However, this kind of approach can sometimes prove deceiving and then it has to be verified and the *v*-th decision has to be modified in order to meet the company's demand.

In the discussion below, we decided to analyse the state of companies' financial requirements at a certain determined state only (the state after 2009).

3 Response model implementation for the questionnaire research in the engineering industry in Slovakia and EU

The backbone of the Slovak economy is manufacturing and its main industry – production engineering (32,5 % of the total industry and 42,3% of manufacturing) and within it automotive industry (20% of all Slovak industry). The branches of industry have decreased considerably not only in Slovakia but also in the whole world as a result of the world crisis. In 2009 the amount of production decreased by 30% to 50% in comparison with the year 2008. The number of companies has decreased by 187, from 715 to 528. Manufacturing of machines and equipment has decreased by 37% and manufacturing of means of transport has dwindled by 29%. Sales in the year 2008 went down by 30% in comparison with the year 2009. Sales reached 13,717 million euros in 2009. Export went down by 26% in 2009 to the level of 13.480 million euros whereas import equalled 9,691 million Euros. The number of employees in the engineering industry decreased by 18% which represents 110,055 employees, the salaries decreased by 27% (in 2009 the labour costs equalled 1.036 million Euros) and the added value by 27% (2.396 million Euros). Investments in the manufacturing of machines and equipment went down by 53%. The share of the added value per employee represented the value of 21.767 million Euros. In 2009 the value of engineering equalled 90.686 million Euros, the revenues amounted to15.889 million Euros, the costs were 15.756 million Euros. A moderate growth in machinery investment was expected in the years 2010-2011and the stronger flow of investments was expected at the turn of 2011-2012. Moreover, more positive developments in external demand as well as growth in export performance are exported as well. 16 million Euros were devoted to research and development in 2009. All in all, Slovakia devoted more than 316,4 million Euros to research and development in the branch of engineering in 2008 which means that the share of this branch in the whole research creates 5,1% and the number of employees in the area of research is 661 which represents 2,1% (more in [10] and [11]). The engineering industry is characterised by its own peculiarities of the capital cycle which includes finances, a long production cycle along with its unequal cost escalation for the work in progress and a short sales cycle of unfinished products. These particularities of companies influence mainly a variable volume; a structure and a turnover period of circulating assets throughout a year and an overall composition of capital. The mechanical engineering sector must constantly seek-new and creative models to manage and withstand the crisis and stay competitive. Engineering production is among the key areas of economic development in the European Union. This branch employed 3,25 million workers in 2007 which accounted for 9,78% of the total EU industry. Moreover, spending on research equalled only two percent of the sector's turnover. Development and production costs are greatly influenced by the increasing technological complexity which results from the search for optimal and efficient solutions of efficient management of the company financial resources. The year 2009 was the worst in the period of crisis, which results from the time economic data analysis. From a statistical point of view, the data for 2009 presents details which go beyond the standard measurement almost in every set of data. Our research was aimed at obtaining information about the use of funding sources in engineering enterprises in Slovakia immediately after the critical year 2009. A questionnaire survey was used to find relations and connections concerning sources of funding in companies orientated towards engineering production. 57 com-

pleted questionnaires were obtained by means of data collection (Slovakia, 2010). Respondents, generally managers of engineering companies, were filling out 17 points of the questionnaire in the presence of the pollster. However, this number can be treated as insufficient taking into account a limited number of companies (there are 528 companies in the Slovak Republic) and the reluctance of managers to cooperate (personal visits and maximum effort was made to obtain the required data). According to the law of large numbers, if the number of chosen units is higher than 30, the method of inductive statistics can be implemented where conclusions are drawn from a part of the set for the whole set treated as the population. In order to obtain the appropriate results, a representative random sample must be created. The choice of 57 companies was not made at random. It was based on the managers' willingness to cooperate. Therefore, a hidden inaccuracy can occur in the results. It is caused by this way of data collection. We restrict ourselves to two items of the questionnaire for the purposes of this paper. Items 8 and 9, which contain sub-items in themselves for each source of financing (altogether 17 sources), detect on the four-ordinal scale the extent to which individual external standard and alternative sources of financing are important while financing a company. The frequency distribution of items 8 and 9 can be briefly seen in Fig. 1. These items result from 55 valid questionnaires (more in [3]).

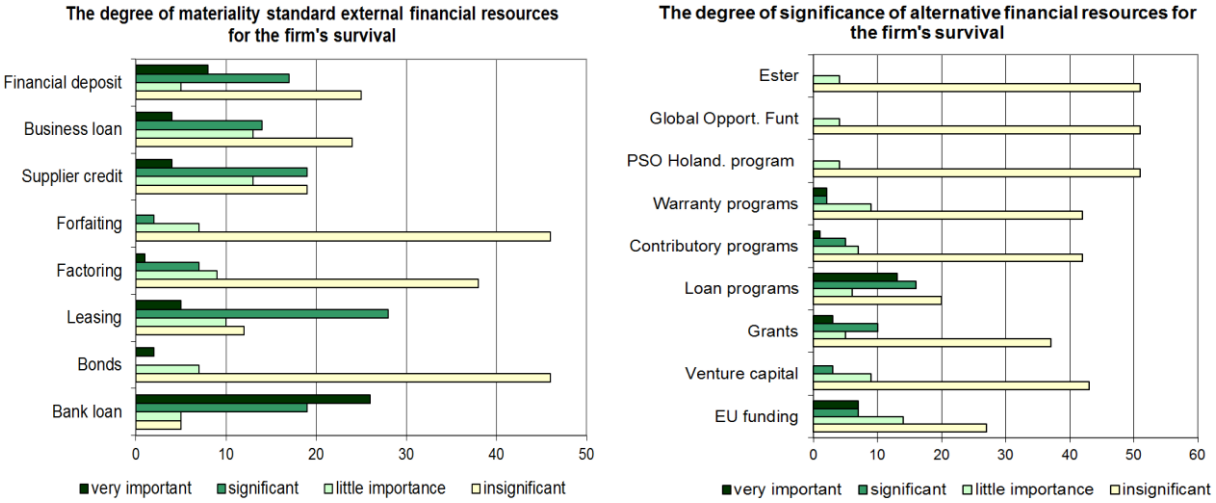

Figure 1 Distribution of selected items

The research hypothesis was formed as follows: We assume that it is possible to create and interpret groups of close sources of financing by means of the extent of their use. While implementing descriptive statistics for evaluating the data from the questionnaire, the individual sources of financing were grouped on the basis of the extent of their use. Since it has been proved that not all 17 sources were implemented, the first 10 sources of financing were used from the point of view of their use to verify the presented research hypothesis (they are grouped as shown in Table 3). These 10 items entered the factor analysis as the manifest variables. The product of these analysis is a lower number of the so-called latent variables which is a linear combination of the original variables. The factor analysis allows us to find latent causes of a data variability (indirectly observed). It is possible to reduce the number of variables by finding latent factors while maintaining maximum information and finding a relationship between observed variables and derived factors (www.spss.cz). Table 1 presents results of the tests by means of which the assumptions for using the factor analysis were validated. The Kaiser-Meyer-Olkin Measure of Sampling Adequacy confirmed the appropriateness of the choice of the method whereas the Bartlett's Test of Sphericity confirmed that the correlation matrix of original manifest variables is not a unit type one.

KMO and Bartlett's Test		
Kaiser-Meyer-Olkin Measure of Sampling Adequacy.		0,682
Bartlett's Test of Sphericity	Approx. Chi-Square	158,111
	df	45
	Sig.	0,000

Table 1 Verification of the assumptions of the factor analysis

The method of principal components was used to find the optimal solution. It generates orthogonal correlated factors (More: knowledgebase at www.spss.com).

Total Variance Explained

Component	Initial Eigen values			Extraction Sums of Squared Loadings			Rotation Sums of Squared Loadings		
	Total	% of Variance	Cumulative %	Total	% of Variance	Cumulative %	Total	% of Variance	Cumulative %
1	3,391	33,909	33,909	3,391	33,909	33,909	2,476	24,760	24,760
2	1,592	15,923	49,832	1,592	15,923	49,832	1,821	18,212	42,972
3	1,225	12,247	62,079	1,225	12,247	62,079	1,603	16,034	59,006
4	,878	8,775	70,854	,878	8,775	70,854	1,185	11,849	70,854

Table 2 Contribution of the factors to explaining the variability

The rotated solution with the Varimax rotation was used for a better interpretation of the extracted factors. Figure 2 shows the layout of the original variables in the space of three principal components after rotation. The rotation does not affect the explained variability as it only corrects the specification of factor loads to clearly assign the variable to the factor (see Table 3).

Rotated Component Matrix[a]

	Component			
	1	2	3	4
SV1 Bank loans	,204	,832	,082	,006
SV3 Leasing	,729	,040	,348	-,293
AV4 Credit programs	,079	,918	,101	,068
SV8 Advances	,095	-,007	,256	,814
SV6 Supplier credit	,113	,065	,847	,200
SV7 Business loan	-,084	,114	,807	,087
AV1 EU funds	,785	,021	-,032	,039
AV3 Subsidies	,719	,217	-,071	,309
SV4 Factoring	,427	,350	,136	,433
AV5 Subsidized programs	,744	,310	-,079	,316

Table 3 Factor loadings

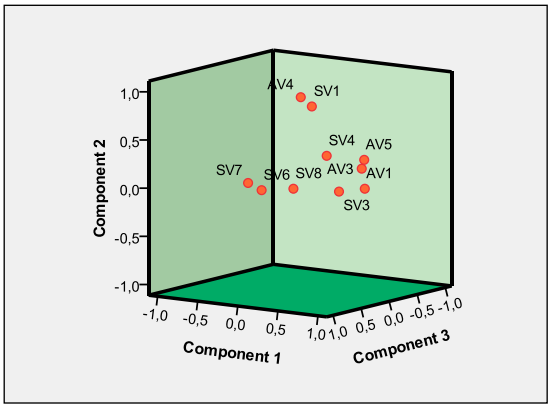

Figure 2 Presentation of the three main components in the rotated space

The sense of organising funding sources which are with its measure of use closer in individual factors can be interpreted as follows:

- **Factor1** is saturated: SV3 – Leasing, AV1 - EU funds, AV3 – Subsidies, AV5 - Subsidized programs. The proper interpretation of the first factor is *the dynamics factor of financing a business entity*.
- **Factor2** is saturated: SV1 - Bank loans, AV4 - Credit programs. The proper interpretation of the second factor is *the loan burden factor of a business entity*.

- **Factor3** is saturated: SV6 - Supplier credit, SV7 - Business loan. The proper interpretation of the third factor is *the factor of supplier-customer lending.*
- **Factor4** is saturated: SV8 – Advances, SV4 – Factoring, however, its assignment here is not entirely clear, this funding source has the same high weight as in the first factor, obviously it is derived from the position SV4 in Figure 2. The proper interpretation of the fourth factor is *the factor of advances and sale of business lending claims.* It was useful to include this factor in the model from the interpretation point of view.

Groups of close sources of corporate financing were created and subsequently interpreted on the basis of the extent of their use by means of implementing the factor analysis method. The research hypothesis is confirmed.

4 Conclusions

One of the ways of obtaining information for research, education and other professional activities is questionnaire research. The goal of the paper is to present one of the general models for questioning by means of questionnaires and the subsequent evaluation of this research as well as the concrete demonstration of the survey realization. The general model represents a possibility of questioning a limited number of respondents and a direct relation of evaluated results with decision-making processes. This practical example was directed to the area of searching for factors influencing the engineering industry in Slovakia and the European Union. The questionnaire was carried out in order to verify the research hypothesis which assumes that it is possible to create and interpret groups of close sources of corporate financing on the basis of the extent of their use.

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Application of non-homogeneous Markov chain analysis to trend prediction of stock indices

Milan Svoboda [1], Ladislav Lukáš [2]

Abstract. The present paper concerns with prolongation of our study aims to predict the stock index trend of various stock indices using Markov chain analysis (MCA). The prediction of the trend using MCA is done using short, medium and long-term data. Each downloaded data is divided into two periods. The first one is used for estimating the trend, whereas the second one is used for comparison and evaluation, too. In the basic framework, we may use either homogeneous MC or non-homogeneous one. When building model we are focused both on various state space discretizations and corresponding construction of transition probability matrices. Non-homogeneous matrices are constructed by linear interpolants between two transition probability matrices at given time steps. These objects represent a core of any MCA and its application. The results of the trend prediction using both versions of MCA are compared. Numerical calculations and computer implementations have been done by Excel and Mathematica modules, which are briefly discussed as well.

Keywords: Markov chain analysis, transition probability matrix, stock index, trend prediction, time series analysis.

JEL Classification: C02, C13, G14, G19
AMS Classification: 90C40, 91B82

1 Introduction

The present study is focused on possibilities to predict stock market indices using Markov chain models (MCA). Since the very beginning we have to note that there is never ending debate as to whether financial asset prices are predictable or not. Academicians, who belief on the EMH, see Fama [3], have been rather sceptic thereabout. The EMH eliminates possibilities to gain higher than average profits in principle. The second reason is that there are also available other negative empirical studies of technical analysis (TA) of the stock market, such as Fama and Blume [4], and Jensen and Benington [5] being referred very often, as well. On the contrary, some recent studies show, see Sweeney [6], and Brock at al. [1], as an example, that methods based upon analysis of past prices and stock volumes are capable to outperforming the market.

The application of MC to stock market analysis is not new. From the recent papers, we refer to Doubleday and Esunge [2], Vasanthi et al. [9], and Zhang and Zhang [10]. However, state spaces used within these papers are relatively simple ones. Now, our study brings a prolongation of our former paper Svoboda and Lukáš [7], where we have presented four Markov chain (MC) models, in general. Particularly, the fourth one having the state space constructed upon the filtered sequence of accumulated yields of the Prague stock exchange index PX was considered as promising one. We focus our study yet to generalization of our previous results considering the time dependence of transition probability matrix, in particular. Hence, we implement non-homogeneous MC models.

In words, MC is a special kind of stochastic process where the next state of the system depends only on the current state and not on the previous ones. A stochastic process, or discrete sequence of random variables $\{X_n\}$, n=1,2,.. , is said to have the Markov property if (1) holds for any finite n, where particular realizations x_n belong to discrete state space $S = \{s_i\}$, i=1,2,..,k. Generally, MC is described by vectors $\boldsymbol{p}(n)$ which give unconditional probability distributions of states assuming $\boldsymbol{p}(0)$ is given, and transition probability matrix which gives conditional probabilities $p_{ij} = P(X_{n+1} = s_j \mid X_n = s_i)$, i,j=1,2,..,k where p_{ij} may depend on n. In such case, we speak about non-homogeneous MC, and on the contrary to homogeneous MC where p_{ij} does not depend on n at all. Development of $\boldsymbol{p}(n)$ is given by recurrence equation (2), where ^{T} denotes transposition, and time dependent transition probability matrix is denoted $\boldsymbol{P}(n)$ in simplified notation, whereas the precise one should be $\boldsymbol{P}(n,n–1)$. In the case of homogeneous MC, this matrix is replaced by $\boldsymbol{P}$ as being time independent one. For more details about theory and applications of MC models, we refer to Lukáš [6].

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$$P\left(X_{n+1}=x_{n+1} \mid X_1=x_1, X_2=x_2,..,X_n=x_n\right)=P\left(X_{n+1}=x_{n+1} \mid X_n=x_n\right) \tag{1}$$

$$\boldsymbol{p}(n)^{\mathrm{T}}=\boldsymbol{p}(n-1)^{\mathrm{T}}\,\boldsymbol{P}(n),\ n=1,2,.. \tag{2}$$

Since we are going to implement the non-homogeneous MC models, we need some results regarding matrix interpolation mappings. In particular, we may consider such models only, which preserve stochasticity of $\boldsymbol{P}(n)$ for any n. Here, we present two lemmas, which enable us to construct suitable $\boldsymbol{P}(n)$ by both linear and quadratic interpolation technique provided the corresponding interpolation matrices be given.

Lemma 1. (Matrix linear interpolation mapping) *If* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *are stochastic matrices then* $\boldsymbol{H}(t)=\boldsymbol{A}+(\boldsymbol{B}-\boldsymbol{A})t$, $t\in R$ *is stochastic matrix, as well, such that* $\boldsymbol{H}(0)=\boldsymbol{A}$ *and* $\boldsymbol{H}(1)=\boldsymbol{B}$.

Proof. *By assumption, it holds* $\sum_{j=1}^{n}a_{i,j}=1$ *and* $\sum_{j=1}^{n}b_{i,j}=1,\ \forall\ i=1,..,n$. *To prove stochasticity of* $\boldsymbol{H}(t)$ *it is sufficient to show that* $\sum_{j=1}^{n}h_{i,j}(t)=1,\forall\ i=1,..,n$ *holds. Let write* $\boldsymbol{H}(t)=\boldsymbol{A}+(\boldsymbol{B}-\boldsymbol{A})t$ *in component-wise form as follows*

$$\sum_{j=1}^{n}h_{i,j}(t)=\sum_{j=1}^{n}(a_{i,j}+(b_{i,j}-a_{i,j}))t=\sum_{j=1}^{n}a_{i,j}+(\sum_{j=1}^{n}b_{i,j}-\sum_{j=1}^{n}a_{i,j})t=1,\forall\ i=1,..,n$$

for any $t\in R$. $\boldsymbol{H}(0)=\boldsymbol{A}$ *and* $\boldsymbol{H}(1)=\boldsymbol{B}$ *is evident. This ends the proof.*

Lemma 2. (Matrix quadratic interpolation mapping) *If* $\boldsymbol{A}$, $\boldsymbol{B}$ *and* $\boldsymbol{C}$ *are stochastic matrices, and* $\tau\in\,]0,1[$ *is given. Then* $\boldsymbol{G}(t)=\boldsymbol{A}+\{(\boldsymbol{B}-(1-\tau^2)\boldsymbol{A}-\tau^2\boldsymbol{C})/(\tau(1-\tau))\}t+\{((1-\tau)\boldsymbol{A}-\boldsymbol{B}+\tau\boldsymbol{C})/(\tau(1-\tau))\}t^2$, $t\in R$ *is stochastic matrix, as well, such that* $\boldsymbol{G}(0)=\boldsymbol{A}$, $\boldsymbol{G}(1)=\boldsymbol{C}$ *and* $\boldsymbol{G}(\tau)=\boldsymbol{B}$.

Proof. *By assumption, it holds* $\sum_{j=1}^{n}a_{i,j}=1$, $\sum_{j=1}^{n}b_{i,j}=1$ *and* $\sum_{j=1}^{n}c_{i,j}=1,\ \forall\ i=1,..,n$. *As in Lemma 1, it is sufficient to show that* $\sum_{j=1}^{n}g_{i,j}(t)=1,\forall\ i=1,..,n$. *Let write* $\boldsymbol{G}(t)$ *in component-wise form as follows*

$$\sum_{j=1}^{n}g_{i,j}(t)=\sum_{j=1}^{n}(a_{i,j}+\frac{1}{\tau(1-\tau)}(b_{i,j}-(1-\tau^2)a_{i,j}-\tau^2c_{i,j})t+\frac{1}{\tau(1-\tau)}((1-\tau)a_{i,j}-b_{i,j}+\tau c_{i,j})t^2),\ \forall\ i=1,..,n$$

for any $t\in R$. *with* $\tau\in\,]0,1[$ *given. Expanding the right hand-side, we may write*

$$\sum_{j=1}^{n}g_{i,j}(t)=\sum_{j=1}^{n}a_{i,j}+\frac{1}{\tau(1-\tau)}\{(\sum_{j=1}^{n}b_{i,j}-(1-\tau^2)\sum_{j=1}^{n}a_{i,j}-\tau^2\sum_{j=1}^{n}c_{i,j})t+((1-\tau)\sum_{j=1}^{n}a_{i,j}-\sum_{j=1}^{n}b_{i,j}+\tau\sum_{j=1}^{n}c_{i,j})t^2\}=$$

$$=1+\frac{1}{\tau(1-\tau)}\{(1-(1-\tau^2)-\tau^2)t+((1-\tau)-1+\tau)t^2\}=1,\forall i=1,..,n.$$

$\boldsymbol{G}(0)=\boldsymbol{A}$ *is evident.* $\boldsymbol{G}(1)=\boldsymbol{A}+(1/(\tau(1-\tau)))\{\boldsymbol{B}-(1-\tau^2)\boldsymbol{A}-\tau^2\boldsymbol{C}+(1-\tau)\boldsymbol{A}-\boldsymbol{B}+\tau\boldsymbol{C}\}=\boldsymbol{C}$, *after a simple technical manipulation, and finally* $\boldsymbol{G}(\tau)=\boldsymbol{A}+(1/(\tau(1-\tau)))\{(\boldsymbol{B}-(1-\tau^2)\boldsymbol{A}-\tau^2\boldsymbol{C})\tau+((1-\tau)\boldsymbol{A}-\boldsymbol{B}+\tau\boldsymbol{C})\tau^2\}=\boldsymbol{B}$, *in a similar way, too. This ends the proof.*

2 Data

In our study, we concern modeling of Prague stock exchange index PX and its trend development by non-homogeneous MC models. We have at our disposal the set of PX day closing prices P_t from Jan. 5, 2004 till Jan. 3, 2013, whereas our previous series spanned period 2004 - 2009, only. We calculate two time series Y_t and K_t thereof, which serve us further to define a state space S for non-homogeneous MC model, too. Y_t is a chain index of day closing prices given by $Y_t = P_t/P_{t-1}$, whereas K_t represents a simple technical index (i.e. accumulated yields, or short-term day basis closing price ratio) defined by (3), where P_t, P_{t-1}, P_{t-2} are day closing prices on day t, t-1, and t-2, respectively. Some leading values of K_t are given in the **Table 1**. Our source data in the form of relative PX day closing prices during 2004-2012 scaled by PX on Jan.5, 2004 as setting 100% are depicted on the **Figure 1**. Inspecting the graph, we may distinguish three different periods. The period of growth from Jan. 2004 till Oct. 2007, period of decrease till Jan. 2009, and finally, period of stagnation till the beginning of 2013.

$$\begin{aligned} K_t &= K_{t-1}Y_t, \quad \text{if } (P_{t-2}\le P_{t-1}\le P_t) \text{, or } (P_{t-2}\ge P_{t-1}\ge P_t) \text{ holds,} \\ K_t &= Y_t, \text{ otherwise.} \end{aligned} \tag{3}$$

t	1	2	3	4	5	6	7	8
P_t	904.0	866.7	893.7	903.6	866.5	825.5	775.3	810.2
K_t		0.959	1.031	1.043	0.959	0.914	0.858	1.045

Table 1 Some values of K_t calculated from P_t

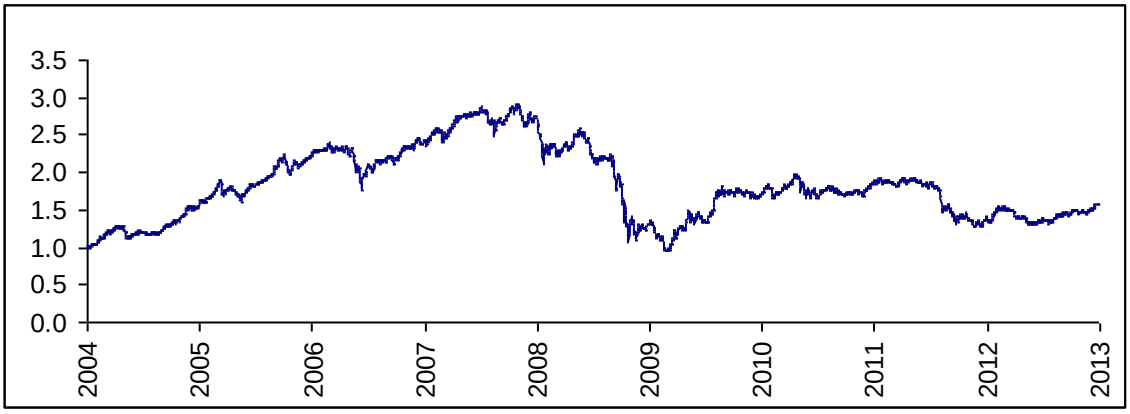

Figure 1 Scaled PX time series P_t/P_0, with $P_0 = P_{\{2004\text{-}01\text{-}05\}}$

3 Markov chain model

First, we define the state space $S = \{s_i\}$, i=1,2,..,8 containing eight states s_i for embedding the calculated time series K_t therein. In order to keep the compatibility with our previous paper [7], we use the same notation for the state space $S = \{D_4, D_3, D_2, D_1, G_1, G_2, G_3, G_4\}$, where D$j$ and Gj, j=1,..,4 denote different levels of growth and decrease of K_t distinguished, which are given by relations (4).

$$\begin{aligned} &D_4\text{: } K_t < 0.97, \quad D_3\text{: } 0.97 \le K_t < 0.98, \quad D_2\text{: } 0.98 \le K_t < 0.99, \quad D_1\text{: } 0.99 \le K_t < 1.00, \\ &G_1\text{: } 1.00 \le K_t < 1.01, \quad G_2\text{: } 1.01 \le K_t < 1.02, \quad G_3\text{: } 1.02 \le K_t < 1.03, \quad G_4\text{: } 1.03 \le K_t\,. \end{aligned} \tag{4}$$

Second, we apply a filtering procedure upon the source time series K_t to get a filtered time series denoted $\mathcal{K}_t$, which is given as $\mathcal{K}_t = \{K_t\}|\mathcal{F}$, where $\mathcal{F}$ denotes our state transition filter, which concerns trend changes within K_t states in particular. Simply, filtering consists of omitting subsequently repeated states, e.g. let $\{D_2\,D_3\,D_3\,G_1\,G_1\,G_1\,G_2\,D_1\,D_1\,G_2\}$ be an un-filtered sequence. By filtering, we get a filtered one $\{D_2\,D_3\,G_1\,G_2\,D_1\,G_2\} = \{D_2\,D_3\,D_3\,G_1\,G_1\,G_1\,G_2\,D_1\,D_1\,G_2\}|\mathcal{F}$. Processing of source PX time series, filtering procedure as well as calculation of all probabilities was performed by MS-Excel with VBA applications. On the **Figure 2**, we show conditional probabilities of transition to general decrease from the state D_2 calculated with sampling window length of 250 days.

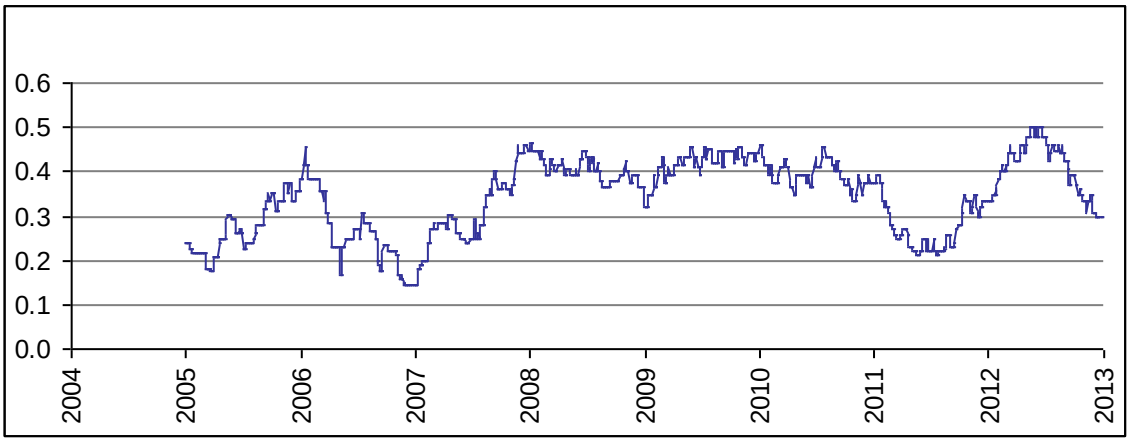

Figure 2 Conditional probabilities $P(\mathcal{K}_{n+1} = D_1 \cup D_2 \cup D_3 \cup D_4 \,|\, \mathcal{K}_n = D_2)$, sampling window of 250 days

Third, we implement Mathematica notebook for all matrix calculations and graphical processing of results. The theoretical framework of our numerical studies is based upon non-homogeneous MC recurrence relation (2). Processing $\mathcal{K}_t$ with sampling window of 250 days (one trading year approximately) we build transition probability matrices denoted $\boldsymbol{P}(05)$, $\boldsymbol{P}(08)$, $\boldsymbol{P}(11)$, and $\boldsymbol{P}(12)$, and state probability distribution vectors $\boldsymbol{p}(05)$, $\boldsymbol{p}(08)$, $\boldsymbol{p}(11)$, and $\boldsymbol{p}(12)$ being assigned to 2005Q1, 2008Q1, 2011Q1, and 2012Q1, respectively. Matrices $\boldsymbol{P}(05)$, $\boldsymbol{P}(08)$, and $\boldsymbol{P}(11)$ serve as interpolation matrices, while $\boldsymbol{P}(12)$ as deviation checking one – see **Figure 3** and **4**.

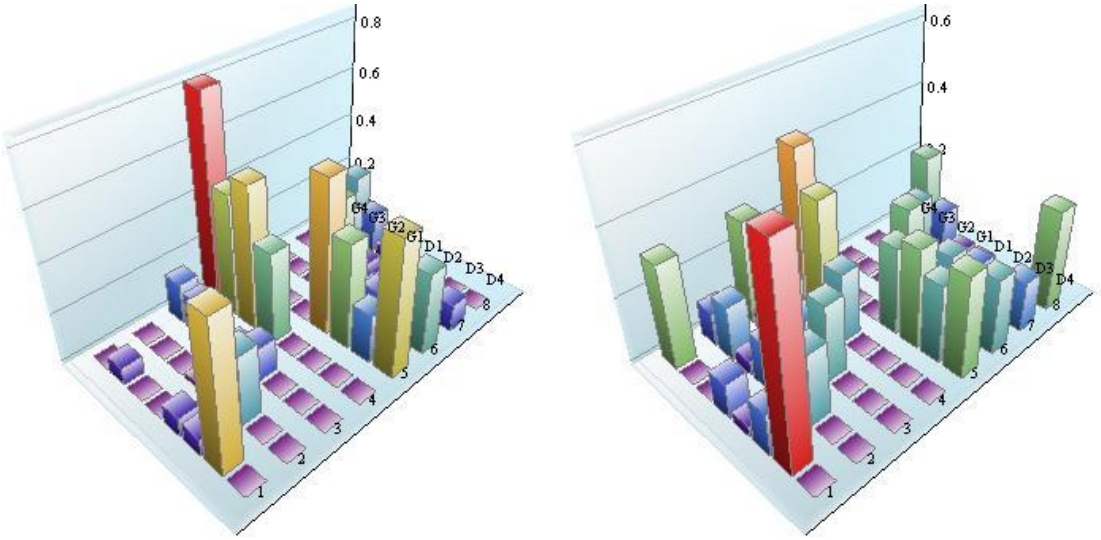

Figure 3 Matrices $\boldsymbol{P}(05)$ – left, and $\boldsymbol{P}(08)$ – right, respectively

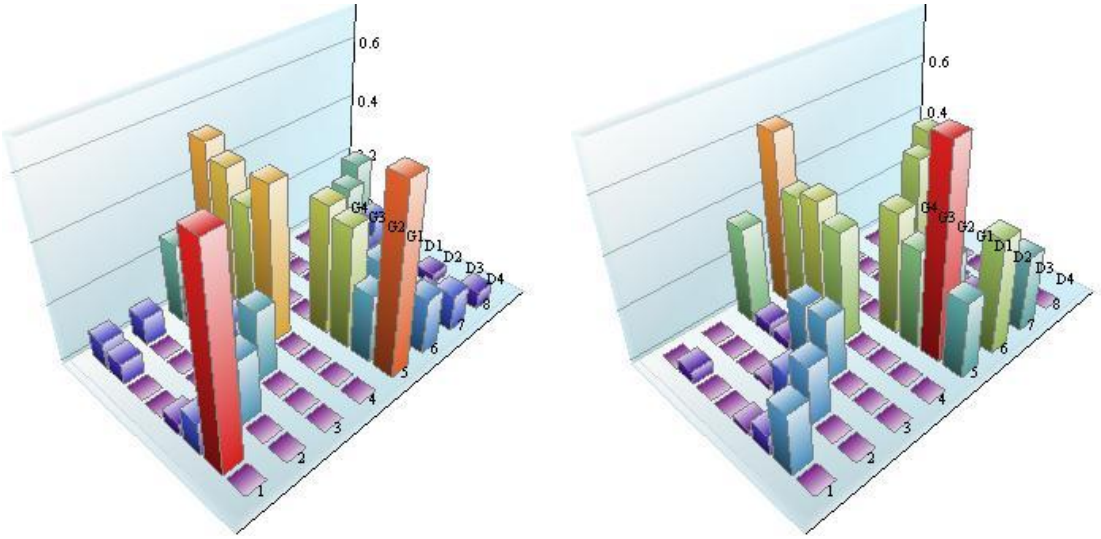

Figure 4 Matrices $\boldsymbol{P}(11)$ – left, and $\boldsymbol{P}(12)$ – right, respectively

We set the non-homogeneous MC models as follows. First, we select the initial vector of state probability distribution $\boldsymbol{p}(0)= \boldsymbol{p}(05)$, and define the time steps n=0, 1,.., 12 adopting a semi-annual scale, i.e. starting at 2005Q1 with n=0, further 2005Q3 with n=1, and ending with 2011Q1 taking n=12. The next two steps with n=13,14 point to 2011Q3 and 2012Q1, respectively, and will be used for prediction. Since we develop two types of matrix interpolation mapping, the linear and quadratic one, we are able to compute two sequences of state probability distribution vectors, which are denoted $\boldsymbol{q}(n)$ and $\boldsymbol{p}(n)$, respectively, whereas the corresponding empirical vectors are denoted $\boldsymbol{r}(n)$.

In order to cover the period between 2005Q1 and 2011Q1, we construct two matrix linear interpolation mappings $\boldsymbol{H}_j(t)$, j=1,2, which are given by (5), and setting joint interpolation condition of both mappings to maintain continuity, i.e. $\boldsymbol{H}_1(1)=\boldsymbol{H}_2(0)=\boldsymbol{P}(08)$. The formulae (5) stand in correspondence with **Lemma 1**, as well.

$$\begin{gathered}\boldsymbol{H}_j(t) = \boldsymbol{A}+(\boldsymbol{B}-\boldsymbol{A})t,\ \ j=1,2,\ \ t\in[0,1],\\ \boldsymbol{H}_1(0)=\boldsymbol{A}=\boldsymbol{P}(05),\ \boldsymbol{H}_1(1)=\boldsymbol{B}=\boldsymbol{P}(08),\quad \boldsymbol{H}_2(0)=\boldsymbol{A}=\boldsymbol{P}(08),\ \boldsymbol{H}_2(1)=\boldsymbol{B}=\boldsymbol{P}(11),\quad \boldsymbol{q}_1(0)=\boldsymbol{p}(05),\ \boldsymbol{q}_2(0)=\boldsymbol{q}_1(1)\end{gathered} \tag{5}$$

In a similar way, however being in correspondence with **Lemma 2**, we construct the matrix quadratic interpolation mapping, which is specified by (6).

$$\begin{gathered}\boldsymbol{G}(t) = \boldsymbol{A}+(-3\boldsymbol{A}+4\boldsymbol{B}-\boldsymbol{C})t+2(\boldsymbol{A}-2\boldsymbol{B}+\boldsymbol{C})t^2,\ t\in[0,1],\\ \boldsymbol{G}(0)=\boldsymbol{A}=\boldsymbol{P}(05),\ \boldsymbol{G}(1/2)=\boldsymbol{B}=\boldsymbol{P}(08),\ \boldsymbol{G}(1)=\boldsymbol{C}=\boldsymbol{P}(11),\quad \boldsymbol{p}(0)=\boldsymbol{p}(05)\end{gathered} \tag{6}$$

Both these mappings are used first, to calculate interim transition probability matrices $\boldsymbol{P}(n)$ at discrete equally distributed points t_n corresponding to n=0,1,..,12. Further using (2), the vectors $\boldsymbol{q}(n)$ and $\boldsymbol{p}(n)$ of state probability distributions are calculated by either matrix linear interpolation mapping or quadratic one. The **Figure 5** show results obtained for interpolation and extrapolation, as well, i.e. for steps n=13,14, and we see the differences.

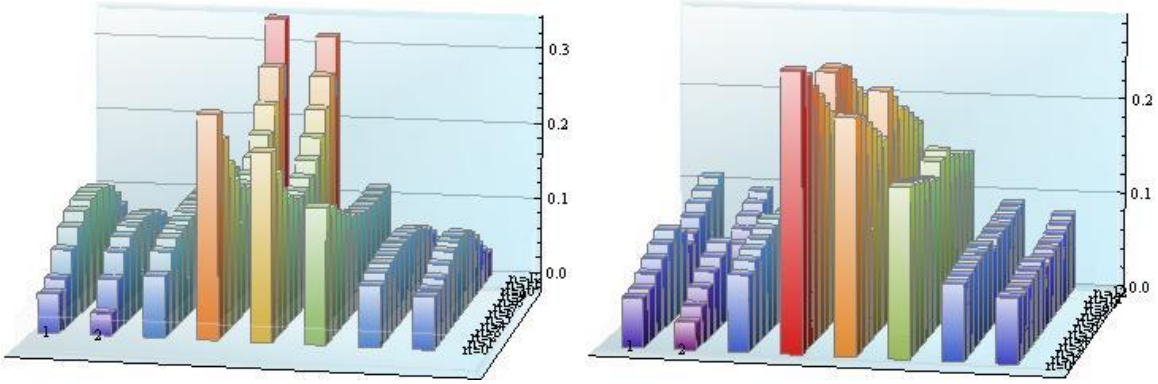

Figure 5 Probability distribution vectors $\boldsymbol{p}(t)$, $\boldsymbol{q}(t)$, at t=2005Q1,..,2011Q1, and 2011Q3, 2012Q1by semi-annual steps using matrix quadratic interpolation mapping – on the left, and piecewise linear mapping – on the right

The **Figure 6** and **7** present difference vectors $\boldsymbol{\Delta}_1(t)= \boldsymbol{q}(t)–\boldsymbol{r}(t)$, $\boldsymbol{\Delta}_2(t)= \boldsymbol{p}(t)–\boldsymbol{r}(t)$, and their norms $||\boldsymbol{\Delta}_1(t)||$, $||\boldsymbol{\Delta}_2(t)||$ calculated at instances 2008Q1, 2011Q1 by mappings (5) and (6) by interpolation, and at 2012Q1 by corresponding extrapolations, where $\boldsymbol{r}(t)$ denote empirical vectors calculated from filtered time series $\mathcal{K}_t$. At the right-most images the value at point ξ=1, which corresponds to t=0, is zero, since $\boldsymbol{q}_1(0)=\boldsymbol{p}(0)= \boldsymbol{r}(0)=\boldsymbol{p}(05)$ by assumption.

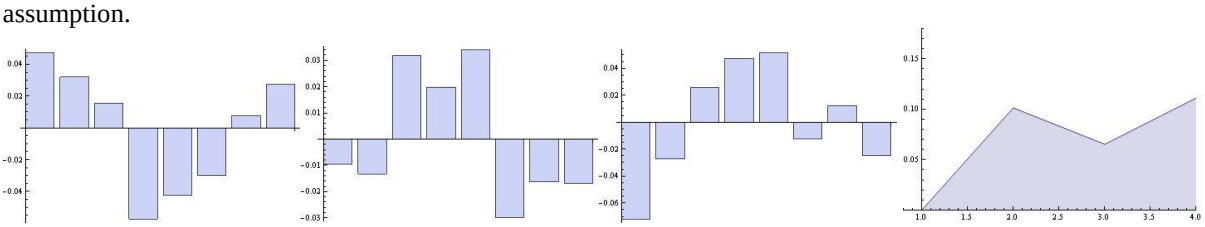

Figure 6 From left to the right, $\boldsymbol{\Delta}_1(t)= \boldsymbol{q}(t) – \boldsymbol{r}(t)$ at t=2008Q1, 2011Q1, 2012Q1, and $||\boldsymbol{\Delta}_1(t)||$ calculated by (5)

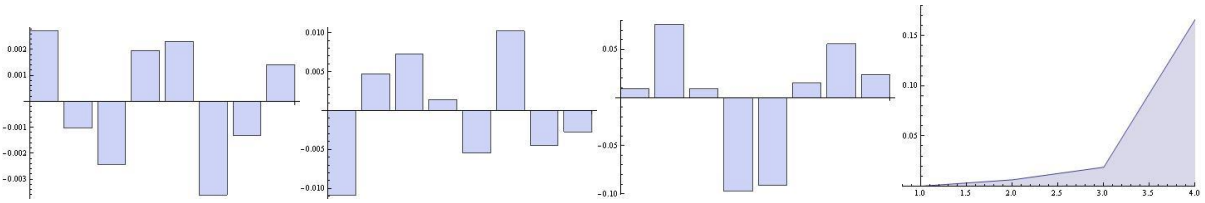

Figure 7 From left to the right, $\boldsymbol{\Delta}_2(t)= \boldsymbol{p}(t) – \boldsymbol{r}(t)$ at t=2008Q1, 2011Q1, 2012Q1, and $||\boldsymbol{\Delta}_2(t)||$ calculated by (6)

Finally, **Figure 8** show difference vectors $\boldsymbol{\Delta}(t)= \boldsymbol{p}(t)–\boldsymbol{q}(t)$ provided the formula (2) is applied with corresponding $\boldsymbol{P}(n)$ being calculated either by (6) for matrix quadratic interpolation mapping, or by (5) for linear ones.

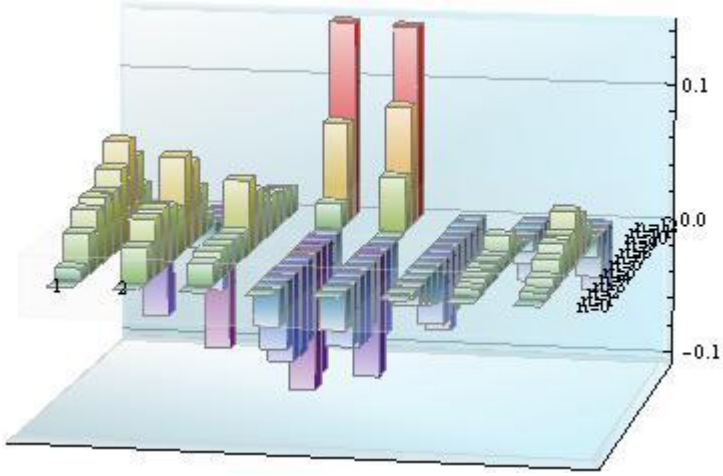

Figure 8 Difference vectors $\boldsymbol{\Delta}(t)= \boldsymbol{p}(t) – \boldsymbol{q}(t)$ at t=2005Q1,..,2011Q1, and 2011Q3, 2012Q1

Calculation of matrices $\boldsymbol{P}(n)$, vectors $\boldsymbol{p}(n)$ and all graphical outputs are realized by Mathematica notebook. The recurrence equation (2) is simply implemented by following commands: `pnkArr={q1};`

```
Do[(q2=q1.P;pnkArr=Append[pnkArr,q2];q1=q2,{k}];
```

While pure function commands are implemented for programming expressions (5) and (6) provided the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are given, as well as τ=1/2, and # stands for an argument t:

```
η1=mtxA + (mtxB-mtxA)#&;

η2=mtxA + (-3*mtxA+4*mtxB-mtxC)# + 2*(mtxA-2*mtxB+mtxC)#*#&;
```

4 Conclusion

We have presented some results of Prague stock exchange index PX analysis using non-homogeneous MC model based upon time dependent transition probability matrices. We have constructed both matrix linear and quadratic interpolation mappings and implemented them numerically in order to analyze evolution of state probability distribution vectors. We hope that such calculations based upon TA and being intended for prediction of trends of financial indices and other financial market prices can contribute to an arsenal of quantitative instruments of financial engineering in theory and practice, as well. Ongoing research will be focused on three main challenging topics

- construction of different state spaces S within MCA,
- problem-oriented filtering procedures of trade sequences,
- development and empirical evaluation of various technical indices for TA, and advanced implementation of non-homogeneous MC, as well.

Acknowledgements

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On a method of estimation of response surface function*

Małgorzata Szerszunowicz[1]

Abstract. One of the tools of statistical quality control, design of experiments allows one to optimise the results of a production process through the use of a function known as response surface function. Classical factorial design of experiments aims at estimating linear or binomial response surface functions. To this end, the method of least squares is used. However, its results may not be sufficient for the correct description of a manufacturing process. Consequently, such an incorrect description may lead to an increase in the costs of, or an improper modification of, the manufacturing process.
The aim of this paper is to use the quantile regression method to estimate the unknown response surface function. The proposed method will be compared with classical factorial designs of experiments; the comparison will be based on the actual results of an experiment.

Keywords: design of experiments, quantile regression, response surface function

JEL Classification: C99
AMS Classification: 62K20

1 Introduction

Statistical methods of quality control are crucial for designing and carrying out a production process. The most important tools of statistical quality control include control charts, acceptance sampling and design of experiments methods.

Design of experiments was first used in the early 20th century by R. A. Fisher in agriculture. An increased interest in methods of designing experiments in industry occurred in the 1960s; design of experiments has been one of the most important stages in planning production processes ever since.

Design of experiments, as a tool which uses probabilistic and statistical methods, requires specific assumptions and the experimenter's proper theoretical and practical knowledge. The correct application of design of experiments not only improves the quality of production processes but also significantly influences their economical results [2].

The use of the classical factorial design of experiments results in descriptions of the relationship between result variable and factors which are characteristic of a given process. This relationship is determined by an unknown response surface function whose parameters are estimated by means of the least squares method.

2 The basis of design of experiments

The effective use of design of experiments methods in practice is associated with the following rules formulated by D. C. Montgomery [4]:

- recognise and define the problem by determining all the aspects, circumstances and potential objectives of the experiment;
- choose factors, define their levels and ranges over which these factors will vary, and explore the possibility of considering them in the experiment;
- select the response variable which provides useful information about the process under study;
- choose a proper experimental design by determining the number of experiments and the possible randomization restrictions;
- perform the experiment;
- analyse the results using statistical methods;
- draw conclusions and formulate recommendations resulting from the analysis of the results for the process under study.

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In order to present the general rules for creating and carrying out designs of experiments, one ought to use appropriate designation. Let us assume that our experiment is a sequence of *n* experimental trials, where single experimental trial is a single result of random variable Y, and the values of factors $X_1, X_2, \ldots, X_m$ are fixed. Let the $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_m$ denote sets of all possible values of factors $X_1, X_2, \ldots, X_m$. The experimental area is a set of points $\mathbf{x} = (x_1, x_2, \ldots, x_m)$, in which $x_i \in \mathbf{X}_i, i = 1, 2, \ldots, m$. Then the set of pairs

$$P_n = \{\mathbf{x}_j, p_j\}_{j=1}^n \tag{1}$$

describes a design with *n* experimental trials, where $\mathbf{x}_j = (x_{1j}, x_{2j}, \ldots, x_{mj})$ and $p_j = \frac{n_j}{n}$, where n_j is a number of experimental trials in point x_j of the experimental area. Moreover, $\sum_{j=1}^{n} n_j = n$ and $\sum_{j=1}^{n} p_j = 1$ for $j = 1, 2, \ldots, n$.

The experiment consists in analysing the influence of a given number of non-random factors $X_1, X_2, \ldots, X_m$ on response variable Y. Moreover, random variable Y may be affected by random factors. This relationship can be presented in the form of the following statistical model [5]:

$$Y(X_1, X_2, \ldots, X_m) = y(X_1, X_2, \ldots, X_m) + \varepsilon \tag{2}$$

where $EY(X_1, X_2, \ldots, X_m) = y(X_1, X_2, \ldots, X_m)$, $E\varepsilon = 0$ and $V\varepsilon = \sigma^2$, where σ^2 is a constant value, independent of the individual factors. The object of the survey is function $y(x_1, x_2, \ldots, x_m)$ called a response surface. The response surface function arguments are realisations of *m* non-random variables $X_1, X_2, \ldots, X_m$. In the classical theory of design of experiments, the model (2) is presented in the form of general linear model $\mathrm{Y} = \mathrm{F}\beta + \varepsilon$, where

$$\mathrm{Y}^{\mathrm{T}} = \left(Y_1 Y_2 \ldots Y_n\right) \tag{3}$$

$$\varepsilon^{\mathrm{T}} = \left(\varepsilon_1 \varepsilon_2 \ldots \varepsilon_n\right) \tag{4}$$

$$\beta^{\mathrm{T}} = \left[\beta_1 \beta_2 \ldots \beta_k\right] \tag{5}$$

$$\mathrm{f}^{\mathrm{T}}(\mathbf{x}) = \left(f_1(\mathbf{x})\, f_2(\mathbf{x}) \ldots f_k(\mathbf{x})\right) \tag{6}$$

$$\mathrm{F} = \begin{bmatrix} f_1(\mathbf{x}_1) & \ldots & f_k(\mathbf{x}_1) \\ \vdots & \ddots & \vdots \\ f_1(\mathbf{x}_n) & \ldots & f_k(\mathbf{x}_n) \end{bmatrix} = \left[\mathrm{f}(\mathbf{x}_1)\, \mathrm{f}(\mathbf{x}_2) \ldots \mathrm{f}(\mathbf{x}_n)\right]^{\mathrm{T}} \tag{7}$$

and $f_i(\mathbf{x}_j) \equiv x_{ij}$, for $i = 1, 2, \ldots k, j = 1, 2, \ldots, n$ [5]. Then the response surface function is defined as equation $\mathrm{y} = \mathrm{F}\beta$. In the case of classical design of experiments, response surface functions with the following interactions of factors are presented in specialist literature:

$$y(\mathrm{x}) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_m x_m + \beta_{12} x_1 x_2 + \ldots + \beta_{12\ldots m} x_1 x_2 \ldots x_m. \tag{8}$$

In justified cases, the experimenter may omit the interactions in the experiment, and then the response surface is

$$y(\mathrm{x}) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_m x_m. \tag{9}$$

or

$$y(\mathrm{x}) = \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_m x_m. \tag{10}$$

The estimation of the parameters of the above response surface functions is effected through an experiment with *m* factors, each on n_i levels. Then the complete experiment requires $n = \prod_{i=1}^{m} n_i$ experimental trials, using the least squares method [5].

3 Quantile regression and design of experiments

The least squares method is a tool which is most commonly used for estimating the parameters of a regression function expressed in the form of a general linear model. One of the assumptions of the least squared method is an assumption of Gaussian distribution of errors. Moreover, the parameters estimated by means of the least squares method show only the expected average relationship between the factors and response variable.

In design of experiments carried out in manufacturing companies, the assumptions of the least squares method, particularly the assumption about the distribution of errors, cannot always be met. Moreover, the interpretation of the estimated parameters of the response surface function can lead to incorrect recommendations for the manufacturing process, which has a considerable influence on the costs and results of the process. It is easy to imagine a situation in which it is important to describe the relationship between the outcome variable and the factors included in the experiment not only with the expected value but also with the extreme values, such as the largest energy or the smallest heat generated during the chemical reaction. Then in order to estimate the parameters of the response surface function describing the relationship between the response variable and the experimental factors, one may use the method of quantile regression.

The concept of quantile regression was introduced by Koenker and Basset in 1978 [1]. Currently, the quantile regression method is a statistical tool widely used in the modelling of economic phenomena.

The purpose of quantile regression is to determine a linear model – on the basis of an *n*-elemental sample of an unknown distribution F – in the following form:

$$Q_\tau(Y \mid X) = X^T \beta_\tau , \tag{11}$$

where $\tau \in (0,1)$ is a fixed for sample $\tau-$quantile, and β_τ is a vector of unknown values of the parameters of a given linear model, and $X = [\mathrm{e} \mid \mathbf{F}]$ where $\mathrm{e} = (1, 1, \ldots, 1)$ and **F** is defined as in (7). In this case, the values of vector β_τ are estimated by means of an estimator in the form of

$$\hat{\beta}_\tau = \min_{b \in R^k} \sum_{i=1}^{n} \rho_\tau (y_i - x_i^T b) , \tag{12}$$

where $\rho_\tau(u) = u(\tau - \mathbf{1}_{(u<0)})$. Then the estimator of model (11) is

$$\hat{Q}_\tau(Y \mid X) = X^T \hat{\beta}_\tau . \tag{13}$$

The procedure for the appointment of the estimator (12) may be written as a linear programming problem [1]. The estimation of a linear model based on the distribution quantiles allows one to describe the dependence of the entire distribution response variable on the independent variables, not just its expected value, which could have a significant influence on the analysis of the results of the experiment.

4 The use of quantile regression for estimating the parameters of response surface function

D. C. Montgomery in [3] presents the data of an experiment whose aim is to determine the relationship between the viscosity of polymer (Y), reaction temperature (X_1) and catalyst feed rate (X_2). This data, obtained from 16 measurements, is presented in Table 1. The response surface function describing the relationship, apart from factor interactions, can be expressed by the following equation:

$$y(\mathrm{x}) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 , \tag{14}$$

and the response surface function with factor interactions is

$$y(\mathrm{x}) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 . \tag{15}$$

Measurement number	**Temperature** X_1	**Catalyst feed rate** X_2	**Viscosity** Y
1	80	8	2256
2	93	9	2340
3	100	10	2426

4	82	12	2293
5	90	11	2330
6	99	8	2368
7	81	8	2250
8	96	10	2409
9	94	12	2364
10	93	11	2379
11	97	13	2440
12	95	11	2364
13	100	8	2404
14	85	12	2317
15	86	9	2309
16	87	12	2328

Table 1 The experimental data

According to classical design of experiments, the least squares method was used to estimate the parameters of response surface function (14) and (15). As a result, the following estimators of response surface function were obtained respectively:

$$y(\mathrm{x}) = 1566{,}08 + 7{,}62x_1 + 8{,}58x_2, \tag{16}$$

$$y(\mathrm{x}) = 1871{,}98 + 4{,}24x_1 - 23{,}63x_2 + 0{,}36x_1x_2. \tag{17}$$

The determined functions allow one to specify the dependence of the expected viscosity on the average changes of temperature and catalyst feed rate. Let us use the data from Table 1 to describe the influence of temperature and catalyst feed rate changes on the high and small values of polymer viscosity. The response surface function – on the assumption that there are no interaction of the factors – is determined by the equation

$$\hat{y}_\tau(x) = \hat{\beta}_{0\tau} + \hat{\beta}_{1\tau}x_1 + \hat{\beta}_{2\tau}x_2 \tag{18}$$

while the response surface allowing for the interaction of the factors is as follows:

$$\hat{y}_\tau(x) = \hat{\beta}_{0\tau} + \hat{\beta}_{1\tau}x_1 + \hat{\beta}_{2\tau}x_2 + \hat{\beta}_{12\tau}x_1x_2 \tag{19}$$

where $\tau \in (0,1)$. The above mentioned equations may be written in the form of model (13), so the parameters of response surface functions (18) and (19) can be estimated according to the quantile regression method, using formula (12). The results are shown in Table 2 and Table 3 respectively for response surface functions (18) and (19) for fixed values of τ.

τ	$\hat{\beta}_{0\tau}$	$\hat{\beta}_{1\tau}$	$\hat{\beta}_{2\tau}$
0,05	1663	6,56	7
0,25	1659,74	6,56	7,41
0,5	1524,32	8,11	8,65
0,75	1568,38	7,81	7,67
0,95	1505,91	8,63	7,46

Table 2 Values of parameters of response surface function (18) for selected values of τ

τ	$\hat{\beta}_{0\tau}$	$\hat{\beta}_{1\tau}$	$\hat{\beta}_{2\tau}$	$\hat{\beta}_{12\tau}$
0,05	1698	6,17	2,63	0,05
0,25	1560,14	7,6	19,86	-0,13
0,5	1921,26	3,78	-30,47	0,43
0,75	1885,94	4,15	-27,74	0,41
0,95	1629,26	7,37	-8,31	0,16

Table 3 Values of parameters of response surface function (19) for selected values of τ

It is worth noting that depending on the predetermined value of τ, the functions differ from each other significantly; in particular they differ from the response surface function estimated by means of the least squares method. The differences between response surface functions (16) and (18), when $\tau = 0{,}05$ and $\tau = 0{,}95$, for the range in question, are shown by Figure 1.

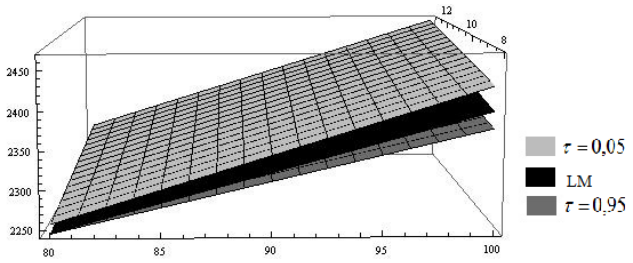

Figure 1 Graphical representation of response surface functions (16) and (18) for $\tau = 0{,}05$, $\tau = 0{,}95$

Similarly, the response surface functions for design of experiment including interactions between factors are illustrated by Figure 2 – for any range of factors – and by Figure 3 – for the range of factors in question.

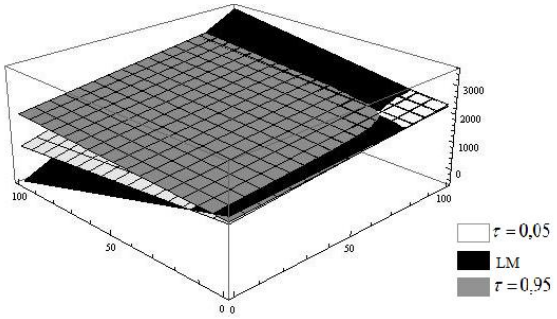

Figure 2 Graphical representation of response surface functions (17) and (19) for $\tau = 0{,}05$, $\tau = 0{,}95$

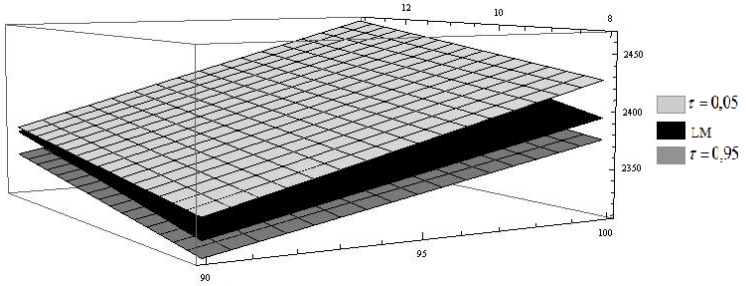

Figure 3 Graphical representation of response surface functions (17) and (19) for $\tau = 0{,}05$ i $\tau = 0{,}95$ for the range of factors in question

The response surface functions obtained by means of the quantile regression method allow one to determine the relationship between the viscosity of polymer and the temperature and catalyst feed rate when the viscosity values are at the levels of the respective quantiles. Thus, thanks to the quantile regression method, it is possible to determine the relationship for the extreme values of response variable, e.g. as 0,05th or 0,95th quantile, which in turn leads to a complete characterisation of response variable. A detailed description of response variable – polymer viscosity, in this case – has an important impact on the costs and production effects of manufacturing processes.

5 Conclusions

Design of experiment methods play an important role in planning production processes. In the classical approach, the most essential phase of describing the relationship between response variable and factors is to determine the form of a response surface function by means of the least squares method. The use of the least squares method for estimating the parameters of response surface requires some assumptions to be met, which in practice cannot always be guaranteed during the implementation of manufacturing processes.

In this article, an alternative method of estimating response surface parameters has been presented. The method of quantile regression allows one to estimate the parameters of response surface functions and to provide results which can supplement information collected in experiments.

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Foreign Direct Investment in the Baltic States: Does it Crowd out or Crowd in Domestic investment?

Zuzana Szkorupová[1]

Abstract. Foreign direct investments are generally considered as one of the factors with positive influence on the economic development of countries in which this investments flow. For this purpose governments of transition countries try to encourage the inflow of foreign direct investment by various measures. Foreign direct investment, with other benefits, brings also many advantages such as new technologies, manufacturing processes, know-how and others. Despite this positive feedback, the theme of foreign direct investment and its impact on the economy is being discussed at many levels. In evaluating the impact of foreign direct investment on development, however, a key question is whether foreign direct investment crowd in domestic investment, or foreign direct investment crowd out domestic investment. The aim of this paper is to examine whether foreign direct investment in the Baltic States (Estonia, Latvia and Lithuania) crowd in or crowd out the domestic investment. For this purpose we apply theoretical model of investment that incorporates foreign direct investment as an exogenous variable. Data were tested with panel regression for the period 1993 – 2009. We found out that for the time period 1993 – 2009 there was an evidence of the effect of crowding out domestic investment by foreign direct investment in the Baltic States.

Keywords: foreign direct investment, domestic investment, crowding in effect, crowding out effect, panel data analysis, Baltic States.

JEL Classification: F21, F23, O16
AMS Classification: 62P20

1 Introduction

Foreign direct investment (FDI) and their effects are discussed on scientific, empirical, political and others levels. Basically we speak about positive and negative effects that might be expressed directly or indirectly. According to Mišun and Tomšík [7] the main positive direct effects are impact of foreign direct investment on economic growth of host country, availability of modern technology, managerial know how or better access to financial fund. Foreign direct investment implement on a green field are the ones that create new work opportunities. Considering transitive economy, an important role plays fiscal income from foreign direct investment performed by method of privatization. In addition, Mišun and Tomšík[7] point out negative direct effects of foreign direct investment. It involves basically pressure on appreciation of domestic currency, enforced increase of money supply and related inflation pressure increase of foreign trade deficit in the area of new technologies import and increase of deficit of current account payment balance as a consequence of profit repatriation by multinational companies. Considering FDI indirect effects, Mišun and Tomšík[7] refer to positive indirect effect of foreign investors on domestic companies when positive effects from foreign companies overfall on domestic market and increase competition on market due to fact that they become business partners of foreign companies. Reverse effect, it means negative indirect effect of foreign direct investment on domestic market is dual economy, the situation when on the one hand there are foreign companies operating in domestic economy with prospect and on the other hand there are domestic companies with inefficient economy and consequently are pushed out from the market by stronger foreign companies.

The aim of this paper is to examine the effects of crowding in and crowding out domestic investment by foreign direct investment in Baltic States. Baltic States includes countries of Estonia, Latvia and Lithuania. For the purpose of crowding in and crowding out effects of domestic investment by foreign direct investment in this paper, we shall apply abbreviation CI – crowding in of domestic investment by foreign direct investment and CO – crowding out of domestic investment by foreign direct investment. Research shall be performed for time period 1993 – 2009. This time period was selected depending on data availability and impact of economic crisis on foreign direct investment and economic growth in period 2009 – 2012. We shall apply annual data and research

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shall be performed according to model Agosin and Mayer [2] and Mišun and Tomšík[7]. This model uses macroeconomic data for estimation of effects of FDI on domestic investment. Model specification and data used to find out the effects of CI and CO shall be stated in the particular chapter. Article is divided into four basic parts. First part deals with literature overview dealing with this issue. Second part pays attention to model specification and data. Third part deals with testing the effects of crowding in and crowding out of domestic investment by foreign direct investment. Fourth part is conclusion.

2 Literature overview

There is a series of empirical studies examining FDI effects on domestic investment in host countries. A model, used to perform research in this article, is formed by Agosin and Mayer [2] who examined effects of crowding in and crowding out of domestic investment in three regions, including Africa, Asia and Latin America. They performed their research with 32 countries for the period 1970 – 1966. In this period they proved effect of crowding in of domestic investment in Asia, effect of crowding out of domestic investment in Latin America and neutral effect of FDI on domestic investment in Africa. Time period subsequently divided into two sections, from 1976 – 1985 and 1986 – 1996. In these periods authors confirmed results reported for the whole time period in Asia and Latin America region. Africa, in both periods showed effect of crowding in of domestic investment by FDI.

Similar research was performed by Mišun and Tomšík[7]. Authors examined crowding in of domestic investment by FDI in Poland, Hungary and Czech Republic for time period 1990 – 2000 (Poland and Hungary) and period 1993 – 2000 (Czech Republic). Results of their research proved the effect of crowding in of domestic investment by FDI in the Czech Republic and Hungary and the effect of crowding out of domestic investment by FDI in Poland. Braunstein and Epstein [4] examined effects of foreign direct investment on domestic investment for time period 1986 – 1999 in China. The authors performed research on the basis of panel data. Results of their work point out the fact that FDI crowd out the domestic investment in China. Backer and Sleuwaegen [3] examined the impact of FDI on domestic entrepreneurs in field of manufacturing industry in Belgium. They performed the research for time period 1990 – 1995. Empirical results of their research show that the proved crowding out effect from the short term point of view might become positive crowding in effect from the long term point of view due to arising business relations between domestic and foreign companies. Driffield and Hughes[5] examined impact of foreign direct investment on domestic investment in Great Britain. Research was done in time period 1984 – 1997. At the end of the study they observed that FDI flow into Great Britain had a positive effect on domestic investor. Upon division of Great Britain into regions they observed that crowding in effect was reflected in south-north part of England and crowding out effect in Scotland, Wales and Northem Ireland. Agosin and Machado [1] examined effects of crowding in and crowding out of domestic investment by FDI in developing countries. Research was done in 12 countries in Latin America, Africa and Asia for time period 1971 – 2000. Result of their research is that FDI have crowding out effect mostly in Latin America. Titarenko [9] examined FDI effect on domestic investment in Lithuania. The author conducted research for time period 1995 – 2004. His research proved that FDI have negative influence on domestic investment and consequently the crowding out effect was approved. Mutenyo et al. [8] examined effects of crowding in and crowding out of domestic investment by FDI in 34 countries from region of Sub-Saharan Africa. Research was done for time period 1990 – 2003. This research proved negative effects of FDI on domestic investment, which means that FDI crowd out the domestic investment. Gallová [6] examined effects of FDI on domestic investment in selected countries of Balkan (Bulgaria, Croatia, Romania and Slovenia) and in the Visegrad Group Countries. The author conducted research for time period 1993 – 2009. Results of her research point out the fact that FDI crowd out the domestic investment in selected countries of Balkan and in the Visegrad Group Countries.

3 Model specification and data

Estimation of crowding in and crowding out of domestic investment by foreign direct investment shall be performed with the model Agosin and Mayer [2] that is modified by Mišun and Tomšík [7] on condition of transitive economies.

Equation used for the purpose of research has the following form:

$$I_{i,t} = \alpha_i + \beta_1 F_{i,t} + \beta_2 F_{i,t-1} + \beta_3 F_{i,t-2} + \beta_4 I_{i,t-1} + \beta_5 I_{i,t-2} + \beta_6 I_{i,t-3} + \beta_7 G_{i,t-1} + \beta_8 G_{i,t-2} + \beta_9 G_{i,t-3} + \\ + \varepsilon_{i,t} \quad (1)$$

Individual variables in an equation are:

- I = domestic investment/GDP;
- F = FDI/GDP;
- G = growth of GDP;
- α = fixed country effects;

- ε = serially uncorrelated random error;
- i = country in panel;
- t = time.

Model is test on annual data for the period of 1993 – 2009. Data about the FDI inflow development, growth of GDP and formation of gross fixed capital were acquired from the World Bank, World Development Indicators. Estimation was performed on the basis of Panel Regression Model in Eviews 7 program. For the assessment of crowding in or crowding out effects of domestic investments by foreign direct investment was used the coefficient in the following form:

$$\pi = \frac{\beta_1+\beta_2+\beta_3}{1-(\beta_4+\beta_5+\beta_6)} \tag{2}$$

Crowding in and crowding out effect of domestic investment by foreign direct investment in an Eviews 7 program is tested by Wald test. Criterion used to determine the crowding in or crowding out effect of domestic investment is the value of π coefficient. In this case arise three possibilities:

- On the basis of Wald test it is impossible to deny hypothesis $H_0: \pi = 1$. This fact means that from the long term point of view the increase of FDI by one unit leads to increase of entire investments equally by one unit.
- In case that $H_0: \pi = 1$ is denied and coefficient $\pi > 1$, we speak about crowding in of domestic investments by foreign investments. From the long term point of view it implies that one additional unit of FDI shall be reflected by more than one additional unit of total investments.
- While $H_0: \pi = 1$ is denied and coefficient $\pi < 1$, it is the evidence that domestic investments are crowded out by foreign investments. From the long term point of view it implies that one additional unit of FDI leads to growth of total investments by less than one unit.

If the long term effect of FDI displays crowding in of domestic investment, we speak about positive side effects of foreign direct investment. Side effects of FDI are negative if the FDI from the long term lead to the crowding out of domestic investments.

4 Testing of Crowding in and Crowding out Effects of Domestic Investment

Testing of crowding in and crowding out effects by foreign direct investment has been performed on the basis of equations specified in the above chapter. Regression analysis based on the equation (1) was performed for Baltic States. Table 1 shows descriptive statistics for each country from Baltic States.

Country	Variable	Average	Median	Maximum	Minimum	Standard deviation
Estonia	F	0.080	0.074	0.211	0.031	0.043
	I	0.284	0.281	0.349	0.218	0.036
	G	0.040	0.071	0.111	-0.140	0.070
Latvia	F	0.044	0.044	0.084	0.003	0.024
	I	0.233	0.242	0.336	0.137	0.064
	G	0.038	0.064	0.122	-0.180	0.074
Lithuania	F	0.032	0.035	0.082	0.004	0.022
	I	0.224	0.223	0.282	0.187	0.023
	G	0.025	0.067	0.102	-0.162	0.083

Table 1 Descriptive statistic

According to Hausman test was used panel regression with fixed effects. In order to perform panel regression it is necessary for the time series being stationary on its own values I(0). For this purpose was used Levin, Lin, Chu test (LLC test). According to test was identified that data are stationary. Table 2 below shows results of panel regression for Baltic States. Figures in parenthesis are t-ratio. Index "a" means: significantly different from zero at the 1 per cent level. Index "c" means: significantly different from zero at the 10 per cent level.

Variable	Baltic States
F_t	0.166
	(1.490)
F_{t-1}	-0.006
	(-0.059)
F_{t-2}	-0.042
	(-0.368)
I_{t-1}	0.172
	(0.823)
I_{t-2}	0.393[c]
	(1.861)
I_{t-3}	-0.090
	(-0.576)
G_{t-1}	0.638[a]
	(5.493)
G_{t-2}	0.066
	(0.516)
G_{t-3}	-0.053
	(-0.644)
Adj. R-square	0.831
DW-statistic	2.138

Table 2 Investment Equations for Baltic States

Testing of crowding in and crowding out effect of domestic investment by foreign direct investment is realized on the basis of Wald test with coefficient stated in equation (2). The test was performed for Baltic States for the period 1993 – 2009. In this region was proved negative FDI effect on domestic investment for a given period – the CO effect. Results are shown in a Table 3. CO is an abbreviation for crowding out effect of domestic investment by foreign direct investment.

Region	Coefficient π	Long-term effect
Baltic States	0.218	CO

Table 3 Effect of FDI on investment

Negative effect of crowding out the domestic investment by foreign direct investment is possible to explain by the fact that foreign companies use services from the same suppliers as their mother companies. It means that services of suppliers are owned by foreign companies. Foreign companies take components necessary for their production from foreign manufacturers who are brought into a host country by foreign companies themselves. Another fact is that companies owned by a foreign possessor are in individual industries so strong and effective that for a domestic investor it is a problem to enforce and conduct business, or cooperate with a multinational company in individual industries.

5 Conclusion

The aim of this paper was to examine effects of crowding in and crowding out of domestic investment by foreign direct investment in the Baltic States (Estonia, Latvia and Lithuania). Research has been conducted for time period 1993 – 2009 according to the annual data obtained from World Bank database. A model Agosin and Mayer [2] and Mišun and Tomšík [7] was used in order to examine particular data. Panel regression was used for calculations necessary to obtain results of the research. Subsequently the effect of crowding in and crowding out of domestic investment by foreign direct investment was tested by Wald test and coefficient π in Eviews 7.

Results of research were evaluated in the last chapter. Positive crowding in effect of domestic investment by foreign direct investment was not proved. On the contrary, foreign direct investment from the long term point of view has negative influence and crowd out the domestic investment in Baltic States.

In literature the foreign direct investment is generally considered to be a positive impulse for economy in a host country, especially in countries with transformation process. Economic growth in a country and growth of export and other positive effects are attributed to foreign direct investment. These facts may be up to certain point true and foreign direct investment are prosperous during the period of transformation process and economy recovery in a country. However, foreign direct investment inflow into a host country should not crowd out business activities of domestic investors, on the contrary, it should crowd in the domestic investment in a positive way. As it was mentioned earlier, crowding out effect of domestic investment by foreign ones is negative for domestic business environment, domestic investors and their possibilities to conduct activities in business area.

There are several explanations of negative crowding out effects domestic investment by foreign direct investment. During the transformation process in Central and Eastern Europe was performed foreign direct investment by way privatization. The foreign investors acquired share in strategic companies in strategic industry (telecommunication, gas industry, manufacturing industry etc.). These companies crowd out domestic investment. Domestic firms are unable to conduct business effectively and be competitive multinational companies. Another reason is the policy of national governments that promote business of foreign investors and on the other hand, domestic companies are without the benefit of business. Next reason can be fact that suppliers from host countries are not attractive for multinational companies. Domestic companies don't fulfill global standards for quality of supplies.

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Estimation of the Production Function Parameters in V4 Economies

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Abstract. We estimate the production function parameters in V4 economies. The first order condition of the firm optimal behaviour states the equality between wage rate and marginal product of the labour. The CES production function generates the logarithmic relation between average and marginal products of the labour. The Cobb-Douglas production function generates the linear relation between average and marginal products of the labour. We estimate the specifications of V4 economies generated by these models. Using logarithmic specification we estimate the value of the substitutability of the inputs parameter and we test a hypothesis of the Cobb-Douglas production function in the corresponding V4 economy. If we do not reject the hypothesis, we estimate the value of the labour share parameter in corresponding V4 economy using linear specification. We use quarterly data of the compensations of employees and gross domestic product.

Keywords: CES and Cobb-Douglas production function, inputs substitution elasticity, labour share, Visegrad Four countries

JEL Classification: C13
AMS Classification: 62H12

1 Introduction

Estimate of the production function parameters has been an important topic on the quantitative research agenda for decades. The most recent approaches are [1], [6], [5] and [2].

In the paper we will use a method of the production function estimation we have applied using Slovak average monthly wages and average product of the labour data in [9]. We will estimate the production function parameters of the Visegrad Four countries (V4, i.e. Czech Republic, Hungary, Poland and Slovakia). The models are based on the long-run first order condition of optimal behaviour of firms that states equality between wage rate and marginal product of the labour. Using more general CES production function we can specify a logarithmic relation between marginal and average products of the labour and using Cobb-Douglas production function we can specify a linear relation between marginal and average products of the labour.

The aim of the paper is to estimate the production function parameters of the Visegrad Four countries (V4, i.e. Czech Republic, Hungary, Poland and Slovakia) using econometric average wage analysis. In the second section of the paper we introduce the models and derive the logarithmic (from the CES production function) and the linear (from the Cobb-Douglas production function) specification for wages. Data and methodology are discussed in the third section of the paper. It includes the estimation methodology of both logarithmic and linear specification, testing of coefficient restrictions in both specifications. The estimates are in the fourth section and the fifth section concludes.

2 Models

Consider the problem of the firms in the economy, which choose capital and labour inputs (K and N) by maximizing the profit function. It is well known that the first-order condition of the problem for labour is that wage rate equals to the marginal product of the labour:

$$w_t = MPN_t \tag{1}$$

We can write the wage condition (1) in both nominal and real units. Official V4 statistical portals publish a corresponding data only in current prices; hence we will assume that all variables are nominal. Using this assumption, this section will be more consistent with the sections dealing with econometric analyses using data.

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We will show that using various known forms of the production function we can express marginal product of the labour (MPN_t) by the average product of the labour (APN_t).

2.1 Logarithmic Specification

Consider the CES Production Function in the form:

$$Y_t = A_t\left[\alpha\left(\kappa K_t\right)^{\gamma} + \left(1-\alpha\right)\left(\nu N_t\right)^{\gamma}\right]^{\frac{1}{\gamma}} \tag{2}$$

where inputs are capital and labour, K_t and N_t, $0 < \alpha < 1$ is the share parameter, γ determines the degree of substitutability of the inputs and A_t is the total factor productivity parameter. The parameters κ and ν depend upon the units in which the output and inputs are measured and play no important role. The value of γ is less than or equal to 1 and can be $-\infty$.

If $\gamma = 1$, the production function is linear and the inputs are perfectly substitutable, if $\gamma = -\infty$, the inputs are not substitutable. Finally, if $\gamma = 0$, the production function is Cobb-Douglas. Using Cobb-Douglas production function we derive the linear specification in the next section.

We assume that the productivity parameter grows constantly with the rate g:

$$A_t = A_0 e^{gt} \tag{3}$$

Let's express the marginal product of the labour from the CES production function (2):

$$MPN_t = A_t\left[\alpha\left(\kappa K_t\right)^{\gamma} + \left(1-\alpha\right)\left(\nu N_t\right)^{\gamma}\right]^{\frac{1-\gamma}{\gamma}}\left(1-\alpha\right)\nu^{\gamma} N_t^{\gamma-1}$$

The log of marginal product of the labour is:

$$\log\left(MPN_t\right) = \log\left(A_t\right) + \frac{1-\gamma}{\gamma}\log\left(x_t\right) + \log\left(1-\alpha\right) + \gamma\log\left(\nu\right) + \left(\gamma-1\right)\log\left(N_t\right) \tag{4}$$

where

$$x_t = \left[\alpha\left(\kappa K_t\right)^{\gamma} + \left(1-\alpha\right)\left(\nu N_t\right)^{\gamma}\right]$$

Let's express the average product of the labour from the CES production function (2):

$$APN_t = \frac{A_t\left[\alpha\left(\kappa K_t\right)^{\gamma} + \left(1-\alpha\right)\left(\nu N_t\right)^{\gamma}\right]^{\frac{1}{\gamma}}}{N_t} \tag{5}$$

The log of average product of the labour is:

$$\log\left(APN_t\right) = \log\left(A_t\right) + \frac{1}{\gamma}\log\left(x_t\right) - \log\left(N_t\right)$$

Combining (5) and (4) we can derive the relation between logs of marginal and average marginal product of the labour in the form:

$$\log\left(MPN_t\right) = \log\left(1-\alpha\right) + \gamma\log\left(\nu\right) + \gamma\log\left(A_t\right) + \left(1-\gamma\right)\log\left(APN_t\right) \tag{6}$$

Substituting logarithm of the marginal product of the labour, (6) and logarithm of the productivity growth, (3) into the logarithm of the wage rate condition (1) we gain the logarithmic specification for wages in the form:

$$\log\left(w_t\right) = \beta_0 + \beta_1\log\left(APN_t\right) + \beta_2 t + u_{gt} \tag{7}$$

where the stochastic term u_{gt} captures the demand, monetary and real shocks and satisfies standard assumptions of the econometric model and coefficients are:

$$\beta_0 = \log\left(1-\alpha\right) + \gamma\log\left(\nu\right) + \gamma\log\left(A_0\right) \tag{8}$$

$$\beta_1 = 1-\gamma \tag{9}$$

$$\beta_2 = \gamma g \tag{10}$$

Unfortunately the logarithmic specification (7) is overidentified, nevertheless an estimation of the specification will provide several information about production function. From the coefficient β_1 we know about substitution abilities of the inputs, the inverse value of the coefficient is the average elasticity of the substitution of the inputs. Moreover we can determine the type of the production function from this value. Using both coefficients β_1 and β_2, we can compute the average productivity growth rate, g.

2.2 Linear specification

Consider more specific case, where $\gamma = 0$ and the production function is in the Cobb-Douglas form:

$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha} \tag{11}$$

Let's go back to the derivation of the marginal product of the labour using average product of the labour. Let's express the marginal product of the labour from the Cobb-Douglas production function (11):

$$MPN_t = (1-\alpha) A_t K_t^{\alpha} N_t^{-\alpha} \tag{12}$$

and the average product of the labour from the same function (11):

$$APN_t = A_t K_t^{\alpha} N_t^{-\alpha} \tag{13}$$

Combining (12) and (13) we express the linear relation between marginal and average product of the labour in the form:

$$MPN_t = (1-\alpha) APN_t \tag{14}$$

Substituting the marginal product of the labour (14) into the wage rate condition (1) we gain the linear specification for wages in the form:

$$w_t = \delta_0 + \delta_1 APN_t + u_{nt} \tag{15}$$

where the stochastic term u_{nt} captures the demand, monetary and real shocks and satisfies standard assumptions of the linear econometric model. We assume the value of the coefficient δ_0 equals to zero and the coefficient δ_1 equals to the average labour share, $\delta_1 = (1 - \alpha)$ and the reminder to unity is the average capital share.

3 Data and Methodology

All data we use are gathered from the portal of the EUROSTAT. We use quarterly not seasonally adjusted GDP measured in current prices and Compensation of Employees – code B1GM and D1 in National Accounts of each country from 1995 to 2012. The obtained data is seasonally adjusted by standard Tramo/Seats procedure.

The stationary analysis of time series is performed by Augmented Dickey-Fuller test (ADF) and Phillips-Perron test (PP). The second unit root test is preferred in case of a high number of the autoregression terms in test equation solving the autocorrelation. The importance of stationarity testing is explained in [7]. The results of this analysis validate our expectations, that each time series have been generated by stochastic processes integrated of order one. So, in estimated regression models we could use the first differences of variables (not allowed in long-run theory) or find co-integrated linear combination of the variables, which is stationary.

The test of co-integration is realized by Johansen's procedure [4]. Engle-Granger's two step method [3] is refused because of the impossibility of direct testing of long-run relationship parameters. We start our analysis with the fourth Johansen deterministic specification of logarithmic model – constant and trend in co-integration equation – according to (7). The estimates are in the Table 1 in the fourth section of the paper. The values in brackets below each estimate are standard deviations of the coefficient. Two asterisks denote statistical significance at the one percent level. Similar denotations are used in all tables (1-4).

We compute lambda trace statistics to verify expected rank of co-integration (which is one). Then we estimate the same specification with the third Johansen deterministic specification – constant and no trend in co-integration equation. The estimates are in the Table 2. Using these two estimates and their maximal likelihood function values we compute the likelihood ratio statistics (see [8]) to verify significance of the trend. These values are in the last column of the Table 1. If we do not reject trend significance hypothesis for any country we can compute the coefficient γ as $1 - \beta_1$ (9) and the average total factor productivity growth rate g as $\beta_1/(1 - \beta_1)$ (10).

If we reject trend significance hypothesis for any country we compute likelihood ratio test for binding restrictions, that $\beta_1 = 1$. The values of this ratio are in the last but first column in the Table 2. If we do not reject this hypothesis the inputs substitutability elasticity is unit and we can calculate the labour share parameter value $(1 - \alpha)$ as $exp(\beta_0)$. The values of this parameter are in the last column in the Table 2.

If we reject trend significance hypothesis and if we do not reject unit inputs substitutability elasticity hypothesis for any country, we use linear specification to estimate the labour share coefficient of this country. We use the second Johansen deterministic specification of linear model – constant and no trend in co-integration equation according to (15). The estimates are in the Table 3.

We compute lambda trace statistics to verify expected rank of co-integration. Then we estimate the same specification with the first Johansen deterministic specification – no constant and no trend in co-integration equation. The estimates are in the Table 4. Using these two estimates and their maximal likelihood function values we compute the likelihood ratio statistics to verify significance of the constant δ_0. We expect rejection of this hypothesis. The values of this ratio are in the last column of the Table 4. The labour share parameter equals the constant δ_1.

4 Results

	β_0	β_1	β_2	$\beta_2 = 0$
CZE	– 0.959	1.007 (0.061) **	0.000 (0.001)	0.021
HUN	– 1.833	1.113 (0.023) **	–0.002 (0.000) **	12.480 **
POL	– 4.864	1.379 (0.284) **	– 0.006 0.005	1.267
SVK	– 0.954	1.012 (0.082) **	– 0.003 (0.002)	0.995

Table 1 The estimates of the fourth Johansen deterministic specification (7)
(with constant and trend in co-integration equation)

It follows from the Table 1 that the trend in the (7) specification is statistical significant only in Hungary. The average value of the coefficient γ equals –0.113. Using (10) we compute the average value of the total factor productivity growth rate, $g = 0.0144$.

	β_0	β_1	β_2	$\beta_1 = 1$	$exp(\beta_0)$
CZE	– 0.882	1	0	2.004	0.414
POL	– 0.945	1	0	0.454	0.389
SVK	– 0.946	1	0	13.532 **	-

Table 2 The estimates of with the third Johansen deterministic specification (7)
(constant and no trend in co-integration equation)

We reject the unit inputs substitution elasticity only for Slovakia. Czech and Polish inputs substitution elasticity is unit.

	δ_0	δ_1
CZE	–64.388	0.432 (0.004) **
POL	–2256.950	0.424 (0.034) **

Table 3 The estimates of the second Johansen deterministic specification (15)
(constant and no trend in co-integration equation)

	δ_0	δ_1	$\delta_0 = 0$
CZE	0	0.431 (0.002) **	0.204
POL	0	0.394 (0.010) **	0.558

Table 4 The estimates of the first Johansen deterministic specification (15)
(no constant and no trend in co-integration equation)

As we expected the constants δ_0 are statistically insifignicant for both Czech Republic and Poland.

5 Conclusion

The Hungarian inputs substitution elasticity is below unit. The Hungarian total factor productivity average quarterly growth rate is about 1.44%.

Using Slovak data we rejected the trend in the (7) specification, however we did not reject the unit inputs substitution elasticity hypothesis. The value of this elasticity is different from one. Rejecting the trend in the (7) specification probably means that the trend (3) does not capture the total factor productivity dynamics. Then (7) specification is not appropriate for Slovak production function estimate.

The Czech and Polish inputs substitution elasticity is unit and Cobb-Douglas production function explain a production process in these countries. The labour average share parameter is 0.414 (using logarithmic specification estimates in the Table 2) or 0.431 (using linear specification in the Table 4) in Czech Republic and 0.389 (Table 2) or 0.394 (Table 4) in Poland.

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Influence of transition matrix in bonus-malus systems on tariff effectiveness in car insurance

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Abstract. In car insurance the process of forming a tariff policy involves two stages. The first stage of a priori tariffication is based on observable risk factors. In Poland these factors are the region of car registration and the car engine displacement. The second stage involves tariffication built on the claim record of the insured. Bonus-malus systems form an a posteriori stage in defining a tariff policy in car insurance. Each bonus malus system is comprised of classes which are defined by the rate of premium and have specified rules of transition. In the paper we explored the influence of changes in transition rules between classes on tariff effectiveness of bonus malus systems. For effectiveness evaluation we used stochastic measures based on the Markov chain theory. The number of insurance claims was modelled with the use of the negative binominal distribution with parameters estimated from the sample of the portfolio of one of Polish insurance companies.

Keywords: bonus-malus system, transition matrix, Markov chain.

JEL Classification: G22
AMS Classification: 60J20, 62P05

1 Introduction

Insurance companies use varies means to compete in local markets, with premium rate being the most important ingredient of their strategies. In majority of European Union countries insurers are free to set their own tariffs. However premium rates should guarantee their financial stability. Bonus malus systems are designed to diversify premium rates in a way that improves financial outcome of an insurance company portfolio. The efficiency of a bonus malus system is determined by premium rates and transition rules between tariff classes. The aim of the paper was to evaluate the influence of changes in transition rules on effectiveness of a bonus malus system. We built our study on four bonus malus systems. The first, basic one, is the PZU system – of the largest Polish insurer, where share in the insurance market is measured by the gross written premium. Three other systems are modifications of the PZU system where the transition rules between classes were altered. Each system was assessed by stochastic measures of effectiveness. We assumed that number of damages followed the negative binomial distribution with parameters estimated from the sample from one of Polish insurers.

2 Ratemaking in car insurance

The premium in car insurance is calculated in a two-stage process. The first stage – a priori ratemaking – involves calculation of a basic premium on the grounds of observable risk factors such as the place of residence, the car engine displacement or its horsepower. At this stage claim severity and claim frequency are estimated with the use of the model-based approach like in Wolny-Dominiak [9]. The tariffication variables may differ from country to country within European Union (EU). Insurers in Poland use the car registration region and the engine displacement. The second – a posteriori – stage is built on the claim record of the insured and involves discounts or surcharges that are applied to the basic premium. Bonus-malus systems are used in an a posteriori stage of defining a tariff policy in car insurance. Each bonus malus system is comprised of classes such that each of them has a different bonus malus coefficient, which is the ratio of the written premium. Rules of transition are used to assign drivers to classes (Lemaire [4]). The idea behind creating a bonus malus system is inclusion of non-observable individual risk factors in the final premium calculation.

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3 Bonus-malus system modelling with the use of Markov chain

Let the number of classes in the bonus-malus system (BMS) be finite and equal to s. Let $S=\{1,2,...,s\}$ denote the set of numbers assigned to tariff classes. By K_t we signify the random variable describing the number of damages caused in the time interval $(t-1,t]$ by a given driver, where $t=1,2,...$ Let us assume that the distribution of the number of damages depends on a damage intensity parameter $\lambda>0$ and $K_1,K_2,...$ are identical independently distributed random variables.

Let C_t be the random variable describing the class to which a given driver belongs in the period $(t-1,t]$, with C_{i_0} being the first (starting) class. Each class has a premium rate, which is the element of the vector $\mathbf{b}=[b_1,...,b_s]$, such that $b_1\geq b_2\geq ...\geq b_s$. A driver trajectory in a bonus-malus system can be modelled by the series of random variables $\{C_1,C_2,...\}$ taking values in the set $\{1,...,s\}$.

Let $T_k(i)=j$ denote the transition of the driver form the class i to the class j if they caused k damages in one year, where $T:S\rightarrow S$, $S=\{1,2,...,s\}$ ($i,j\in S$, $k=0,1,2,...$). The function defined in such way is called the transition function, whereas transition rules can be expressed as k binary matrices

$$\mathbf{T}(k)=[t_{ij}(k)]=\begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1s} \\ \vdots & \vdots & \ddots & \vdots \\ t_{s1} & t_{s2} & \cdots & t_{ss} \end{bmatrix}, \tag{1}$$

where $t_{ij}^{(k)}=t_{ij}(k)=\begin{cases}1 \text{ dla } T_k(i)=j \\ 0 \text{ dla } T_k(i)\neq j\end{cases}$ for $i,j\in S,\ k=0,1,2,...$

The bonus-malus system model for one insured entity with a constant damage intensity parameter $\lambda>0$ is a homogenous Markov chain $\{C_t\}_{t\in N}$ with the state space $S=\{1,2,...,s\}$, the transition matrix of the form

$$\mathbf{M}(\lambda)=\sum_{k=0}^{\infty}p_k(\lambda)\mathbf{T}_k \tag{2}$$

and the probability of transition of the insured $p_{ij}(\lambda)$ from the tariff class C_i to the tariff class C_j in one year which equals to

$$p_{ij}(\lambda)=\sum_{k=0}^{\infty}p_k(\lambda)t_{ij}^{(k)}, \tag{3}$$

where $p_k(\lambda)$ is the probability of k claims in a year for the given insured.

The main problem in modelling bonus-malus systems with the use of Markov chains is finding the stationary (equilibrium) distribution of the chain under the assumption of its homogeneity and ergodicity. For each irreducible ergodic Markov chain there exists one equilibrium distribution of the form:

$$\mathbf{a}(\lambda)=[a_1(\lambda),...,a_s(\lambda)], \tag{4}$$

where $a_j(\lambda)=\lim_{n\rightarrow\infty}p_{ij}^n(\lambda)$ and $p_{ij}^n(\lambda)$ is the transition probability of the insured in n years from the class C_i to the class C_j. The equilibrium distribution can be found by solving the equation:

$$a_j(\lambda)=\sum_{i=1}^{s}a_i(\lambda)p_{ij}(\lambda),\quad j=1,...,s, \tag{5}$$

where $\sum_{j=1}^{s}a_j(\lambda)=1$. The element $a_j(\lambda)$ of the vector $\mathbf{a}(\lambda)$ denotes the fraction of the insured that belong to the class C_j after reaching the equilibrium state by the system or the probability that the insured belongs to the class

C_j in n periods, when the number of periods convergent to infinity. On the above assumptions the vector $\mathbf{a}(\lambda)$ can be calculated as a normalized left eigen vector of the transition matrix $\mathbf{M}$.

The application of Markov chain theory to modelling bonus-malus systems can be found in papers by Bonsdorff [1], Lemaire [3], Lemaire [4], Mahmoudvand et al [5], Niemiec [6]. The premium rates in bonus-malus systems can be modelled with Bayesian methods, which are presented in the paper by Pekasiewicz [7].

4 Chosen measures of effectiveness of bonus-malus systems

One of the criteria used for bonus-malus system evaluation is its transactional elasticity, where elasticity is understood in the same way as the price elasticity of the demand. The ideal bonus-malus system has the elasticity equal to one, so the premium levels change at the same rate as the damage intensity of the insured.

There are a number of effectiveness measures of bonus-malus systems. The most commonly used measure evaluating a bonus-malus system is the Loimarant effectiveness $\eta(\lambda)$, which is the elasticity of the mean stationary premium with respect to the claim frequency (Lemaire [4]), given by

$$\eta(\lambda) = \frac{dB(\lambda)}{B(\lambda)} \Big/ \frac{d\lambda}{\lambda}, \tag{6}$$

where the expected stationary premium for the single period, after the system reaches the state of equilibrium equals to

$$B(\lambda) = \sum_{i=1}^{s} a_j(\lambda) \cdot b_j . \tag{7}$$

The Loimarant effectiveness allows for assessment of merit-rating in the system, which should reward claim-free years by bonuses and penalize at-faults accidents by surcharges. This measure informs to what extend the driver claim record is included in the premium calculation. In the ideal case $B(\lambda)$ should be an increasing function of λ, such that $\eta(\lambda) = 1$.

The values of the elasticity function (6) depend on the risk parameter λ. For that reason elasticity functions are often compared or, as an alternative, the so called joint elasticity is calculated by numerically solving the integral

$$\eta = \int_0^{\infty} \eta(\lambda)\pi(\lambda)d\lambda . \tag{8}$$

Let us assume that the number of damages in a single accident has the Poisson distribution and the intensity parameter is gamma distributed, which implies that the overall number of accidents has the negative binomial distribution. As the density of the gamma distribution $\pi(\lambda)$ and the elasticity function converge to zero for λ convergent to infinity, the integral in the (8) may be approximated by the integral

$$\eta = \int_0^{w} \eta(\lambda)\pi(\lambda)d\lambda \tag{9}$$

The integral in the formula (9) can be calculated by the trapezium method (Stoer, Bulirsch [8]), which gives the approximation in the form

$$\eta = \int_0^{w} \eta(\lambda)\pi(\lambda)d\lambda \approx \sum_{i=1}^{w-k} \frac{1}{2k}\left[\eta\left(\frac{i-1}{k}\right)\pi\left(\frac{i-1}{k}\right) + \eta\left(\frac{i}{k}\right)\pi\left(\frac{i}{k}\right)\right] \tag{10}$$

In this paper we assumed w=3 and k=500. Increasing the values of w and k enhances the approximation accuracy on the ninth decimal point.

Lemaire [3] proposed a measure of bonus-malus system effectiveness called the relative stationary average level which is given by

$$RSAL(\lambda) = \frac{B(\lambda) - \min_j(b_j)}{\max_j(b_j) - \min_j(b_j)}. \tag{11}$$

The measure given by the formula (11) informs about the relative position of the insured with the average claim record in case when the value 0 is assigned to the lowest premium and the value 1 to the highest. This measure does not have an optimal value which would be acknowledged in the literature. According to its author, in the ideal case the value of this indicator should equal 0.5 for the average claim record. Low values of *RSAL* indicate that the system is imbalanced and over time majority of the insured is going to belong to the classes with highest discounts. Large values of this measure correspond to the uniform distribution of the insured among BMS classes.

5 Empirical study

The analyzed bonus-malus systems are presented in Tables 1-4. All considered systems reward claim-free years by discounts. Lack of a damage claim in a given year corresponds to the transition to the next class and a higher discount on the anniversary of the contract (e.g. from the class BM 3 to the class BM 4).

In the PZU system a claim in a given year entails an increase of a number of degrees in the bonus-malus system and transition to the former class (by one class upwards in the Table 1) with a higher premium at the renewal of the policy (e.g. from the class BM 3 to the class BM 2). BMS I is system where one claim corresponds to the increase on a bonus-malus scale by one class and two damage claims or more entail the transition two classes upwards. BMS II is more restrictive for the insured. In case of one damage the insured moves two class upwards in Table 3. Three damage claims result in the transition three classes upwards.

The most restrictive system is BMS III. Here one damage in a year implies loss of every discount and transition to the starting class i.e. the fifth class (BM 5) in Table 4. In case of a driver with no discount one claim results in a transition to the first class with maximum surcharge (BM 1) in Table 4. In this system, for two or more damages the insured is also given the maximum premium surcharge, so again goes to the first class (BM 1) in Table 4.

In the study we assumed that the number of damages has the negative binomial distribution with the intensity parameter $\lambda = 0,043$, which was estimated from the sample of car insurance portfolio of one of the biggest Polish insurance companies. The value of the parameter is similar to the damage intensity parameter for the whole car insurance market in Poland. The negative binomial distribution is used in majority of literature regarding models of the number of damages in car insurance (e.g. Ibiwoye et al [2], Lemaire [3]).

BM class	Premium rate [%]	Number of claims per year	
		0	1 and more
1	200	2	1
2	150	3	1
3	130	4	1
4	115	5	2
5	100	6	3
6	90	7	4
7	80	8	5
8	80	9	6
9	70	10	7
10	60	11	8
11	50	12	9
12	50	13	10
13	40	14	11

Table 1 BMS PZU

BM class	Premium rate [%]	Number of claims per year		
		0	1	2 and more
1	200	2	1	1
2	150	3	1	1
3	130	4	2	1
4	115	5	3	2
5	100	6	4	3

6	90	7	5	4
7	80	8	6	5
8	80	9	7	6
9	70	10	8	7
10	60	11	9	8
11	50	12	10	9
12	50	13	11	10
13	40	14	12	11

Table 2 BMS I

BM class	**Premium rate [%]**	**Number of claims per year**		
		0	**1**	**2 and more**
1	200	2	1	1
2	150	3	1	1
3	130	4	1	1
4	115	5	2	1
5	100	6	3	2
6	90	7	4	3
7	80	8	5	4
8	80	9	6	5
9	70	10	7	6
10	60	11	8	7
11	50	12	9	8
12	50	13	10	9
13	40	14	11	10

Table 3 BMS II

BM class	**Premium rate [%]**	**Number of claims per year**		
		0	**1**	**2 and more**
1	200	2	1	1
2	150	3	1	1
3	130	4	1	1
4	115	5	1	1
5	100	6	5	1
6	90	7	5	1
7	80	8	5	1
8	80	9	5	1
9	70	10	5	1
10	60	11	5	1
11	50	12	5	1
12	50	13	5	1
13	40	14	5	1

Table 4 BMS III

6 Effectiveness evaluation of considered bonus-malus systems

The values of the measures of effectiveness for the bonus-malus systems are presented in Table 5 and in Figure 1.

	$\eta(\lambda)$	η	$B(\lambda)$	$RSAL(\lambda)$
PZU	0,128164	0,126651	0,410854	0,006783
BMS I	0,062188	0,061454	0,404827	0,003016
BMS II	0,131691	0,130135	0,411124	0,006952
BMS III	0,597433	0,590377	0,561531	0,100956

Table 5 Values of measures of effectiveness of bonus-malus systems

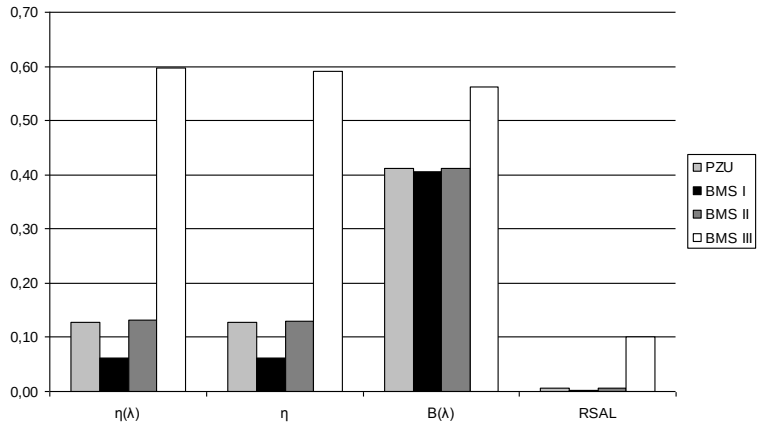

Figure 1 Values of measures of effectiveness of bonus-malus systems

The conducted study showed that introduction of more restrictive rules for drivers that cause damages increases the system effectiveness. The PZU system as well as the systems BMS I and BMS II have low effectiveness and the insured tend to concentrate in the last class. The last of all considered systems offers best tariffication rules. The expected premium is higher and concentration in classes with premium discounts is lower.

7 Conclusion

The research conducted on the basis of data from the largest Polish insurer showed that the transition rules between bonus-malus classes have a clear effect on the transition probabilities in the Markov chain. In consequence the tariff effectiveness of the bonus-malus system changes. The more restrictive transition rules, the more effective the system. A restrictive bonus malus system offers more accurate recognition of a driver record and insurance policies do not tend to concentrate in classes with maximum discount.

Another factor that influences bonus-malus system effectiveness, which was not analyzed in this paper, is values of the coefficients that correspond to different classes used in a bonus-malus system.

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Effects of Exchange Rate Volatility on Poland´s Trade Flows

Jana Šimáková[1],

Abstract. The aim of the paper is to investigate the impact of exchange rate volatility on bilateral trade flows between Poland and its major trading partners. It is expected that increased exchange rate volatility increases the risk of exporters and reduces foreign trade. For this purpose we employ extended trade gravity model approach. In the basic form of the gravity equation, trade volume between a pair of countries is modeled as an increasing function of their sizes (GDP) and a decreasing function of the distance between the two countries. Additional factors included in extended model are population, dummies for common border; membership in EU and EMU and proxy for exchange rate volatility. The measure of exchange rate volatility is estimated by standard deviation. This paper explores relationship between trade and exchange rate uncertainty using quarterly data over the period 1997:1 – 2012:2. In order to obtain the objective result, we use the panel data regression with 19 trading partners. Based on a gravity model that controls for other factors likely to determine bilateral trade, the results suggest that nominal exchange rate volatility of Polish zloty has a significant negative effect on bilateral trade over the sample period.

Keywords: bilateral trade, exchange rate, gravity model, panel data, volatility.

JEL Classification: C51, F14, F31
AMS Classification: 62M10

1 Introduction

The intensity of Poland's participation in the international trade is still at a relatively low level. Although the volume of Poland's international trade has been increasing since the joining of the EU in 2004, it is constantly characterised by a long-term negative trade balance. The reducing of Poland's negative trade balance has been largely affected by the foreign direct investment inflow since the companies with foreign ownership account for a substantial share in Poland's exports.This situation raises makes Poland's economy more vulnerable to any adverse changes in other economies.

Both, exports and imports, solidly depend on exchange rates and their development. According to Abeysinghe and Yeak [1] , policy prescriptions have generally assumed that currency depreciation stimulate exports and curtail imports, while currency appreciation are detrimental to exports and encourage imports. Generally is assumed, that higher exchange rate volatility leads to higher transaction costs for traders and is followed by decrease of foreign trade. Theoretical analyses of this relationship have been conducted by Hooper and Kohlhagen [9], who argue, that if changes in exchange rates are unpredictable it means uncertainty about companies' profits and reduces the benefits of foreign trade. Even if hedging in the forward markets were possible, there are limitations and costs which are especially considerable for small firms. According to latest studies made by Taglioni [14] and Ozturk [12], it can be stated, that the adverse effect of exchange rate volatility on trade flows, if it exists, is not large. On the other hand, De Grauwe [8] pointed out that the dominance of income effects over substitution effects on international trade can lead to a positive relationship between trade and nominal exchange rate volatility, because an increase in exchange rate volatility raises the expected marginal utility of export revenue and therefore can induce increasing of exports.

Findings differ across the studies as well for aggregation reason. In the IMF's study [10] on exchange rate volatility and trade flows can be found conclusion that there is no obvious negative relationship between aggregate exchange rate volatility and aggregate trade. When the research is turned to bilateral trade, we do find evidence that exchange rate volatility seems to more affect bilateral trade than the aggregate one. Evidence on the researched relationship between exchange rate volatility and trade flows is characterized as heterogenous as the results tend to be sensitive to the choices of sample period, model specification, proxies for exchange rate volatility and countries.

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The aim of the paper is to investigate the impact of exchange rate volatility on bilateral trade flows between Poland and its major trading partners. Panel data used in this study covers period from 1997 to 2012 and 19 trading partners.

Hence, this study provides additional evidence on the effect of exchange rate volatility on trade flows in the context of emerging market after the most turbulent part of economic transformation. One aspect of this transformation was a transition from fixed exchange rate arrangement into a crawling peg and recently to a free-float regime. In addition, Poland is interesting objective to study because international trade serves as a major channel of economic integration within the Group of Visegrad countries or the EU as a whole, because usually, international trade tends to be a driver of the economy in countries neighbouring with economies with open trade regimes, high presence of multinational companies and large volume of re-exports. The fact that this example fits to Poland can be illustrated with increasing share of merchandise trade on Poland's GDP. The latest data od the World Bank shows, that it is 76.8 % as compared to 36.5 % in 1997.

2 Gravity equation of foreign trade

To estimate the impact of exchange rate volatility on foreign trade in this paper is used a gravity model, which is a simple empirical model for analyzing bilateral trade flows. Despite this approach was often criticized for insufficient theoretical foundations, this drawback has been eliminated in the recent years.

The original gravity equation is based on Newton´s gravity law equation:

$$F_{ij} = g\frac{m_i m_j}{d_{ij}^2}$$

where F_{ij} is the value of gravity force, $m_{i(j)}$ is the weight of object *i (j)*, d_{ij} represents the distance between the objects and g is the gravity constant.

Based on this equation, the gravity model of trade analogous describes the force of gravity and explains the flow of trade between a pair of countries as being proportional to their economic "weight" (national income) and inversely proportional to the distance between them. The model has a lineage that goes back to Tinbergen [16] and Pöyhönen [13], who specified the gravity model equation as follows:

$$X_{ij} = \delta\frac{GDP_i^{\beta_1} GDP_j^{\beta_2}}{D_{ij}^{\theta}}$$

where δ, β and θ are the parameters of the modified equation, X_{ij} is the bilateral trade between countries (dependant variable), $GDP_{i(j)}$ represents income of respective trading partner *i(j)* (independent variable) D_{ij} is the distance between these two countries (independent variable) and δ is constant.

Trade theorists have found the model to be consistent with theories of trade based upon models of imperfect competition and with the Hecksher-Ohlin model. For example Carrere [7] points out its microeconomic foundation. The gravity equation can be formally derived within an imperfectly competitive set up with increasing returns to scale and firm-level product differentiation as well as within a perfect competition set-up with product differentiation at the national level [2]. Countries with a larger economy tend to trade more in absolute terms as they have larger demand respectively supply. Higher distance depresses the bilateral trade as it represents higher costs for transportation, higher shipment time, and higher costs for searching trading opportunities. Distance can be used as proxy for cultural difference as well [6].

The basic gravity equation is frequently extended to incorporate other factors stimulating or reducing of bilateral trade flows. As an additional determinant of trade there is often used a population size of respective countries. Generally coefficient for country population is expected to be positive, since bigger market in the recipient country is expected to demand more goods. And population of the export country is expected to be able to supply more as the population grows in size. Recent models also include many dummy variables that can affect transaction costs. For example common border, language or memberships in custom union are supposed to decrease transaction costs and to promote trade [2]. Therefore inserting these variables in the model we obtain:

$$X_{ij} = e^{\delta}\frac{GDP_i^{\beta_1} GDP_j^{\beta_2} Pop_i^{\beta_3} Pop_j^{\beta_4 \sum_l dum_l^{\gamma} + \varepsilon_{ij}}}{D_{ij}^{\theta}}$$

where *POPi(j*) stands for population of country *i(j)* participating in bilateral trade and *dum* represents dummy variables in addition to equation (3).

3 Extended gravity model with exchange rate volatility

To analyze effects of exchange rate volatility on international trade flows of Poland we employ an augmented gravity model equation. In this paper we express bilateral trade flows as a function:

$$X_{ij} = \alpha GDP_i^{\beta_1} GDP_j^{\beta_2} Pop_i^{\beta_3} Pop_j^{\beta_4} D_{ij}^{\beta_5} e^{\beta_6 V(ER)_{ij}} \prod_l dum_l^{\gamma_1} u_{ij}$$

where additional factor ER_{ij} is the spot exchange rate than $V(ER)_{ij}$ is its volatility and α, β_k, γ_l are the unknown parametres of the model, u_{ij} is the error term. In order to add the exchange rate volatility into the equation, we follow Tichý [15] and Baldwin et al. [13] who pointed out that this relationship is not linear but convex and to avoid error due to rounding during data transformation, the volatility is used in exponent. Common border; membership in EU and EMU are incorporated in variable *dum*.

Transforming this function to log-linear form, we get an extended gravity model equation:

$$lnX_{ij} = \alpha + \beta_1 lnGDP_I + \beta_2 GDP_j + \beta_3 lnPop_i + \beta_4 lnPop_j + \beta_5 lnD_{ij} + \beta_6 lnV(ER)_{ij} + \sum_l \gamma_l dum_l + u_{ij}$$

Logarithmic transformation helps to reduce skewness and heteroscedasticity and to stabilize variability. The stability of regressors is needed in empirical estimation.

3.1 Data description

The models are estimated by using quarterly data over the period 1997:1 – 2012:2. The dependent variable in the model is a volume of bilateral trade between Poland and its trade partner, which is given as a sum of export and import flows of respective country. Trade flows are obtained from the OECD statistics of international trade. The data are in current prices and denominated in the US dollars. The countries selection is based on the share of total international trade turnover and their list can be seen in table 1.

Main trading partners of Poland		
Austria	Hungary	Slovenia
Belgium	Ireland	Spain
Czech Republic	Italy	Sweden
Denmark	Luxembourg	Switzerland
Finland	Netherlands	United Kingdom
France	Norway	
Germany	Portugal	

Table 1 Country list for regression

GDP for every country is also obtained from OECD statistics of national accounts, calculated by expenditure approach in millions of US dollars in current prices. Time series for population is acquired from Eurostat. Distance, common border and common language between Poland and its trading partner are used based on GeoDist database made by Mayer and Zignago [11]. They made available the exhaustive set of gravity variables, in particular bilateral distances measured using city-level data. Exchange rates are obtained from OECD statistics of international trade and are the only variables on monthly frequency. The basic statistical description of data is stated in table 2.

Variable	Observations	Mean	Minimum	Maximum
lnX_{ij}	1178	19.30405	15.10395	22.94386
$lnGDP_i$	1178	12.68463	9.729657	15.01207
$lnGDP_j$	1178	13.16610	12.70339	13.64407
$nPOP_i$	1178	9.283552	6.035242	11.32105
$lnPOP_j$	1178	10.55209	10.54831	10.55948
lnD_{ij}	1178	6.968083	6.247390	7.923146
$lnV(ER)_{ij}$	1178	6.968083	0.327031	2.361932
Common border (*CB*)	= 1, if trading partner shares a common border with Poland; = 0 if not			
EU member (*EU*)	= 1, if trading partner is a member of EU; = 0 if not			
EMU member (*EMU*)	= 1, if trading partner is a member of EMU; = 0 if not			

Table 2 Data description

3.2 Measuring of exchange rate volatility

To measure the exchange rate volatility in this paper we have used standard deviation of the first difference, based on monthly average nominal exchange rates of the period of 1997:1 to 2012:6. We employ following formula:

$$V(ER)_{ij} = \sqrt{\frac{\sum_{m=1}^{n}(ER_{ij,m} - ER_{ij,t})^2}{n}}$$

In this paper is used nominal exchange rate, as nominal and real exchange rates tend to move closely together and the choice is not likely to the econometric results. As literature review made by Auboin and Rutha [4] states, the probability that the variability of nominal exchange rates did not translate into that of the real exchange rate would be small, occurring only during exceptionally high periods of domestic inflation. In empirical studies, both variables are generally tested. In addition, using nominal exchange rate avoids bias from changes in price levels via spurious correlation.

3.3 Estimation of gravity model

As Arvas [3] states, standard gravity models usually employ cross-section data to estimate trade patterns in a given year, or averaged data. We employ panel data regression to avoid the risk of choosing an unrepresentative year and to monitor unobservable individual effects between trading partner.This can provide additional insights to trading relationships. In addition using of panel data for estimation brings mostly significant relation between international trade and exchange rate volatility.

Before estimating ordinary least squares (OLS) based methods on panel data, it is needed to determine data set effects as random or fixed. The fixed effects are when the heterogeneity in the model is unobservable, but correlated with any variable included in model. On the other hand, the heterogeneity in random effects is also unobservable, but it is not correlated with any other variable. In this case we follow Tichý [15] again. The Breusch-Pagan Langrage multiplier test is used and the test criteria is calculated from equation:

$$LM = \frac{nT}{2(T-1)}\left[\frac{\sum_{i'1}^{n}.(\sum_{t'1}^{T}e_n)^2}{\sum_{i=1}^{n}.\sum_{t=1}^{T}e_{it}^2} - 1\right]^2$$

where T is the length of time series, n is the number of unit in cross-section dimension and e_{it} is a residuum.

4 Empirical results

The estimation of the Breusch-Pegan Langrage multiplier test revealed random effects. Thus the extended gravity model is estimated by Generalized least squares for panel data. The dependent variable in the model is total trade turnover between Poland and its trading partners. We have included 19 cross-sections and 62 periods. Total panel observations are 1178. The value of adjusted R-squared in this model is 92.61 %. The results of this estimation can be seen in table 3.

Variable	Coefficient	Std. Error	t-Statistic	Prob.
$lnPOP_i$	0.260188	0.115524	2.252247	0.0245
$lnPOP_j$	-21.51528	1.948769	-11.04045	0.0000
$lnGDP_i$	0.651731	0.099215	6.568904	0.0000
$lnGDP_j$	1.779169	0.070401	25.27178	0.0000
lnD_{ij}	-1.292551	0.204634	-6.316417	0.0000
$lnV(ER)_{ij}$	-0.018150	0.005184	-3.501307	0.0005
EU	0.249139	0.030013	8.301059	0.0000
EMU	-0.143396	0.022194	-6.461127	0.0000
CB	0.563128	0.296643	1.898338	0.0579
C	220.8696	20.25225	10.90593	0.0000

Table 3 Regression results

Except parameter of dummy variable common border, all estimated parameters are statistically significant. We can observe expected positive impact of foreign population, Poland´s and foreign GDP, membership in European Union. On the other hand, there is confirmed supposed indirect relationship between trade volume distance and volatility of exchange rates. Results show, that the population of Poland has negative instead of positive impact. The same direction of relationship is revealed with membership in European monetary union.

5 Conclusion

The aim of the paper was to investigate the impact of volatility of Polish zloty on bilateral trade flows between Poland and its major trading partners. For this purpose we employed extended trade gravity model approach. We included 19 trading partners into the panel data analysis and the results suggest that the nominal exchange rate volatility of Polish zloty has a significant negative, but weak effect on bilateral trade over the sample period. Therefore, the results indicate that an active exchange rate policy aimed to influence exchange rate development is not supposed to promote any notable improvement of Poland´s international trade.

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Logistic and ARIMA models in the Estimation of Life Expectancy in the Czech Republic

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Abstract. The aim of the presented paper will be the calculation of the estimated life expectancy at birth for males and females in the Czech Republic using selected levelling functions (Gompertz–Makeham, Kanistö and Thatcher) and using ARIMA model and Random Walk with Drift, constructed for the time series modelling. The levelling functions for modelling mortality rate are nonlinear, thus difficult to solve, ARIMA models are in turn based on entirely different principles (stochastic process and backward looking expectations). Subsequently it will be pointed to differences that arise when using both methods and the differences that occur when we compare the estimates that are published by the Czech Statistical Office. Given that the estimates of life expectancy at birth using different approaches are similar, then based on ARIMA model and Random Walk with Drift there will be extrapolated to the near future (2012–2030). Presented alternative estimates will be correlated with predictions which are calculated by the Czech Statistical Office and it will be easier to obtain them, because they are based purely on a statistical approach, which does not require the additional demographic information, which are expensive to obtain.

Keywords: Life Expectancy, ARIMA, Gompertz–Makeham, Kanistö, Thatcher

JEL Classification: C61, C63
AMS Classification: 90C30

1 Introduction

Mortality and its development always has been very interesting topic (not only for demographers). The mortality trend is one of the most important indicators of standard of living (see e.g. Bérenger and Verdier-Chouchane [2]). If people live longer, it means that mortality is going to be better (Arltová et al. [1]). The reason for increasing life expectancy could be e.g. better health care (Jia et al. [15]). The second reason is greater interest in healthy life style. On the other hand the increase in values of life expectancy means population aging (Gavrilov and Gavrilova [11] or Boleslawski and Tabeau [3]). Because more and more people live to the highest ages, it is very important to have the best idea of the trend of mortality at the highest ages. In the previous years it was not so important, because only a few of them live to the highest ages. It is also important to note that data about mortality at the highest ages is unreliable and mortality of oldest persons is different from the younger ones. Therefore it is necessary to use some of the existing models for extrapolating specific mortality rates at the highest ages (see e.g. Burcin et al. [7]).

The aim of the article is to use selected models (Gompertz–Makeham, Thatcher and Kanistö) to smooth the specific mortality rates (see e.g. Gompertz [12], Makeham [17] or Thatcher et al. [19]) and use them to calculate the life expectancy at birth for Czech males and females using available data from 1970 to 2011. Data is published by the Czech Statistical Office (CZSO) and available with the annual frequency. Because the calculation of balanced specific mortality rates (and consequently the life expectancy at birth) are computationally difficult, the values of life expectancy from the life tables will be estimated by an alternative approach – "ex-post" by random walk model with drift and by ARIMA approach (see e.g. Šimpach and Langhamrová [18] or Torri [20]). These estimates will be equivalent to the estimates that provide selected logistic models. Due to the statistically significant estimates of life expectancy there will be performed the extrapolation to subsequent 19 periods, i.e. until 2030. These estimates of the future values will be confronted with the estimates that are published by the Czech Statistical Office as predicted values of life expectancy at birth in low, medium and high variant. It will be shown that the low variant of CZSO population projection best corresponds with the estimates by ARIMA model for both males and females. The estimates of random walk model with drift are for both males and females situ-

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ated in the middle of the estimates of low and medium variant of the CZSO population projection. For the analysis it will be first used software DeRaS for levelling mortality rates (Burcin et al. [8]) and consequently the Statgraphics Centurion XVI to calculate the optimized models of random walk with drift and ARIMA models. For calculate the empirical values of life expectancy at birth it will be used a classical approach of calculation of the life tables.

2 Methodology

For the description of mortality development is most often used an indicator known as life expectancy. Nowadays, the development of mortality for the oldest persons (and thus the development of life expectancy) enters to the forefront of research interest (Booth [4]). The extension of life expectancy is still in progress. It is important to note that for the examining of mortality at the highest ages we cannot use the empirical data. It is caused by the different development of mortality trend in the highest age groups compared to younger ones. For the own calculation of development of life expectancy is necessary to smooth the specific mortality rates by some of available models. In the past, the most-used was the Gompertz–Makeham (G–M) model (see e.g. Lagerås [16] or Ekonomov and Yarigin [10]), which can be expressed as

$$\mu_x = a + bc^x \tag{1}$$

where μ_x is the intensity of mortality at age x, x is the age and a, b, and c are parameters. This model is suitable for levelling the specific mortality rates from 60 to 85 years. Due to increased life expectancy in the most populations indicates that the G–M is not the most appropriate model. Nowadays more and more preferred models for levelling mortality rates are logistic models (Gavrilov and Gavrilova [11]). But it is important to know that logistic models are very optimistic (i.e., they provide higher values of life expectancy than other models). This difference is apparent mainly from 60 years. In this paper, we introduce two another models from Thatcher and Kannistö. Due to the recommendations by Boleslawski and Tabeau [3], the Thatcher model we can express as

$$\mu_x = \frac{z}{1+z} + \gamma \tag{2}$$

where $z = \alpha e^{\beta x}$, x is the age and α, β and γ are the parameters of model. Kannistö model we can express as

$$\mu_x = \frac{e^{[\theta_0 + \theta_1 .(x-80)]}}{1+e^{[\theta_0 + \theta_1 .(x-80)]}} \quad \text{for} \quad x \geq 80, \tag{3}$$

where μ_x is the intensity of mortality at age x, θ_0 and θ_1 are the parameters of model, which assumes values > 0. Research papers shows that the Kannistö model is the best suitable for description of mortality at the end of human life (Gavrilov and Gavrilova [11]). Therefore it is suitable especially for the highest ages (i.e. for persons older than 80 years). Life expectancy is obtained as an output indicator from mortality tables. The calculation is carried out in several steps. First, we calculate the specific mortality rates as

$$m_x = \frac{M_x}{\overline{S}_x}, \tag{4}$$

where M_x is the number of deaths at the exact age x and $\overline{S}_x$ is the middle number of living. Between the specific mortality rate and the intensity of mortality is valid the followed relationship

$$m_x \approx \mu(x + \frac{1}{2}). \tag{5}$$

Next, we will calculate the probability of death as

$$q_0 = \frac{M_0}{\alpha N_t^v + (1-\alpha) N_{t-1}^v} \quad \text{for} \quad x = 0, \tag{6}$$

where M_0 is the number of deaths at the age 0, α is the proportion of lower elementary file of deceased and N_t^v, respectively $N_{t\text{-}1}{}^v$ is the number of live births in year t, respectively in year t-1. The probability of death

$$q_x = 1 - p_x \tag{7}$$

is valid for $x > 0$. The calculation of the probability of survival is given by

$$p_0 = 1 - q_0 \quad \text{for} \quad x = 0 \tag{8}$$

and by

$$p_x = e^{-m_x} \quad \text{for} \quad x > 0. \tag{9}$$

Next part of the calculation relates to tabular (i.e. imaginary) population. First, we select the initial number of live births in tabular population: $l_0 = 100\,000$. Based on knowledge of the probability of survival, we are able to calculate the number of survivors in the further exact ages by

$$l_{x+1} = l_x p_x, \tag{10}$$

where l_x is the number of survivors at the exact age x from the default file of 100 000 live births of tabular population. The number of deaths of tabular population is given by

$$d_x = l_x q_x. \tag{11}$$

Next we calculate the number of lived years (L_x), respectively the number of years of remaining life (T_x) as

$$L_0 = l_0 - 0.85 d_0 \quad \text{for} \quad x = 0 \tag{12}$$

(where 0.85 = α is the proportion of lower elementary file of deceased) and

$$L_x = \frac{l_x + l_{x+1}}{2} \quad \text{for} \quad x > 0, \tag{13}$$

respectively

$$T_x = T_{x+1} + L_x. \tag{14}$$

Finally we obtain the life expectancy as

$$e_x = \frac{T_x}{l_x}. \tag{15}$$

In the next part of the calculation it will be used the ARIMA approach by authors Box and Jenkins for modelling of time series, which will be supplemented by the model of random walk with drift. Drift will be optimized using computational system Statgraphics Centurion XVI (version 16.1.11). ARIMA time series modelling is based on analysis of past trend of the time series and does not use an additional information that is in other circumstances required to extrapolate the life expectancy at birth.

3 Results

Based on the data published by the CZSO and using the conventional calculation of life tables there were calculated values of life expectancy at birth for males and females in the Czech Republic from 1970 to 2011. These values have been levelled in DeRaS software using Thatcher, G–M and Kanistö model. Levelled values for males are graphically shown in Fig. 1, levelled values for females are shown in Fig. 3. On the basis of the methodological approach of the authors Box and Jenkins [5] we identified the model ARIMA (1,0,0) without constant for the times series "life expectancy at birth – males" and further the random walk model with drift (see for instance Hughes [13]), where the drift was optimized at the value 0.209533. The estimates of the parameters of the ARIMA (1,0,0) model without constant are given in Tab. 1. For the time series "life expectancy at birth – females" we identified the model ARIMA (0,2,1) without constant (see Tab. 2) and further the random walk model with drift, where the drift was optimized at the value 0.189104. The diagnostic tests of the models indicate, that

the non-systematic component of the model is not auto-correlated, is homoscedastic and has normal distribution at the 5% significance level (see for instance Breusch and Godfrey [6], Darnell [9] and Jarque and Bera [14]).

Parameter	Estimate	Stnd. Error	t	P-value
AR(1)	1.00255	0.000612525	1636.75	0.0000

Table 1 ARIMA (1,0,0) model without constant for life expectancy of males. Source: authors' calculations

Parameter	Estimate	Stnd. Error	t	P-value
MA(1)	1.02508	0.00587664	174.432	0.0000

Table 2 ARIMA (0,2,1) model without constant for life expectancy of females. Source: authors' calculations

Using ARIMA models and random walk with drift there were estimated ex-post the values of life expectancy at birth for males in years 1970–2011 (which are graphically illustrated in Fig. 2) and the values of life expectancy at birth for females in years 1970–2011 (which are graphically illustrated in Fig. 4). These levelled values are highly correlated with the values that were calculated by Thatcher, Gompertz–Makeham and Kanistö model. From CZSO population projections from 2009 were acquired the estimates of life expectancy at birth for males and females in low, medium and high variant. Based on ARIMA models without constant and random walk models with drift for males and females there were constructed predictions of life expectancy for the period 2012–2030. These values were graphically compared in Fig. 5 from which it appears that the values predicted by ARIMA models without constant are highly correlated with the values published in low variant of CZSO population projection.

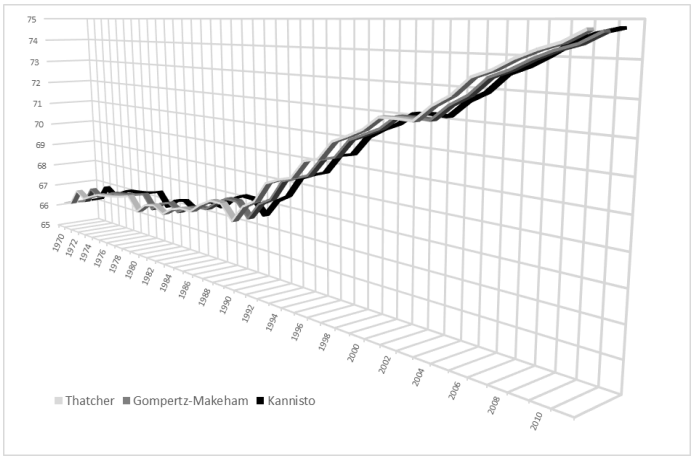

Figure 1 Estimated life expectancy at birth – males from 1970 to 2011 by Thatcher, Gompertz–Makeham and Kanistö model. Source: authors' calculations

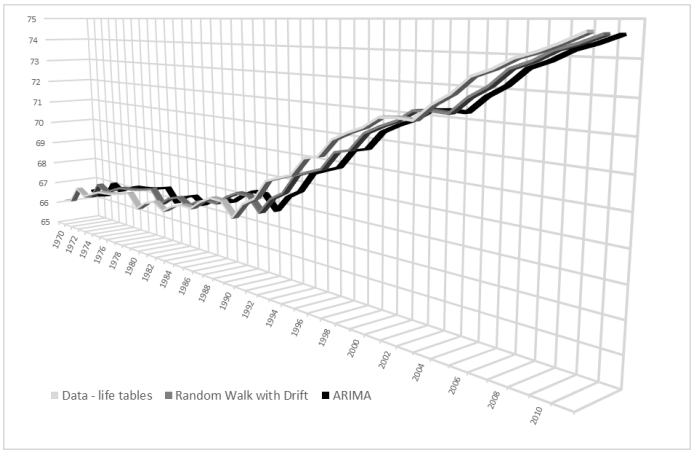

Figure 2 Life expectancy at birth from life tables and estimated life expectancy at birth – males from 1970 to 2011 by random walk model with drift and ARIMA model. Source: CZSO, authors' calculations

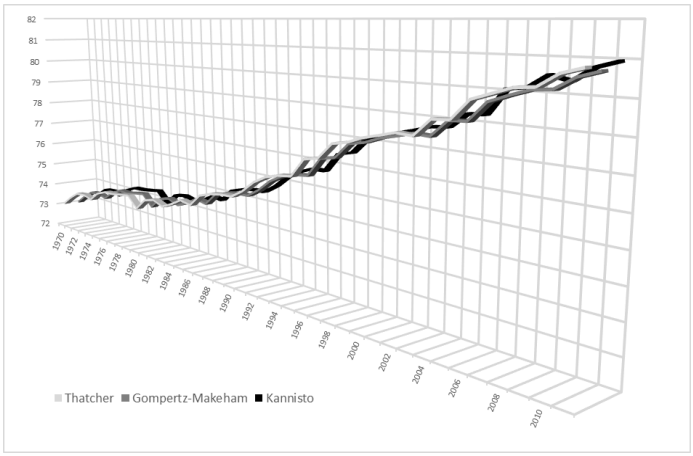

Figure 3 Estimated life expectancy at birth – females from 1970 to 2011 by Thatcher, Gompertz–Makeham and Kanistö model. Source: authors' calculations

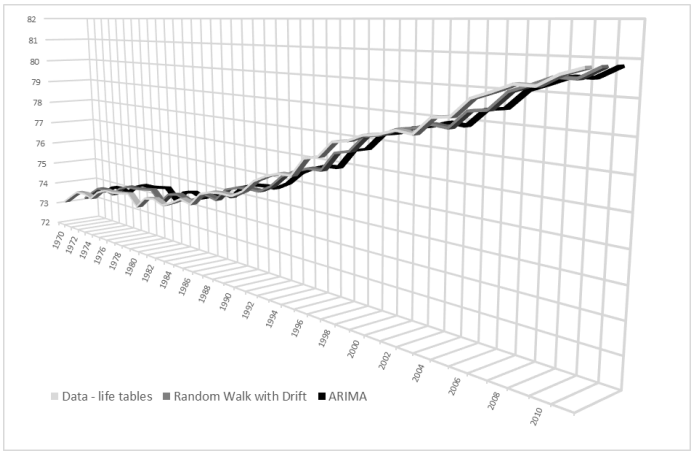

Figure 4 Life expectancy at birth from life tables and estimated life expectancy at birth – females from 1970 to 2011 by random walk model with drift and ARIMA model. Source: CZSO, authors' calculations

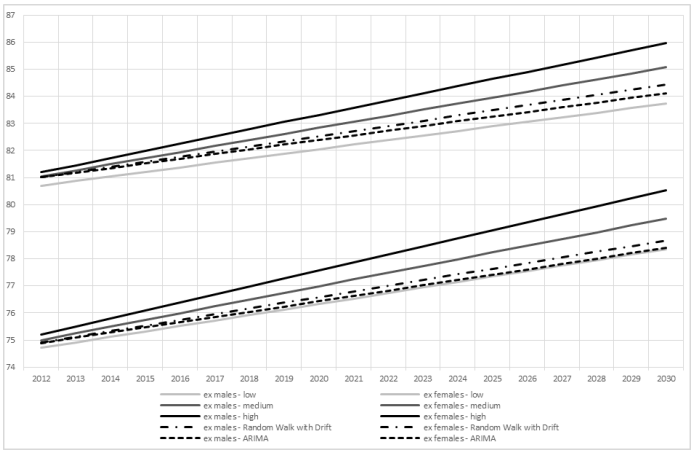

Figure 5 Expected life expectancy at birth from CZSO population projection for males (bottom) and females (top) in low, medium and high variant and an extrapolation of life expectancy at birth by random walk model with drift and ARIMA model for males (bottom) and females (top). Source: CZSO, authors' calculations

The parameters of the Kanistö and Thatcher model were estimated using the nonlinear regression, so their suitability was assessed on the basis of the corrected coefficient of the multiple determination. In the case of Kanistö model, the values were estimated slightly higher than in the case of Thatcher model.

4 Conclusion

The aim of the presented paper was the calculation of life expectancy at birth for males and females in the Czech Republic from 1970 to 2011 using selected levelling functions (Gompertz–Makeham, Kanistö and Thatcher) and using ARIMA model and Random Walk with Drift. It was shown that all approaches provide similar results, but ARIMA models and the random walk models with drift require less computation. Moreover, based on them it is possible to construct the predictions for the future. Predictions by ARIMA and random walk models do not require expensive input data and provide comparable results with low variant by CZSO, which calculation is incomparably more difficult. This provides time and money savings. Therefore, the challenge for future research is to use this approach for constructing cheaper predictions.

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Assessing the impact of standard of living on the life expectancy at birth using Vector Autoregressive Model

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Abstract. The aim of the paper is to evaluate the impact of living standards (LS) on the life expectancy at birth. The LS of the population can be described using different variables. Therefore the first part deals with the questions how to alternatively express the LS and whether a significant correlation exists between the imaginary LS variable and other chosen variables used to express it. As there is a problem with quantifying the LS with specific variables or a specific coefficient, the development of monitored life expectancy is modelled by trend function. The LS is one of the key variables that significantly affect the trend of particular demographic indicators such as life expectancy. Using certain assumptions, we can express the quality of life and its level, which should affect the development of life expectancy. However, as it is shown using Granger causality and VAR model, these general assumptions has limited usage. The life expectancy in the last quarter of 2011 was influenced by its values from previous quarter, last year capacity in elderly homes and the percentage of population living in houses connected to water supply system and drains five quarters ago. The impact of other variables was negligible. In summary, in case of CR, it is difficult to express the indicators, which can at the statistical significance level express and influence the quality of life and the life expectancy.

Keywords: alternative prediction, life expectancy at birth, standard of living, VAR
JEL Classification: C10, C13, C53

AMS Classification: 62M10

1 Introduction

Ranking of the countries, regions, firms, etc. according to their performance has become frequented topic of various researches, surveys or analysis. The main concern is usually to create a ranking according to the development level of states. Scholars use various variables, criteria and indicators in order to assess which country does economically better. The reason is, as Dowrick et al. [11] states in their article that "International comparisons of living standards (LS) and development are inescapable. Public opinion, inasmuch as it is represented by the media, has an insatiable appetite for world or regional rankings. Politicians and policy analysts regularly use changes in these rankings as a basis for assessing the efficacy of national policies." Nonetheless, what is the reliability of these rankings? Are the used ranking criteria meaningful? We argue that usage of measures to rank the countries' performance is justified only in those cases when appropriate measures are used. The outcome of the performance of the country should not be an end in itself. For example *GDP per capita* is a good of the outcome of the economy recalculated on one person, but does not have direct impact on the life of people (Meisel and Vega [20]). On the other hand, *income per person* indicator does better as these are real money (despite being averaged) which could be used for consumer expenditures of the person. We believe that indicators from the demographic area are the best (with reference to Arltová et al. [2]). We selected as a measure of the outcome an indicator of life expectancy at birth (e_0^0). It enables to capture not only the economic situation of a country (with higher GDP per capita, the countries tend to have longer e_0^0, see e.g. George and Beller [14] or Jia et al. [19]), but also the health of the society and social, hygienic and environmental conditions of a particular country. We consider e_0^0 to be an appropriate measure of the development and performance of a particular country.

1.1 Proxy variables of the standards of living

At the beginning, the proxies for LS must be defined. It is difficult to express the LS objectively. However, subjective assessment is also not without complications. Each individual considers important different features of life. As Binder and Coad [4] noted "Individual's mental well-being or happiness depends on a complex vector

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of factors, ranging from individual determinants (e.g. self-esteem, optimism) to socio-demographic (such as gender, age, education, or marital status), economic (such as income, status, or unemployment), situational (such as health, social relationships), and even institutional factors". When the comparison of LS is needed, the gross domestic product (GDP) belongs to the often used indicators. "GDP per capita is the most commonly used indicator to compare wealth among countries and is a measure of well-being and development exclusively based on material wealth. However, insufficient income is merely one dimension of under-development, so development cannot be understood by only taking into account economic performance" (Bérenger and Verdier-Chouchane [3]). Dowrick [11] marked, that "the most frequent criticism is based on the observation that standard GDP indexes are more properly regarded as partial measures of aggregate output than as indicators of wellbeing. No allowance is made for environmental differences, domestic activities, or production and consumption externalities." Therefore alternative measures were elaborated by international institutions or scholars. For example "UNDP Human Development Index (HDI) constructed in 1990 was intended to be more comprehensive indicator than per capita income for comparing the well-being of countries" (Bérenger and Verdier-Chouchane [3]). Fleurbaey and Gaulier [13] developed a "measure of LS for international comparisons, based on GDP per capita, in PPP, and incorporate corrections for international flows of income, labour, risk of unemployment, healthy life expectancy, household demography, inequalities and sustainability." From other economic variables can be utilize customer price index (CPI) to measure LS. Nevertheless, the economic indicators are not always good proxies of LS. As Gibson at al. [15] pointed out in their analysis on the situation of Russia during transition period, there was a bias – "the decline in LS has been substantially less than what is inferred by looking at official statistics on real output." Neumayer [21] suggests using as proxy variables for the LS: e_0^0, infant survival rates, literacy rates among the adult population, the combined primary, secondary and tertiary educational enrolment ratio as well as telephone mainlines and television set availability per capita. He considers the telephone availability as a measure of communication facilities and television of entertainment possibilities. However, currently the communication media are advanced. Therefore, instead we use as a proxy the equipment of the households with computers (Šimpach, Langhamrová [22]). "In welfare economics, individual well-being is traditionally conceptualized by the satisfaction of an individual's preferences, and the usual proxy to measure this satisfaction has been income" (Binder and Coad [4]). Day and Dowrick [9] concentrated in their survey on length of life. They had selected indicators which determinate the mortality such as food, medical services, education, environmental conditions, social customs and income distribution. Therefore, we use in our research also some of these indicators in order to assess if they really influence e_0^0 and hence justify their usage in researches. Curi [7] in his study clustered 125 countries according to their LS which were not expressed by any particular indicator. He used a component analysis where 26 indicators from social, economic, demographic, health, cultural and educational area instead. Other possible methodologies which can be used to express LS are Totally Fuzzy Analysis (TFA) and Factorial Analysis of Correspondences (FAC). These two were used by Bérenger and Verdier-Chouchane [3] in order to construct indicators of LS and quality of life which they further utilized as indices of well-being. Similarly various indexes exist in order to express the LS in more complex way. Despite their esteem to capture various aspects of LS, "the problem is that these indicators are based on the aggregation of various sub-indexes of social performance, arbitrarily weighted." Bérenger and Verdier-Chouchane [3] use to express LS by: standard of health - particularly public health expenditure (% of GDP), improved water source (% of population with access), physicians (per 1 000 people); standard of education: age dependency ratio (dependents to working-age population), public spending on education, total (% of GDP), net primary enrolment (%) and *material well-being*: vehicles (per 1 000 people), roads paved (% of total roads), television sets (per 1 000 people). There is a problem of causality relations between variables. Despite the fact, that the proxies for LS were chosen with the assumptions that they influence e_0^0, the dependence could be mutual. It is necessary to consider (as it is recommended by Binder and Coad [4]) the interrelations among explanatory variables. Our indicators from the area of economy, social and facilities used in this study are introduced in the chapter Methodology. We suppose in all cases that the higher is the value of particular characteristics, the higher is e_0^0. The impact is explicitly assessed using Granger causality and VAR model which are discussed in chapter Results. In the chapter Discussion we are arguing that the indicators commonly used as the expression of the LS do not have statistically significant impact on the outcome – i.e. the e_0^0. Final chapter summarizes the conclusion and states limitations of the utilized approach to the analysis as well as the challenges for the future research.

2 Methodology and Material

In order to examine the impact of the different variables representing the LS, a VAR model was constructed (Booth [5]). This method enables to clearly see which characteristics have statistically significant impact. This type of model is frequently used by researchers. However, its utilization in the area of social policy is rare. For example Binder and Coad [4] used a "panel VAR model to examine the coevolution of changes in mental well-being and changes in income, health, marital status and employment status for the British Household Panel Survey (BHPS) data set. As this method uses panel data, it allows "to simultaneously analyse the impact of the aforementioned factors on each other." The selection of the indicators for our analysis was limited to some ex-

tend due to the data availability. Not all indicators are gathered since 1996 and in quarterly division. Several proxies from the demographic and economic area were considered as same as various types of facilities (see Table 1). For the consequent analysis it is also important that values of the variables has the same development in time i.e. also increase overtime as same as e_0^0. Therefore, the plot of the values was drawn (see Figure 1).

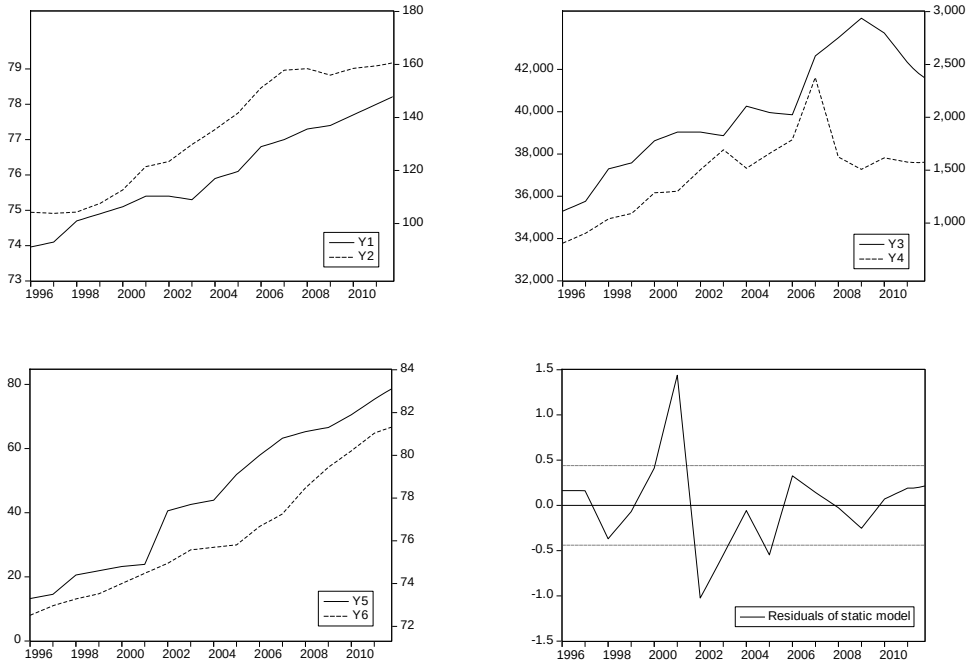

Figure 1 Selected time series and residuals of static regression model. Source: CZSO, authors' construction

Indicators area	Mark	Indicator	Units
Demographic	Y1	Life expectancy at birth (for total population)	years
Economic	Y2	Labour productivity	per 1 hour worked, c.p. in CZK
	Y3	Housing construction in the Czech Republic	three-room flats
Social	Y4	Capacity in elderly homes	number of beds
Facilities	Y5	Population living in houses connected to water supply system and drains	%
	Y6	Computer facilities of Czech households	%

Table 1 Indicators selected for the analysis. Source: authors' construction

To make the analysis as actual as possible we used the time series from 1st q. of 1996 until 4th q. of 2011. Only data for e_0^0 were available throughout all range. Therefore other values (from 1st quarter 2011 until 4th quarter 2011) had to be forecasted using Box-Jenkins methodology (e.g. Arlt, Arltová [1]). It is based on the assumption that the considered time series is stationary. Particularly augmented Dickey-Fuller [10] test (ADF) was used as it enables testing even in case where the error terms ε_t are correlated. The calculation of Y_t is expressed in equation

$$\Delta Y_t = B_1 + B_2 t + B_3 Y_{t-1} + \sum_{i=1}^{m} \alpha_i Y_{t-i} + \varepsilon_t, \tag{1}$$

where ΔY_t is the first difference of the examined variable, t is trend variable, ε_t is a pure white noise error term, m is the maximum length of the lagged dependent variable. The basic assumption of a possible correlation between the time series is that they have a similar trend. At the beginning we must verify whether the examined time series are stationary, because the analysis of the relationship between them makes sense only if they are integrated of the same order. The ADF tests show, that the time series are non-stationary, I (1), see Table 2. The relationship between non-stationary time series is associated with the problem of spurious regression (Granger, Newbold, [17]). Differentiation between co-integration regression and spurious regression is performed by Engle–Granger [12] co-integration test based on the analysis of residues of static regression model. If residues of the model are stationary I (0), then the time series are co-integrated, if the residues are non-stationary I (1), it is a case of spurious regression. From the Figure 1 is evident (t_{ADF} = -3.571947, Prob. 0.0094), that the residues of the static model are stationary I (0), which excludes the possibility of spurious regression. Because the direction

of causality of the individual time series and whether the residuals in Table 3 are auto-correlated, is not entirely clear we estimate the short-term relationships between the time series by multi-dimensional VAR model.

Y_t		$\Delta_2 Y_t$	
t_{ADF}	Prob.	t_{ADF}	Prob.
0.225631	0.9720	-8.645782	0.0000
-1.359391	0.5964	-7.672481	0.0000
-1.980265	0.2946	-7.662394	0.0000
-2.333508	0.1650	-6.792391	0.0000
-0.474059	0.8887	-7.681558	0.0000
0.479981	0.9847	-7.681562	0.0000

Table 2 The unit root test of time series Y_t and second differences. Source: authors' calculation

The general model VAR (p) can be written in the form:

$$\mathbf{Y}_t = \mathbf{c} + \sum_{i=1}^{p} \mathbf{\Phi}_i \mathbf{Y}_{t-i} + \boldsymbol{\varepsilon}_t, \tag{2}$$

where $\mathbf{c}$ is l x 1 dimensional vector of constants, $\mathbf{\Phi}_i$, $i = 1, 2, \ldots, p$ are l x l dimensional non-random matrices of AR parameters and $\boldsymbol{\varepsilon}_t$ is l-dimensional process of white noise. On the basis of diagnostic tests displayed in Table 3) (Breusch, Godfrey [6], Jarque, Bera [18], Darnell [8]) it can be concluded that the distribution of residuals is mostly non-normal with exception of population living in houses connected to infrastructure. The autocorrelation was present only at the fourth lag of the explained variable. Usage of heteroscedasticity and autocorrelation correct (HAC) errors did not change the results substantially. As a result of that, we neglect the autocorrelation and use the original model. The variance of residuals is constant and finite. The VAR (5) model

$$\mathbf{Y}_t = \mathbf{c} + \mathbf{\Phi}_1 \mathbf{Y}_{t-1} + \mathbf{\Phi}_2 \mathbf{Y}_{t-2} + \mathbf{\Phi}_3 \mathbf{Y}_{t-3} + \mathbf{\Phi}_4 \mathbf{Y}_{t-4} + \mathbf{\Phi}_5 \mathbf{Y}_{t-5} + \boldsymbol{\varepsilon}_t, \tag{3}$$

or in matrix form:

$$\begin{aligned}
\begin{bmatrix} Y_{1,t} \\ Y_{2,t} \\ \vdots \\ Y_{6,t} \end{bmatrix} &= \begin{bmatrix} \phi_{1,11} & \phi_{1,12} & \cdots & \phi_{1,16} \\ \phi_{1,21} & \phi_{1,22} & \cdots & \phi_{1,26} \\ \vdots & \vdots & \cdots & \vdots \\ \phi_{1,61} & \phi_{1,62} & \cdots & \phi_{1,66} \end{bmatrix} \begin{bmatrix} Y_{1,t-1} \\ Y_{2,t-1} \\ \vdots \\ Y_{6,t-1} \end{bmatrix} + \begin{bmatrix} \phi_{2,11} & \phi_{2,12} & \cdots & \phi_{2,16} \\ \phi_{2,21} & \phi_{2,22} & \cdots & \phi_{2,26} \\ \vdots & \vdots & \cdots & \vdots \\ \phi_{2,61} & \phi_{2,62} & \cdots & \phi_{2,66} \end{bmatrix} \begin{bmatrix} Y_{1,t-2} \\ Y_{2,t-2} \\ \vdots \\ Y_{6,t-2} \end{bmatrix} + \cdots \\
&\cdots + \begin{bmatrix} \phi_{5,11} & \phi_{5,12} & \cdots & \phi_{5,16} \\ \phi_{5,21} & \phi_{5,22} & \cdots & \phi_{5,26} \\ \vdots & \vdots & \cdots & \vdots \\ \phi_{5,61} & \phi_{5,62} & \cdots & \phi_{5,66} \end{bmatrix} \begin{bmatrix} Y_{1,t-5} \\ Y_{2,t-5} \\ \vdots \\ Y_{6,t-5} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \vdots \\ \varepsilon_{6,t} \end{bmatrix},
\end{aligned} \tag{4}$$

can be rewritten in the multidimensional form:

$$\begin{aligned}
Y_{1,t} &= \phi_{1,11}Y_{1,t-1} + \phi_{2,11}Y_{1,t-2} + \cdots + \phi_{5,11}Y_{1,t-5} + \phi_{1,12}Y_{2,t-1} + \phi_{2,12}Y_{2,t-2} + \cdots + \phi_{5,12}Y_{2,t-5} + \\
&\quad \phi_{1,13}Y_{3,t-1} + \phi_{2,13}Y_{3,t-2} + \cdots + \phi_{5,13}Y_{3,t-5} + \phi_{1,14}Y_{4,t-1} + \phi_{2,14}Y_{4,t-2} + \cdots + \phi_{5,14}Y_{4,t-5} + \\
&\quad \phi_{1,15}Y_{5,t-1} + \phi_{2,15}Y_{5,t-2} + \cdots + \phi_{5,15}Y_{5,t-5} + \phi_{1,16}Y_{6,t-1} + \phi_{2,16}Y_{6,t-2} + \cdots + \phi_{5,16}Y_{6,t-5} + \varepsilon_{1,t} \\
Y_{2,t} &= \phi_{1,21}Y_{1,t-1} + \phi_{2,21}Y_{1,t-2} + \cdots + \phi_{5,21}Y_{1,t-5} + \phi_{1,22}Y_{2,t-1} + \phi_{2,22}Y_{2,t-2} + \cdots + \phi_{5,22}Y_{2,t-5} + \\
&\quad \phi_{1,23}Y_{3,t-1} + \phi_{2,23}Y_{3,t-2} + \cdots + \phi_{5,23}Y_{3,t-5} + \phi_{1,24}Y_{4,t-1} + \phi_{2,24}Y_{4,t-2} + \cdots + \phi_{5,24}Y_{4,t-5} + \\
&\quad \phi_{1,25}Y_{5,t-1} + \phi_{2,25}Y_{5,t-2} + \cdots + \phi_{5,25}Y_{5,t-5} + \phi_{1,26}Y_{6,t-1} + \phi_{2,26}Y_{6,t-2} + \cdots + \phi_{5,26}Y_{6,t-5} + \varepsilon_{2,t} \\
&\vdots \\
Y_{6,t} &= \phi_{1,61}Y_{1,t-1} + \phi_{2,61}Y_{1,t-2} + \cdots + \phi_{5,61}Y_{1,t-5} + \phi_{1,62}Y_{2,t-1} + \phi_{2,62}Y_{2,t-2} + \cdots + \phi_{5,62}Y_{2,t-5} + \\
&\quad \phi_{1,63}Y_{3,t-1} + \phi_{2,63}Y_{3,t-2} + \cdots + \phi_{5,63}Y_{3,t-5} + \phi_{1,64}Y_{4,t-1} + \phi_{2,64}Y_{4,t-2} + \cdots + \phi_{5,64}Y_{4,t-5} + \\
&\quad \phi_{1,65}Y_{5,t-1} + \phi_{2,65}Y_{5,t-2} + \cdots + \phi_{5,65}Y_{5,t-5} + \phi_{1,66}Y_{6,t-1} + \phi_{2,66}Y_{6,t-2} + \cdots + \phi_{5,66}Y_{6,t-5} + \varepsilon_{6,t}
\end{aligned} \tag{5}$$

where $\phi_{1,11}$ = 1.304608 (0.19097) [6.83147], $\phi_{5,14}$ = -7.34E-05 (3.2E-05) [-2.28539], $\phi_{5,15}$ = 0.210248 (0.06088) [3.45350], $\phi_{1,21}$ = -6.942611 (2.71596) [-2.55623], $\phi_{1,22}$ = 2.203656 (0.26276) [8.38656], $\phi_{1,23}$ = 0.011441 (0.00383) [2.98806], $\phi_{1,24}$ = -0.002720 (0.00079) [-3.45388], $\phi_{5,24}$ = -0.001226 (0.00046) [-2.68586], $\phi_{1,25}$ = 3.117852 (1.13408) [2.74924], $\phi_{1,33}$ = 1.258935 (0.19049) [6.60886], $\phi_{5,35}$ = 100.5042 (43.0764) [2.33316], $\phi_{4,36}$ = 67.97408 (27.4159) [2.47937], $\phi_{5,36}$ = -70.49117 (14.7622) [-4.77510], $\phi_{1,41}$ = -2600.202 (1057.27) [-2.45935], $\phi_{4,41}$ = 4160.659 (1671.61) [2.48901], $\phi_{5,41}$ = -2956.861 (887.590) [-3.33134], $\phi_{1,42}$ = 317.4339 (102.288) [3.10334], $\phi_{1,43}$ = 3.284038 (1.49050) [2.20331], $\phi_{4,43}$ = -6.465611 (2.16904) [-2.98087], $\phi_{5,43}$ = 4.242376 (1.69592) [2.50152], $\phi_{1,45}$ = 1250.880 (441.476) [2.83341], $\phi_{1,55}$ = 1.248863 (0.32356) [3.85979], $\phi_{1,62}$ = -0.300833 (0.14559) [-2.06631], $\phi_{1,64}$ = 0.000921 (0.00044) [2.11096], $\phi_{1,65}$ = -1.649565 (0.62837) [-2.62516], $\phi_{5,65}$ = -1.305652 (0.47973) [-2.72161], $\phi_{1,66}$ = 1.300585 (0.16400) [7.93039] and $\phi_{5,66}$ = 0.382087 (0.16440) [2.32407]. Standard errors are in ( ) ant *t*-statistics are in [ ].

Normality tests			Autocorrelation tests			Heteroskedasticity test	
Y_t	Jarque-Bera	Prob.	Lags	LM-Stat	Prob.	Chi-sq (df)	Prob.
Y_1	11.77285	0.0028	1	36.18057	0.4602	1225.971 (1176)	0.1515
Y_2	11.06396	0.0040	2	49.06088	0.0720		
Y_3	11.53477	0.0031	3	36.37753	0.4511		
Y_4	11.32136	0.0035	4	178.3569	0.0000		
Y_5	0.278317	0.8701	5	23.73942	0.9419		
Y_6	11.04043	0.0040					

Table 3 The diagnostic tests of VAR (5) model. Source: authors' calculation

3 Discussion

The e_0^0 in current quarter is influenced by its values from last quarter, last year capacity in elderly homes and the percentage of population living in houses connected to water supply system and drains five quarters ago. The impact of other variables is negligible. On the other hand e_0^0 significantly influences the labour productivity (LP) in previous period, capacity in elderly homes one, four and five quarters ago. The results imply that commonly used indicators are not always good proxies of the LS as they do not have significant impact on the result expressed by e_0^0. However, the analysis revealed other relations among the indicators. LP is influenced by capacity in elderly homes one and five quarters ago and by population living in houses connected to water supply system and drains in a previous quarter. The number of newly build three-room flats depends not only on the number in previous quarter, but also on population living in houses connected to water supply system and drains and computer facilities (CF) five quarters ago. The later influenced the flats also with one year lag. The social indicator – capacity in elderly homes was dependent not only on the e_0^0 (which is in line with logical assumptions), but also on LP, housing constructions in three-room flats (both one period ago), housing constructions one year and five quarters ago and finally by percentage of population living in houses connected to infrastructure in previous quarter. Surprisingly despite the fact that it had the significant impact on other variables the population living in houses connected to infrastructure facilities was influenced only by itself with one lag.

I/1996-IV/2011	χ^2	Prob.	*I/1996-IV/2011*	χ^2	Prob.	*I/1996-IV/2011*	χ^2	Prob.
$Y_2 \rightarrow Y_1$	7.404649	0.1922	$Y_1 \rightarrow Y_2$	14.14956	0.0147	$Y_1 \rightarrow Y_3$	7.902832	0.1617
$Y_3 \rightarrow Y_1$	8.492336	0.1311	$Y_3 \rightarrow Y_2$	18.05111	0.0029	$Y_2 \rightarrow Y_3$	1.774210	0.8794
$Y_4 \rightarrow Y_1$	17.75160	0.0033	$Y_4 \rightarrow Y_2$	16.65255	0.0052	$Y_4 \rightarrow Y_3$	4.554422	0.4726
$Y_5 \rightarrow Y_1$	27.58778	0.0000	$Y_5 \rightarrow Y_2$	13.94631	0.0160	$Y_5 \rightarrow Y_3$	18.06461	0.0029
$Y_6 \rightarrow Y_1$	17.24373	0.0041	$Y_6 \rightarrow Y_2$	5.989158	0.3073	$Y_6 \rightarrow Y_3$	50.75533	0.0000
$Y_1 \rightarrow Y_4$	26.32671	0.0001	$Y_1 \rightarrow Y_5$	4.890978	0.4293	$Y_1 \rightarrow Y_6$	18.88592	0.0020
$Y_2 \rightarrow Y_4$	17.30898	0.0039	$Y_2 \rightarrow Y_5$	6.705082	0.2435	$Y_2 \rightarrow Y_6$	19.16702	0.0018
$Y_3 \rightarrow Y_4$	25.60349	0.0001	$Y_3 \rightarrow Y_5$	2.210761	0.8193	$Y_3 \rightarrow Y_6$	7.691130	0.1741
$Y_5 \rightarrow Y_4$	10.15793	0.0709	$Y_4 \rightarrow Y_5$	0.962332	0.9655	$Y_4 \rightarrow Y_6$	11.59659	0.0408
$Y_6 \rightarrow Y_4$	20.54387	0.0010	$Y_6 \rightarrow Y_5$	1.107516	0.9534	$Y_5 \rightarrow Y_6$	20.33432	0.0011

Table 4 The results of Granger causality test. Source: authors' calculation

Finally, CF, in spite they influenced only one indicator (housing construction), were affected (except by itself one and five quarters ago) by LP, capacity in elderly homes and percentage of population living in houses connected to water supply system and drains – all one quarter ago. The last one had also influence five quarters ago. Despite that the proxies for LS were chosen with the assumptions that they influence the e_0^0, the opposite direction of dependence was found. The results of Granger test (displayed in Table 4), through which we proved the causal effect of selected time series in Granger sense (Granger [16]), more or less correspond with the results of the models above. Only difference is that VAR hadn't shown that the CF significantly influence the e_0^0. LP and housing construction shouldn't be used as the indicators of LS as they don't directly influence the e_0^0. However, they can have indirect impact. LP influence capacity in elderly homes and CF which jointly cause the e_0^0. Simi-

larly housing constructions are caused by population living in houses connected to water supply system and drains and by CF. The mutual causality was found between e_0^0 and capacity in elderly homes, and between e_0^0 and CF. This causality enabled Fleurbaey and Gaulier [13] to use as measure of LS the e_0^0.

4 Conclusion

Used methods enabled to assess the lag of the influence and the causality among particular variables. VAR showed that the e_0^0 in current period was influenced by its values from last quarter, last year capacity in elderly homes and the percentage of population living in houses connected to water supply system and drains five quarters ago. The effect of other used variables was not significant. Hence, usage of these indicators as an expression of LS is limited. Granger test proved also mutual dependence of e_0^0 and capacity in elderly homes and CF. As a result of that, these indicators can be utilize in the analysis of LS in the CR. The selection of indicators which could have expressed the LS was limited by the type of the progress of the time series. As the e_0^0 shows a raising trend, only those indicators which have the same development were included into the system. We included only those time series that met requirements. Therefore, the challenge for future research is to find methodology enabling wider usage of proxies of LS.

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Methods of deriving factors influencing housing tenure choice in the Czech Republic

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Abstract. One of the key decisions to be made by any household is the choice between owning or renting their home, which is known in specialized literature as tenure choice. The paper examines the potential factors affecting housing tenure choice decision (rent or own) in the Czech Republic. To derive these factors we employ traditional approach via econometric model with logistic regression (probit model) drawing on sample data. In addition, we consider possible pros and cons of introduction of other approaches e.g. hierarchical model of tenure choice (Boehm [5]). Analysis covers period of 2005-2011 and makes use of the data coming from investigation EU-SILC in the Czech Republic. Preliminary results of the basic (probit) model testify to the fact that tenure choice in Czech Republic is influenced by the factors similar to those in other countries – household income, marital status of the household head and household size. On the other hand, the influence of other demographic characteristics of the household head (gender or age) has not been confirmed. Analysis of the tenure choice (i.e. conditional probabilities of choosing housing type) will be accompanied by detailed analysis of households who have moved during the analyzed period.

Keywords: housing, tenure choice, factors

JEL Classification: D12, P36, R21
AMS Classification: 91C

1 Introduction

The quality of family life is affected by the place where the family lives, its location, its adequate floor area enabling sufficient privacy for the household members and many other aspects. That is why one of the key decisions made by any household is the choice between owning and renting their home, which is known in specialized literature as tenure choice. This issue has drawn the attention of researchers for many years. They focus mostly on identifying the determinants which are relevant to the decision of households to own or rent their housing as well as on establishing the degree of their impact. A host of authors view tenure choice as an independent decision. Nevertheless, a more common opinion has it that the choice of home type is only one part of all the other decisions households have to make in relation to consumption and investment (Turner, O'Neal [13]). Thus tenure choice is typically analyzed in connection with household mobility (Boehm [4]; Krumm [10]; Kan [9]), housing attributes (dwelling characteristics), household attributes or as an element of consumption and investment decisions (Artle,Varaiya [2]; Henderson,Ioannidis [8]). Presented paper uses two of above mentioned approaches housing attributes and household attributes.

It was Boehm [5] who concentrated on the issue of mutual links between tenure choice and the so-called housing attributes. More specifically, he studied the links between tenure choice (owning or renting) and other household's requirements, such as the quality and size of the house. Boehm suggests that existing tenure choice analysis does not include all aspects of a family's housing choice. In addition to the choice between owning and renting, households have to solve other aspects too, e.g. home size, its location, availability of services, etc. For this purpose he made a three-level hierarchic model. 1. A household decides about the type of housing (tenure choice: own/rent). 2. A household makes a decision about the size of home (large or small). 3. A decision about location is made (neighborhood: high income/low income). At each level of hierarchy the choice of a specific housing characteristic is estimated as based on the previous choice made, brought to a higher level. At the same time he studied the impact of selected variables, which could affect the decision-making at particular levels of the model. They include headings such as Family Size, Age of Head, Marital Status, Permanent Income, Wealth, Prior House Value, Race, Relative Price Variables etc.

The general assumption, that a household is more likely to choose a larger size and higher quality housing because of its increasing income, was confirmed. It was pointed out, however, that this assumption is met with

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a dependence on certain household characteristics - e.g. the household with a higher number of members seeks a larger size of housing but not necessarily a home of higher quality. Analyses of the roles of other variables in the decision-making of households demonstrated (in accordance with the above statements) that their impact can significantly differ depending on the household character.

An alternative approach was adopted by Andersen [1], who assessed household preferences for housing quality using a questionnaire survey. He divided the housing characteristics into four groups: the dwelling itself (size and number of rooms, home amenities, house type, yard or garden, view, air pollution, light, noise), its surroundings (exterior of other houses and streets, environment – green spaces, criminality, social environment), social infrastructure, (shops, restaurants, cultural facilities, sports facilities), location and transport (distance to work/school, transport accessibility, urbanization). He concluded that tenure choice is greatly affected by such factors as environment without crime and good access to public transport suitable for bringing up children. However, the survey results have to be interpreted with caution, because some of the preferences may not be realistic or feasible and they can be rather biased by each household's current housing situation.

Tenure choice is probably most frequently examined in connection with household attributes. A wide range of these characteristics can be categorized into two basic groups, namely socio - economic and demographic characteristics. Burgess [6] added one more group, namely location factors. In his work he analyses impacts of all these determinants on tenure choice of households and also poses a question on whether the impact of individual variables differs depending on the gender of the household head. However, his attention was not on a traditional household (married couples), but in keeping with current social trends, he focused on the households headed by a single man or a single woman. According to Burgess, the probability of living in one's owned home is significantly influenced by the head's age, household size and current income in both types of households (male headed/female headed). A statistically significant but negative impact has been confirmed in case of welfare income and living within the city centre. Second income and employment status proved to have no influence.

Bazyl [3] was also involved in analysing the impact of the above factors on tenure choice. She labeled them socio-demographic characteristics of households and compared their importance across several European countries. Performed analyses led her to a conclusion that marriage proved to be an important factor influencing the decision to buy a home in all analysed countries without exception. In most countries the fact that the household head is a citizen of that country (i.e. they are not foreigners) can be even more significant than the marriage itself. In most countries the odds of homeownership grows with higher age of the household head and his/her income. Bazyl gradually expanded her basic model – in the second one she included a type of location, i.e. whether a household is urban (living in the city) or rural (living in the country). She found out that homeownership is more likely in the countryside. In the third model she included only renters living in houses owned privately. The fourth model investigated only recent movers. This last case demonstrated that the impact of income on tenure choice is many times greater than the model involving all households.

Despite many differences in the conducted analyses, all the researchers studying tenure choice in connection with household attributes agreed that the level of income has the most profound impact on the outcome of household decisions.

The issues of tenure choice were touched on by Czech authors as well. Lux and Sunega [11] assessed the influence of the form of ownership (i.e. only tenure) on mobility. Using family accounts statistics, Tsharakyan and Zemčík [12] studied whether the rent deregulation had an impact on households renting behaviour or their ownership status (renters vs. owners). None of the above papers is devoted to the analysis of tenure choice itself.

Our paper has combined two of the above mentioned approaches towards the exploration of tenure choice. Based on the results of the econometric model, we have defined the factors that affect tenure choice of Czech households over the period 2005-2011. Our aim is to assess the suitability of the given method for investigating tenure choice. To our knowledge this is the first analysis of tenure choice trends with connection to housing expenditures in the Czech Republic. The robustness of the results is tested on the subsample of moving households over studied period. Special attention is paid to households moving from rental sector (particularly with price-regulated tenure). With respect to rent deregulation in the Czech Republic in 2007-2012 we expect massive moving in the sector. The analysis of behavior of these households can evaluate real influence of the factors derived by econometric model.

2 Material and Methods

As shown above, our approach relies on the quantitative analysis of sampled data. In order to analyse tenure choice we have applied an econometric model in accordance with the most frequent approaches. The model relies on a form of regression logit model (e.g. Bazyl [3], Ulker [11]). This approach allows us (using the so-

called odds ratios) to capture the individual influences of each set of potential factors that may affect tenure choice. The model enables to calculate the conditional probability of the choice of a particular type of housing depending on a given factor, provided that the values of other factors are constant. In this paper we have used a probit model, which has the following formula (Wooldridge [15]):

$$log\frac{\pi}{1-\pi} = \beta_0 + \beta_1 X_1 + \cdots + \beta_k \mathrm{X_k} \quad (1)$$

where π : Probability of homeownership for a given value of x and takes the form of normal cumulative distribution function:

$$\pi = \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$$

x_k : Explanatory variables (see below)

β_k Coefficients estimated by the probit model.

The values of the coefficients β_k express the effect of each factor on the tenure choice and at the same time show its direction. We refer to homeownership as the default choice, because the values of the coefficients β_k are related to this option. Positive values of the coefficient β_k therefore indicate that higher values of factor x_k increase the likelihood of rental. On the other hand, negative values indicate that there is a greater likelihood of choosing home ownership in case of high values of factor x_k. Then for those factors with a dichotomous nature (most often Yes/No), lower values of the factor mean 'No' and its higher values 'Yes'.

The analyses included in this paper are based on the data pertaining to Czech households collected by the CSO (The Czech Statistical Office). The data were collected under sample surveys of income and living conditions of households between 2005 and 2010, under the Living Conditions Programme (called hereinafter the "EU-SILC"). This data set contains information on the social structure of households, their incomes and expenditures. The dates relate to the date of the investigation, i.e. the defined date in the spring of that year, only incomes are listed for the previous year. In order to convert the sample of households to the entire Czech Republic the coefficient of the "PKOEF" is used, which expresses the weighting of each surveyed household.

In the data sets, households are divided into five groups according to the form of ownership. For the purposes of our analysis, however, the relevant division is the category of home ownership or that of rented housing. Homeownership comprises of three different forms - living in an owned house, in a flat in personal ownership, and a flat in cooperative ownership. Lease and rental of the whole flat fall into the category of rented housing.

After calculations for the entire population of the CR it was ascertained that the percentage of homeownership significantly exceeds the percentage of rented housing and keeps increasing over time. The ownership percentage grew from 73.6 % in 2005 to 80.6 % in 2011 as presented in the Table 1. If we de-compose the percentage of movers, we found that majority of movers come from rental sector (90 %).

	2008	**2009**	**2010**	**2011**
Owners	76.0	76.8	79.1	80.6
Tenant	24.0	23.2	20.9	19.4
- market rent	5.7	6.2	5.5	14.4
- regulated rent	18.3	17.0	15.4	4.9
Total	100.0	100.0	100.0	100.0
- movers	6.4	7.9	5.0	-

Table 1 Household by tenure status (%)

This development was the logical result of gradual privatisation of the housing stock in the past. Privatisation, however, is slowly becoming a thing of the past, and so a question emerges: What factors have caused this development in recent years? To what extent have demographic and socioeconomic characteristics affected household's choice of the type of housing? Which of these characteristics has been statistically the most significant? Has the structure of the households in both sectors changed in any way?

Our research relies on the following assumptions or hypotheses: (H1) - "Home ownership is preferred in households where the household head has a higher than basic education and is married." (H2) - "Rented housing

is preferred by households with a lower income and that are "incomplete", specifically, divorced persons with children."

3 Results and Discussion

In line with the above hypotheses, we focused on the examination of the impact of the demographic and socioeconomic characteristics of the households on tenure choice. To some extent, the range of potential factors is determined by the used data source (Commission Regulation EC No 1983/2003 [7]). Potential factors mostly comprise variables monitored within SILC survey (i.e. our source of data), or variables that can be derived from monitored variables.

In accordance with the proposed hypotheses we have concentrated on the household heads – their age, gender and marital status and also on the household from the perspective of its economic status, disposable income or number and age of children. Since we believe that the factors influencing tenure choice are similarly projected into the structure of housing costs, we base our analysis on the same range of variables.

Results of the regression analysis examining tenure choice determinants are summarized in Table 2. Marginal effects β_k are computed not only for the year 2011 but they also cover a four-year time span (2008-2011). This enables to assess trends of development of these effects.

	All households / Marginal effect (β_k)**				**Movers / Marginal effect (β_k)****		
Factor (x_k)	**2008**	**2009**	**2010**	**2011**	**2008 - 2009**	**2009 – 2010**	**2010 – 2011**
Constant	-5.185	-6.556	-6.528	-6.468	-4.076	-4.851	-4.720
Household size (number of persons)	-0.201	-0.153	-0.131	-0.109	-0.111	-0.111	-0.092
Number of economically active persons	-0.014	-0.028	-0.002	-0.018	-0.021	-0.047	-0.058
Number of self-employed	-0.134	-0.094	-0.070	-0.035	-0.084	-0.007	-0.128
Age of household head	-0.013	-0.018	-0.014	-0.015	-0.012	-0.014	-0.010
Gender of household head (1=male 2=female)	-0.030	-0.051	-0.011	-0.027	0.011	-0.022	-0.028
Education in the household (1=both primary school, 0-higher)	0.321	0.367	0.295	0.351	0.324	0.237	0.253
Household head is single*	-0.216	-0.211	-0.088	-0.170	-0.148	-0.115	-0.087
Household head is married*	-0.481	-0.478	-0.363	-0.440	-0.382	-0.267	-0.334
Household head is divorced*	0.014	-0.063	-0.008	-0.084	-0.001	0.009	-0.004
Household with children*	0.008	0.021	-0.094	-0.195	0.010	-0.136	-0.151
Children under two years*	-0.032	-0.213	-0.023	0.045	-0.032	-0.095	-0.011
Fully unemployed household	0.266	0.186	0.108	-0.012	0.409	0.250	-0.017
At least one retiree	-0.037	0.045	-0.024	-0.108	-0.019	0.058	-0.118
Household head works in public sector*	-0.142	-0.118	-0.079	-0.180	-0.098	0.047	-0.077
Household resides in Prague*	0.319	0.176	0.141	0.336	0.119	0.272	0.226
Region	0.007	0.004	0.002	0.003	0.005	0.003	0.003
Type of settlement**	-0.251	-0.244	-0.253	-0.214	-0.268	-0.175	-0.195
Total floor area per person	-0.024	-0.024	-0.022	-0.021	-0.018	-0.019	-0.019
Household income (disposable)	-0.261	-0.346	-0.373	-0.378	-0.210	-0.274	-0.277

*dichotomous variables take values 1 (Yes) / 0 (No)

***variable can take values: 1 (Prague), 2 (region), 3 (town), 4 (village)*

****All values are statistical significant at 99% level (t-statistics are upon request by authors) excluded the coefficient market* ^{n}

Table 2 Probit Estimation Results for Housing Tenure Choice (1=owner)

As expected, one of the most significant factors affecting the choice of type of housing in 2011 (as in other years) was the marital status of the household head (β^{2011}=-0,440). If the household head was married, the odds of choosing homeownership increased. The analogous assumption, i.e. that single or divorced persons would more likely prefer rented housing, was not confirmed. On the contrary, the singles were more likely to choose homeownership even if the strength of this factor was far weaker. The second strongest factor was the net disposable income of households (β^{2011}=-0,378). Higher-income households tend to prefer homeownership to renting. The importance of this factor was growing over the covered period. Among other strong (and statistically significant) factors, affecting tenure choice in 2011, was education of household members (β^{2011}=0,351) and residence in Prague (β^{2011}=0,336). Our findings showed that households with lower level of education and Prague households were more likely to choose rented housing. This fact can be linked with a high proportion of rented flats as compared with privately owned flats (or houses) in the capital city and so renting is a much faster and simpler way to acquire a new home there. Prague is also a university city, where many students and mainly graduates obviously prefer renting.

Others factors affecting tenure choice can be found. Rented housing was chosen with high probability by the households with a lower number of persons (β^{2011}=-0,109), lower education of the head β^{2011}=0,351) or residence in a smaller town (β^{2011}=-0,214). So the results corroborate the assumption that homeownership is affordable only for higher-income households, which usually corresponds to higher education. This housing type is preferred by families (not individuals), i.e. a higher number of persons per household, and also by households, whose members work, since it is necessary to make regular mortgage payments (i.e. the past significance of the number of economically active persons grew much weaker in 2011). For the first time children within a household became significant towards choosing homeownership in 2011. The influence of other factors, i.e. the number of self-employed, age or gender of the household head and the total floor area per person was not established.

Tenure choice model was applied also to the subsample consisted of the households that has moved during the period 2008-2011. Results show that also for such households (that had to decide on the choice of the tenure) there are the same four key factors that influence the decision: marital status of the household head (married), the net disposable income of households, education of households' members and residence in Prague. Compared to the whole sample there are differences in the relative strength of the factors. The most important factor that influences the tenure choice decision for moving households is again marital status of the household head - marriage ($\beta^{2010\text{-}2011}$=-0,334). The same applies for the education of the household ($\beta^{2010\text{-}2011}$=0,253) while the influence of both factors is somewhat weaker than for the whole sample. Again, if the moving household is from Prague, there is high probability that the household will move to rental sector ($\beta^{2010\text{-}2011}$=0,226).

Besides the four key factors other factors that influence the behaviour of really moving households can be found. Interesting is the development of the influence of the rate of economic activity in the household. While in the whole sample is the factor of low importance, moving households with higher economic activity more likely choose ownership of the house (or flat).

Table 2 also indicates the development trends of the individual factors in tenure choice over the period 2008-2011. Throughout this period the following factors had a decisive effect on tenure choice: income of the household; education of its members; whether the head of the household was married or not. If the head of the household was married, the members of the household had a higher than basic education and a higher income, it implies the choice of homeownership. Our results in this respect correspond to the results of similar foreign studies. Ulker [11] in his article showed that there was a highly significant influence of socio-economic characteristics, such as income and education on the tenure choice of households in the USA. Both Ulker and Bazyl [3] confirmed a decisive impact of marriage on tenure choice of households throughout all monitored European countries.

4 Conclusion

As in other countries, many factors influence the decisions of Czech households to live in rented housing or in their own home. Although the motive of homeownership as a long-term investment cannot be neglected, most studies report that the greatest influence is exercised by demographic and socio-economic composition of household. Our results corroborate this assumption to a considerable degree. During 2008-2011 tenure choice was particularly influenced by household income, education of its members, and marital status of the household head (married, single or divorced), residence in Prague or elsewhere. By contrast, gender or age of the household head, children or retired persons living in the household proved to be insignificant. The results greatly support our hypotheses proposed at the beginning of our research. We have succeeded in confirming the hypothesis that higher education of household members (including the household head) and marriage of the household head are more likely to lead to homeownership. However, the second part of our hypothesis has only partly been supported. Based on our results we argue that households with lower income prefer rented housing. On the other hand,

no support for argument that incomplete households (e.g. divorced persons with children) do prefer renting has been found.

The robustness of the results has been tested via application the model to subsample of households who really decided on the type of tenure. The overall results (i.e. factors that influence the tenure choice) are similar to ones for the whole sample. On this basis we argue that probit model that we used for analysis of the tenure choice can be treated as appropriate method of such kind of analysis.

Of course, the interpretation of the submitted results must be treated with caution. The EU-SILC is a sample survey, in which 11,249 households were examined out of the total of 4,018,288 in the year 2011. The source data are only statistical estimates, which are burdened with an unspecific error. We believe, however, that despite these limitations the results presented are relevant and conclusions can be drawn from them.

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Optimal localization of investment means for increasing networks capacity in indeterminate financial conditions

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Abstract. The article deals with an optimal assignment of investment means into a given network. The networks capacity expressed by the maximal flow is considered to be the objective criterion. To find out the maximal flow in a network, for example, the universal Ford - Fulkerson algorithm or a linear mathematical model can be employed. But both approaches assume defined capacity of all edges. In our problem we know the capacity of edges as well. But contrary to the previous problems we consider that the edges capacity can be increased. The increase of the edges capacity can be done because we have some investment means which can be used for increasing the edges capacity. But in the practice, it often happens that same quantity values, having the character of constants in the model, cannot be exactly given. The article includes a mathematical model enabling to work with this indeterminacy. In the article we choose an approach, where the indeterminacy is modelled using fuzzy numbers and fuzzy sets.

Keywords: maximal flow problem, mathematical programming, fuzzy linear programming.

JEL Classification: C61
AMS Classification: 90C05

1 Introduction

For every network (transport networks, electrical networks etc.) the calculation of its capacity is one of the basic problems. Graph theory and mathematical programming offer some methods enabling the capacity calculation of such networks. The network capacity is limited with defined capacity of individual edges. It often happens that the network capacity is not sufficient and it is necessary to increase it. The network capacity can be increased through the investment means.

In our article we show the method which can be employed to increase the network capacity through the effective allocation of the investment means which are invested in order to increase the capacity of selected network edges. Our situation is complicated because the network capacity will be increased in terms of the indeterminacy. The indeterminacy reflects the fact that the costs necessary for the increase of the edge capacity are not exactly defined.

First of all let us define the elementary maximal flow problem.

2 The maximal flow problem

One of the basic problems of graph theory is the problem of searching the maximal flow in a graph. It deals with the problem about the transport planning on the net edges. The goal is to maximize the overall flow which passes through the net.

But at first let us define the elementary problem. Let us assume that the directed acyclic graph $G(V;H)$ is known. In this graph the edge set H represents the real net sections (roads, railway lines, tubes etc.) and the vertex set V represents the places, where the edges crosses each other (crossroads).

There are three vertex types in the graph. The vertexes with the edges which only exit the vertex (so called sources) represent the first type. The vertexes with the edges which only enter the vertex (so called sinks) form

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the second type. The vertexes with some edges exiting the vertex and some edges entering the vertex are called transit vertexes and represent the third type.

In our situations let us suppose that the solved graph will have one source and one sink only. This approach is current in solving such type of problems because we can transform every graph with more sources and sinks into the graph with one source and one sink. It is demonstrated in figure 1.

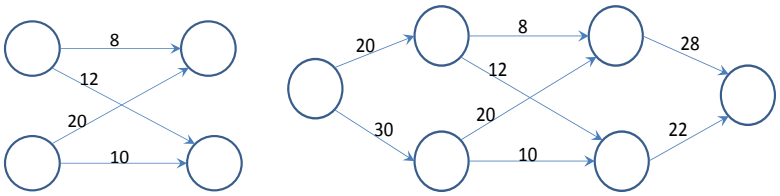

Figure 1 The transformation of the graph with more sources and sinks into the graph with one source and one sink

In the solved graph we know the capacity for every edge $h \in H$. Let us suppose the vertexes have the sufficient capacity. In the case that the vertexes capacity is known, we substitute every original vertex with two new vertexes – an input vertex and an output one. The newly created vertexes will have the unlimited capacity. These newly created vertexes will be connected with a new directed edge leading from the input vertex into the output one. The capacity of the newly created edge will be the same as the original vertex capacity. The real edges entering the original vertex have to be finished in the new input vertex. The real edges exiting the original vertex we lead from the new output vertex. Figure 2 demonstrates an example of such transformation.

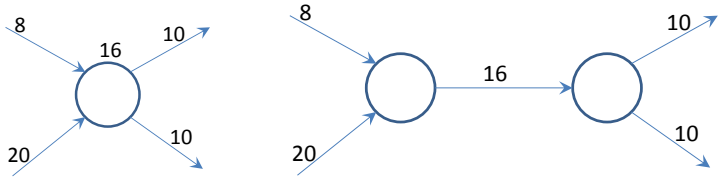

Figure 2 The transformation in the case the vertex with limited capacity

For the net flow the following characteristics has to be held, see Plesník [1]. The flow cannot be negative, the flow has to be integer and the flow cannot exceed the edge capacity. Let us denote with symbols v_0 the source, v_n the sink, v_i the transit vertexes for $i = 1,\ldots,n-1$, $y(h)$ the flow on the edge $h \in H$, H the edge set, $H_{v_i}^-$ the set of edges entering the vertex v_i and $H_{v_i}^+$ the set of edges exiting the vertex v_i. Then it has to hold:

$$\sum_{h \in H_{v_i}^-} y(h) = \sum_{h \in H_{v_i}^+} y(h) \quad \text{for } i = 1,\ldots,n-1, \tag{1}$$

$$\sum_{h \in H_{v_o}^-} y(h) = \sum_{h \in H_{v_n}^+} y(h). \tag{2}$$

The group of equations (1) ensures conservation of flows for the transit vertexes. Equation (2) ensures conservation of flows for the relation the source – the sink. In the past several algorithms for solving this problem were created, for example Ford-Fulkerson algorithm. But this problem can be also solved using mathematical programming, see Cenek-Klima-Janáček [2]; this approach was chosen to solve the problem. Before presenting the mathematical model for solving the maximal flow problem, let us introduce the rules according to which we will denote the vertexes in the solved graphs.

The source will be denoted by the symbol 0, the sink by the symbol *n*. The transit vertexes will be numbered in ascending order from the value 1 to $n-1$. The transit vertex will be labeled by a number only if all edges entering the vertex exit vertexes which have already been numbered. From the chosen denomination of vertexes it results that there will not be any edge which would exit the vertex and which would be labeled by higher number than the number of vertex which the edge enters. It will have a positive influence on the form of constraints used in the model; constraints will be simpler.

In order to create the mathematical model let us denote:

k_{ij} … the capacity of edge exiting the vertex v_i and entering the vertex v_j,

x_{ij} … the flow through the edge exiting the vertex v_i and entering the vertex v_j,

a_{ij} … the matrix element. When the edge exits the vertex v_i and enters the vertex v_j then $a_{ij} = 1$, in the opposite case $a_{ij} = 0$. The matrix elements are constant because they represent the facts not the decisions.

The mathematical model of basic problem (without the supplementary investments) we can formulate in the following way.

$$\max\ f(x) = \sum_{j=1}^{n} a_{0j} x_{0j} \tag{3}$$

subject to

$$a_{ij} x_{ij} \le k_{ij} \text{ for } i = 0,\ldots,n-1 \text{ and } j = 1,\ldots,n. \tag{4}$$

$$\sum_{i=0}^{n-2} a_{ij} x_{ij} = \sum_{i=2}^{n} a_{ji} x_{ji} \text{ for } j = 1,\ldots,n-1. \tag{5}$$

$$\sum_{j=1}^{n} a_{0j} x_{0j} = \sum_{i=0}^{n-1} a_{in} x_{in}, \tag{6}$$

$$x_{ij} \ge 0 \text{ for } i = 0,\ldots,n-1 \text{ and } j = 1,\ldots,n. \tag{7}$$

Function (3) represents the optimization criterion – the maximal flow on the edges exiting the source. The group of constraints (4) ensures that the capacity of each edge will not be exceeded. The group of constraints (5) ensures conservation of flows for the transit vertexes. Condition (6) ensures conservation of flows for the relation the source – the sink. The group of constraints (7) defines domain of definition of variables used in the mathematical model. A different version of the mathematical model solving the same problem is presented in Cenek-Klima-Janáček [2].

3 Main problem and its theoretical solving

Let us suppose that we want to increase the maximal flow in a given net by means of money investments. We know how much money C we can invest (given budget) and further it is known for each edge how much money from our budget we need in order to increase the edge capacity by one unit. This value we denote c_{ij}. In order to model it we have to add a new group of variables y_{ij} into the original model (3) – (7). These variables will model the values by which the edge capacities will be increased. Further, into the model the following constraints have to be added. The first constraint ensures that we do not exceed the given budget by investment activity in order to increase the maximal flow in the net. The constraint is in the form:

$$\sum_{i=0}^{n-1} \sum_{j=1}^{n} a_{ij} c_{ij} y_{ij} \le C. \tag{8}$$

The second group of constraints ensures the logical relations among the variables x_{ij} and y_{ij}. These relations can be reached by the modification of constraints (4). Original constraints (4) have to be modified into the form:

$$a_{ij} x_{ij} \le k_{ij} + a_{ij} y_{ij} \text{ for } i = 0,\ldots,n-1 \text{ and } j = 1,\ldots,n. \tag{9}$$

In addition, the obligatory constraints for the variable y_{ij} have to be added into the model. That means:

$$y_{ij} \in Z_0^+ \text{ for } i = 0,\ldots,n-1 \text{ and } j = 1,\ldots,n. \tag{10}$$

In the final mathematical model the value of function (3) will be maximized subject to constraints (5) – (10).

But in the practise, it often happens that same quantity values, having the character of constants in the model, cannot be exactly given. Let us consider that in our case the values c_{ij} will be uncertain. The fact that the c_{ij} values are not quite precise given can be caused by several reasons. For example, we must realize a lot of factors depend on the individual driver´s behaviour. It is not possible to describe these factors with the precisely given relations, formulas and functional relations etc.

In mathematical modelling this uncertainty of input data can be modelled in more ways. In our approach we will model these uncertain values by means of fuzzy numbers. For our purposes we will employ triangular fuzzy numbers in order to model the uncertainty. There are at least three following reasons to choose triangular fuzzy numbers. During optimisation we have not to do the fuzzy numbers deformation, for example defuzzification (substitution of the interval with only one value). Literature relating the problem offers the verified approaches enabling to work with triangular fuzzy numbers. And finally, computations with triangular fuzzy numbers are not complicated.

Now, let us fuzzyficate the model (3) and (5) – (10). From the original model implies that the fuzzy coefficients will be found in single constraint (8) only. Because of it we formally write it at first. Constraint (8) will be transformed into the form:

$$\sum_{i=0}^{n-1}\sum_{j=1}^{n} a_{ij}\tilde{c}_{ij}y_{ij} \le C \text{ for } i=0,\ldots,n-1 \text{ and } j=1,\ldots,n. \tag{11}$$

The symbols $\tilde{c}_{ij}$ in constraint (11) represent triangular fuzzy numbers substituting the original coefficients values c_{ij}. From the view of fuzzy models categorization - see Teodorović-Vukadinović [3] - it will be the model in which the fuzzy coefficients are found in constraints of the model. Constraint (11) contains the sum of triangular fuzzy numbers on the left side of the inequality. Applying the rules of fuzzy arithmetic - see Teodorović-Vukadinović [3] - original constraints (11) will be transformed into form:

$$\sum_{i=0}^{n-1}\sum_{j=1}^{n} a_{ij}\left[c_{ij}^{3}-\left(c_{ij}^{3}-c_{ij}^{2}\right)q\right]y_{ij} \le C,$$

where:

c_{ij}^{2} … the average value of fuzzy number $\tilde{c}_{ij}$,

c_{ij}^{3} … the right boundary of fuzzy number $\tilde{c}_{ij}$,

q … the level of satisfaction.

Using the approach proposed Teodorović-Vukadinović [3] we must implement a constraint into the modified model which ensures reaching the satisfactory profit. In our case it is the satisfactory level of the maximal flow in the graph. The satisfactory level of the maximal flow will be ensured by adding the following constraint

$$\sum_{j=1}^{n} a_{0j}x_{0j} \ge f_{pessimistic} + q\left(f_{optimistic} - f_{pessimistic}\right),$$

where:

$f_{pessimistic}$ … the pessimistic value of the satisfactory level of the maximal flow,

$f_{optimistic}$ … the optimistic value of the satisfactory level of the maximal flow.

The value $f_{optimistic}$ is calculated from model (3), (5) – (7) and (9) – (11), where in constraint (11) the symbol $\tilde{c}_{ij}$ equals c_{ij}^{1} for $i=0,\ldots,n-1$ and $j=1,\ldots,n$. The symbol c_{ij}^{1} represents the left boundary of fuzzy number $\tilde{c}_{ij}$. The value $f_{pessimistic}$ is calculated from model (3), (5) – (7) and (9) – (11), where in constraint (11) the symbol $\tilde{c}_{ij}$ equals c_{ij}^{3} for $i=0,\ldots,n-1$ and $j=1,\ldots,n$. The symbol c_{ij}^{3} represents the right boundary of fuzzy number $\tilde{c}_{ij}$.

In our modified model we have to change the objective function. Now, the level of satisfaction will be maximized. The fuzzy mathematical model will be in the following form:

$$\max\ f(x,y,q)=q \tag{12}$$

subject to

$$\sum_{j=1}^{n} a_{0j}x_{0j} \geq f_{pessimistic} + q\left(f_{optimistic} - f_{pessimistic}\right), \tag{13}$$

$$a_{ij}x_{ij} \leq k_{ij} + a_{ij}y_{ij} \text{ for } i=0,\ldots,n-1 \text{ and } j=1,\ldots,n, \tag{14}$$

$$\sum_{i=0}^{n-2} a_{ij}x_{ij} = \sum_{i=2}^{n} a_{ji}x_{ji} \text{ for } j=1,\ldots,n-1, \tag{15}$$

$$\sum_{j=1}^{n} a_{0j}x_{0j} = \sum_{i=0}^{n-1} a_{in}x_{in}, \tag{16}$$

$$\sum_{i=0}^{n-1}\sum_{j=1}^{n} a_{ij}\left[c_{ij}^{3} - \left(c_{ij}^{3} - c_{ij}^{2}\right)q\right]y_{ij} \leq C, \tag{17}$$

$$x_{ij} \geq 0 \text{ for } i=0,\ldots,n-1 \text{ and } j=1,\ldots,n, \tag{18}$$

$$y_{ij} \in Z_0^{+} \text{ for } i=0,\ldots,n-1 \text{ and } j=1,\ldots,n, \tag{19}$$

$$q \geq 0. \tag{20}$$

Function (12) represents the optimization criterion – level of satisfaction. The meaning of constraints (14) – (17) was explained in the text above. But this model is not linear in constraint (17). To solve it we will use for example Tanaka-Assai iterative approach, see Tanaka-Assai [4] or Janáček et al [5]. During the iterate Tanaka-Assai approach, the model (3), (13) – (19) is solved. The basic value $q \in (0;1)$ is chosen and the model is solved. When there is an admissible model solution on the basic level of satisfaction $q \in (0;1)$, the value q is increased by the chosen value $\Delta q > 0$ and the model is resolved. When there is not an admissible solution then the calculation is finished. When we do not reach an admissible solution on the basic level of satisfaction q, the value q is decreased by the chosen value $\Delta q > 0$ and the model will be solved again. The value q will be decreased to the time before the first admissible solution is reached.

4 Numerical experiment

Let us consider the following graph (see figure 3) in which we would like to decide about the investment allocation into the single edges so that the maximal flow value will be increased as much as possible.

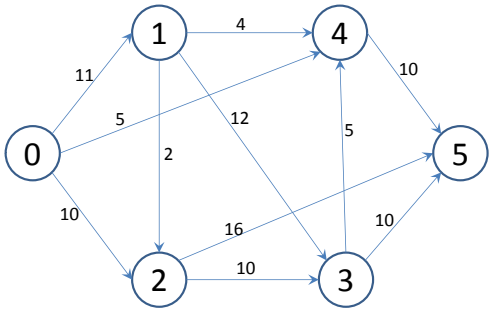

Figure 3 The graph for the numerical experiment

Using the model (1) – (5) we get the maximal flow which is equal to 26.

1	**2**	**3**	**4**	**5**	**1**	**2**	**3**	**4**	**5**
v_0	v_1	1.8	2	2.2	v_2	v_3	0.9	1	1.1
v_0	v_2	2.7	3	3.3	v_2	v_5	5.4	6	6.6
v_0	v_4	4.5	5	5.5	v_3	v_4	6.3	7	7.7
v_1	v_2	3.6	4	4.4	v_3	v_5	0.9	1	1.1
v_1	v_3	7.2	8	8.8	v_4	v_5	8.1	9	9.9
v_1	v_4	5.4	6	6.6	-	-	-	-	-

Table 1 The input data for the numerical experiment

Now let us suppose the situation that we have the budget equal to 100 units of money which can be invested in order to increase the edge capacities. The costs for edge capacity increase by 1 unit of money are summarized in table 1.

The first column defines the initial vertex of the edge, the second one the terminal vertex of the edge. The third one presents the minimal costs that are necessary in order to increase the edge capacity by 1 unit (the left boundary of fuzzy number), the fourth one the average costs for increasing the edge capacity by 1 unit and the last one the maximal costs for increasing the edge capacity by 1 unit (the right boundary of fuzzy number). To be able to solve the model (1), (11) – (17) we have to calculate the values $f_{optimistic}$ and $f_{pessimistic}$ at first. They will be calculated by solving the model (1), (3) – (5) and (7) – (9) using the left and right boundaries of fuzzy numbers. For our example it was calculated that $f_{optimistic} = 55$ and $f_{pessimistic} = 51$. Now we apply the Tanaka-Assai approach on the model (1), (11) – (17). Increasing the value q we found out that the last admissible solution was obtained on the level of satisfaction $q = 0{,}25$. For this value the maximal flow equal to 52 was reached. The investment money should be allocated on the single edges as follows (see table 2).

From	To		From	To	
v_0	v_1	14	v_2	v_3	6
v_0	v_2	57	v_3	v_5	16

Table 2 The investment money should be allocated on these edges

5 Conclusion

The presented article deals with the problem of maximal flow increasing in the graph; the increase is caused by additional investments in order to increase edge capacities. The maximal flow problem was defined many years ago and its solving is already known for a long time. The contribution of our paper can be seen in two levels. The first one is that the article includes the modification of the original problem enabling increase of edge capacities caused by additional investments. The second one is that information about how much money we need to increase the edge capacity by 1 unit is not considered to be precisely given. That is why the article includes also the approach using the fuzzy number optimisation. The proposed fuzzy model was solved by the Tanaka-Assai method.

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A note on NIG-Levy process in asset price modeling: case of Estonian companies

Dean Teneng[1]

Abstract. The purpose of this note is two folds. First, to correct mistakes relating to terminology and analysis of results in Teneng [7]. Second, to extend results by showing returns of companies trading on Tallinn Stock Exchange between 01 January 2008 and 01 January 2012 cannot be modeled by NIG distribution; both in cases where closing prices can and cannot be modeled by NIG distribution. Thus, the NIG-Levy process cannot be used to forecast the future prices of these assets.

Keywords: NIG, Levy process, Jumps, forecasting, goodness of fits.

JEL Classification: C51, C44
AMS Classification: 60G50, 60G15

1 Introduction

Of late, many processes have been suggested to replicate price trajectories. Famous among is Brownian motion, known to have serious modeling limitations like light tails, inability to effectively capture and model jumps, stochastic volatility and a host of others [1, 5].

Levy process based models, of which Brownian motion is a special case, seek to eliminate shortcomings of Brownian motion based models. They clearly can distinguish between large and small jumps, and do not necessarily have continuous paths [4]. Two major classes of Levy processes are of interest here: Jump diffusion and infinite activity Levy processes. For jump diffusion cases, jumps are considered rare events and in any finite interval, there are only finitely many jumps. Opposite case applies to infinite activity models i.e. in any finite interval, there are infinitely many jumps. Infinite activity models are our focus, and particularly the NIG case. NIG case was implemented with daily closing data of some companies trading on Tallinn Stock Exchange in [7] and extended for their returns in this note.

The Black-Scholes model makes use of the exponential of Brownian motion with drift [1, 4, 5]. This ensures its probability distribution is lognormal. This is especially interesting when considering log returns which are not the case in this note. I work simply with one period returns. One period returns have little difference with log returns but for long time scales [3]. As well, lognormal model has continuous paths but cannot model jumps; one of the reasons why Levy process based models are attractive here.

This note is intended as a complement to [7]. The theory and model selection procedure are the same, but applied to return data here. I fit one period returns of companies trading on the Tallinn Stock Exchange between 01 January 2008 and 01 January 2012 with the NIG distribution and select good models following the procedure in [7]. I observe that none of these returns can be appropriately modeled with NIG-distribution; a pre-condition for forecasting with NIG-Levy process [4]. Hence, NIG-Levy process cannot be used to forecast the closing prices of any of these assets.

Path properties of Levy processes are studied in [4]. NIG-Levy process has been demonstrated to model and forecast closing prices of indexes [6]. This clearly confirms that classes of asset returns exhibit unique characteristics even though most exhibit general stylized facts [2, 3]. This is particularly important as most practitioners classify stock market data as financial data and apply general set of modeling criteria without considering the uniqueness of each asset class. Failing to model and forecast stock prices is a clear indication that the returns of stocks (companies) exhibit unique stylized facts which can differentiate them from returns of other asset classes.

Section two recaps the definition of NIG- Levy process and outlines model selection strategy. Corrections to [7] and results of modeling returns with NIG-distribution are the subject of section 3.

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2 NIG Levy process and model selection strategy

Definition 1 (Levy process). *A real valued stochastic process $X_t = X(t,\omega), t \geq 0, \omega \epsilon \Omega$ defined on a probability space (Ω, F, P) is called a one dimensional Levy process if it satisfies the conditions: $X_0 = 0$ a.s., X has independent and stationary increments with stochastically continuous paths, and X has paths continuous from the right with left-sided limits. In fact, for any time t > s, the distribution of the increments $(X_t - X_s)$ depends only on the length of the interval (t-s) and $(X_t - X_s)$ is independent of $(X_u, u \leq s)$*. Standard and linear Brownian motion, Poisson and compound Poisson processes are examples of Levy processes [4]

Definition 2 (NIG-Levy process). *NIG-Levy process with parameters $\alpha, \beta, \mu, \delta$ and condition $-\alpha < \beta < \alpha$ denoted $NIG(\alpha, \beta, \delta, \mu)$ can be defined as follows:*
Consider a bivariate Brownian motion (u_t, v_t) starting at point $(u, 0)$ and having constant drift vector (β, γ) with $\gamma > 0$ and let z denote the time at which v_t hits the line $v = \delta > 0$ for the first time (u_t, v_t are assumed independent). Then letting $\alpha = \sqrt{\beta^2 + \gamma^2}$, the law of u_z is $NIG(\alpha, \beta, \delta, \mu)$ [4]. This distribution has probability density of the form

$$f_{NIG}(x; \alpha, \beta, \delta, \mu) = \frac{\alpha\delta}{\pi} e^{\left(\delta(\alpha^2-\beta^2)^{1/2} - \beta(x-\mu)\right)} \frac{K_1\left(\alpha(\delta^2 + (x-\mu)^2)^{1/2}\right)}{(\delta^2 - (x-\mu)^2)^{1/2}} \quad (1)$$

where $K_1(x) = \frac{x}{4}\int_0^\infty e^{\left(t + \frac{y^2}{4t}\right)} t^{-2}\, dt$ is a modified Bessel function of the third kind. $\delta > 0$ is scale, $\beta \geq 0$ is symmetry, μ is location and $\alpha > 0$ is tail heaviness.

Modeling with NIG-distribution entails estimating the parameters $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}, \boldsymbol{\mu})$ of the NIG-distribution and performing classical goodness of fits test. This is just density modeling i.e. checking whether empirical and theoretical distribution functions are comparable using any known classical goodness of fits test like Kolmogorov-Smirnov. Our interest in this note is on returns. This is particular because if returns can be modeled with NIG-distribution, then we can make accurate forecast of future prices by drawing return values from the probability distribution of returns. These drawn return values can be added to the stock price. With these, we can compare forecasted price with known price and check RMSE values both in-sample and out of sample. This will equip us with other forecasting method. Our results can then be compared to standard Random Walk. Below, we outline model selection strategy.

Käärik and Umbleja proposed method for selecting best models

1. choose a suitable class of distributions (using general or prior information about the specific data) ;
2. estimate the parameters (by finding maximum likelihoods);
3. estimate goodness of fit;
   a) visual estimation,
   b) classical goodness-of-fit tests (Kolmogorov-Smirnov, chi-squared with equiprobable classes),
   c) probability or quantile-quantile plots.

3 Corrections to [7] and returns modeling

3.1 Corrections

In [7], the closing prices of assets trading on the Tallinn stock exchange from 01 January 2008 to 01 January 2012 were fitted with the normal inverse Gaussian (NIG) distribution[2]. Interpretation of results concluded Baltika and Ekpress Grupp were suitable candidates for NIG-Levy asset model. Unfortunately, there were mistakes in terminology and analysis. First correction deals with the definition of general Levy process i.e. the independence criteria (pg. 2). It is suppose to read

$$for\ all\ s, t \geq 0, X_{s+t} - X_s\ is\ independent\ of\ X_u, u \leq t \quad (1)$$

i.e. independent increments. Second correction is related to analysis of data (Pg. 4). From Table 1 in this note, we can clearly see that Ekpress Grupp has a very small Kolmogorov-Smirnov (KS) test p-value (0.012). This means there is no significance at five percent level and we reject it. As well, the graphs were not very clear and I have included an updated version (Figure 1) to display correctly the goodness of fits. Thus, following model selection procedure outlined above, only Baltika can have its closing prices modeled with NIG-distribution, despite my being able to estimate NIG-distribution parameters for it and other companies.

[2] See [6, 7] for exposition on NIG distribution.

Company	alpha	beta	dealt	mu	skew	kurtosis	KS p	KS d
Arco Vara	468.9	468.86	0.03	0.02	0.38	-1.53	$<10^{-5}$	0.23
Baltika	7.06	6.62	0.22	0.52	1.67	-1.53	0.06	0.06
Ekpress Grupp	2.68	2.15	0.49	0.85	1.70	2.53	0.012	0.07
Harju Elekter	3.2	-2.07	0.72	2.95	-0.82	-0.05	0.0003	0.09

Table 1 Estimated NIG Parameters, Skews, Kurtoses, KS test results for NIG models (Daily closing prices)

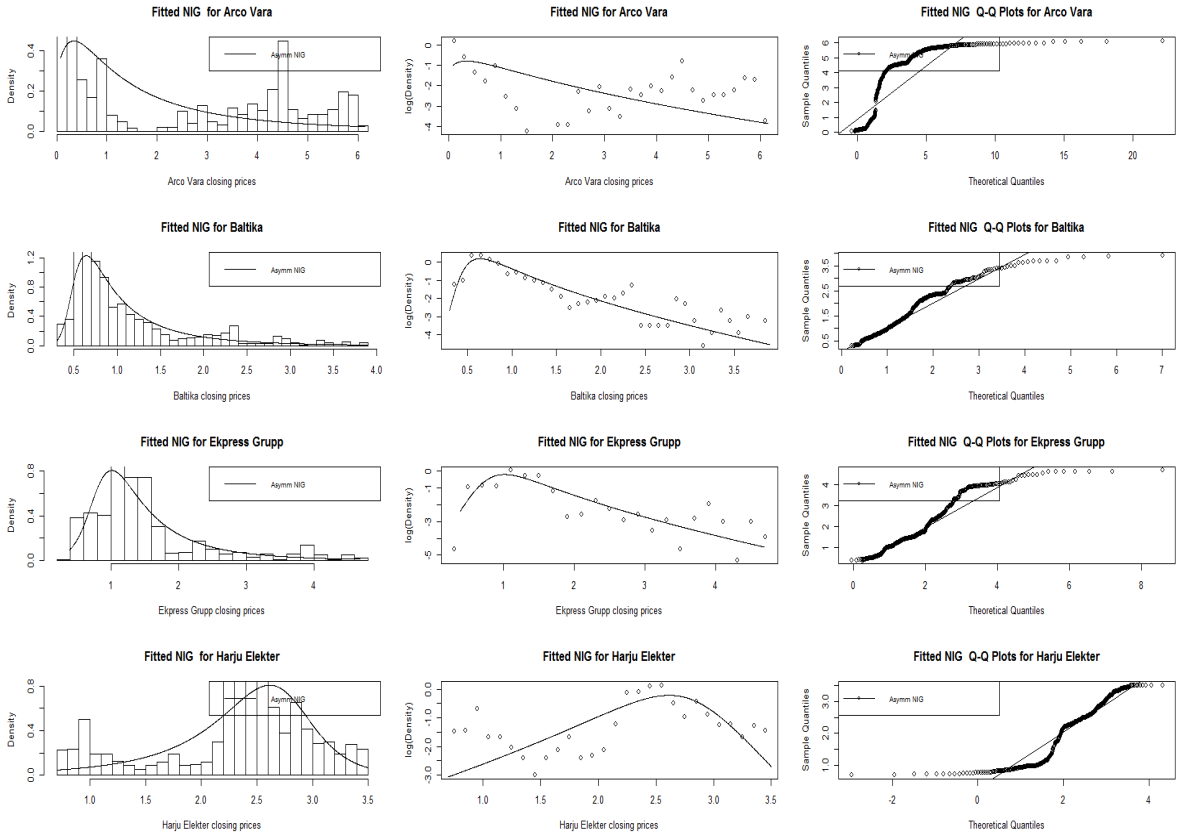

Figure 1 Fitted NIG density, log densities and Q-Q plots for Baltika, Arco Vara, Harju Elekter and Ekpress Grupp (Daily closing prices)

3.2 Modeling returns

For closing prices to be forecast-able with NIG-Levy process, the returns should be NIG distributed [4]. In this note, one period returns are fitted with NIG-distribution and classical goodness of fits test performed to validate the model. I start by looking at skews and kurtosis which clearly suggest using a heavy tail distribution. Then estimating NIG-distribution parameters from return data (i.e. alpha, beta, delta, mu). Having these estimated parameters seem to suggest distribution is a candidate for good model. Looking at plots (Figure 2) especially log density plots, they are not good fits. Classical goodness of fits test considering five percent level of significance also confirm bad model. Results of these are displayed in Table 2 and plots of these are displayed in Figure 2 (only four of them could be plotted due to poor estimates of parameters). Following procedure outlined above, I reject all return models. This is despite result above that closing prices of Baltika can be modeled with NIG-distribution. Hence, these companies cannot have their returns modeled with NIG-distribution. Not being able to have the returns modeled with NIG-distribution means we cannot predict future asset price values with NIG-Levy process [4]. In other words, the precondition of forecasting with NIG-Levy process (i.e. asset returns being

NIG-distributed) is not met by any of these companies trading on Tallinn Stock Exchange between 01 January 2008 and 01 January 2012.

Company	alpha	beta	delta	mu	skew	kurtosis	KS p	KS d
Arco Vara	0.96	0.756	0.015	-0.0017	31.47	994.45	$<10^{-5}$	0.1433
Baltika	20.923	0.858	0.032	-0.003	-0.06	2.07	0.002	0.082
Ekpress Grupp	15.59	0.7032	0.024	-0.002	0.253	2.308	0.0002	0.095
Harju Elekter	14.32	0.774	0.015	-0.0008	0.247	3.098	$<10^{-5}$	0.128
Viisnurk	6.001	0.227	0.011	$7.5e^{-05}$	-0.168	4.05	$<10^{-5}$	0.18
Olympic Entertainment Grupp	19.58	1.565	0.238	-0.003	0.017	4.52	0.006	0.0762
Silvano Fashion Grupp	9.203	0.516	0.0189	-0.0006	0.195	4.19	$<10^{-5}$	0.1088
Tallink	22.19	2.27	0.02	-0.0022	0.817	5.7	$<10^{-5}$	0.1385
Tallinna Kaubamaja	21.17	-0.384	0.007	-0.0004	-0.844	10.42	$<10^{-5}$	0.1108
Trigon	1.097	-1.097	$5.5e^{-12}$	$6.03e^{-21}$	0.28	7.45	$<10^{-5}$	0.3145
Nordecon	24.34	2.48	0.025	-0.0037	0.13	2.053	0.002	0.0841

Table 2 Estimated NIG Parameters, Skews, Kurtoses, KS test results for NIG distribution models (returns)

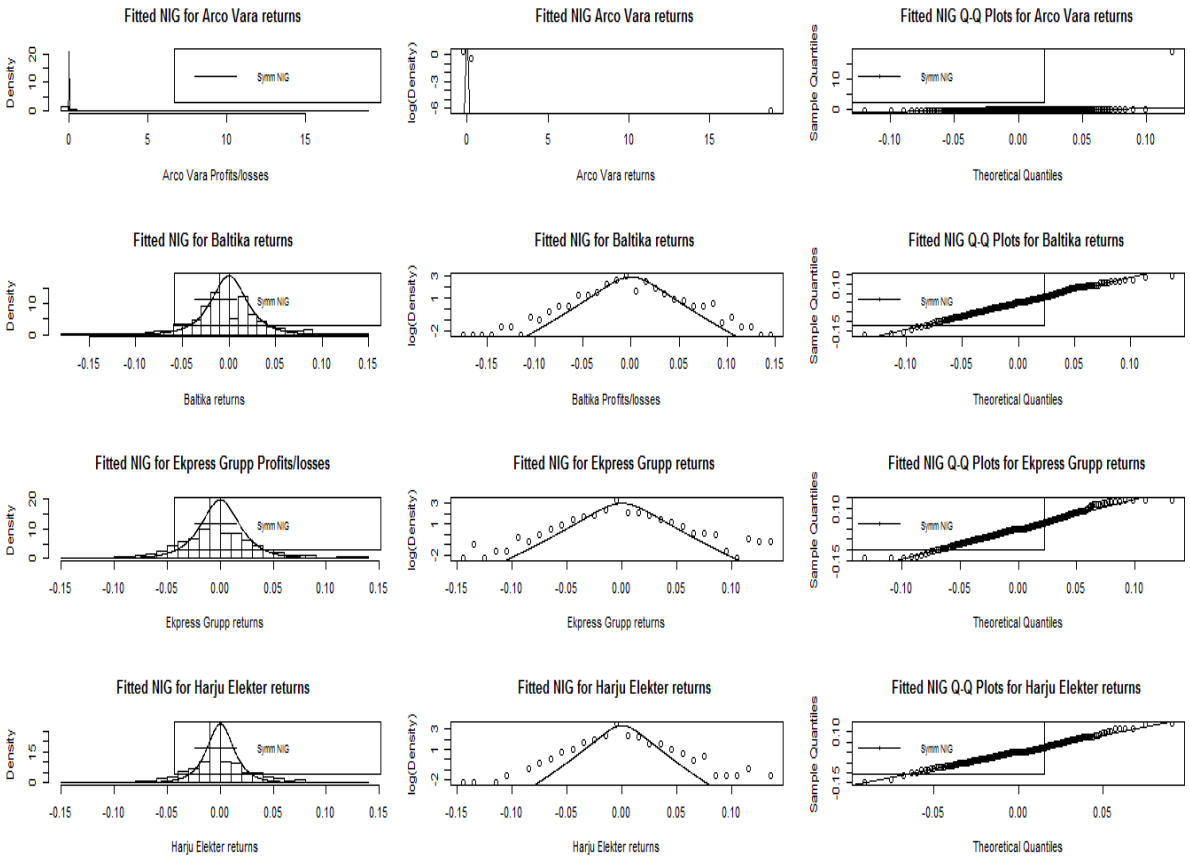

Figure 2 Fitted NIG density, log densities and Q-Q plots for Baltika, Arco Vara, Harju Elekter and Ekpress Grupp (returns)

4 Conclusion

Results of Teneng [7] have been updated to conclude that closing prices of Baltika (company trading on the Tallinn Stock Exchange between 01 January 2008 and 01 January 2012) can be modeled with NIG distribution, but its returns cannot. Other companies considered can have neither their closing prices nor their returns modeled by NIG-distribution. These lead to the conclusion that none of these companies can have their future prices forecasted with NIG-Levy process. This clearly confirms that classes of asset returns exhibit unique characteristics even though most exhibit general stylized facts. This is particularly important as most practitioners classify stock market data as financial data and apply general set of modeling criteria without considering the uniqueness of each asset class. Failing to model and forecast stock prices is a clear indication that the returns of stocks (companies) exhibit unique stylized facts which can differentiate them from returns of other asset classes.

Acknowledgements

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Outperforming the naïve Random Walk forecast of foreign exchange daily closing prices using Variance Gamma and normal inverse Gaussian Levy processes

Dean Teneng[1]

Abstract. This work demonstrates that forecast of foreign exchange (FX) daily closing prices using the normal inverse Gaussian (NIG) and Variance Gamma (VG) Levy processes outperform the naïve Random Walk model. We use the open software R to estimate NIG and VG distribution parameters and perform several classical goodness–of -fits test to select best models. Seven currency pairs can be forecasted by both Levy processes: TND/GBP, EGP/EUR, EUR/GBP, EUR/JPY, JOD/JPY, USD/GBP, and XAU/USD, while USD/JPY and QAR/JPY can be forecasted with the VG process only. RMSE values show that NIG and VG forecast are comparable, and both outperform the naïve Random Walk out of sample. Appended R-codes are original.

Keywords: Levy process, NIG, VG, forecasting, goodness of fits, foreign exchange.

JEL Classification: C13, C15, C46, C53
AMS Classification: 60G50, 60G15

1 Introduction

There are many models purporting to describe the evolution of exchange rates as a function of different macro-economic fundamentals like prices, money, interest rates, productivity differentials, government debts, terms of trade, net foreign assets, etc. These models are typically limited by the fact that all macro-economic fundamentals cannot be included; thus such models are usually a simplification of the economy [1]. Fundamentals are capable of explaining exchange rates, but hardly are good in forecasting. Also, factors relating to the behavior of market participants are usually neglected; something captured, modeled and replicated in the framework of standard asset pricing model in [8]. It must be pointed out that the famous Meese-Rogoff [3,10] paper advocating the Random Walk model as the best model for exchange rate forecasting was never really about ex-ante forecasting in its true sense i.e. using time *t* information to forecast exchange rates at time *t+1*. In fact, Meese-Rogoff's regressions were more about concurrent explanations, with the only "forecasting" element being their reliance on ex-ante data to estimate equation parameters [3, 10].

Micro based models in contrast, focus on the processes through which information affect prices. First, transaction flows contain information relevant to fundamentals and second, market markers adjust prices based on information in such a way that these are reflected in spot prices [1]. Our approach is inspired by Charles Engel and Kenneth West (2004) [3] who suggests an approach based on asset prices. They advocate that if prices are I(1), the exchange rate will follow a process arbitrarily close to a Random Walk (RW). This is because I(1) processes can be split into RW and stationary processes. The class of Levy processes we use here can be approximated by RW's. These Levy processes have dynamics capable of capturing FX dynamics arising from exceptional circumstances.

Modeling and forecasting daily prices with NIG-Levy process has been demonstrated for index prices in [14] and for FX in [15]. Considering that FX prices often exhibit unique qualities not typical of other assets prices (see [9],[11] for detailed exposition), we extend results of [15] by demonstrating that seven currency pairs can be forecasted by both NIG and VG processes: TND/GBP, EGP/EUR, EUR/GBP, EUR/JPY, JOD/JPY, USD/GBP, and XAU/USD, while USD/JPY and QAR/JPY can be forecasted with the VG process only. We also report that forecast of daily closing prices of USD/GBP for a sixty day period using NIG and VG processes outperform the naïve Random Walk approach looking at RMSE values.

The section below reviews the construction of NIG and VG processes starting with the definition of a general Levy process, time changes and approximation of Levy processes by RW's. Next, we propose an implement a forecasting mechanism. Appended R codes are original.

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2 NIG and VG processes

Definition 1 (Levy process). *A real valued stochastic process $X_t = X(t,\omega), t \geq 0, \omega \epsilon \Omega$ defined on a probability space (Ω, F, P) is called a one dimensional Levy process if it satisfies the conditions: $X_0 = 0$ a.s., X has independent and stationary increments with stochastically continuous paths, and X has paths continuous from the right with left-sided limits. In fact, for any time t > s, the distribution of the increments $(X_t - X_s)$ depends only on the length of the interval (t-s) and $(X_t - X_s)$ is independent of $(X_u, u \leq s)$*. Standard and linear Brownian motion, Poisson and compound Poisson processes are examples of Levy processes [6,13].

Definition 2 (Infinite divisibility). *A real value random variable X is infinitely divisible if for each $n \in N$, there exist a sequence of i.i.d. random variables $X_{1,n}, X_{2,n}, \ldots, X_{n,n}$ such that X is distributed as $X_{1,n} + X_{2,n} + \ldots + X_{n,n}$.* Complete characterization of an infinitely divisible distribution in terms of its characteristic exponent is given by the Levy-Khintchine theorem defined below.

Theorem 1 (Levy-Khintchine theorem). *A random variable X_t is infinitely divisible if and only if there exist a triplet (b, c, ν) such that the characteristic exponent of X_t is of form $Ee^{iuX_t} = e^{t\psi(u)}$ where*

$$\psi(u) = ibu - \frac{(uc)^2}{2} + \int_{-\infty}^{+\infty} \left(e^{iux} - 1 - iux1_{\{|x|\leq 1\}}\right) \nu(dx) \tag{1}$$

$b, c \in \mathcal{R}$ and b is a drift term, c a diffusion term and ν a positive measure on $\mathcal{R}/\{0\}$ such that $\int_{\mathcal{R}/\{0\}} (1 \wedge |x|^2)\nu(dx) < \infty$. $\psi(u)$describes the path properties of a Levy process [13]. A well known result of Levy processes is that any Levy process can be represented as an independent sum of Brownian motion ($\mu t + \sigma B_t$) and a compound Poisson-like process [12]. Without dwelling on the technicalities and restriction imposed on $\psi(u)$, we turn our focus to random walk approximation before stating another important theorem necessary for understanding time changes.

Random Walk approximation: Let us consider the Levy process X_t and apply the Levy-Khintchine formula above. Suppose $c = 0$ in $\psi(u)$ and we have an infinite Levy measure ν. Fixing the time domain $[0, T), n \geq 1$ and letting $h = \frac{T}{n}$, we generate increments $\Delta_i^h X = X(jh) - X\big((j-1)h\big)$ as independently identical random variables [8] with distribution $P_h(.) = P(X(h) \epsilon .),\ j = 1, \ldots, n-1$. Let

$$X^h(t) = \begin{cases} 0, \ if\ 0 \leq t < h, \\ \Delta_1^h X + \cdots + \Delta_j^h X, \ if\ jh \leq t < (j+1)h, \end{cases}$$

then the process $\{X^h(t)\}_{0\leq t<T}$ is a RW approximation to $\{X(t)\}_{0\leq t<T}$. This approximation illuminates the connection with I(1) processes.

Theorem 2. *Let X_t for $t \geq 0$ be a Levy process with characteristic exponent ψ and let τ_s, $s \geq 0$ be an independent subordinator[2] with characteristic exponent Φ. Then the process $Y_s = X_{\tau_s}$ is again a Levy process with characteristic exponent $\Phi o \psi$*. This theorem is the basis of definition and use of time changes below[7].

Definition 3 (Time Change). *Consider the Levy process $X(t), t \geq 0$ and let T(t) be a subordinator. Then the process described by $X(t) = Z(T(t))$ is called a subordinated process, with the process Z called a directing process.* This directing process usually provides the link between business time and real time i.e. at real time $t \geq 0$, $T(t)$ of business time units passed and the value of our asset is positioned at $Z\big(T(t)\big)$. The directing process we use is Brownian motion. This is because Levy processes are semi-martingales and any semi-martingale can be written as a time changed Brownian motion. There are also many directing processes that can be used. A detailed analysis is given in [7].

Deviating slightly, the no-arbitrage assumption implies the existence of a probability measure under which the discounted stock prices are martingales. This means stock prices are martingales under real probability measure; implying $Ln(S(t)) = Z(T(t))$. This clearly represents how asset prices respond to information arrival [6]. Some days, very little good or bad news is released causing trading to typically slow and causing prices to barely fluctuate. Other days, trading is brisk and price evolution accelerates as traders adjust their expectations to arrival of new information [2,6]. Thus, the price process is seen as instantaneously and continuously adjusting to exogenous demand and supply shocks [7]. This accounts for some of the reasons why uncertainty in the economy is often represented by a filtered probability space $(\Omega, F, F_t, \mathrm{P})$ where F_t is the filtration of information available at time *t* and *P* is the real probability measure [2, 6]. There is also a net flow of information from long to short time scales and the behavior of long-term traders influences those of short-term traders. This is the basis for [8]. Let

[2] A subordinator is a strictly non-decreasing Levy process.

us proceed with defining the inverse Gaussian distribution and Gamma process; basis to define and understand the NIG and VG processes.

Definition 4 (Inverse Gaussian distribution). *A random variable X has inverse Gaussian (IG) distribution with parameters υ and λ if its density is of form*

$f(x;\upsilon,\lambda)=\sqrt{\frac{\lambda}{2\pi x^3}}e^{-\frac{\lambda(x-\upsilon)^2}{2x\upsilon^2}}$ *and $\lambda > 0$ is shape parameter, $\upsilon > 0$ is its mean while $x > 0$.*

Definition 5 (Gamma process). *The density of a gamma (G) process*[3] *with mean rate t and variance υt is given by*

$f(g)=\frac{g^{(\frac{t}{\upsilon}-1)}e^{\left(-\frac{g}{\upsilon}\right)}}{\Gamma\left(\frac{t}{\upsilon}\right)\upsilon^{\frac{t}{\upsilon}}}$ *where $\Gamma(x)$ denotes the classical gamma function.*

Armed with definitions 1-5 and theorems 1 and 2, we are now ready to fully define NIG and VG processes.

NIG process: Consider a Brownian motion *Z* with drift θ and volatility σ. If we consider a random time change which follows $IG(t;\upsilon,1)$, NIG process can be defined as

$$X_{NIG}(t;\sigma,\upsilon,\theta)=\theta IG(t;\upsilon,1)+\sigma Z\big(IG(t;\upsilon,1)\big). \tag{2}$$

$Ee^{iuX_{NIG}(t)}=e^{\left(-t\sigma\sqrt{\left(\frac{\upsilon}{\sigma}\right)^2+\frac{\theta^2}{\sigma^4}-\left(\frac{\theta}{\sigma^2}+iu\right)^2}-\frac{\upsilon}{\sigma}\right)}$. Using the substitutions $\beta=\frac{\theta}{\upsilon^2},\sigma=\delta,\alpha^2=\left(\frac{\upsilon}{\sigma}\right)^2+\frac{\theta^2}{\sigma^4}$, the NIG process can be rewritten as

$$X_{NIG}(t;\alpha,\beta,\delta,\upsilon=0)=\beta\delta^2 IG\big(t;\delta\sqrt{\alpha^2-\beta^2},1\big)+\delta Z\left(IG\big(t;\delta\sqrt{\alpha^2-\beta^2},1\big)\right), \tag{3}$$

more suitable for computations. With (3), we get the mean = $\frac{\beta\delta}{\sqrt{\alpha^2-\beta^2}}$, variance = $\alpha^2\delta(\alpha^2-\beta^2)^{\frac{-3}{2}}$,

skew = $\frac{3\beta(\alpha\delta)^{-1}}{4\sqrt{\alpha^2-\beta^2}}$ and kurtosis = $3\left(1+\frac{\alpha^2+4\beta^2}{\delta\alpha^2\sqrt{\alpha^2-\beta^2}}\right)$.

VG process: Alternatively, if we consider a random time change which is a Gamma process with mean rate unity and variance rate , we get VG process as

$$X_{VG}(t;\sigma,\upsilon,\theta)=\theta G(t;\upsilon,1)+\sigma Z\big(G(t;\upsilon,1)\big). \tag{4}$$

$Ee^{iuX_{VG}(t)}=\left(\frac{1}{1-i\theta\upsilon u+\upsilon\frac{(\upsilon\sigma)^2}{2}}\right)^{\frac{1}{\upsilon}}$ with mean θ, variance = $\theta\upsilon^2+\sigma^2$, skew = $2(\upsilon\theta)^2\theta+3\theta\upsilon(\sigma)^2$, and

kurtosis = $3variance^2+12\sigma+3\upsilon\sigma^4$.

It is easy to see that with proper parameters (σ,υ,θ), the processes 3 and 4 propagate with time. This means if we know (σ,υ,θ) at time *t*, we can let the process propagate to time *t+1*, providing us with values out of sample. This model intrinsically assumes there will be no changes to the values of estimated parameters before time *t+1* or beyond. Using the same estimated parameters for long periods i.e. beyond a day in our case will not reflect the arrival of new information, neither ongoing changes in the market and will put traders at risk. We consider daily closing prices only and forecast of next day closing prices on a rolling basis. Data used for parameter estimation are for daily closing prices between 29/11/2007 and 22/07/2011; covering parts of the recent financial crises. In sample forecast is of little appeal to us. We use data from 29/11/2007 to 28/04/2011 (900 data points) as our training sample and the remaining (60 data points) as test sample. Below, we outline our procedure for out of sample forecast.

3 Model selection, analysis and forecasting

We follow a similar procedure as in [14, 15] and it is outlined in simple terms as follows:
1) Select distribution based on known characteristics of data. 2) Estimate parameters of distribution. 3) Carry out classical goodness of fits test i.e. Kolmogorov-Smirnov and Anderson Darling. 4) Select good models i.e. those passing classical goodness of fits test. 5) Develop forecasting codes. 6) Check RMSE values.

[3] A gamma process is a random process with independent gamma distributed increments.

Table A.2 outlines the summary statistics of our return data. With these, it is clear that our distribution must be heavy tailed, and just like the NIG, the VG has semi heavy tails. Parameters are estimated using nigFit,vgFit functions of the ghyp and VarianceGamma packages in R-version 2.15.3 respectively. These can be varied so as to get better fit. Tables A.1 and A.3 display estimated parameters, Kolmogorov-Smirnov and Anderson-Darling test results for VG and NIG processes respectively. Looking at the P-values, there is clear significance at the 5% level; quite intriguing as these two distributions are quite different. Forecast is of 60 values are done on a rolling basis. RMSE[4] values are then calculated. Using our basic code, results for NIG and VG forecast respectively are

```
> RiMSE(60,100)
[1] 0.003184383
> RiMSE(60,1000)
[1] 0.003246613
> RiMSE(60,10000)
[1] 0.003235002
```

```
> RiMSE(60,100)
[1] 0.00330961
> RiMSE(60,1000)
[1] 0.003242958
> RiMSE(60,10000)
[1] 0.003230029
```

Random walk forecast is estimated using the "forecast" package of the newly released R-Version 3.0.1 in RStudio interface.Simple code for RW forecast is

```
> USD.GBP1 <- read.delim("C:/Users/Rehoboth/Desktop/Exchange rates/USD-GBP1.txt")
>   View(USD.GBP1)
> library(forecast)
> plose1<-USD.GBP1$CLOSE
> dlose1<-ts(plose1)
> fit1<-rwf(dlose1[1:900],h=60)
> accuracy(fit1,dlose1[901:960])
                      ME       RMSE          MAE       MPE      MAPE      MASE
Training set 0.0001263626 0.00497182 0.003595106 2.032913 57.925703 1.000000
Test set     0.0177733333 0.01891583 0.017773333 2.873276  2.873276 4.943758
                   ACF1
Training set 0.05147218
Test set             NA
```

It is clear from the RMSE values above that both NIG and VG forecast outperform the naive RW. It is also crystal clear that forecast from NIG and VG Levy processes are comparable up to four decimal places. This results are a clear indication on the inefficiency of RW model as compared to more robust estimates.

FX	USD/ JPY	EGP/ EUR	EUR/ GBP	EUR/ JPY	JOD/ JPY	QAR/ GBP	TND/ GBP	USD/ GBP	XAU/ USD
vgC	0.0003	-3.11e-5	1.83e-5	0.002799	-4.149e-4	-0.0011	-0.001	-0.001	0.0052
sigma	0.0069	7.941e-3	5.98e-3	0.008827	7.846e-3	0.0074	0.0074	0.0074	0.0109
theta	0.0001	5.152e-5	-1.13e-4	-0.00298	8.918e-5	0.0014	0.0015	0.0016	-0.004
nu	1.1803	.2997	.1586	0.2743	.2500	0.3015	0.3311	0.30	0.1256
KS D	0.035	0.04	0.039	0.028	0.042	0.033	0.037	0.048	0.048
KS P	0.572	0.3998	0.4318	0.8275	0.3404	0.6469	0.4997	0.199	0.1991
AD P	0.4448	0.3166	0.3891	0.4804	0.30697	0.5066	0.424	0.244	0.191

Table A.1 Estimated VG parameters, Kolmogorov-Smirnov (KS) and Anderson-Darling (AD) test values

FX	Mean	Variance	Skew	Kurtosis
EGP/EUR	-4.29e-5	3.865e-6	-0.431	11.05
EUR/GBP	-0.0002	8.20e-5	-0.262	1.1422
EUR/JPY	-0.0028	7.75e-5	1.204	15.10
JOD/JPY	0.00048	0.00012	0.1752	5.491
QAR/GBP	-1.62e-6	4.034e-7	0.219	7.141
TND/GBP	0.0003	5.875e-5	0.5061	2.604
USD/GBP	0.000297	5.86e-5	0.5031	2.6171
XAU/USD	0.000789	0.000127	-0.31995	1.9890
USD/JPY	-0.00013	4.189e-5	0.274	2.715

Table A.2 Summary statistics of FX returns

Code fragment for forecasting:

This code is used after first estimating relevant parameters by maximum likelihoods. *s0* is the last closing price while m is the number of prices we average. We set m=10,000 for all the computations above.

[4] RiMSE below is just RMSE. We use this in our program code to calculate RMSE values.

VG forecast: library(VarianceGamma); library(fBasics)
forecaster1=function(s0,m){ matah<- rep(0,m); for (i in 1:m){ matah[i]<-(1+rvg(1, vgC = 0, sigma = 1, theta = 0, nu = 1,param = c(vgC,sigma,theta,nu)))*s0}; return(mean(matah))}
NIG forecast [14, 15]: forecaster=function(alpha,beta,delta,mu,s0,m){ matah<- rep(0,m); for (i in 1:m){ matah[i]<-(1+rnig(1,alpha,beta,delta,mu))*s0}; return(mean(matah))}

FX	TND/ GBP	EGP/ EUR	EUR/ GBP	EUR/ JPY	JOD/ JPY	USD/ GBP	XAU/ USD
alpha	151.045	205.1585	201.7434	173.3203	124.4768	*151.2416*	85.0122
beta	16.631	-0.9756	0.2773	-30.845	-0.805378	16.2792	-8.948
delta	0.00865	0.0130	0.007583	0.0136	0.00831	0.00865	0.01088
mu	-0.00066	7.88e-05	-0.0001	0.00227	-0.000271	-0.00064	0.00195
KS D	0.024	0.0311	0.0361	0.0361	0.019	0.0371	0.0431
KS P	0.9351	0.7214	0.5348	0.5348	0.9936	0.4991	0.3124
ADP	0.62032	0.36019	0.53921	0.48450	0.62478	0.40622	0.36348

Table A.3 Estimated NIG parameters, Kolmogorov-Smirnov (KS) and Anderson-Darling (AD) test values

4 Conclusion

We have demonstrated that seven currency pairs: TND/GBP, EGP/EUR, EUR/GBP, EUR/JPY, JOD/JPY, USD/GBP, and XAU/USD, can be forecasted with both the NIG and VG processes, while USD/JPY and QAR/JPY can be forecasted with the VG process only. We have also demonstrated that NIG and VG forecast beat the naïve Random Walk approach. Further research will concentrate on exploring and forecasting with other Levy processes and comparing these with traditional forecasts results.

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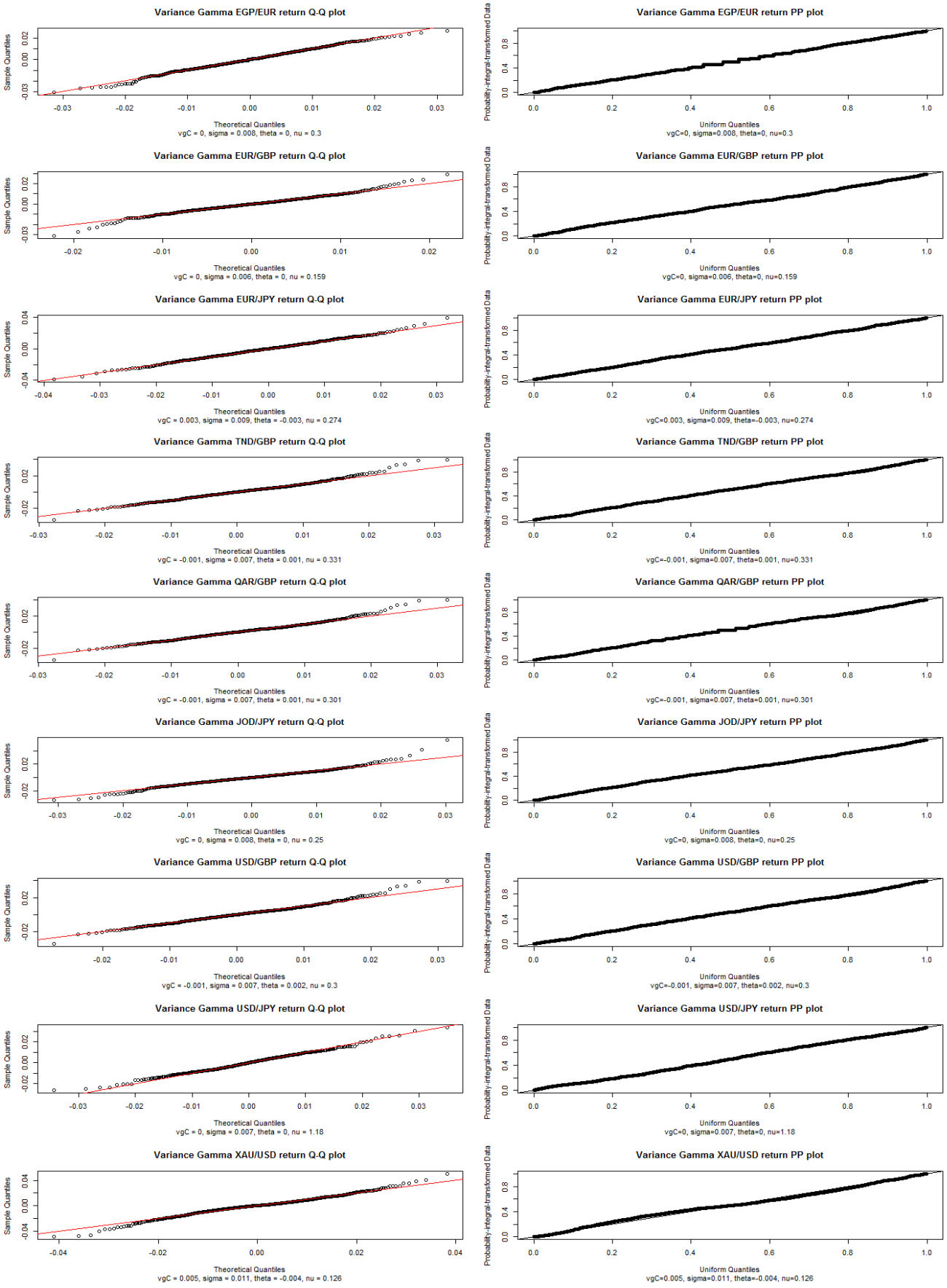

Figure A Variance Gamma QQ and PP plots for nine FX currency pairs

Decision making based on tropical algebra

Hana Tomášková[1], Martin Gavalec[2]

Abstract. In AHP approach to multi-criteria decision problem, the relative importance of alternatives is computed from preference matrices, which come from experience and can possibly be inconsistent. Standardly, the preference vector is computed as the eigenvector of the preference matrix by methods of linear algebra. Alternative use of non-standard methods in tropical algebra is considered in this paper. The preference matrix will be processed by the methods used in max-prod algebra and other tropical algebras. Given preference matrix will be transformed by the tropical operations, until a steady state is reached. The eigenvector of the matrix then describes the steady state preferences and respects all preference relations contained in the original matrix. Efficient algorithms for computing eigenvectors in the tropical algebra are described. The method is illustrated by numerical examples and compared with the linear algebra approach. The consistent and inconsistent cases are considered.

Keywords: preference matrix, tropical algebra, AHP decision making, eigenvectors

JEL classification: C44
AMS classification: 90C15

1 Introduction

Analytical Hierarchy Process is a mathematical model of multi-criteria decision making that uses the decomposition of a complex unstructured situation into simpler parts - the hierarchical system. Using subjective pairwise comparisons numerical values are assigned to individual components, showing their relative importance. One of the fundamental questions in AHP decision making is how to find the appropriate preference matrix for a set of alternatives.

The subjective preferences given by human experts are often inconsistent and do not reflect the deep relations between the processed notions, see [13, 14]. The standard approach to finding the relative importance vector out of an incosistent matrix uses an eigenvector of the preference matrix computed by the methods of linear algebra, [17]. Possible use of other non-standard methods in other algebras, such as tropical or fuzzy algebra, is considered in this paper. Differently from binary operations plus and product in classical linear algebra, the tropical algebra uses binary operation maximum and plus, or maximum and product, or similar combinations with minimum instead of maximum. Similarly, the fuzzy algebra uses the operation maximum together with some fuzzy triangular norm, e.g. the Gödel, Łukasiewicz, product, or drastic t-norm, [3, 10]. Properties of various tropical and fuzzy algebras were studied by many authors, with useful applications. Tropical algebras are important in the study of discrete events systems, the steady states of which correspond to eigenvectors of max-plus or max-prod matrices, see [1, 2, 4, 8, 12]. The eigenvectors of max-min matrices are useful in cluster analysis or in fuzzy reasoning, the eigenvectors of some fuzzy algebras are described in [6, 7].

In this paper the preference matrices are studied by methods used in tropical algebra, in particular in the max-prod algebra. The original preference matrix is processed by max-prod operations, until a steady state is reached. The eigenvector of the matrix then describes the steady state preferences and, therefore, it respects all relations hidden in the original matrix. Efficient algorithms for computing eigenvectors in

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the tropical algebra are used. The method is illustrated by numerical examples and compared with the linear algebra approach. The consistent and inconsistent cases are considered separately.

2 AHP decision making

Analytic Hierarchy Process (AHP) is a method developed for creating structured models of multi-criteria decision problems. The method helps to find an alternative which suits best the given needs of the deciding person. Analyzing the set of possible alternatives, the AHP method finds the one with the best rating, based on the structure of the problem and given preferences. Saaty formulated the principles of AHP in late 1970s [15], and the method has been broadly studied and applied in many cases since the time, [5, 11, 14]. The method combines mathematical and psychological aspects, starting with defining the structure of the problem, then quantifying the relative preferences, computing the priorities and finally computing the evaluation of all considered alternatives. First of all, the multi-criteria decision problem is converted into a hierarchy of sub-problems and each of the sub-problems is then independently analyzed. The criteria of the sub-problems in the hierarchy may have very heterogeneous nature, they may be precisely or vaguely defined, with crisp or fuzzy parameters, formal or intuitive, etc. The relative preferences of heterogeneous criteria are then quantified by human decision-maker using the ability to compare various aspects of the problem. The decision maker systematically compares the criteria in pairs and quantifies the relative importance either by available data or by intuitive judgment. The relative preferences found by pairwise comparisons are then used to compute weights (priorities) for every part of the hierarchy model. The evaluation computed for all decision alternatives then shows their relative strength from the point of view of the entire problem. It is the advantage of AHP that even considerably diverse criteria can be used in the model, and that not only exact data but also human judgments can be applied to describe various aspects of the problem, [16].

Formally, AHP is expressed by matrices and matrix operations are used to find and evaluate the best alternative:

Let $A_1, A_2, .., A_n$ be a set of variables. The quantified judgments on pairs of variables A_i, A_j are represented by an $n \times n$ matrix $A = (a_{ij})$, $i, j = 1, 2, ..., n$. Entries a_{ij} are defined as follows: if $a_{ij} = a$, then $a_{ji} = 1/a$; $a \neq 0$ for all i, j. As A_i is considered to be of equal relative intensity to itself, then $a_{ii} = 1$ for all i.

$$A = \begin{pmatrix} 1 & a_{12} & \cdots & a_{1n} \\ 1/a_{12} & 1 & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 1/a_{1n} & 1/a_{2n} & \cdots & 1 \end{pmatrix}$$

Then, relative rankings of variables will be obtained by computing an eigenvector X of the matrix A.

$$AX = \lambda X$$

Finally, by matrix multiplication of matrices of relative rankings of criteria and alternatives relative rankings matrix, total ranking of particular alternatives is obtained.

3 Tropical algebras

Two most frequently used tropical algebras are the max-plus and the max-prod algebra. In this paper we work with preference matrices in the multiplicative form. It is natural, therefore, to process the preference matrix in the max-prod algebra. By max-prod algebra we understand a linear structure on a linearly ordered set $\mathcal{R}$ of real numbers together with the binary operations $\oplus$ = maximum and $\otimes$ = product, similarly as the ordinary addition and multiplication operations are used in the classical linear algebra. The operations $\oplus$ and $\otimes$ are extended to matrices and vectors in a natural way. We should remark that the max-prod algebra is isomorphic to max-plus algebra, with the operations maximum and addition. The eigenvalue of a given max-plus or max-prod matrix and the eigenvectors can be efficiently described by considering cycles in specifically evaluated directed graphs.

The eigenproblem in the max-prod algebra $(\mathcal{R}, \oplus, \otimes)$ can be formulated as follows. Given matrix $A \in \mathcal{R}^{n \times n}$, find $\lambda \in \mathcal{R}$ and $X \in \mathcal{R}^n$ such that

$$A \otimes X = \lambda \otimes X$$

It is a well-known fact that the eigenvalue λ can be computed as the maximal geometric cycle mean in the complete directed graph $G(A)$ with n nodes, in which the edges are evaluated by the matrix inputs, e.i. the weight of the edge (i,j) is $w(i,j) = a_{ij}$ for every i,j. If $C = (v_0, v_1, v_2, \ldots, v_k)$ with $v_0 = v_k$ is a cycle of length $k > 0$ in $G(A)$, then the weight of C is the value $w(C) = \prod_{i=1}^{k} w(v_{i-1}, w_i)$, and the geometric cycle mean $\overline{w}(C) = w(C)^{1/k}$, [13]. Hence $\lambda = \max_{C \text{ cycle in } G(A)} \overline{w}(C)$. The maximal geometric cycle mean can be computed in $O(n^3)$ time be slightly modified Karp's method, see [9].

When the eigenvalue λ has been computed, then we denote $B = A \otimes \lambda^{-1}$ and compute the matrix $B^\star = I \oplus B \oplus B^2 \oplus \cdots \oplus B^{n-1}$, which is called Kleene star. The columns of the Kleene star matrix with diagonal value 1 are the fundamental eigenvectors of B with eigenvalue value 1, and they are also the eigenvectors of the original matrix A with eigenvalue λ.

4 Consistent matrix

In this section we define the consistency of matrices. Consistent matrix is a form of preference matrix where the following conditions are satisfied.

1. $a_{ii} = 1$
2. $a_{ij} = 1/a_{ji}$
3. $a_{ij} * a_{jk} = a_{ik}$

It should be remarked that Kleene star matrix $B^\star$ is consistent. Conversely, if a preference matrix A is consistent, then it is equal to its Kleene star matrix, moreover, all eigenvectors are multiples of each other. Thus, for a consistent matrix there exists only one fundamental eigenvector, which gives the priority vector, after normalization.

Example 1. The eigenproblem for a given matrix $A \in \mathcal{R}^{n\times n}$ in extremal (max-prod) algebra consists of finding a vector $x \in \mathcal{R}^n$ (eigenvector) such that the equation $A \otimes x = \lambda \otimes x$ holds true. The solutions are found by LibNOM (Library of Nonstandard Optimization Methods - developed at the University of Hradec Králové, Czech Republic, 2012).

$$A = \begin{pmatrix} 1 & 1/2 & 3 \\ 2 & 1 & 6 \\ 1/3 & 1/6 & 1 \end{pmatrix}$$

$\lambda = 1$ and eigenspace of fundamental eigenvector found by Kleene star method ($B^\star = I \oplus B \oplus B^2$) in LibNOM is

$$B^\star = \begin{pmatrix} 1 & 0.5 & 3 \\ 2 & 1 & 6 \\ 0.3333 & 0.1667 & 1 \end{pmatrix} \quad X = \begin{pmatrix} 0,3 \\ 0,6 \\ 0,1 \end{pmatrix}$$

5 Inconsistent matrix

For an inconsistent matrix A the priority vector is found in the following steps.

1. Find the eigenvalue λ of matrix A. The eigenvalue is maximal geometric mean weight of cycle in the corresponding graph, computation is done by the modified Karp's algorithm
2. Converte A into matrix $B = A \otimes \lambda^{-1}$ to get the eigenvalue equal to one.
3. Compute the Kleene star matrix $B^\star = I \oplus B \oplus B^2 \oplus \cdots \oplus B^{n-1}$, a consistent matrix.
4. Normalize the fundamental eigenvector, any column from $B^\star$. The normalized column is the priority vector.

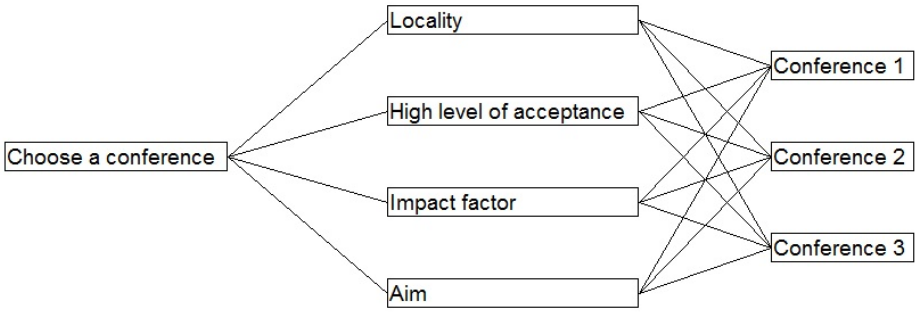

Figure 1 Hierarchy graph

Example 2. The example is based on an actual problem: how to choose a conference for presentation of research results. We prepare a 4 attributes and 3 alternatives, as is shown in a picture bellow.

The problem is situated as a way, how to choose a conference. We defined a four levels

- Locality - exotic, holiday areas, etc.,
- High level of acceptance,
- Impact factor - means the outputs of the conference, from nonindexed proceedings to journal with impact factor,
- Aim - if the conference is very close to my work.

We anonymised the conference, but there is a list of short information which will describe the instance.

Conference 1 is located on very interesting places, in most cases oceanic places, the level of acceptance is almost 95%, the outputs have almost zero value for current scientific scoring system, the conferences are wide, with many scientific areas.

Conference 2 is located on interesting places, not necessary close to ocean, the level of acceptance is high in case of not so wide target scientific population. The outputs have moderate importance, but the level of papers and discussions on the conference allows you to prepare a journal paper. The aim is very close.

Conference 3 is located at normal places and it is easy to travel to it. The level of acceptance is higher. The outputs of the conference are indexed in required databases and the aim is close.

In the example are used two programs Criterium Decision Plus and our LibNOM working with the extremal algebra max-prod.

Both approaches use a pairwise comparisons, where the priorities will be derived from a series of measurements: pairwise comparisons involving all the nodes. The nodes at each level will be compared, two by two, with respect to their contribution to the nodes above them. The results of these comparisons will be entered into a matrix which is processed mathematically to derive the priorities for all the nodes on the level.

Intensity of Importance	Definition
1	Equal importance
3	Moderate importance
5	Strong importance
7	Very strong importance
9	Extreme importance

The following matrices represent the preference matrices with transferred weight.
Locality matrix is inconsistent matrix, has a $\lambda = 1.3104$. High level of acceptability matrix is inconsistent matrix, has a $\lambda = 1.3264$.

$$\text{Locality} = \begin{pmatrix} 1 & 6 & 8 \\ 1/6 & 1 & 3 \\ 1/8 & 1/3 & 1 \end{pmatrix} \quad \text{HLA} = \begin{pmatrix} 1 & 9 & 7 \\ 1/9 & 1 & 1/3 \\ 1/7 & 3 & 1 \end{pmatrix}$$

Impact factor matrix is inconsistent matrix, has a $\lambda = 1.1587$. Aim matrix is inconsistent matrix, has a $\lambda = 1.1587$.

$$\text{IF} = \begin{pmatrix} 1 & 1/9 & 1/7 \\ 9 & 1 & 2 \\ 7 & 1/2 & 1 \end{pmatrix} \quad \text{Aim} = \begin{pmatrix} 1 & 1/8 & 1/6 \\ 8 & 1 & 3 \\ 6 & 1/3 & 1 \end{pmatrix}$$

The fundamental eigenvectors are

$$\text{Locality X} = \begin{pmatrix} 0.7612 \\ 0.1662 \\ 0.0726 \end{pmatrix} \quad \text{HLA X} = \begin{pmatrix} 0.7854 \\ 0.0658 \\ 0.1488 \end{pmatrix}$$

$$\text{IF X} = \begin{pmatrix} 0.0572 \\ 0.5969 \\ 0.3458 \end{pmatrix} \quad \text{Aim X} = \begin{pmatrix} 0.0623 \\ 0.6527 \\ 0.2851 \end{pmatrix}$$

Criteria matrix is inconsistent matrix, has a $\lambda = 1.1006$ and normalized fundamental eigenvector is

$$\text{Criteria} = \begin{pmatrix} 1 & 1/6 & 1/9 & 1/3 \\ 6 & 1 & 1/2 & 2 \\ 9 & 2 & 1 & 4 \\ 3 & 1/2 & 1/4 & 1 \end{pmatrix} \quad X = \begin{pmatrix} 0.0524 \\ 0.2857 \\ 0.5191 \\ 0.1428 \end{pmatrix}$$

The linear algebra approach as an output of the program Criterium Decision Plus is below.

$$\text{Locality} = \begin{pmatrix} 0.761 \\ 0.166 \\ 0.073 \end{pmatrix} \quad \text{HLA} = \begin{pmatrix} 0.785 \\ 0.066 \\ 0.149 \end{pmatrix} \quad \text{IF} = \begin{pmatrix} 0.057 \\ 0.597 \\ 0.346 \end{pmatrix} \quad \text{Aim} = \begin{pmatrix} 0.062 \\ 0.653 \\ 0.285 \end{pmatrix}$$

$$\text{Criteria} = \begin{pmatrix} 0.051 \\ 0.282 \\ 0.526 \\ 0.141 \end{pmatrix}$$

By comparing the obtained values we see that both methods give almost identical results in this case.

6 Conclusions

Relative importance of alternatives is computed from the (possibly incosistent) preference matrix in AHP multi-criteria decision problem as the eigenvector of the preference matrix by the methods used in max-prod algebra. Given preference matrix is processed by max-prod operations, until a steady state is reached. The eigenvector of the matrix then respects all preference relations expressed in the preference matrix. Efficient algorithms for computing eigenvectors in the tropical algebra are described.

Further research in this direction will consider possible use of other non-standard algebras, such as tropical or fuzzy algebra. Binary operation maximum and plus, or similar combinations with minimum instead of maximum could be used. Similar approach will involve the fuzzy algebras with the operation maximum together with some of fuzzy triangular norms, e.g. the Gödel, Łukasiewicz, product, or drastic t-norm.

Acknowledgements

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Labour Market Modelling within a DSGE Approach

Jaromír Tonner[1], Stanislav Tvrz[2], Osvald Vašíček[3]

Abstract. The goal of this paper is to contribute to the understanding of the dynamics of main labour-market variables and their relevance for macroeconomic modelling and forecasting in a period of recession. We consider two ways of incorporating labour market dynamics into the New Keynesian DSGE framework. The first one is rather straightforward and it is based on Jakab and Kónya (2009). The second one is more sophisticated and it is based on Christiano *et al.* (2011). In both cases we extend the DSGE framework with search and matching frictions. Due to the early stage of our research we mainly focus on the literature review in this paper but we also provide some preliminary empirical results. In-sample forecasts generated on calibrated models are employed to evaluate their predictive power.

Keywords: DSGE models, labour market, search and matching, frictions.

JEL classification: E32, J64
AMS classification: 91B64

1 Introduction

Majority of current DSGE models does not embed an explicit labour market with unemployment linked to real economic activity. In standard New Keynesian DSGE models, movements in the labour market are captured by varying hours worked (intensive margin) or choosing whether or not to participate on the labour market at all (extensive margin). Many authors (e.g. Blanchard [5], or Christiano *et al.* [7]) point out the limitations of that approach as unemployment is an important indicator of aggregate resource utilization and an important focus of the policy debate.

Incorporating labour market imperfections into DSGE models is one of currently most discussed issues on the field of macroeconomic modelling. The research intensified during the last decade when a lot of authors implemented a labour market with search and matching (Mortensen and Pissarides [19]) into New Keynesian framework. Effects of such labour market frictions within DSGE models with flexible wages but sticky prices are analyzed in Walsh [23] and Trigari [22]. Christoffel *et al.* [8] extend this work by relaxing the assumption of flexible wages by introducing wage rigidity in the framework. Also, Gertler *et al.* [14] and de Walque *et al.* [9] analyze models with staggered wage bargaining. Another strand of research turned to more normative issues analyzing implications of labour market frictions for the monetary policy, examples are Blanchard and Galí [6], Faia [11] or Thomas [21].

Galí [13] uses a framework where the unemployment results from presence of market power in the labour market implying a positive average wage markup (gap between prevailing wage and the disutility of work for the marginal worker). On the other hand, fluctuations in the unemployment rate are associated with variations in the average wage markup due to the presence of nominal wage rigidities.

A radically different approach to labour market modelling was initiated in a series of papers by R. Farmer and his co-authors (see Farmer [12] as a survey of the early beginning of this research agenda). These papers try to formalize the ideas of Post Keynesian economics in a modern framework of DSGE models and attribute the labour-market fluctuations and output fluctuations to self-fulfilling prophecies, which formalize the Keynesian idea of animal spirit. It seems that this approach is able to explain some data features, which are hard to explain in a standard New Keynesian framework. However, to the

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best knowledge of the authors, this approach has been tested only on small stylized models and not on elaborate models suitable for practical forecasting.

1.1 Czech labour market

During past seven years, there has been some interesting research on the Czech labour market. Several papers deal with labour-market rigidities. Hence, they have implications for fluctuations of labour-market variables over the cycle. For example, Babecký *et al.* [2] use a survey at the firm level to investigate the determinants of the wage and price formation in the Czech firms. They find efficiency wage models relevant for the wage setting. Babecký *et al.* [3] investigate asymmetric response of the labour demand by firms in good times and during the crisis.

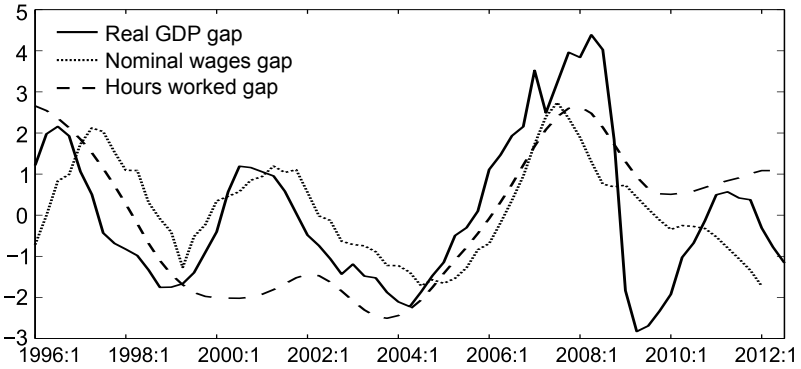

Figure 1 Cyclical behaviour of the Czech labour market

In Figure 1, the percentage deviations of real GDP, nominal wages and hours worked from their respective trends are depicted.[1] Since the development of labour market variables usually lags behind the business cycle, the nominal wages were shifted by three quarters and hours worked by one quarter back in time. This figure illustrates the comovement of nominal wages and hours worked with the business cycle and the explaining potential of the labour market variables. Most of the variation in hours worked is generated by the extensive margin of labour supply. To capture both margins of the labour supply we will extend our model with explicit modelling of the labour market.

2 Model structure

Since it is the concept of search and matching frictions that is becoming a standard in the modelling of the labour market, we decided to study this type of models in greater detail. Namely, we chose the models introduced by Jakab and Kónya [18] and Christiano *et al.* [7]. In this section we briefly describe the structure of these DSGE models.

2.1 Labour market modelling á la Jakab and Kónya

Jakab and Kónya [18] incorporate a labour market with search and matching frictions of Mortensen and Pissarides [19] and staggered wage setting to the standard small open economy DSGE setup.

New jobs are created when unemployed workers and open job vacancies meet on the labour market. Number of newly formed pairs is given by Cobb-Douglas matching function with constant returns to scale

$$m_t = \sigma_m \varepsilon_t^m v_t^\sigma u_t^{1-\sigma}, \tag{1}$$

where m_t is the number of new matches, v_t is the number of open vacancies, u_t is the number of unemployed and ε_t^m is a shock in the efficiency of matching. Employment n_t follows the law of motion

$$n_t = (1-\rho_t)n_{t-1} + m_t, \tag{2}$$

[1] The real GDP and nominal wages were detrended with the use of Hodrick-Prescott fitler ($\lambda = 1600$) and hours worked were demeaned.

where ρ_t is exogenous separation rate. And the development of unemployment u_t is described by

$$u_t = L_t - n_{t-1}, \tag{3}$$

where L_t is the exogenously given labour force.

Optimal wage is negotiated each period as a solution to the Nash bargaining. The optimal wage w_t^* is, hence, a maximizer of

$$\max_{w_t^*}[W_t(w_t^*) - U_t]^\eta J_t(w_t^*)^{1-\eta}, \tag{4}$$

where W_t is a value of the job and U_t is a value of being unemployed for the worker, J_t is the value of the job for the employer and η is the bargaining power of workers. However, only a fraction $(1-\gamma_w)$ of existing jobs is allowed to renegotiate their wages each period. Remaining fraction γ_w are remunerated by the average wage of last period. Similarly, a fraction $(1-\theta_w)$ of new hires gets to negotiate their wage while the remaining share θ_w gets the average wage of last period. Therefore, the average wage develops according to

$$w_t = \frac{m_t}{n_t}[\theta_w w_{t-1} + (1-\theta_w)w_t^*] + \frac{(1-\rho)n_{t-1}}{n_t}[\gamma_w w_{t-1} + (1-\gamma_w)w_t^*]. \tag{5}$$

2.2 Labour market modelling á la Christiano, Trabandt and Walentin

Christiano *et al.* [7] include the labour market search and matching framework of Mortensen and Pissarides [19]) and, more recently, Hall [15], [16], [17] and Shimer [20]. In their model, the probability of finding a job is increasing in search effort, and imperfect risk sharing among individuals is a consequence of the unobservability of effort.

The framework is integrated into the baseline new Keynesian model of a small open economy which includes physical capital and monetary factors. A key feature of this model is that there are wage-setting frictions, but they do not have a direct impact on ongoing worker employer relations as long as these are mutually beneficial. However, wage-setting frictions have an impact on the effort of an employer in recruiting new employees. Accordingly, the setup is not vulnerable to the Barro [4] critique that wages cannot be allocational in ongoing employer-employee relationships (see Hall [17]).

There are three main differences between the Christiano *et al.* [7] labour market modelling and Jakab and Kónya [18]. First, Jakab and Kónya [18] assume wage-setting frictions of the Calvo type, while Christiano *et al.* [7] instead work with Taylor-type frictions. Second, Jakab and Kónya [18] shut down the intensive margin of labour supply in their empirical specification, while Christiano *et al.* [7] allow for variation in this margin. And third, Jakab and Kónya [18] model the separation rate as exogenous, while Christiano *et al.* [7] allow for endogenous separation of employees from their jobs.

In the standard new Keynesian model, the homogeneous labour services are supplied to the competitive labour market by labour contractors who combine the labour services of households who monopolistically supply specialized labour services (see Erceg *et al.* [10]). Christiano *et al.* [7] labour market model dispenses with the specialized labour services abstraction and the accompanying monopoly power, which is commonly modelled as time-varying (wage-markup shocks). The reason for this modelling choice is that Christiano *et al.* [7] do not think this type of union monopoly power, nor its high frequency time-variation, accurately describes the labour market. Labour services are instead supplied to the homogeneous labour market by "employment agencies" - a modelling construct best viewed as a goods producing firm's human resources division.

Each employment agency retains a large number of workers. At the beginning of each period a fraction of workers are randomly selected to separate from the agency and go into unemployment. Also, a number of new workers arrive from unemployment in proportion to the number of vacancies posted by the agency in the previous period. After separation and new arrivals occur, the nominal wage rate is set. Then idiosyncratic shocks to workers' productivities are realized and endogenous separation decisions are made. The nominal wage paid to an individual worker is determined by Nash bargaining, which occurs once every N periods. Each employment agency is permanently allocated to one of N different cohorts. Cohorts are differentiated according to the period in which they renegotiate their wage. Since there is an equal number of agencies in each cohort, $1/N$ of the agencies bargain in each period. The intensity of labour effort is determined efficiently by equating the worker's marginal cost to the agency's marginal benefit. The efficient provision of labour on the intensive margin implies an important difference to Erceg *et al.* [10] where a direct link between the sticky wage and hours worked is assumed instead.

3 Empirical results

In Figure 2, in-sample forecasts of the Czech National Bank's core prediction model (see Andrle *et al.* [1]) and its alternative specification with the labour market á la Jakab and Kónya [18] are depicted. Sums of squared prediction errors are reported for both model specifications. We found that the the model with explicitly modelled labour market performs comparably to the baseline model. The model with labour market (LM) outperforms the baseline specification (noLM) in forecasting of the nominal wages, exchange rate and to a lesser extent also GDP and inflation. The baseline model offered slightly better forecasts of interest rate and consumption. Obviously, the main advantage of the model with labour market is the ability to forecast unemployment and other labour market variables.

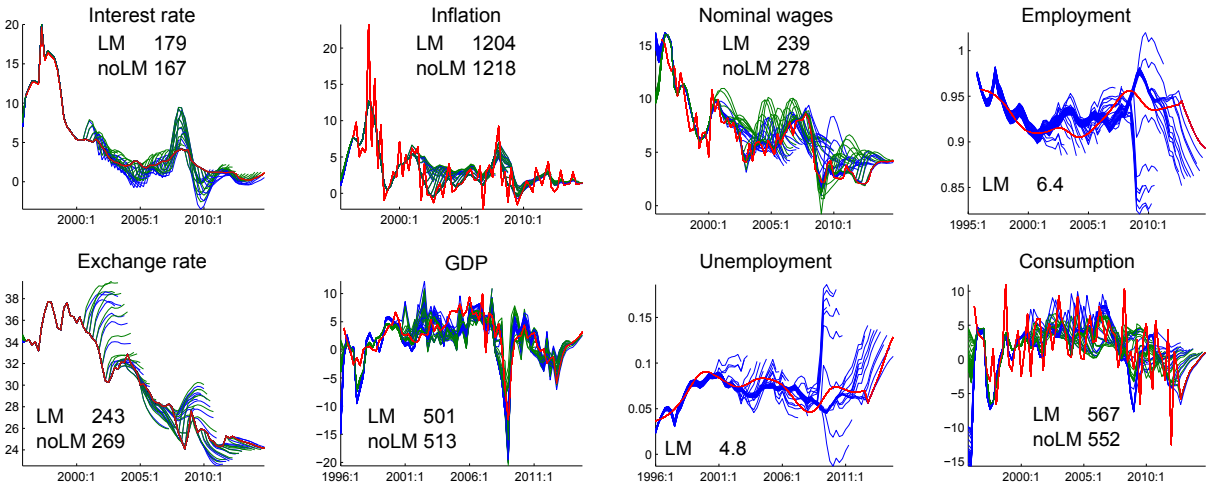

Figure 2 In-sample forecasts

We also calculated the in-sample forecasts of the model specification with the labour market á la Christiano *et al.* [7] and compared it to the baseline model. Surprisingly enough, we found the predictive power of the extended model to be worse than in case of the baseline model for all common variables including the nominal wages.

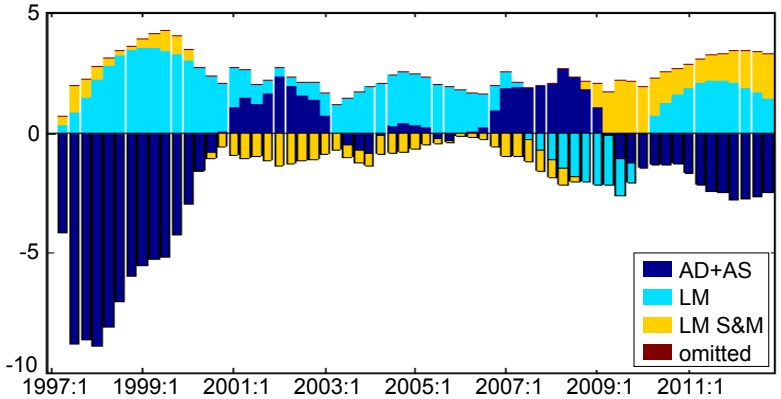

Figure 3 Shock decomposition of unemployment

Explicit modelling of the labour market and introduction of new type of structural labour market shocks enables micro-founded shock decomposition of the unemployment. Figure 3 shows a shock decomposition of the unemployment gap suggested by the model with labour market á la Jakab and Kónya [18]. While in the baseline g3 model we had only the wage markup and labour supply shock (LM) that originated in the (implicit) labour market, the model with search and matching frictions contains additional matching and separation shocks (LM S&M). Remaining demand and supply shocks (AD+AS) are aggregated in the shock decomposition in order to highlight the labour market. We found that the LM

S&M shocks played only marginal role in the past and that it affects the unemployment more significantly since the recent crisis. Overall, the LM shocks probably play more important role in explaining the development of unemployment. Also, the role of LM S&M shocks in explaining the development of other important economic variables (interest rate, inflation, exchange rate etc.) is marginal and often dominated by the LM shocks.

4 Conclusion

It was the goal of this paper to contribute to the understanding of the dynamics of the labour market and to evaluate its relevance for macroeconomic modelling and forecasting. We performed a comprehensive review of the literature related to the topic of labour market modelling within a DSGE framework and outlined current approaches to this problem. We found that the labour market is most often modelled with the use of search and matching frictions. Following this trend, we chose the models of Jakab and Kónya (2009) and Christiano *et al.* (2011) for further analysis. In both of these DSGE models, the labour market is explicitly modelled with the use of search and matching framework that introduces several interesting labour market variables into the DSGE model – including unemployment. Due to the early stage of our research we provided only preliminary empirical results. Using the Czech National Bank's core prediction model and its alternative specifications with explicit labour market, we calculated in-sample forecasts to evaluate the predictive power of different model specifications. We found that the straightforward approach of Jakab and Kónya [18] is quite successful. The model was able to produce acceptable forecasts of the main labour market variables while overall predictive power of the baseline model remained unimpaired. Micro-founded shock decomposition of the unemployment gap showed another advantage of explicit labour market modelling. Additional structural shocks defined in the labour market allowed us to describe the development of the labour market and its effects on the rest of the economy in greater detail. In case of elaborate approach of Christiano *et al.* [7], the results were less encouraging as overall predictive capability of the model worsened in comparison to the baseline model (for the common variables). However, as the model structure is quite complicated the results have to be treated with caution. Further empirical analysis of these models is needed to be able to evaluate the relevance of the labour market for macroeconomic modelling and forecasting in its complexity.

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Non-parametric estimation of a GARCH-M model with heuristic

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Abstract. We propose a non-parametric technique with the use of a heuristic called differential evolution to estimate the parameters of a GARCH-M model. In comparison to some parametric technique, this approach can ensure several objectives at once. First, it can avoid the inconsistency problem which may rise when the distribution assumption is misspecified. Second, this approach can easily escape from a local extreme to reach a global solution while dealing with the maximum likelihood estimation (MLE) task. Finally, it can also more effectively control the required properties of the estimates as they come from a well selected population. The suitability of our approach is verified on modeling the CZK/USD and CZK/EURO forward exchange rate premium of period from 2007 to 2012.

Keywords: Non-parametric GARCH-M, maximum likelihood estimation, differential evolution, forward exchange rate premium

1 Introduction

Volatility is an important information input for risk management. It can be modeled by a generalized autoregressive conditional heteroskedasticity model (GARCH). To estimate the model's parameters, the maximum likelihood method is often used and an apriori distribution assumption is required. If the distributional specification is incorrect, the inconsistency of estimates may occur. There are also other two problems connected when estimating the model's parameters. The first one is the local optimum problem as common optimization methods can only find local, but not global optimum. The second one is the uncontrolability of the size of the estimates so that they can guarantee the stability of the model as well as the positiveness of the unconditional variance. As it is very difficult to reach this three-fold objective by using parametric estimation technique, our goal in this contribution is to propose a nonparametric GARCH model estimation procedure. This procedure would first eliminate the specification issue since the distribution of the error component are estimated from the data. Then as we use heuristic optimization technique, we can reliably deal with the local optimum as well as control for the size of the estimates as they are chosen from a well-defined population of parameters complied with the requirements put on them. To our knowledge, this approach has not been used in the literature so far [2]. We also verify the applicability of our method on the GARCH-M specification with data on forward exchange rate of the Czech crowns against Euro and the U.S. dollar in the period 2007 - 2012. The obtained results are also compared with those obtained by a traditional optimization method. The rest of the paper is structured as follows. In the next section we will describe how a non-parametric estimation procedure with heuristic works. In the section after that one we report the results of using this procedure to estimate forward exchange rate premium. Finally, we wrap up our work with some concluding remarks.

2 Model GARCH and its non-parametric estimation with heuristic

In 1982 Engle [4] came up with a model autoregressive conditional heteroscedasticity, whose specification consists of two equations: the conditional mean equation and conditional equation. Bollerslev [3] generalized Engle's ARCH model to the so called GARCH model by including q delayed terms of conditional variance to get a GARCH specification. Engle, Lilien and Robins [5] include the GARCH term into the mean equation to get the so called ARCH-M model. If GARCH terms are added to the conditional

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variance equation, we have a fully specified GARCH-M model as follows:
Mean equation:

$$y_t = X_t^T b + \delta h_t + \epsilon_t. \tag{1}$$

Variance equation:

$$h_t = \alpha_0 + \sum_1^p \alpha_i \epsilon_{t-i}^2 + \sum_1^q \beta_j h_{t-j}. \tag{2}$$

GARCH-M model parameters are often estimated by the maximum likelihood method when p and q are relatively small numbers. To use this method, we have to assume the distribution of the random component. Usually, the distribution is assumed to be normal. Later, with respect to the leptokurtic nature of financial data, other distributions like t-distribution and GED distribution are used. Once the distribution is specified, the corresponding likelihood function $L(\theta\ |X, y)$ can be constructed. The vector of parameters in demand θ is such that satisfies the following condition:

$$\theta = \underset{\theta}{\operatorname{argmax}}\ \log(L(\theta\ |X, y)). \tag{3}$$

As the (log)likelihood function is often a complicated one, it is almost impossible to find the model parameters analytically. Therefore a Newton-like numerical optimization method is used to solve this task. So far the procedure suggested by Berndt, Hall, Hall and Hausman [1] seems to be the most comfortable algorithm for this purpose. At times, the derivatives of log-likelihood function are calculated numerically [7] instead.

Parametric estimation of model GARCH-M is comfortable. As the estimation is based on a prior distributional form of the random component, the estimates may be inconsistent if this assumption is misspecified. Therefore non-parametric estimation approach is used to solve this problem. In this approach, the functional shape of the distribution is estimated from the data as well as the parameters of the model. Based on the definition of density probability for a continuous random variable X with density distribution $f(x)$ we have:

$$P(a \leq X \leq b) = \int_a^b f(x)\ dx, \tag{4}$$

The probability density function $f(x)$ can be estimated from observations $(x_1, x_2, ..., x_n)$, which are deemed to be independent realization of random variable X. For random component ϵ_t, equation (4) can be approximated as follows:

$$P(x - h \leq X \leq x + h) = \int_{x-h}^{x+h} f(x)dx \approx 2hf(x), \tag{5}$$

where h is a small positive number. Equation (5) can be rearranged to

$$\hat{f}(x) = \frac{1}{2h} \frac{\#\text{of points} \in (x - h, x + h)}{n}. \tag{6}$$

where n is the total number of observations. Equation (6) is nothing but the definition of a histogram and we know a histogram is a rough estimate of the density distribution of a random variable. Histogram provides basic information about the type of random variable, its skewness, kurtosis. However, a histogram is only a rough estimate of the density because it is a step function. If one of these intervals is empty, a histogram is also a discontinuous function. To get the smoothness and continuity of a estimated density, we rearrange equation (6) as follows:

$$\hat{f}(x) = \frac{1}{n} \sum_1^n w(x - x_i, h), \tag{7}$$

where $(x_1, x_2, ..., x_n)$, the realization of random variable X and $w(t, h)$ is the weighting function defined as follows:

$$w(t, h) = \begin{cases} \frac{1}{2h} & \text{if } |t| < h, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Equation (7) shows that the estimated smoothed probability density function is a weighted average of those observations that fall into symmetrical vicinity of the point at which we estimate and the weights

are determined by weight function $w(t, h)$. Weighting function also called kernel (core) function, which we denote as $\mathcal{K}(x)$. There are quite a large number of kernel functions which should be symmetric and bounded. The most commonly used functions are normal, Epanechnikov and triangular kernels. Equation (7) shows that principally, characteristics of the estimated density distribution $\hat{f}(x)$ depends both on the choice of the type of kernel function and the chosen length of the interval around the point the density is estimated at. Actually, the estimated density does not depend so much on the choice of the kernel, but it quite strongly depends on the length of the smoothing window [6]. The wider the window, the smoother the estimated density, but this also means that the deviation from the true density will be larger and vice versa. The selection of the length of this window must be a compromise between these two factors.

Suppose we have estimated the density distribution of the random component GARCH-M model in the form:

$$\hat{f}(\epsilon_t) = \frac{1}{Th} \sum_{1}^{T} \mathcal{K}\left(\frac{\epsilon - \epsilon_t}{h}\right), \tag{9}$$

then estimate model parameters GARCH-M means finding the parameter vector θ, which maximizes the value of the logarithmic likelihood function:

$$\log L = \sum_{1}^{T} \log \left[\frac{1}{Th} \sum_{1}^{T} \mathcal{K}\left(\frac{\epsilon - \epsilon_t}{h}\right)\right] \tag{10}$$

where $\epsilon_t = y_t - X_t T^b - \delta h_t$. The optimal solution for the logarithm of likelihood function (10) can be found with traditional optimization methods. Function (10) is a function with a relatively large number of independent variables with a complex development, so it is possible that it is a multimodal function. As a result, the optimal solution when found may not be a global one, but just a local one. To solve this problem, we use a heuristic called differential evolution algorithm for finding the optimal solution.

Differential evolution is a relatively new heuristic technique for finding the global optimum of functions of several variables. It was suggested by Storn and Price [8] at the end of the last century. It is a simple model of Darwin's theory of evolution of populations and many concepts of the evolution theory like an individual, a population, a generation, evolution, a parent, a descendant, crossover, mutation are used. When searching for optimum of function $f(x)$ in a non-empty domain $D = \{x \in R^d : a \leq x \leq b\}$, we start with a random population of N individuals, that is, with set $P = (x_1, ..., x_n) \subset D$. Then for each vector x_k of old population P we create a mutant y using three mutually different individuals r_1, r_2, r_3 of population P as follows:

$$y = r_1 + F \cdot (r_2 - r_3) \tag{11}$$

where $0 < F \leq 1$ is a parameter influencing the extent of mutation. The mutation technique is called random evolution. An alternative procedure for the generation of a new generation of targeted mutations is such that we select the best individual x_{best} of P (the best value of objective function f) of the population and four different individuals from population r_1, r_2, r_3, r_4 from P as follows:

$$y = x_{\text{best}} + F \cdot (r_1 + r_2 - r_3 - r_4). \tag{12}$$

The resulting mutant y does not have to be an element of the D. In this case, we flip it back using one or more of the boundary hyperplanes D. The position correction is called mirroring. When crossing parents x_k with a mutant y, genetic dominance parameter of a mutant is captured by $0 \leq C \leq 1$ and each component of a random mutant z is generated by the following formula:

$$z_j = \begin{cases} y_j & \text{for } rnd_j < C \\ x_j & \text{for } rnd_j \geq C, \end{cases} \tag{13}$$

where $j = 1, ..., d$ and rnd_j is a random number from a uniform distribution on the interval [0, 1]. A mutant wins the clash with his parent if $f(z)$ is better than $f(x_k)$. Otherwise, the parent wins. One thus gets a new population of $Q \subset D$. By this way, we will gradually get a final population which we is better than the original one.

A differential evolution algorithm has three important parameters: N, F and C. The number of individuals N in the population for which the following rule must hold: the higher N is, the easier the selection of individuals is. But it also makes the calculation more time-consuming. Parameter F, which is also called the differential weighting procedure, allows the optimizer to jump from one local area to

another. Parameter C determines how a new generation will look like. The final results of a differential evolution algorithm is very dependent on the choice of these parameters. Storm and Price [8], and Tvrdik [9] recommend $N = 4d$, $F = 0.8$ and $C = 0.5$.

3 Verification and its results

The above described procedure for nonparametric estimation of GARCH-M model coefficients will be applied to estimate the risk premium of forward exchange rate of the Czech crown against Euro and U.S. dollar. According to the covered interest rate parity theory version, the following relationship must hold true:

$$1 + r = \frac{F_t^{t+1}}{S_t}(1 + r^f), \tag{14}$$

where r is the yield of assets denominated in the domestic currency, r^f is the yield of assets denominated in foreign currency, S_t is the spot exchange rate at time t, F_t^{t+1} is forward exchange rate agreed at time t and valid at time $t + 1$. Taking the logarithm of equation (14) and after a small adjustment to get:

$$f_t^{t+1} - s_t = r - r^f \tag{15}$$

where $f_t^{t+1} = \log F_t^{t+1}$ and $s_t = \log S_t$. Similarly, we can derive the formula for the expected spot exchange rate in the uncovered interest rate parity theory version:

$$\mathrm{E}_t s_{t+1} - s_t = r - r^f \tag{16}$$

where $\mathrm{E}_t s_{t+1}$ is the expected (log) spot exchange rate at time t to time $t + 1$. For a long time it was considered that the forward exchange rate should be an impartial estimate of the future spot exchange rate. However, the observed systematic deviations from a forward rate and future spot rate, this deviation should be risk premium of the forward exchange rate and the risk premium rp_t is defined as follows:

$$rp_t = r - r^f - (f_t^{t+1} - s_t). \tag{17}$$

Forward exchange rate premium was first modeled by a linear model. But a linear model tends to be only able to identify the deviation of forward rate from hypothesis which postulates that forward exchange rate should be an unbiased predictor of future spot rate. On the other hand, it fails to reliably quantify the size of this deviation. Another way of modeling the forward risk premium is based on the assets pricing model. It assumes that the future price of any asset, discounted by a stochastic discount factor must equal to its current price. This conclusion must hold for currency and the exchange rate which is the price of foreign currency expressed in domestic currency. As a result, the risk premium must be a function of future exchange rate volatility, the volatility of the stochastic discount factor, and covariance of both. If we assume that the discount factor is stable, then the risk premium of forward exchange rate is a function only of the exchange rate volatility. With this simplified assumption, we can model the risk premium of forward exchange rate GARCH-M model with the following specifications:
conditional mean equation:

$$rp_t = b_0 + \delta h_t + \epsilon_t, \tag{18}$$

conditional variance equation:

$$h_t = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta_1 h_{t-1}. \tag{19}$$

To estimate parameters of the GARCH-M model for modeling risk premia of forward exchange rate, we use the following time series: two series of daily spot exchange rate EUR / CZK and USD / CZK. In addition, four series of daily forward points of Euro and USD for three months and six months are used. We add these points to the values of spot rates to get daily series of forward rates EUR / CZK and USD / CZK for three months and six months. Because in the Czech Republic are not available daily returns of government bonds with maturity of 3 months and 6 months, we use the interbank market in Prague interest rate PRIBOR for three and six months. The corresponding rates for foreign currencies are the EURIBOR and LIBOR for U.S. dollar for the period of three and six months. All data are from period from May 2007 to February 2012. From these data we generate four series of risk premia of forward exchange rate EUR / CZK and USD / CZK according to the relation (17), which we denote as RPE3, RPE6, RPU3, RPU6. To check the stationarity of these series, the modified Dickey-Fuller test for unit root is performed.

GARCH-M model for the risk premium of forward exchange rate of the Czech crown against Euro and U.S. dollar specified by equations (18) and (19) is estimated by maximum likelihood. The corresponding

likelihood function is defined by (19), where $\epsilon_t = \Delta rp_t - \delta h_t$ and h_t is defined by (19). The first values $\epsilon_1 = h_1 = \frac{1}{T}\sum_1^T \Delta rp_t^2$. Since the choice of the kernel is not so important for the estimation procedure, the normal kernel is selected as a kernel function. When it comes to the length of the smoothing window, it is recommended in the literature to value $1.06hT = h_t^{-\frac{1}{5}}$. We respect this recommendation. The vector of coefficients is a five component vector: $\theta = (b_0, \delta, \alpha_0, \alpha_1, \beta_1)^T$, thus we have $d = 5$. The coefficients $\alpha_0, \alpha_1,$ and β_1 must be non-negative and their sum can not be greater than 1; for b_0 and δ, no other a priori assumptions are required. The values maximizing the likelihood function are estimated with the use of differential evolution, whose parameters are: $N = 10d = 50$, $F = 0.8$ and $C = 0.5$, as recommended in the literature. For each series of optimization 10,000 independent runs are performed. With regard to the accuracy of estimates, because it is the method of maximum likelihood estimates, they should have asymptotic properties. The results of nonparametric estimation procedure for coefficients of GARCH-M model on the risk premium of forward exchange rate of the Czech crown against Euro and U.S. dollar are shown in Table 1. The results of parametric estimation procedure for coefficients of GARCH-M model for the risk premium of forward exchange rate of the Czech crown against Euro and U.S. dollar are shown in Table 2. In both tables the standard deviation of estimates are included in square brackets. The results show that non-parametric estimation results differ substantially from

Table 1 Neparametric estimation results of GARCH-M model for risk premium

	RPE3	RPE6	RPU3	RPU6
b_0	7.4440e-4	2.0619e-4	1.9696e-3	8.7083e-4
	[0.0054-4]	[0.2218e-4]	[NaN]	[1.6771e-4]
δ	7.8239	4.1048	3.6916	4.3378
	[0.5718e-4]	[0.3356e-4]	[0.0045-6]	[0.0014-0]
α_0	8.5751e-4	7.9337e-4	5.5706e-4	5.2224e-4
	[0.0038e-4]	[0.0001-4]	[1.0317e-6]	[NaN]
α_1	0.0867	0.0296	4.6487e-3	5.0721e-3
	[0.0647e-4]	[0.0246e-4]	[3.5106e-6]	[3.0359e-5]
β_1	0.8560	0.9598	0.9886	0.9862
	[0.0638e-4]	[0.0012-4]	[8.6481e-6]	[6.6635e-9]
$\log L$	212.80	130.81	462.25	598.10

Table 2 Parametric estimation results of GARCH-M model for risk premium

	RPE3	RPE6	RPU3	RPU6
b_0	-0.0023	0.0004	0.0003	0.0012
	[0.0008]	[0.0009]	0.0005	[0.0012]
δ	4.4603	0.2441	-0.0738	-0.1182
	[1.4660]	[0.5325]	[0.6956]	[0.4105]
α_0	5.55e-6	2.85e-05	3.15e-06	6.33e-05
	[9.86-7]	[0.0001-4]	[1.44-06]	[8.69-06]
α_1	0.0547	0.1817	0.1840	0.1540
	[0.0049]	[0.0063]	[0.012]	[0.0100]
β_1	0.9378	0.8502	0.8523	0.8404
	[0.0043]	[0.0048]	[0.0085]	[0.0081]
$\log L$	122680	103108	186522	229112

the results of a parametric estimation for model GARCH-M. According to the value of the likelihood function obtained by nonparametric estimation of GARCH-M model for the risk premium of forward exchange rate of the Czech crown against Euro and U.S. dollar are higher than the values of this function estimated by a parametric estimation (parametric estimates after substituting into the likelihood function for nonparametric estimation gives lower values of the likelihood function, so they are suboptimal in terms of non-parametric estimate). Also, coefficient δ in the conditional mean equation is statistically significant if estimated non-parametrically in all cases, while this coefficients are statistically insignificant (the only

exception is the case of RPE3). Another difference between the two methods of estimation is that while we can very well control for stability condition imposed on coefficients $\alpha_0, \alpha_1,$ and β_1 when estimating them non-parametrically with differential evolution, in the case of parametric estimation of coefficients of GARCH-M model using traditional optimization methods we cannot do so. As a result, with non-parametric estimation, the sum of the coefficients of $\alpha_0, \alpha_1,$ and β_1 is always less than 1, with traditional optimization technique that condition is not fulfilled in the case of series RPU3 and RPE6. We are aware that the estimated coefficients of a GARCH model is always associated with a number of problems. The estimation of GARCH-M model is even more complicated because the conditional variance equation appears in the conditional mean. However, our proposed method of estimation of the GARCH-M model parameters seems to be reliable. Therefore, the risk premium of forward exchange rate of the Czech crown against Euro and U.S. dollar are statistically significantly quantified.

4 Conclusion

Parameters of a GARCH-type model are usually estimated by maximum likelihood method using parametric estimation technique. This approach is comfortable, but it can face some issues like inconsistency due to distributional misspecification, local optimum and inability to control the size of estimates. We propose an alternative nonparametric approach where both the coefficients of the model and the random components are estimated simultaneously from the data using differential evolution to solve the optimization problem. The suitability of our approach is validated for modeling forward exchange rate premium, and exchange rate of the Czech crown against the U.S. dollar and the Euro common currency GARCH-M model in the period from 2007 to 2012. The same model is also estimated traditionally. The results show that our approach seems to be superior to the traditional approach. It can identify a statistically significant effect on the conditional variance risk premium of forward exchange rate in all cases in the period under review, while the traditional approach this effect recorded in only two cases. The volatility in the exchange rate affects the amount of risk premium on a forward exchange rate of the Czech crown against U.S. dollar and the Euro, which is consistent with the theory.

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Nonlinear DSGE model of the Czech economy with time-varying parameters

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Abstract. In this paper, we study the changes in the structure and behaviour of a small open economy in a period of global recession and subsequent return to the long-run equilibrium. An example of the Czech real economy represented by nonlinear dynamic stochastic model of a general equilibrium with financial accelerator is employed to explore this problem. The development of time-varying structural parameters is identified using the second order approximation of a nonlinear DSGE model. The model is estimated with the use of unscented particle filter. It is the prime goal of this paper to identify the most important changes of the structural parameters and interpret them in terms of behaviour of representative economic agents. We focus on the subset of parameters that are generally considered deep, i.e. time-invariant. Changes of these parameters during the period of global recession would imply that the structure of the economy together with its behaviour changed as a consequence of the recession. The changing behaviour of the economy is described by time-varying impulse response functions.

Keywords: nonlinear DSGE model, time-varying parameters, unscented particle filter.

JEL classification: E32, E44
AMS classification: 91B64

1 Introduction

Having witnessed the turbulent period of financial and economic crisis of 2007–2009, we find it desirable to look back and investigate the possible structural changes of the Czech economy that might have occurred in that period. Between 2008Q3 and 2009Q3 the Czech economy experienced the biggest contraction in recent history and the real GDP declined by approximately 5 %. Since 2011Q3, the Czech economy is stuck in a shallow recession that is one of the longest lasting in recent history. The aim of this paper is to identify the most important changes of the structural parameters during and after the recent crisis and to interpret these changes in terms of behaviour of representative economic agents. We also seek to find out to what extent was the economic development in this period influenced by the structural changes and related changes in the behaviour of the economic system in comparison to the exogenous shocks.

The question of structural stability of the macroeconomic system was investigated for example by Fernández-Villaverde and Rubio-Ramirez [2]. The authors studied the stability of structural parameters of a medium-scale nonlinear DSGE model of the U.S. economy. Significant changes in values of structural parameters were reported. Yano [7] proposed a new method of studying time-varying structural parameters with the use of Sequential Monte Carlo methods and identified the most important changes of the structural parameters of a DSGE model of the U.S. economy. Vašíček *et al.* [6] studied the structural changes of the Czech economy and found evidence of parameter drifting.

In this paper, we estimate a DSGE model of the Czech small open economy with time-varying parameters. We perform a filtration of a nonlinear model with the use of unscented particle filter (UPF) to identify the unobserved trajectoeies of the time-varying parameters. Significant changes of parameter values are interpreted as structural changes of the economy.

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2 The model

Since we focus on the period of financial and subsequent economic crisis, a DSGE model with financial frictions is used for the analysis. In our paper we use the model framework developed by Shaari [3] that includes financial accelerator mechanism proposed by Bernanke *et al.* [1]. This appropriately complex medium-sized model of a small open economy incorporates important real as well as nominal rigidities and allows us to describe the Czech economy in a reasonable detail. Structure of the model is quite standard, therefore, we will describe only the most important features of the model.

The model contains households, entrepreneurs, retailers, central bank and foreign sector. The households receive wages for supplied labour, government transfers, profits made by retailers and domestic and foreign bonds returns. Domestic bonds pay fixed nominal return in domestic currency while foreign non-contingent bonds give a risk adjusted nominal return denominated in foreign currency. The debt-elastic risk premium contains exogenous AR(1) component of risk-premium or uncovered interest parity shock. The households then spend their earnings on consumption and domestic and foreign bonds acquisition.

2.1 Entrepreneurs

The entrepreneurs play two important roles in the model. They run wholesale goods producing firms and they produce and own the capital. Market of intermediate goods as well as capital goods market is assumed to be competitive. The wholesale goods production is affected by domestic productivity AR(1) shock and the capital goods production is subject to capital adjustment costs. Entrepreneurs finance the production and ownership of capital K_t by their net-worth N_t and borrowed funds. Cost of borrowed funds is influenced by borrower's leverage ratio via external finance premium,

$$EFP_t = \left(\frac{N_t}{Q_{t-1}K_t}\right)^{-\chi}, \tag{1}$$

where Q_t is real price of capital or Tobin's Q and χ is financial accelerator parameter. To maximize profit, the entrepreneurs choose the optimal level of capital and borrowed funds. Each turn a proportion $(A_t^{NW}\varsigma)$ of entrepreneurs leaves the market and their equity $(A_t^{NW}\varsigma)V_t$ is transferred to households in a form of transfers. A_t^{NW} is a shock in entrepreneurial net worth. It influences the development of net worth by changing the bankruptcy rate of entrepreneurs. Its logarithmic deviation from steady state is assumed to evolve according to AR(1) process. ς is the steady-state bankruptcy rate.

2.2 Retailers

Next, there are two types of retailers in the model. Home goods retailers and foreign goods retailers. Both types of retailers are assumed to operate in conditions of monopolistic competition. Home good retailers buy domestic intermediate goods at wholesale price and sell the final home goods to the consumers. Foreign good retailers buy goods from foreign producers at the wholesale price and resell the foreign goods to the domestic consumers. The difference between foreign wholesale price expressed in domestic currency and final foreign goods price, i.e. deviation from law of one price is determined by exogenous AR(1) shock. By Calvo-type price setting and inflation indexation of the retailers the nominal rigidities are introduced into the model.

2.3 Central bank

The central bank determines the nominal interest rate in accordance with forward/backward-looking Henderson McKibbin Taylor interest rate rule. Deviations of interest rate from the interest rate rule are explained as monetary policy i.i.d. shocks.

2.4 Foreign sector

The foreign economy variables - real output, CPI inflation and nominal interest rate, are modelled using a structural VAR(1) model.

2.5 Time-varying parameters

All the model parameters are considered time-varying with the exception of shock autoregression parameters and standard deviations. Time-varying parameters are defined as unobserved endogenous variables with following law of motion

$$\theta_t = (1 - \alpha^\theta) \cdot \theta_{t-1} + \alpha^\theta \cdot \overline{\theta} + \nu_t^\theta \tag{2}$$

where θ_t is a general time-varying parameter, $\overline{\theta}$ is initial value of this parameter, α^θ is an adhesion parameter and $\nu_t^\theta \sim N(0, \sigma_\nu^\theta)$ is exogenous innovation in the value of parameter θ_t. Setting of the adhesion parameter α^θ influences the tendency of the time-varying parameter θ_t to return to its initial value $\overline{\theta}$. With $\alpha^\theta = 0$, the time-varying parameter would be defined as random walk, while with $\alpha^\theta = 1$, the parameter would be white noise centred around the initial value $\overline{\theta}$. For the purposes of this exercise, we set the adhesion parameter for all the time-varying parameters to a value of 0.25.

3 Estimation technique

Unscented particle filter (UPF) is used to identify the unobserved states of the DSGE model, including the time-varying parameters, in this paper. In this section we briefly describe the main principles of this nonlinear particle filter.

3.1 Unscented transformation

Since the UPF works with the unscented transformation (UT), we will first describe the principles of this transformation. UT is a method of calculating the statistics of a nonlinear transformation of a random variable. UT estimates are accurate up to the second order of Taylor expansion of the transformation function. Suppose that we have an n-dimensional random variable x with mean $\overline{x}$ and covariance matrix P_x. To calculate the statistics of its nonlinear transformation $y = f(x)$ we have to calculate a set of sigma points and weights $\{X_i, W_i\}_{i=0}^{2n}$. To capture the mean and covariance of random variable x, the sigma points and weights have to be chosen in a following way

$$X_0 = \quad \overline{x} \qquad\qquad W_0 = \frac{\kappa}{(n+\kappa)} \qquad i = 0 \tag{3}$$

$$X_i = \quad \overline{x} + \left(\sqrt{(n+\kappa)P_x}\right)_i \qquad W_i = \frac{1}{2(n+\kappa)} \qquad i = 1, \ldots, n \tag{4}$$

$$X_i = \quad \overline{x} - \left(\sqrt{(n+\kappa)P_x}\right)_{i-n} \qquad W_i = \frac{1}{2(n+\kappa)} \qquad i = n+1, \ldots, 2n, \tag{5}$$

where κ is a scaling parameter and $\left(\sqrt{(n+\kappa)P_x}\right)_i$ is the ith column of the matrix square root of $(n+\kappa)P_x$. Sum of weights W_i is equal to one. Mean and covariance matrix of y can then be described as $\overline{y} = \sum_i W_i f(X_i)$ and $P_y = \sum_i W_i (f(X_i) - \overline{y})(f(X_i) - \overline{y})^T$.

3.2 Nonlinear state-space model

In this subsection we introduce the notation used to describe the nonlinear state-space system. The state transition is described by the transition equation

$$x_t = g(x_{t-1}, w_{t-1}), \tag{6}$$

where $x_t \in \mathbb{R}^{n_x}$ is the vector of unobserved states and $w_t \in \mathbb{R}^{n_w}$ is the process noise with covariance matrix Q. Observations are related to the unobserved states by the measurement equation

$$y_t = h(x_t, v_t), \tag{7}$$

where $y_t \in \mathbb{R}^{n_y}$ is the vector of observations and $v_t \in \mathbb{R}^{n_y}$ is the measurement noise with covariance matrix R.

3.3 Unscented particle filter

Unlike basic Kalman filter that is optimal only for linear systems with Gaussian noise, the unscented particle filter is a more sophisticated tool that can be used even for nonlinear state-space systems with non-Gaussian. In this section, we provide only the basic principles of the algorithm. A detailed description can be found for example in Van Der Merwe *et al.* [5].

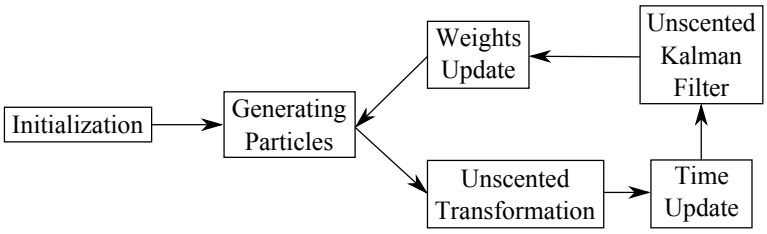

Figure 1 Unscented particle filter

Figure 1 contains a diagram of the UPF algorithm. In a condensed form, the UPF algorithm can be described as follows:

1. Initialization: $t = 0$, set the prior mean $\overline{x}_0$ and covariance matrix P_0 for the state vector x_t.
2. Generating particles: Draw a total of N particles $x_t^{(i)}$, $i = 1, \ldots, N$ from distribution $p(x_t)$ with mean $\overline{x}_t$ and covariance matrix P_t and update the mean and covariance matrix of augmented state vector $x_t^a = \begin{bmatrix} x_t & w_t & v_t \end{bmatrix}^T$ ($x_t^a \in \mathbb{R}^{n_a}$, $n_a = n_x + n_w + n_y$).
3. Unscented transformation: Calculate sigma points and weights $\{X_i, W_i\}_{i=0}^{2n_a}$ for the random vector x_t^a.
4. Time Update: $t = t+1$, for each particle ($i = 1, \ldots, N$) propagate the particle into future with the use of sigma points and transition and measurement equation and calculate means $\overline{x}_{t|t-1}^{(i)}$, $\overline{y}_{t|t-1}^{(i)}$ and covariance matrices $P_{t|t-1}^{(i)}$, $P_{y,y}^{(i)}$ and $P_{x,y}^{(i)}$.
5. Unscented Kalman filter: For each particle ($i = 1, \ldots, N$) calculate $K_t^{(i)} = P_{x,y}^{(i)} \left(P_{y,y}^{(i)}\right)^{-1}$, $\overline{x}_t^{(i)} = \overline{x}_{t|t-1}^{(i)} + K_t^{(i)}(y_t - \overline{y}_{t|t-1}^{(i)})$ and $P_t^{(i)} = P_{t|t-1}^{(i)} - K_t^{(i)} P_{y,y}^{(i)} \left(K_t^{(i)}\right)^T$
6. Weights update: for each particle ($i = 1, \ldots, N$) draw a sample $x_t^{(i)}$ from $q(x_t^{(i)}|x_{0:t-1}, y_{1:t}) = = N(\overline{x}_t^i, P_t^{(i)})$ and evaluate the importance weight $w_t^i \propto \frac{p(y_t|x_t^i)p(x_t^{(i)}|x_{t-1}^{(i)})}{q(x_t^i|x_{0:t-1}, y_{1:t})}$. For all particles together, normalize the weights and calculate $\overline{x}_t = \sum_i w_t^{(i)} x_t^{(i)}$ and $P_t = \sum_i w_t^{(i)} (x_t^{(i)} - \overline{x}_t)(x_t^{(i)} - \overline{x}_t)^T$. Continue by step 2 until $t = t_{max}$.

In our application we performed 12 runs of the UPF with 30.000 particles each for the second order approximation of the nonlinear DSGE model.

3.4 Initial values

Before the application of the UPF algorithm we estimated the model with constant parameters to obtain estimates of autoregression parameters and standard deviations of structural shocks that are considered constant even in the UPF. Posterior means of the structural parameters were used as initial values of the time-varying parameters ($\overline{\theta}$) in the UPF estimation. Standard deviations of time-varying parameter innovations (σ_ν^θ) were set proportional to the standard deviations of posterior estimates of the model with constant parameters. Constant model parameters were estimated using Random Walk Metropolis-Hastings algorithm as implemented in Dynare toolbox for Matlab. Two parallel chains of 1.000.000 draws each were generated during the estimation. First 50 % of draws were discarded as burn-in sample. The scale parameter was set to achieve acceptance rate around 30 %. Estimation of the model with constant parameters is described in detail in Tvrz *et al.* [4].

4 Data

Quarterly time series of eight observables were used for the purposes of estimation. These time series cover the period between 1996Q1 and 2012Q4 and contain 68 observations each.

For the domestic economy, time series of real aggregate product, consumer price index (CPI), 3-month PRIBOR and Prague stock exchange (PSE) PX index were used. These seasonally adjusted time series were obtained from the database of Czech Statistical Office and PSE. The time series of PX stock market index was used as a proxy for the entrepreneurial net worth.

The foreign economy was represented by eleven founding countries of the Euro area plus Greece that joined the monetary union in 2001. For these twelve economies, the seasonally adjusted time series of real aggregate product, CPI index and 3-month EURIBOR was used. Also, time series of CZK/EUR real exchange rate was used. These data were obtained from the Eurostat database.

Original time series were transformed prior to estimation so as to express the logarithmic deviations from steady state. Relative changes of the CPI indices were demeaned. The logarithmic deviations of remaining observables from their steady state were calculated with the use of Hodrick-Prescott filter with parameter $\lambda = 1600$.

5 Empirical results

In this section we present and discuss the most interesting empirical results obtained by the UPF estimation. Figure 2 contains the trajectories of the most variable time-varying parameters.[1] The dashed line represents the initial values of time-varying parameters ($\overline{\theta}$).

Parameters of the financial frictions were identified among the most time-variable parameters. According to our results, elasticity of the external finance premium χ declined since 2006 and quite sharply in 2009. This development can be interpreted in terms of commercial banks behaviour during the crisis in a following way. At first the sensitivity of banks to the leverage ratio of entrepreneurs declined because of favourable economic conditions during the economic boom of 2006–2008. Sharper decline of χ in 2009 is probably related to the changes in the selection process of the credit applicants. A segment of applicants that were perceived to be more risky was rejected by the banks during the crisis and only the investment projects with best prospects were granted funding. Similar arguments can be used to explain the trajectory of capital adjustment costs parameter Ψ_I. Increased bankruptcy rate ς during the economic crisis of 2009 can be directly attributed to the economic crisis.

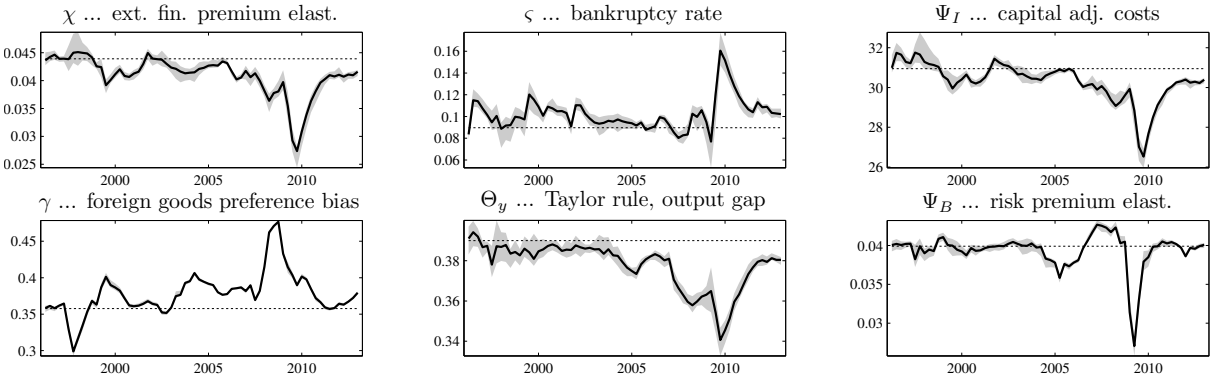

Figure 2 Selected time-varying parameters

Development of foreign goods preference bias γ can be related to the changing openness of the Czech economy and the development of international trade. Foreign goods preference increased during 2007–2008 and declined after 2008 with the onset of worldwide economic difficulties and downturn in international trade.

[1] Three measures of time-variability of time-varying parameter trajectories $\{\theta_t\}_{t=1}^{68}$ were considered: $\frac{\text{std}(\theta_t)}{\text{mean}(\theta_t)}$, $\frac{\max(\theta_t)-\min(\theta_t)}{\text{mean}(\theta_t)}$ and $\frac{\text{mean}(|\theta_t-\theta_{t-1}|)}{\text{mean}(\theta_t)}$.

Parameters of the Taylor interest rule are quite stable and can be considered deep with the exception of the output gap weight Θ_y. Decline of this parameter after 2006 suggests that the central bank reacted to the development of the output gap less and less vigorously.

Also, the parameter of risk premium elasticity Ψ_B shows distinct development during the economic crisis of 2009. Remaining structural parameters, including the consumption habit of households, Calvo parameters of domestic and imported goods, inflation indexation of prices, weight of inflation and smoothing parameter of the Taylor rule, are much more stable and can be considered time-invariant (deep).

6 Conclusion

It was the goal of this paper to identify the most important structural changes of the Czech economy during and after the financial and economic crisis of 2007–2009. A nonlinear DSGE model of a small open economy with financial frictions and time-varying parameters was estimated by the unscented particle filter for this purpose. We found some evidence for structural changes of the economy during the period of interest. We identified several parameters, including the parameters of financial frictions, that showed interesting development during the financial and economic crisis. Remaining parameters stayed relatively stable and can be, therefore, considered deep.

By studying the time-varying impulse response functions of the model economy we saw that the changing structural parameters influenced its behaviour during the crisis to some extent. However, since the UPF estimation identified the structural shocks similarly to the model with constant parameters, the structural shocks probably played dominant role. More detailed analysis of the question of changes in the behaviour of the economic system with time-varying structural parameters is left for further research.

Acknowledgements

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A Note on Computation of Causal Effect in Economic Models

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Abstract. One of the phenomena connected with the field of economics is an existence of causal relations among entities. Economic theories provide information needed for the creation of causal diagrams (the diagrams with variables connected by links of causal character). The question then arises concerning the change of model variables when an external intervention appears (in the context of the paper an intervention is to be understood as a setting of the fixed value or values on the set of variables in the model by an external agent). The paper aims at the analysis and presentation of means used to measure the causal effect (i.e., the changes in related variables in response to the intervention on specific set of variables) in a causal diagram. Among the significant features of the presented method ranks the ability to compute (under certain conditions) considered causal effect without necessity to employ all model variables. The impact of this methodology is important in situations with unmeasured variables involved. Additionally, this type of model provides a possibility to deal with scenarios where controlled experiments are impossible to be performed because of the costs or either ethical or legal obstacles. The methodology used in the paper is based on Judea Pearl's approach to causality.

Keywords: causal effect, do calculus, intervention, unmeasured variables.

JEL classification: C50
AMS classification: 62H99

1 Introduction

The contribution considers an economic application of the Pearl's approach to the causality modeling elaborated into three subsequent models. The presented application discusses the causal relations among the selected set of economic variables (i.e., production R, unemployment rate U, gross domestic product G, tax policy T, investment incentives I, demand decrease D, key business leaving their enterprizes in the country B). Throughout the paper, all variables are considered as discrete with finite number of values. Also the economic problem itself and the connections between variables in the considered models are highly simplified in order to facilitate the explanations and reduce the extent of computations. For the sake of presentation of dealing with unmeasured quantities, the variable crisis C is included in the models.

In the process of the causal model design a disputable question sometimes appears – "Which variable has a role of the cause and which one is the effect?" (see, e.g., the relation between GDP and unemployment rate). The structure of models was set up on the basis of the fact that the cause precedes the effect, together with the influence of temporal order according to [4].

But first of all, let us clarify a few basic notions concerning the modeling of causal relations. A *directed acyclic graph* (DAG) is a graph with all edges directed and containing no directed cycles. A *causal structure* of a set of variables V is a DAG with each node corresponding to a distinct element of V and each arrows representing a direct functional relationship among corresponding variables.

A *causal model* can be (according to [6] or [7]) specified as a pair $M = \langle D, \Theta_d \rangle$ (D is a causal structure, Θ_D is a set of parameters compatible with D). The parameters Θ_D assign a function $x_i = f_i(pa_i, u_i)$ to

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each $X_i \in V$ and a probability measure $P(u_i)$ to each U_i, where pa_i are particular values of variables from PA_i, i.e., of the parents of X_i in D and where each U_i is a random disturbance (*unmeasured variable*) distributed according to $P(u_i)$, independently of all other U.

An *intervention* (or an *action*) denoted $do(X = x)$ is setting the fixed value by an external force on the specific set of variables in the diagram. This is often shortened to $do(x)$ or $\hat{x}$ (see, e.g., [7]).

Given two disjoint sets of variables X and Y, a *causal effect* [8] of X on Y is a function from X to the space of probability distributions on Y and is denoted by $P(y|\hat{x})$. For each realization x of X, $P(y \mid \hat{x})$ gives the probability of $Y = y$ induced by deleting from the causal model all equations corresponding to variables in set X and substituting $X = x$ in the remaining equations.

Provided that all variables are observable, the post-intervention distribution (therefore also the causal effect) can be expressed from the pre-intervention probabilities according to the Formula (1) for a truncated factorization in case of intervention on variables from set S (see, e.g., [8])

$$P(x_1, \ldots, x_n \mid \hat{s}) = \begin{cases} \prod_{i|X_i \notin S} P(x_i \mid pa_i) & \text{for } x_1, \ldots, x_n \text{ consistent with } s, \\ 0 & \text{for } x_1, \ldots, x_n \text{ inconsistent with } s. \end{cases} \tag{1}$$

In contrary to the preceding text the following Theorem 1 holds only for singletons (not for sets of variables). When unmeasured variables are involved, the following four graphical conditions of Galles and Pearl (see [3]) can be used to identify a causal effect.

Theorem 1 (Galles and Pearl 1995)**.** *Let X and Y denote two singleton variables in a causal model characterized by a DAG G. A necessary*[1] *and sufficient condition for identifiability of $P(y \mid \hat{x})$ is that G satisfies one of the four conditions:*

1. *There is no back-door path from X to Y in G; that is $(X \perp\!\!\!\perp Y)_{G_{\underline{X}}}$.*
2. *There is no directed path from X to Y in G.*
3. *There exist a set of nodes L that blocks all back-door paths from X to Y so that $P(l \mid \hat{x})$ is identifiable.*
4. *There exist sets of nodes Z_1 and Z_2 such that:*
   - *(i) Z_1 blocks every directed path from X to Y (i.e., $(Y \perp\!\!\!\perp X \mid Z_1)_{G_{\overline{Z_1}\,\overline{X}}}$);*
   - *(ii) Z_2 blocks all back-door paths between Z_1 and Y (i.e., $(Y \perp\!\!\!\perp Z_1 \mid Z_2)_{G_{\overline{X}\,\underline{Z_1}}}$);*
   - *(iii) Z_2 blocks all back-door paths between X and Z_1 (i.e., $(X \perp\!\!\!\perp Z_1 \mid Z_2)_{G_{\overline{X}}}$);*
   - *(iv) Z_2 does not activate any back-door paths from X and Y (i.e., $(X \perp\!\!\!\perp Y \mid Z_1, Z_2)_{G_{\overline{Z_1}\,\overline{X(Z_2)}}}$).*

Where $G_{\overline{X}}$ is a graph obtained from G by deleting all arrows pointing to nodes in X and $G_{\underline{Z_1}}$ is a graph constructed from G by deleting all arrows emerging from nodes of Z_1. In the context of graph $G_{\overline{Z_1}}$ of condition (iv), the symbol $X(Z_2)$ denotes a set of X-nodes that are not ancestors of any Z_2 node in $G_{\overline{Z_1}}$. The notion of back-door path *means that we consider only paths with an arrow pointing at the first of considered pair of sets (namely in Condition 1 paths with an arrow pointing to X).*

2 Modeling of intervention on tax policy

2.1 A basic model

Let us study a model of effects of tax changes and introduction of investment incentives on production, unemployment and gross domestic product (see Figure 1).

Namely, we consider the influence of change in tax policy T on gross domestic product G. First of all let us express the joint probability distribution of the model $P(c, t, i, r, u, g)$ in the terms of dependency structure imposed by the graph of the considered causal model. Hence

$$P(c, t, i, r, u, g) = P(c) \cdot P(t|c) \cdot P(i|c) \cdot P(r|t, i) \cdot P(u|r, c) \cdot P(g|t, r, u).$$

[1] A variant of Theorem 1 published by Galles and Pearl [3] was formulated as a sufficient condition. But Huang and Valtorta [5] showed that it can be formulated as both necessary and sufficient.

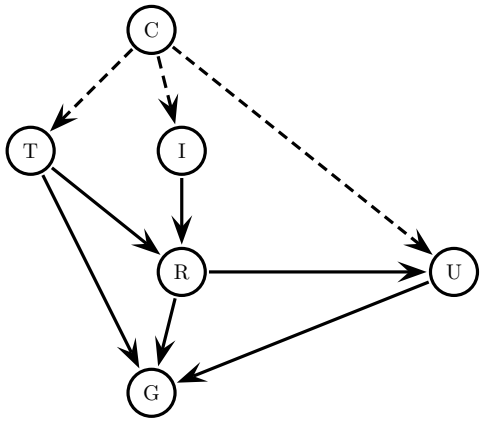

Figure 1 A graph of causal model shows the influence of tax changes and introduction of investment incentives on production, unemployment and gross domestic product.

Now, the effect of intervence on the tax policy $\hat{t}$ removes the factor T and we obtain the conditional distribution of all other variables given the intervention on taxes

$$P(c, i, r, u, g|\hat{t}\,) = P(c) \cdot P(i|c) \cdot P(r|t, i) \cdot P(u|r, c) \cdot P(g|t, r, u).$$

Let us derive the desired effect of intervence on the tax policy on gross domestic product via the summation over C, I, R and U

$$\begin{aligned} P(g|\hat{t}\,) &= \sum_{c,i,r,u} P(c) \cdot P(i|c) \cdot P(r|t, i) \cdot P(u|r, c) \cdot P(g|t, r, u) = \\ &= \sum_{i,r,u} P(p|t, i) \cdot P(g|t, r, u) \sum_{c} P(c) \cdot P(i|c) \cdot P(u|r, c). \end{aligned} \tag{2}$$

Let us focus on an elimination of the unmeasured variable (crisis) from the corresponding sum in the preceding Formula (2)

$$\sum_{c} P(c) \cdot P(i|c) \cdot P(u|r, c)$$

which can be equivalently expressed as

$$\begin{aligned} \sum_{c} P(c) \cdot P(i|c) \cdot P(u|r, c) &= \sum_{c} P(u|r, c) \cdot P(i, c) = \\ &= \sum_{c} P(u|r, c) \cdot \sum_{t'} P(c, t', i) = \\ &= \sum_{c} \sum_{t'} P(u|r, c) \cdot P(c|t', i) \cdot P(t', i). \end{aligned}$$

Notice that the following independencies are induced by a causal diagram in Figure 1

$$C \perp\!\!\!\perp R \,|\, T, I; \qquad U \perp\!\!\!\perp T, I \,|\, C, R$$

and therefore

$$\begin{aligned} \sum_{c} P(c) \cdot P(i|c) \cdot P(u|r, c) &= \sum_{c} \sum_{t'} P(u|r, c, t', i) \cdot P(c|r, t', i) \cdot P(t', i) = \\ &= \sum_{t'} P(u|r, t', i) \cdot P(t', i) \end{aligned}$$

Hence, we can express Formula (2) in the following manner

$$P(g|\hat{t}\,) = \sum_{i,r,u} P(p|t, i) \cdot P(g|t, r, u) \sum_{t'} P(u|r, t', i) \cdot P(t', i)$$

where the unmeasured factor of crisis C was eliminated and a possibility to enumerate the economic effect of an intervention on tax policy T is provided.

2.2 A basic model revised

In order to increase the precision, we will continue with an augmented model where a factor of business leaving their enterprizes is included (see left part of Figure 2). Again, our goal is to express an effect of the tax policy intervention on the resulting gross domestic product.

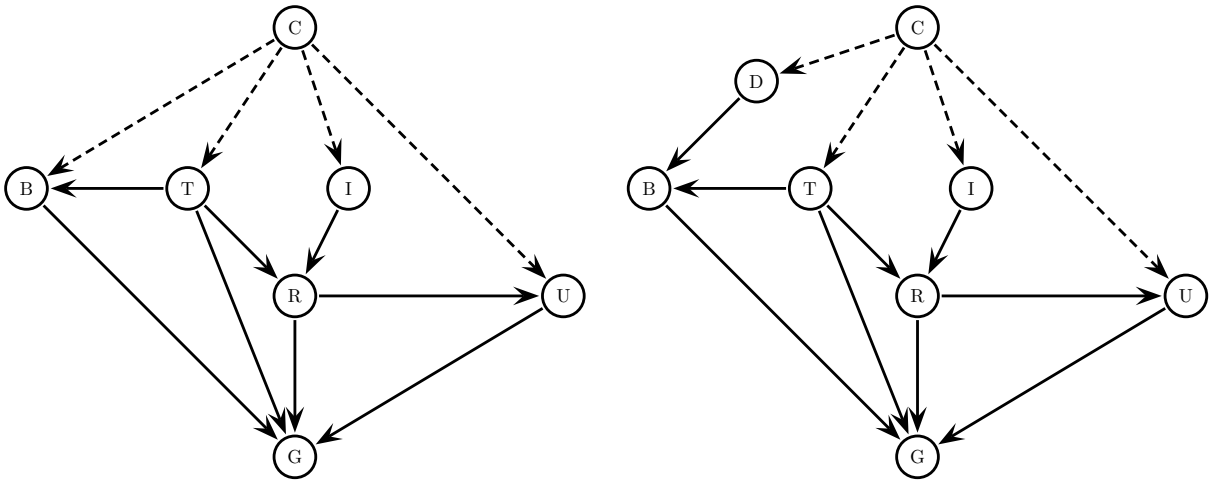

Figure 2 The left causal structure includes an effect of business leaving B (the effect of the intervention in tax policy T on the resulting gross domestic product G is impossible to enumerate under given conditions). The right model presents a possibility to fix this model by insertion of an additional measured variable (decrease of demand D).

Using the above presented technique of sequence of equivalent expressions we will not succeed in elicitation of the intervention effect and elimination of the influence of unmeasured variable.

This impossibility to identify a causal effect of intervention within the *do* calculus (see [3]) and in general (see [5]) can be inferred from the Theorem 1 in the following way:

Condition 1 is not satisfied because of the back-door paths existence, one example is a back-door path

$$G \leftarrow R \leftarrow I \leftarrow C \rightarrow T.$$

Condition 2 doesn't hold due to the existence of directed path between T and G (more precisely, it is a directed edge).

When it comes to Condition 3, let us first consider, e.g., a set $L = \{B, I, U\}$. In order to compute $P(b, i, u|\hat{t}\,)$ we obtain

$$P(c, b, i, r, u, g|\hat{t}\,) = P(c) \cdot P(b|c, t) \cdot P(i|c) \cdot P(r|T, i) \cdot P(u|r, c) \cdot P(g|b, t, r, u).$$

Marginalization over C, R and G, reorganization and factoring out gives

$$P(b, i, u|\hat{t}\,) = \sum_{r,g} P(r|t, i) \cdot P(g|b, t, r, u) \underbrace{\sum_{c} P(c) \cdot P(b|c, t) \cdot P(i|c) \cdot P(u|r, c)}_{(*)}.$$

But we are not able to simplify the formula for $P(b, i, u|\hat{t}\,)$ due to inability to perform a summation over the unmeasured variable crisis in $(*)$. The same problem arises for all possible choices of set L (i.e., $L = \{B, R, U\}$, $L = \{B, R, I, U\}$, etc.).

Non existence of any set that blocks directed path from T to G implies that even Condition 4 cannot be satisfied.

We can see that none of the conditions can be applied and therefore the causal effect of intervention cannot be inferred.

2.3 A possible solution

A solution providing the possibility to compute an effect of intervention in the considered case is to introduce another measurable variable into the model. A proposed causal structure can be seen in the right part of Figure 2.

In this augmented model we can see that the identification of causal effect of intervention is possible using the Theorem 1. In contrary to the above case we can consider $L = \{B, I, U\}$ and we obtain

$$P(b, i, u|\hat{t}\,) = \sum_{d,r,g} P(b|d,t) \cdot P(r|t,i) \cdot P(g|b,t,r,u) \underbrace{\sum_{c} P(c) \cdot P(d|c) \cdot P(i|c) \cdot P(u|r,c)}_{(**)}.$$

Unlike the previous example we are able to sum up the unmeasured variable crisis from the expression $(**)$ can be rearranged using the independencies in the model

$$C \perp\!\!\!\perp R \,|\, T, I; \qquad U \perp\!\!\!\perp T \,|\, C, R; \qquad U \perp\!\!\!\perp I \,|\, C, R; \qquad U \perp\!\!\!\perp D \,|\, C; \qquad D \perp\!\!\!\perp I \,|\, C$$

and we obtain

$$\begin{aligned}
\sum_{c} P(c) \cdot P(d|c) \cdot P(i|c) \cdot P(u|c,r) &= \sum_{c} P(i|c) \cdot P(d|c) \cdot P(u|c,r) = \\
&= \sum_{c} P(i|c) \cdot P(d|c,i) \cdot P(u|c,r) = && (D \perp\!\!\!\perp I \,|\, C) \\
&= \sum_{c} P(d,i,c) \cdot P(u|c,r) = \\
&= \sum_{c,t'} P(c,d,t',i) \cdot P(u|c,r) = \\
&= \sum_{c,t'} P(c|d,t',i) \cdot P(d,t',i) \cdot P(u|c,r) = && (C \perp\!\!\!\perp R \,|\, T, I;\ U \perp\!\!\!\perp I \,|\, C, R; \\
& && U \perp\!\!\!\perp D \,|\, C;\ U \perp\!\!\!\perp T \,|\, C, R) \\
&= \sum_{c,t'} P(c|d,t',i,u) \cdot P(d,t',i) \cdot P(u|c,r,d,t',i) = \\
&= \sum_{t'} P(d,i,t') \cdot P(u|r,d,i,t'). && (3)
\end{aligned}$$

Now the Condition 3 is satisfied and the causal effect is identifiable. Therefore, we can compute it using a direct elicitation in a similar manner like in a basic model (Subsection 2.1).

Let us start with a joint probability distribution stemming from the causal structure depicted in the right part of Figure 2

$$P(c,d,b,t,i,r,u,g) = P(c) \cdot P(d|c) \cdot P(t|c) \cdot P(b|d,t) \cdot P(i|c) \cdot P(r|t,i) \cdot P(u|r,c) \cdot P(g|b,t,r,u)$$

and elicit the effect of intervention on the model

$$P(c,d,b,i,r,u,g|\hat{t}\,) = P(c) \cdot P(d|c) \cdot P(b|d,t) \cdot P(i|c) \cdot P(r|t,i) \cdot P(u|r,c) \cdot P(g|b,t,r,u).$$

The effect of intervence on the tax policy on gross domestic product follows

$$\begin{aligned}
P(g|\hat{t}\,) &= \sum_{c,d,b,i,r,u} P(c) \cdot P(d|c) \cdot P(b|d,t) \cdot P(i|c) \cdot P(r|t,i) \cdot P(u|c,r) \cdot P(g|b,t,r,u) = \\
&= \sum_{d,b,i,r,u} P(b|d,t) \cdot P(r|t,i) \cdot P(g|b,t,r,u) \sum_{c} P(c) \cdot P(d|c) \cdot P(i|c) \cdot P(u|c,r). && (4)
\end{aligned}$$

The rearrangement in Formula (4) and separate summation over crisis variable C was performed in order to remove the effect of unmeasured variable. This step will be performed in the following text. But the sum $\sum_c P(c) \cdot P(b|c,t) \cdot P(i|c) \cdot P(u|c,r)$ from Formula (4) was already expressed above and we derived

Formula (3). Now, putting it back into the Formula (4) we can express the effect of intervention on tax policy on the gross domestic product in the following way

$$P(g|\hat{t}) = \sum_{d,b,i,r,u} P(r|d,t) \cdot P(r|i,t) \cdot P(g|b,t,r,u) \sum_{t'} P(d,i,t') \cdot P(u|r,d,i,t')$$

where we once more stress that the unmeasured factor of crisis C was eliminated.

3 Conclusion

Although our rather toy model is only a rough simplification, we showed that only small alterations of model can completely change our ability to compute a causal effect of an intervention. Particularly, we presented that introduction of a "business leaving" node ruins our ability to compute a causal effect. This inability is shown using the Galles and Pearl theorem (Theorem 1).

In contrary, in Subsection 2.3 we present that an introduction of one additional variable can enable the direct computation of the considered causal effect. This stems from the fact that this change of model fulfils (at least) one of the conditions in the Theorem 1.

Let us notice that the employment of the presented apparatus requires low-dimensional joint and conditional probabilities which can be in general estimated from the survey data and economic yearbooks. But frequently the required distributions are not available. In such cases expert estimates can be used or additional methods need to be applied (e.g., a search of joint probability distribution with given marginals using a famous IPF procedure [2]).

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Analysing spurious interactions between rating factors

Jiří Valecký[1]

Abstract. It is the known generally that omitting interactions may result in the inaccurate parameter estimates and in misinterpreting or incorrect assessing impact of the independent variable on the outcome. In addition, mismodelling variable (assuming linear relation instead of non-linear one) or omission of another adjustment variable may result in generating spurious interactions in the model. For these reasons, it is necessary to pay attention to correct transformation of independent variables and thoroughgoing examination of all identified potential interactions. Presuming a linear negative-binomial model of claim frequency for given policy insurance portfolio, we demonstrate the effect of these two issues on the model and we analyse the altered model to examine the potential interactions. First and foremost, using treatment-effect plot and likelihood ratio test, we identify the potential interaction between risk factors and thereafter we perform the sensitivity analysis to assess the stability of interaction on variable transformation and on inclusion of other adjustment variables into the model. Finally, we conclude which interactions may be considered as spurious and which should not be included into the model.

Keywords: negative-binomial model, fractional polynomial, interaction, treatment-effect plot, stability analysis, motor hull insurance, claim frequency.

AMS Classification: 62 J 12, 62 P 05
JEL Classification: C31, C58, G22

1 Introduction

The pure premium is usually determined in accordance with undertaken risk which implies that the annual premium paid should be determined in accordance with risk behaviour of given policyholder and other relevant rating factors. For this purpose, one can perform a tariff analysis based on the generalized linear model for claim frequency and claim severity separately [2] or construct a tweedie model to model pure premium directly, see [12] or [18]. Generally, the former is preferred because the separated analysis gives more insight into how the rating factors affect the pure premium, see [15] for more details.

In spite of the fact that trend of differentiating premiums according to the undertaken risk has been observable in recent decades, some relevant rating factors are not allowed to use for that purpose any more. For instance, it is very common that the insurers set the annual premium in motor hull insurance in compliance with the volume of an engine. Moreover, many of them also distinguish the size of district where the client lives or even respect the client‘s age. On the other hand, using of gender for differentiating purposes is prohibited since 21.12.2012 on the basis of judgement of EU’s Court of Justice.

Due to this fact, it is growing stronger to gain more accurate models for making of fairer insurance premium to acquire additional competition advantage or just calculate more precise pure premiums. Both reasons comprise not only seeking new raring factors but also constructing more precise models though employing of methods of non-linear modelling and modelling interactions among rating factors.

On the one hand, it is facile to identify two-term interaction, see [8] or [1] for more details, but on the other hand, addition to being difficult to handle three- or more-term interactions due to their complexity, see [4], the interaction models are difficult to interpret. Many econometricians and statisticians misinterpret the coefficients of the interaction term in non-linear models because the marginal effect of changing the interaction terms is dependent on interacted variables, see [1]. Moreover, also assessing statistical significance via traditional z-test cannot be conducted for the same reason, i.e. the dependency on the level of interacted variables. On the contrary, [6] argues that evaluating marginal effect is not necessary informative and is also difficult to interpret it in terms of the relationships among the variables.

General concept and consideration of assessment and testing for interactions are described in many textbooks for different types of outcome data (i.e. [20], [10], [5], [3] [19]). Thus, the question does not only consist in

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generating interactions to improve model fit, but also in examination of them with effort to reveal spurious interactions which are induced mostly by mismodelling, see [13] for more details about simulation study, or omitting influential variable [17].

Thus, we focused on identification of spurious interactions in presumed model for claim frequency of given motor-hull insurance portfolio. The aim of this paper is to determine interactions between rating factor under interest and to perform their analysis. We focus on detecting spurious interactions caused by omitting of other influential covariates and by mismodelling. The paper is organized as follows. The general model for claim frequency, fractional polynomials and interaction terms are described in Section 2. Section 3 consists of identification of potential and spurious interactions. Section 4 concludes the paper.

2 Identifying effect of rating factors on insured accident

Next, for the purpose of modelling claim frequency by interaction model, we describe firstly the negative-binomial model, in which continuous covariates may be modelled by fractional polynomial and to which is added interaction term.

2.1 Model for Claim Frequency

We presume linear multiplicative model for claim frequency. This model respects the heterogeneity in insurance portfolio across the policyholders using the gamma distribution. The model applied is known in insurance theory as negative-binomial which is widely used when overdispersion occurs. Except for that, our model is based on the log link function, which is not canonical, but is mostly applied by practitioners, [9], and is known as NB2 model.

The negative-binomial model (NB2) is usually derived from the Poisson model with gamma heterogeneity where the gamma noise has a mean of 1. Let's write a distribution function in the form of

$$f\left(y;\alpha,\mu\right)=\frac{\exp\left(-\alpha_i\mu_i\right)\left(\alpha_i\mu_i\right)^{y_i}}{y_i!}, \tag{1}$$

where μ_i is an individual intensity $\exp\left(x_i\beta\right)$ and α_i is a random heterogeneity factor $\exp\left(\varepsilon_i\right)$. Considering various time exposition w_i, the Equation (1) is rewritten in the form of

$$f\left(y;\alpha,\mu\right)=\frac{\exp\left(-\alpha_i w_i\mu_i\right)\left(\alpha_i w_i\mu_i\right)^{y_i}}{y_i!}. \tag{2}$$

From the Equation (2) we obtain the negative binomial p.d.f., for derivation without exposition see [7],

$$f\left(y;\mu,\alpha\right)=\frac{\Gamma\left(y_i+1/\alpha\right)}{\Gamma\left(y_i+1\right)\Gamma\left(1/\alpha\right)}\left(\frac{1}{1+\alpha w_i\mu_i}\right)^{\frac{1}{\alpha}}\left(1-\frac{1}{1+\alpha w_i\mu_i}\right)^{y_i} \tag{3}$$

and general log-likelihood function

$$\ell\left(\mu;y,\alpha\right)=\sum_{i=1}^{n}\left\{\begin{array}{l} y_i\ln\left(\frac{\alpha w_i\mu_i}{1+\alpha w_i\mu_i}\right)-\left(\frac{1}{\alpha}\right)\ln\left(1+\alpha w_i\mu_i\right)+\ldots \\ \ldots+\ln\Gamma\left(y_i+\frac{1}{\alpha}\right)-\ln\Gamma\left(y_i+1\right)-\ln\Gamma\left(\frac{1}{\alpha}\right)\end{array}\right\}. \tag{4}$$

Inserting the inverse link function in the form of $\mu_i=\exp\left(x_i\beta\right)$ into (4), we obtain the log-likelihood function

$$\ell\left(\mu;y,\alpha\right)=\sum_{i=1}^{n}\left\{\begin{array}{l} y_i\ln\left(\frac{\alpha w_i\exp\left(x_i\beta\right)}{1+\alpha w_i\exp\left(x_i\beta\right)}\right)-\left(\frac{1}{\alpha}\right)\ln\left(1+\alpha w_i\exp\left(x_i\beta\right)\right)+ \\ +\ln\Gamma\left(y_i+\frac{1}{\alpha}\right)-\ln\Gamma\left(y_i+1\right)-\ln\Gamma\left(\frac{1}{\alpha}\right)\end{array}\right\}, \tag{5}$$

where the parameters α,β are estimated using the Newton-Raphson or IRLS algorithm.

2.2 Fractional polynomials

The expression $x_i\beta = g(x)$ in the log-likelihood function (5) may not necessary be linear. One of the techniques how to handle nonlinearity is using fractional polynomials (FPs). Thus, let's define the nonlinear function and rewrite the expression $g(x)$ into the form of

$$g(x) = \beta_0 + \sum_{j=1}^{J} \beta_{1j} F_j(x_1) + \beta_2 x_2 + ... + \beta_K x_K , \tag{6}$$

where $F_j(x_1)$ is a particular type of power function. The power p_j could be any number, but [16] restricts the power to be among the set $S \in \{-2;-1;0,5;0;0,5;1;2;3\}$, where 0 denotes the log of the variable. The remaining functions are defined as

$$F_j(x_1) = \begin{cases} x_1^{p_j}, p_j \neq p_{j-1}, \\ F_{j-1}(x_1)\ln(x_1), p_j = p_{j-1}. \end{cases} \tag{7}$$

Finally, we note that the identification and comparison of the most appropriate FPs could be made by the sequential procedure, [16], or closed test procedure, [14], which is generally preferred over.

2.3 Interaction terms

In literature, there are two definitions and interpretations of the interaction effect. Firstly, [10], [19] and others define the interaction effect as changing the slope parameter of one variable depending on the level of the other variable. On the other hand, [1] interpret the interaction effect as marginal effect of interacted x_1 and x_2 via cross-derivative of probability function $F(\boldsymbol{\beta}; \mathbf{x}_i)$.

Thus, let us assume the interaction between variables x_1 and x_2 which both may be categorical or continuous and let variable x_1 be a rating factor under interest. The changing slope parameter of x_1, in accordance with the first definitions, is depending on the level of x_2, thus

$$\beta_{x_1 \times x_2} = \beta_1 x_1 + \beta_{12} x_1 x_2 \text{ for all } x_2 . \tag{8}$$

It is obvious that the coefficient $\beta_{x_1 \times x_2}$ varies dependently on the level of x_2.

The former definition of the interaction is proposed by [1] who defined the interaction effect on the estimated cross-derivative of the terms in the interaction, not on the coefficient of the interaction term, thus

$$\frac{\partial^2 F(\boldsymbol{\beta}; \mathbf{x}_i)}{\partial x_1 \partial x_2} = \beta_{12} F'(\boldsymbol{\beta}; \mathbf{x}_i) + (\beta_1 + \beta_{12} x_2)(\beta_2 + \beta_{12} x_1) F''(\boldsymbol{\beta}; \mathbf{x}_i). \tag{9}$$

We can see from Equations (8) and (9) that the interaction term can be non-zero in both cases even if $\beta_{12} = 0$. Except for that, the interaction effect can have different signs for different values of covariates, thus the sign of β_{12} is ambiguous.

3 Check of interactions in the model

In this section, we demonstrate the effect of other covariates included in the model and effect of mismodelling on occurrence of spurious interactions. We presume linear NB2 model intended for modelling claim frequency of given motor hull insurance portfolio and estimated on data sample encompassing characteristics of policies during the year 2008 (42 190 observations). The explanatory variables are following: vehicle age (*agecar*), engine volume (*volume*) and engine power (*kw*), owner's age (*ageman*), vehicle price (*price*), number of citizens in a region (*nocit*), company car (*company*), gender of policyholder (*gender*), type of fuel (*fuel*) and district area (*district*).

To show the causation of including influential covariates into the model and generating spurious interactions, we firstly omitted the presumed NB2 model and constructed two term models for all 43 pairs of variables mentioned above. Thus, we obtained 43 different main-effect model, to which is added one interaction term and which is tested against the original two term model without interaction by LR test, for p-values see **Table 1**.

	agecar	*volume*	*kw*	*ageman*	*price*	*nocit*	*company*	*fuel*	*gender*	*district*
agecar	-									
volume	0.2269	-								
kw	**0.0063**	**0.0000**	-							
ageman	0.3239	0.4563	0.9077	-						
price	0.5865	1.0000	**0.0000**	0.0782	-					
nocit	**0.0140**	1.0000	0.6328	0.6274	1.0000	-				
company	**0.0008**	**0.0023**	**0.0118**	-	0.1811	0.2125	-			
fuel	0.9987	0.9747	0.9981	0.7225	1.0000	0.9318	**0.0004**	-		
gender	0.2847	0.4619	0.1692	0.4738	0.0953	0.5629	-	0.5785	-	
district	0.1845	0.8689	0.6366	0.7450	0.7640	0.0638	0.9945	0.9945	0.8109	-

Table 1 p-values of LR test: two term interaction models against two term model without interaction

According to the results in the table above, 8 of the 43 interactions are significant at the 0.05 level. To reveal the spurious interactions caused by neglecting some influential covariate in the model, we consider our presumed linear NB2 model and we extend it by adding one of the 43 interactions. All 43 interaction NB2 models where tested against the original presumed one, see **Table 2** for the results.

	agecar	*volume*	*kw*	*ageman*	*price*	*nocit*	*company*	*fuel*	*gender*	*district*
agecar	-									
volume	0.1385	-								
kw	**0.0309**	**0.0002**	-							
ageman	0.1994	0.1065	0.1937	-						
price	0.4368	1.0000	0.2038	0.5033	-					
nocit	0.1539	1.0000	0.7731	0.6741		-				
company	**0.0465**	**0.0436**	0.1531	-	0.9294	0.4872	-			
fuel	0.7661	0.9998	0.9998	0.1049	0.9464	0.9859	**0.0134**	-		
gender	0.7405	0.0563	0.0539	0.4381	0.2359	0.6083	-	**0.0435**	-	
district	0.5366	0.9574	0.7440	0.1595	0.8403	0.2019	0.3738	0.9950	0.8796	-

Table 2 p-values of LR test: presumed linear NB2 model against 43 pairs of interaction models

We could see that, due to the respecting other covariates in the model, 6 of the 43 interactions are significant at the 0.05 level. The interactions between *nocit* × *agecar*, *company* × *kw* and *price* × *kw* may be assessed as spurious due to the neglecting other covariates.

In the final step, to reveal spurious interactions due to the mismodelling of covariates and applying the closed test procedure, we adjusted the linear NB2 model which we term non-linear NB2. The analysis gives FP powers -0.5 for *kw*, (0, 3) for *ageman* and linear terms for the other continuous variables. Again, we extended the non-linear NB2 model by adding one of the 43 interactions and tested against the original, see **Table 3**.

	agecar	*volume*	*kw*	*ageman*	*price*	*nocit*	*company*	*fuel*	*gender*	*district*
agecar	-									
volume	0,6426	-								
kw	0,2541	0,6617	-							
ageman	**0,0188**	0,5769	0,8284	-						
price	0,0559	1,0000	0,1256	0,1658	-					
nocit	0,1368	1,0000	0,3095	0,1543	1,0000	-				
company	**0,0284**	0,2047	0,4016	-	0,6576	0,2479	-			
fuel	0,7140	0,9999	0,9995	0,1414	0,9983	1,0000	**0,0170**	-		
gender	0,8658	0,2070	0,3236	**0,0127**	0,4733	0,5861	-	0,0676	-	
district	0,5627	0,8995	0,5436	0,2110	0,7128	0,1961	0,2841	0,9945	0,9030	-

Table 3 p-values of LR test: non-linear NB2 model against 43 pairs of non-linear interaction models

Only 4 of 43 interactions are still significant at level 0.05 and *kw* × *agecar*, *kw* × *volume*, *company* × *volume* can be termed spurious. The other interactions, *ageman* × *agecar*, *company* × *agecar* and *gender* × *ageman* are assessed below.

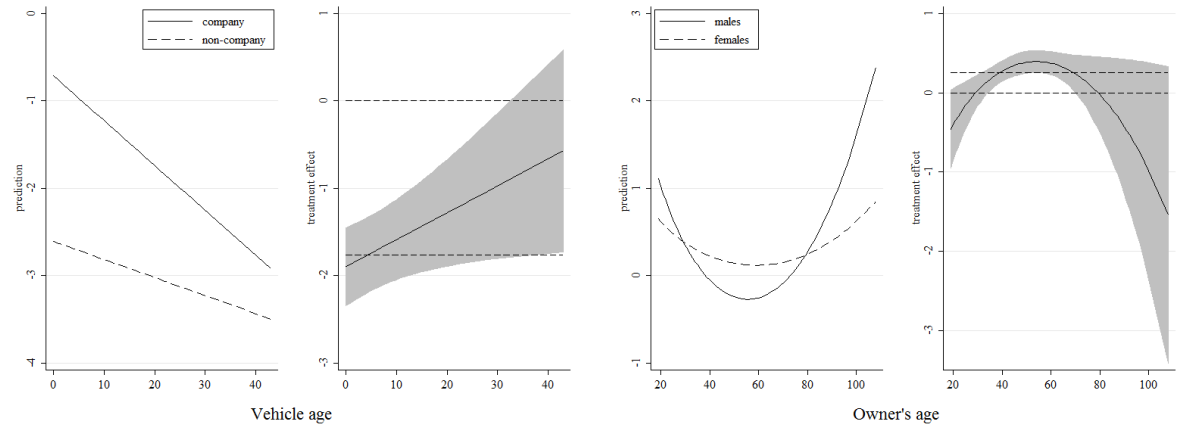

Figure 1 Analysis of *agecar* × *company* and *gender* × *ageman*. Left panels: solid line, estimated effect of *agecar* (*ageman*) for company cars (for males); dashed line, estimated effect of *agecar* (*ageman*) for non-company cars (for females). Right panels: effect of company (*gender*), with 95% pointwise CI. Horizontal dashed lines denote zero and the main effect of company (*gender*) in the absence of interactions.

We could see the different slope of function for non-/company car and different shape for males/females. It is obvious that the slope of company car is different from non-company and the claim frequency slopes down slower for non-company ones. The difference between males and females for various *ageman* is also obvious. The treatment-effect plot (estimated difference between the levels of binary variable) shows that the effect of *company* is not constant (dashed line) and is diminishing with increasing *agecar* (solid line). On the other hand, the effect of *gender* is more complex. In addition to changing the sign of slope parameter, the effect is diminishing and growing stronger again when the *ageman* increases.

To check continuous × continuous interactions, we divided *ageman* and *agecar* into four quartile and we estimated the effect on claim frequency for *agecar* (*ageman*) over each quartile of *ageman* (*agecar*), see the Figure 2.

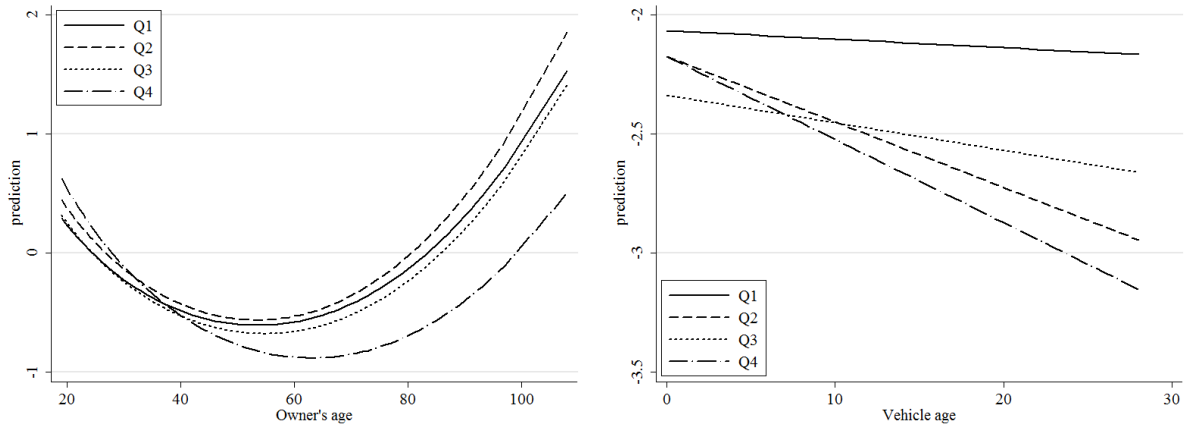

Figure 2 Graphical representation of *agecar* × *ageman* interaction. Left panel show results for quartiles of the distribution of *agecar* and right panel for the same quartile for *ageman*.

The different effect of *ageman* on claim frequency is not so obvious because the effect of *ageman* for comparatively newer cars (quartile Q1-Q3) is very similar. Only the effect of *ageman* for old car (Q4) stands out and the difference is apparent for older drivers. On the contrary, the difference of effects of *agecar* over *ageman* is more apparent. We could see that response gradient on *agecar* is much steeper for old (Q4) policyholder than for young (Q1) drivers.

4 Conclusion

The paper was devoted to identifying and analysing potential and spurious interactions. We examined the effect of omitting influential variable and mismodelling on the existence of spurious interaction in the presumed model intended for claim frequency modelling of given motor hull insurance portfolio. In our analysis, we revealed 8 of 43 potential significant interactions and we revealed that some of them are spurious due to the omission of other influential variable and some due to the mismodelling.

On the basis of analysis performed in the paper, we can conclude on the significance level 0.05 that interactions appropriate for including into the model are following: (1) owner`s age × vehicle age, (2) gender × owner`s age, (3) company car × vehicle age and (4) company car × fuel category. Within further analysis, we proved that the response gradient of vehicle age for non-/company car differs and the effect of binary variable indicating a non-company car on the outcome is diminishing with increasing vehicle age. We also confirmed different effect of owner`s age over males/females. The effect of being female is negative for lower and higher owner`s age and positive for middle age compared to males of the same age whereas the effect of being female on the outcome is positive only if the interaction is not considered. Finally, the analysis applied to interaction of vehicle age × owner`s age also confirmed the interaction which could be useful to improve model fit. In addition to identifying of significant different effect of owner's age over the older cars and new or newish cars, we also found out the different effects of vehicle age on the claim frequency over the changing owner's age.

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Factor analysis of managerial approaches in a sample of Czech companies

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Abstract. The main purpose of the paper is to outline a preliminary factor structure of managerial approaches in a sample of 98 Czech companies operating in various sectors (mechanical engineering, electrical engineering, civil engineering, development and production of computer software and hardware, and consumer industry). Managerial approaches were assessed using a 42-item questionnaire based on the EFQM model. The EFQM model recognizes five "enabler" areas of managerial approaches. Therefore, the theoretical prediction is that five factors will be discovered. However, we discovered nine underlying factors. The factors seem to represent Communication with stakeholders, Leadership, Improvement management, Innovations, Pro-customer motivation of employees, Cash-flow management, Cooperation with partners, Improvement as a priority, and Information security. Thus, we can observe that the initial areas suggested by the EFQM model theory are mapped onto nine empirical factors. The fact that the empirical factors do not exactly correspond to the EFQM areas is no cause for concern, though – for example Quanzi et al. (1998) [Critical factors in quality management and guidlines for selfassessment: The case of Singapore. *Total Quality Management*, *9*, 35–55] indentified 16 empirical factors. It is useful to do factor analysis of managerial approaches for Czech companies, not just to rely on past findings, as we discuss in the paper.

Keywords: factor analysis, managerial approaches, EFQM, empirical study.

JEL Classification: C38, L20
AMS Classification: 62H25

1 Introduction and theoretical background

In this paper we will analyze managerial approaches in 98 Czech companies with factor analysis.

We will first provide a few introductory notes on factor analysis (for more details see [18]). Factor analysis is a method for exploring complex datasets, i.e. datasets with a large number of variables. Specifically, it can be used to identify the underlying structure of complex datasets by clustering interrelated variables into common factors. A factor can be sometimes understood as a latent variable (e.g. employees' productivity) that expresses something common or similar to the measured variables contained in it. Factor analysis can be also used to develop more reliable questionnaires and assessment techniques [5]. Factor analysis has been used in various areas of economics, see e.g. [8], [14], [16], [21].

We assessed managerial approaches in 98 Czech companies using a 42-item questionnaire. Technically, each of the items can be said to measure one variable. Hence, each company will be described by 42 variables.

The 42-item questionnaire is partially inspired by the EFQM model (see [3]). EFQM model is an operational framework to implement Total Quality Management (see [2], [19]). There are five main areas – the enabler criteria – measured by the EFQM model: Leadership, Strategic management (or Policy and strategy), Human resources (or People), Partnership and other resources, and Processes.

Hence, we can assume that these five areas are measured by our 42-item questionnaire. We can also assume that these five areas capture the underlying structure of managerial approaches in the companies we studied. However, these assumptions are based mainly on past experience and/or expert knowledge. With factor analysis we can discover the *actual* underlying structure of managerial approaches in the 98 companies. This latent factor structure can be quite similar to the structure suggested by the EFQM model (the five areas), but there can be

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important differences. For example we might discover just four significant factors, not five – one of the EFQM areas could be "swamped" by the remaining areas (specifically, one of the areas can share so much variance with one or more of the areas that it does not emerge as a factor per se in the factor analysis, but is contained in one or more of the remaining factors).

In the study of managerial approaches usually only relationships between certain selected criteria are examined, see e.g. [7], [10], [12], [20]; for an overview see [2]. Factor analysis, on the other hand, allows us take all the interrelations among all measures of management quality into account.

[6] used factor analysis and found that the five EFQM areas mentioned above – the enabler criteria – have a common underlying factor.

[2] arrived at a similar result (i.e. the existence of a common general factor underlying the five enablers) using multidimensional structural equation modeling. In addition, using factor analysis [2] confirmed that individual enablers were unidimensional, i.e. the items related to the respective areas (Leadership, Strategic management, Human resources, Partnership and other resources and Processes) had just one underlying factor each.

Using factor analysis on responses to an EFQM model-based questionnaire, [4] discovered 10 underlying dimensions (factors) of quality management. Similarly, [17] discovered eight factors of quality management, [1] found eight factors, and [15] identified 16 factors (see [4] for an overview).

The identified factors partially overlap with the criteria of the EFQM model. Sometimes, however, new aspects of quality management are uncovered with factor analysis. For example the 10 dimensions in [4] were Leadership, Quality planning, Communication, Training, Specialist training, Suppliers management, Customer focus, Process management, Continuous improvement, and Learning.

The main purpose of the present paper is to demonstrate the usefulness of factor analysis and outline a preliminary factor structure of managerial approaches in a sample of Czech companies.

2 Method

The survey was performed in 98 Czech companies of different size (M = 353.4 employees, SD = 1082.1). The companies operate in different sectors (mechanical engineering, electrical engineering, civil engineering, development and production of computer software and hardware, and consumer industry).

Obtaining a more homogenous sample can be seen as one of the tasks for future research, because the factor structure might be sensitive to the size and type of the company (on the other hand [2] showed that mixing service and manufacturing firms in one sample does *not* confound the results). Note that factor analysis has high requirements with respect to sample size, therefore obtaining a homogenous sample can be quite difficult.

The companies were approached by our trained student-collaborators who interviewed the companies' managers. A structured interview technique comprising the 42-item assessment questionnaire was used. The items are available from the first author upon request. They are not reproduced here to spare space (just five examples are given below). Obtaining information based on respondents' perceptions is common in organizational research (see [11], [13]).

As we mentioned in the introduction the 42-item model is loosely based on the EFQM model, hence it covers the five EFQM areas:

1. Leadership (items LEAD1-LEAD8), e.g. "Managers define and develop the function, vision, culture and values of the organization."
2. Strategic management (items STR1-STR6), e.g. "The current and future needs and expectations of parties involved (including customers) are satisfied."
3. Human resources (items HR1-HR8), e.g. "HR planning is well documented and is in agreement with the strategy and with the organizational and process structure."
4. Partnership and other resources (items PAR1-PAR8), e.g. "Mutual development of the company and its partners is promoted."
5. Processes (items PROC1-PROC11), e.g. "System of quality management is built and/or certified according to the norm ISO 9001. It is implemented and fully functional."

Fulfilling/satisfying the tasks/criteria described by each item was scored on a scale 0-100 (with 0 being the worst possible score and 100 the best). This is in line with the usual practice in management audit and consulting (in fact, the 42-item model has been used by the first author and his colleagues – auditors and consultants – in a number of companies in the past).

As far as the factor analysis is concerned, we use principal component analysis as the extraction method and direct oblimin as the rotation method. We replace missing data with mean for the respective item (the missing data are not normally distributed, thus it is not recommendable to exclude the cases with missing data from the analysis).

3 Results and discussion

Kaiser-Meyer-Olkin measure of sampling adequacy has the value .868 for our data, which verifies the data have good factorability (i.e. there appears to be some underlying structure in the data). KMO is an index for comparing the magnitudes of the observed correlation coefficients to the magnitudes of the partial correlation coefficients. The minimum acceptable value of KMO is .5, values of .8 and above are considered very good. The KMO was also calculated for individual items and was usually above .8 (for two items it was between .7 and .8).

Using Bartlett's test of sphericity we confirm that our data are adequate to be analyzed using factor analysis, $\chi^2(861) = 2362.9$, $p < .001$. Bartlett's test tells us that sufficiently large correlations exist between items, which is necessary for factor analysis to work.

By Kaiser's criterion (i.e. eigenvalue of each factor is above 1) we extracted nine factors. The factors are listed in Table 1. The total percentage of variance in the data explained by the nine factors is 74.3%.

Factor	Factor interpretation	Eigenvalue before rotation	Percentage of variance explained by unrotated factor	Eigenvalue after rotation
Factor 1	Communication with stakeholders	18.762	44.670	5.700
Factor 2	Leadership	2.849	6.783	10.069
Factor 3	Improvement management	1.790	4.263	5.815
Factor 4	Innovations	1.645	3.916	5.529
Factor 5	Pro-customer motivation of employees	1.436	3.419	6.491
Factor 6	Cash-flow management	1.322	3.148	6.882
Factor 7	Cooperation with partners	1.200	2.857	8.804
Factor 8	Improvement as a priority	1.114	2.652	6.365
Factor 9	Information security	1.095	2.606	8.155

Table 1 Summary of the factor matrix

The total percentage of explained variance is naturally the same for rotated and unrotated factors. Note, however, that in the rotated solution the relative importance of the factors is equalized (compare the eigenvalues before and after rotation in Table 1).

Table 2 gives the item loadings to the rotated factors. More precisely, the pattern matrix is reproduced here – the pattern matrix indicates the unique contribution of an item to each factor (only loadings with magnitude higher than .4 are displayed).

Item	Factor 1	Factor 2	Factor 3	Factor 4	Factor 5	Factor 6	Factor 7	Factor 8	Factor 9
HR3	.621								
PROC20	-.495				-.414				
HR6	.471								
STR3	.434								
STR10	.403								
STR7									
STR4									
HR1									
LEAD2		-.734							
LEAD8		-.696							
LEAD7		-.684							
LEAD5		-.672							
LEAD10		-.611							
LEAD3		-.568							

LEAD9	-.553							
LEAD1	-.526							
STR1	-.472							
PROC4		.814						
PROC3		.656						
PAR10		.446						
PROC1			-.627					
PROC19			-.624					
PROC7			-.598					
PROC13				-.627				
HR11				-.579				
PROC18				-.509				
STR2								
PAR4					-.812			
PAR5					-.643			
PAR6					-.562			
HR4								
PAR1						.781		
PAR2						.747		
PROC11						.409		
PAR3								
PAR14							.655	
HR9							.620	
PROC5							.566	
HR8	-.441						.503	
HR2								
PAR12								.804
PROC17								.748

Table 2 Items loading the factors – rotated pattern matrix

Based on the content of the questionnaire items that load on each factor we can interpret what each factor represents. For example Factor 2 quite clearly represents several aspects of leadership (eight Leadership items, one Strategy item and one HR item load on this factor). Our interpretation of the factors is given in Table 1. For example Factor 1 represents Communication with stakeholders.

We can observe that the initial areas predicted by theory (i.e. the five enabler areas of the EFQM model) are mapped onto nine factors. The nine factors thus usually represent either a sub-area or a mixture of sub-areas of the initial five areas/EFQM criteria.

The main purpose of the present paper was to demonstrate the usefulness of factor analysis and outline a preliminary factor structure of managerial approaches in a sample of Czech companies. In this preliminary attempt at description of factor structure of managerial approaches we identified nine factors (underlying variables) of managerial approaches in the selected companies.

It is useful to do this analysis for Czech companies, because the results of previous exploratory factor analyses cannot be statistically extrapolated to Czech companies (see [4]). And since there might be cultural differences in quality management practices in different countries (see e.g. [9]) a mere heuristic/logical extrapolation from previous research performed in different countries could be culturally biased.

[2] argue that understanding the structure of managerial approaches can help us assess the application of quality management in companies. We can for example notice that items from different areas of the EFQM model are contained in one underlying variable (a factor). This indicates that actions/situations described by these items are in fact closely related and should not be treated in managerial practice as independent and unrelated. How should the managers practically implement this kind of findings is clearly a topic for another investigation (but see e.g. [4], [11] for some suggestions).

Future research should try to corroborate our findings using alternative methodologies (e.g. experiments). We are currently collecting additional data with the intention to re-calculate the factor structure for a larger sample of companies.

4 Conclusions

The main purpose of the paper was to outline a preliminary factor structure of managerial approaches in a sample of 98 Czech companies. Using PCA on data obtained with a questionnaire based on the EFQM model we identified nine latent factors, which seem to represent Communication with stakeholders, Leadership, Improvement management, Innovations, Pro-customer motivation of employees, Cash-flow management, Cooperation with partners, Improvement as a priority, and Information security. These findings can be useful, because the results of previous exploratory factor analyses cannot be statistically extrapolated to Czech companies and because understanding the latent structure in managerial approaches can for example help us assess the application of quality management in the companies.

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Generalized closures of rough set theory and formal concept analysis

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Abstract.

Often, both in practice and theory, it happens that it is impossible to describe some sets precisely. The paper examines the mathematical background of methods for handling such situations. In particular, the paper is concerned with the possibilities of describing a set qualitatively by non-numeric techniques that are used in the theory of rough sets and in the analysis of formal concepts. The paper demonstrates that the basic mathematical ideas underlying these techniques are closely related to ideas appearing in various studies of set-valued set-functions of extended topology, Galois correspondences, Boolean algebras with operators, and modal logics.

Keywords: extended topology, generalized closures, rough set, set approximation.

JEL classification: C65, C44
AMS classification: 03E99, 68T37, 06C06, 54C60

1 Introduction

Uncertainty is a part of our life and our knowledge is generally incomplete. Therefore, various types of uncertainty should be included in formal models, and reasoning in such models should be able to cope with uncertainty. As a consequence, we have to approximate: sometimes because the complexity of objects and sets under investigation necessitates various types of simplification and idealization, sometimes because insufficient information or other sources of uncertainty make it impossible to describe the relevant sets and functions precisely. The general idea of approximations has been used since antiquity, and also many theoretical results and applications of computer science and mathematics of the present time are based on approximation of functions by other functions and approximation of sets by other sets. However, to obtain useful approximations, the sets under investigation must be endowed with some reasonable mathematical structure.

One way of handling situations with imperfect information about subsets of objects from a given set U (often called the universe of discourse) is to assign to the claim "an object x belongs to a subset X of U" a number expressing some degree of belief. For example, we can assign to it a probability, or a membership value, or an interval of real numbers. Another way is to approximate the sets in question by sets that can be described precisely through available information. The basic idea of the latter approach is to use available information about the subsets of U for introducing, for every subset X of U, two subsets: the set of elements from U that (on the basis of available information) necessarily belong to X, and the set of elements from U that (on the basis of available information) possibly belong to X. Available information is often provided through a binary relation on U. It is known that there is bijection between the set of binary relation on U and a special set of mappings from the power set of U to itself, see [1]. Therefore, it is possible to construct and study approximation of sets with the help of mappings from the power set of U into itself.

Let $\mathcal{P}(U)$ denote the potential set of U and let f and g be mappings from $\mathcal{P}(U)$ into itself. If $f(X)$ is to represent the set of elements that necessarily belong to X and $g(X)$ the set of elements that possibly belong to X, then it is natural to require that the mappings f and g have the property: For each $X \subseteq U$,

$$f(X) \subseteq X \subseteq g(X) \tag{1}$$

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Frequently, one has to deal also with more general situations where information is provided by a binary relation between two different sets U and V. Then it is possible to use it for defining several mappings from $\mathcal{P}(U)$ to $\mathcal{P}(V)$ and from $\mathcal{P}(V)$ to $\mathcal{P}(U)$ which are which are useful in the formal concept analysis and which are closely related to the Galois connection between partially ordered sets $(\mathcal{P}(U), \subseteq)$ and $(\mathcal{P}(V), \subseteq)$. For example, if R is a binary relation between U and V and the mappings $f_R : \mathcal{P}(U) \to \mathcal{P}(V)$ and $g_R : \mathcal{P}(V) \to \mathcal{P}(U)$ are defined by

$$\begin{aligned} f_R(X) &= \{v \in V : \text{ there exists } u \in U \text{ such that } uRv\} \\ g_R(Y) &= \{u \in U : \text{ for each } v \in V, \text{ we have } v \in Y \text{ whenever } uRv\} \end{aligned}$$

then, in the formal concept analysis, $f_R(X)$ and $g_R(Y)$ define the span of X and the content of Y, respectively; for further details, see [1].

2 Approximations based on equivalences

2.1 Pawlak's Approximation Space

Let U be a nonempty finite set, let E be an equivalence relation on U, and let P_E be the partition of U induced by E. Following Pawlak [6] and [7], we define lower and upper approximations of subsets of Uas follows:

- The *E-lower approximation* of a subset X is the union of those members of P_E that are subsets of X.
- The *E-upper approximation* of a subset X is the union of those members of P_E that contain at least one element of X.

Using these approximations we introduce the following notion of a *definable* subset of U:

- A subset of X is called *E-definable* if its E-lower approximation is equal to its E-upper approximation.

Instead of introducing the definable sets by means of lower and and upper approximation, we can first introduce definable sets and then to define the corresponding lower and upper approximations by means of definable sets as follows:

- A subset X of U is *E-definable* if it is either empty or a member of P_E or the union of two or more members of P_E.
- The *E-lower approximation* of X is the union of those E-definable subsets of U that are subsets of X.
- The *E-upper approximation* of X is the intersection of those E-definable subsets of U that include X.

It can easily be verified that, for every equivalence relation E on U, both approaches lead to the same collection of E-definable sets and the same collections of E-lower and E-upper approximations. Moreover, the lower approximation of X is the greatest definable set included in X and the upper approximation of X is the least definable set including X.

Let $\mathcal{D}_E$ be the collection of all definable subsets of U. In terms of set-to-set mappings, the lower and upper approximations of subsets of U are values of mappings f_E and g_E of $\mathcal{P}(U)$ into itself defined by

$$\begin{aligned} f_E(X) &= \bigcup\{A \in \mathcal{D}_E : A \subseteq X\} \\ g_E(X) &= \bigcap\{A \in \mathcal{D}_E : A \supseteq X\} \end{aligned}$$

or, equivalently,

$$\begin{aligned} f_E(X) &= \bigcup\{A \in \mathcal{P}_E : A \subseteq X\} \\ g_E(X) &= \bigcup\{A \in \mathcal{P}_E : A \cap X \neq \emptyset\} \end{aligned}$$

It can easily be verified that, for each X, we have:

- The inclusions in (1) are satisfied; that is, $f_E(X) \subseteq X \subseteq g_E(X)$.
- The sets $f_E(X)$ and $g_E(X)$ are definable.
- The set $f_E(X)$ is the greatest definable set included in X and the set $g_E(X)$ is the least definable set including X.
- The values of f_E and g_E at X are related by $g_E(X) = U \setminus f_E(U \setminus X)$ and $f_E(X) = U \setminus g_E(U \setminus X)$. In other words, the mappings f_E and g_E are not mutually independent.

There often exist different subsets of U with the property that their lower approximations are the same and, at the same time, their upper approximations are also the same. Such subsets are mutually indistinguishable in terms of approximations based on equivalence relations. It is therefore natural to use the lower approximations and upper approximations defined by equivalence relation E for introducing a binary relation (let us denote it by $\sim_E$) on the power set of U by requiring that $X \sim_E Y$ holds for subsets X and Y of U if and only if $f_E(X) = f_E(Y)$ and $g_E(X) = g_E(Y)$. This relation is an equivalence relation on the power set of U. The equivalence classes of $\sim_E$ are called the *rough sets*[1] in (U, E), and the ordered pair (U, E) is often called *Pawlak's approximation space*.

Example 1. Let U be the set $\{a, b, c\}$ and let E be the equivalence on U whose only equivalence classes are $\{a, c\}$ and $\{b\}$. Simple calculations (see Järvinen [5] for details) show that the collection of rough sets consists from the following six sets: $\{\emptyset\}; \{\{a\}, \{c\}\}; \{\{b\}\}; \{\{a, b\}, \{b, c\}\}; \{\{a, c\}\}; \{U\}$.

Remark 1. Because the rough sets are equivalence classes, we can represent every particular rough set by one of its members. It follows that rough sets can also be represented by ordered pairs $(f_E(X), g_E(X))$ where X is an arbitrary member of the rough set in question. This makes it possible to introduce a partial order in into the set of rough sets such that it becomes a complete Stone algebra, see [8].

Remark 2. Another point of view is to connect the mappings f_E and g_E with operators $\Box$ and $\Diamond$ used in the modal logic $S5$ for a modality of universal character (necessarily) and its existential dual (possibly). It turns out that these operators are exact counterparts of the lower approximation mapping f_E and upper approximation mapping g_E; see, for example, [12] and [9]. Consequently, we can translate the notions and results from the rough set theory to those of the modal logic $S5$.

2.2 Partition Topologies

In the basic Pawlak's model (U, R), it is assumed that the universe of discourse U is a nonempty finite set and R is an equivalence relation on U. However, there are a number of situations of practical relevance and of theoretical interest for which one or both of these assumptions are too restrictive. Fortunately, many of the results obtained for Pawlak's basic model are valid in (or can be extended to) more general cases.

Let U be a nonempty (not necessarily finite) set. As previously, let U be an equivalence relation on U, and let D_E be the partition of U induced by E. It can easily be verified that (also in this more general case) the collection of E-definable sets has the following properties:

- Both the empty set $\emptyset$ and the universe of discourse $\mathcal{U}$ are E-definable.
- The union of the members of each sub-collection of E-definable sets is an E-definable set.
- The intersection of the members of each finite sub-collection of E-definable sets is an E-definable set.

In other words, the collection of E-definable set is a topology on U. We denote it by τ_E. This topology is a very special, it belongs in the class of so called *partition topologies*. The partition topologies are characterized by the fact that every open set is also closed, and vice versa. Moreover, except for the partition topologies generated by trivial partitions, the partition topologies are not T_0-spaces, because

[1] Notice that rough sets are not subsets of U but subsets of $\mathcal{P}(U)$.

every nontrivial partition has some member which contains two or more elements neither of which can be separated from the other. It is known that, for a finite nonempty set U, there are one-to-one correspondences between the set of all equivalence relations on U and the set of partition topologies and the set of all regular topologies.

It turns out that if E is an equivalence on $\mathcal{U}$, then the collection of subsets of $\mathcal{U} \times \mathcal{U}$ that include E is a uniformity for $\mathcal{U}$ and the topology for $\mathcal{U}$ induced by this uniformity coincides with topology τ_E. Hence the difference between Pawlak's approximation space (U, E) and the topological space (U, τ_E) is only terminological. In particular, for each subset X of U, we have:

- X is definable if and only if it is open.
- X is definable if and only if it is closed.
- The lower approximation of X is the interior of X.
- The upper approximation of X is the closure of X.
- X is definable if and only if its interior is equal to its closure.

Remark 3. Let T be a tolerance relation on U, that is, T is reflexive and symmetric. We define x/T to be the set of those elements y in U for which $(x, y) \in T$. If a tolerance T is not transitive, then two distinct x/T and y/T may have common elements. Therefore, the collection $\{x/T : x \in U\}$ of subsets of U is not necessarily a partition of U. However, by reflexivity of T, the collection $\{x/T : x \in U\}$ is always a **covering** of U. Let T be a tolerance relation in U, let X be a subset of U. Let us define lower and upper approximations, and definable sets as follows:

- The *T-lower approximation* $\underline{T}X$ of X is the set of those x from U for which $x/T \subseteq X$.
- The *T-upper approximation* $\overline{T}X$ of X is the set of those x from U for which x/T intersects X.
- X is *T-definable* if $\underline{T}X = \overline{T}X$.

Let E_T be the intersection of all equivalences in $\mathcal{U}$ that include T. It turns out that E_T is an equivalence, and the collection of T-definable sets is the same as the collection of E_T-definable sets. Therefore, for each tolerance T, the collection of T-definable sets forms a partition topology. Nevertheless, there is a significant difference between approximations based on tolerances and approximations based on equivalences: the latter are always definable while the former may be undefinable.

3 Hammer's Extended Topology

Let $\mathcal{U}$ be a nonempty set. There is a one-to-one correspondence between binary relations on $\mathcal{U}$ and mappings $f : \mathcal{P}(\mathcal{U}) \to \mathcal{P}(\mathcal{U})$ that satisfy $f(\emptyset) = \emptyset$ and $f(\bigcup_{i\in I} X_i) = \bigcup_{i\in I} f(X_i)$. Therefore, we can obtain an insight into the system of approximations of rough set theory also by studying properties of set-valued set-functions. Basic properties of such functions were studied in detail by Preston C. Hammer in a series of papers on the extended topology published in the early 1960's; as examples, see [3] and [4], and Gniłka [2] for further developments.

Let f be an arbitrary mapping from $\mathcal{P}(\mathcal{U})$ into itself. Define the *dual* f_d of f by $f_d(X) = \mathcal{U} \setminus f(\mathcal{U} \setminus X)$. It turns out that the following properties are equivalent:

$$f \text{ is isotonic,}$$

$$f(X) \cup f(Y) \subset f(X \cup Y) \text{ for all } X, Y \in \mathcal{U},$$

$$f(X) \cap f(Y) \supset f(X \cap Y) \text{ for all } X, Y \in \mathcal{U},$$

$$f_d \text{ is isotonic,}$$

$$f_d(X) \cup f_d(Y) \supset f_d(X \cup Y) \text{ for all } X, Y \in \mathcal{U},$$

$$f_d(X) \cap f_d(Y) \subset f_d(X \cap Y) \text{ for all } X, Y \in \mathcal{U},$$

$$X \cap Y = \emptyset \text{ implies } f(X) \cap f_d(Y) = \emptyset.$$

In Hammer's system of extended topology the isotonic functions that are also *non-shrinking* in the sense

$$X \subset f(X) \text{ for all } X$$

are called *expansive* functions. It can easily be seen that if f is expansive, then f_d is *contractive*; that is, f_d is isotonic and *non-enlarging* in the sense

$$f_d(X) \subset X \text{ for all } X.$$

We have seen that the upper approximations are expansive and lower approximations are contractive. Moreover the upper approximations are *subadditive* in the sense that

$$f(X) \cup f(Y) \supset f(X \cup Y) \text{ for all } X, Y \in \mathcal{U},$$

and the lower approximations are *submultiplicative* in the sense

$$f(X) \cap f(Y) \subset f(X \cap Y) \text{ for all } X, Y \in \mathcal{U}.$$

The fact that the isotonic approximations are exactly those with the property

$$X \cap Y = \emptyset \text{ implies } f(X) \cap f_d(Y) = \emptyset$$

is useful in many problems because it provides necessary condition for the disjointness of sets.

For example, the optimality of a feasible solution of an optimization problem can be expressed by stating that the intersection of certain sets is empty.

In contrast to the approximations used in the theory of rough sets, the approximations used in optimization are usually neither expansive nor contractive. Properties of such approximations have been studied by many authors and their application proved to be useful in deriving optimality conditions in various areas of optimization theory. For example, in [10] and [11], the isotonic approximations with the property

$$f(X) = X \text{ for } X \in \{\emptyset, \mathcal{U}\}$$

are called *external* approximations, provided they are subadditive, and *internal* approximations, if they are submultiplicative.

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Loan to asset ratio and its determinants in the Visegrad countries

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Abstract. The aim of this paper is to identify determinants of the loan to asset ratio of commercial banks in the Visegrad countries. The analysis is carried out with the panel data regression analysis with fixed effects for the period from 2000 to 2011. Although the Visegrad countries have a lot in common, different factors determined the loan to asset ratio in individual countries and also the direction of the influence of some factors differ. Both bank specific and macroeconomic factors matters.

Keywords: loan to asset ratio, panel data regression analysis, Visegrad countries.

JEL Classification: C23, G01, G21
AMS Classification: 62P20

1 Introduction

During global financial crisis, financial sector has gone through a dramatic re-appraisal of the liquidity risk. Focusing on the Visegrad countries, liquidity problems could be seen also in Hungarian, Polish and Slovak banking sector: three Hungarian banks (OTP Bank, FHB Mortgage Bank and MFB – Magyar Fejlesztési Bank) were provided with a loan from the government in March 2009; besides, FHB Mortgage Bank was granted a HUF 30 billion equity raise [11]. In the Polish banking sector, the deterioration in the macroeconomic situation weakened the functioning of the interbank market, increased the cost of money on the market and deepened the gap between deposits and loans [17]. Liquidity of Slovak banks sharply decreased in 2009 and 2010, mainly due to the changes in the interbank market activities and because of the fact that economic crisis adversely effected sectors in which Slovak banks have significant credit exposures [15].

It is evident that liquidity risk of banks is very up-to-date and important topic. The aim of this paper is therefore to identify determinants of the loan to asset ratio of commercial banks in the Visegrad countries. Although all Visegrad countries are characterized by a universal banking model and the financial system is traditionally based on banks and credit markets, activities of banks in the financial markets significantly differ. In spite of its importance, the empirical evidence of bank liquidity in these countries is very poor and the most recent period has not been studied yet (the study of [8] uses data only for the pre-crisis period 1994-2004). The contribution of this paper is therefore obvious.

The paper is structured as follows. The next section defines bank liquidity and characterizes methods of its measurement. Section 3 describes methodology and data used. Section 4 contains results of the analysis. The last section captures concluding remarks.

2 Bank liquidity and methods of its measurement

Liquidity is the ability of bank to fund increases in assets and meet obligations as they come due, without incurring unacceptable losses [3]. Liquidity risk, e.g. the risk that a bank would not have enough liquidity, arises from the fundamental role of banks in the maturity transformation of short-term deposits into long-term loans. According to [16], the term liquidity risk includes central bank liquidity risk (which is highly unlikely as it is a risk that central bank would not be able to supply the liquidity needed to the financial system), funding liquidity risk (which captures the inability of a bank to service their liabilities as they come due) and market liquidity risk (which relates to the inability of trading at a fair price with immediacy). These types of liquidity risk are intensively interconnected. In normal times, these linkages promote a virtuous circle in financial system liquidity, guaranteeing the smooth functioning of the financial system. In turbulent times, the linkages can be distorted and there is a possibility of reverting from a virtuous to a vicious circle in the economy.

Liquidity risk can be measured mainly by liquidity ratios which are various balance sheet ratios which should identify main liquidity trends. These ratios reflect the fact that bank should be sure that appropriate, low-cost funding is available in a short time. This might involve holding a portfolio of assets than can be easily sold (cash

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reserves, minimum required reserves or government securities), holding significant volumes of stable liabilities (especially deposits from retail depositors) or maintaining credit lines with other financial institutions. Various authors like [1], [6] or [13] provide various liquidity ratios. Most common measure of liquidity risk is a share of liquid assets on total assets, a share of liquid assets on deposits and short term borrowing, a share of liquid assets on deposits, a share of loans on deposits or an interbank ratio which is a ratio of dues from banks over dues to banks. We will focus on the last liquidity ratio which is the loan to asset ratio. It is the share of loans on total assets. It is an indirect way how to measure liquidity risk as it indicates what percentage of the assets of the bank is tied up in illiquid loans. Therefore the higher this ratio the less liquid the bank is.

3 Methodology and data

In order to identify determinants which affect loan to asset ratio in the Visegrad countries we use the panel data regression analysis (1):

$$LAR_{it} = \alpha + \beta' \cdot X_{it} + \delta_i + \varepsilon_{it} \tag{1}$$

where LAR_{it} is loan to asset ratio for bank i in time t, X_{it} is vector of explanatory variables for bank i in time t, α is constant, β' is coefficient which represents the slope of variables, δ_i represents fixed effects in bank i, and ε_i means the error term. We choose the model with fixed effects because banks in our sample are not randomly chosen. This was confirmed also by results of the Hausman test.

It is evident that the most important task is to choose the appropriate explanatory variables. Although liquidity problems of some banks during the global financial crisis re-emphasized the fact that liquidity is very important for the functioning of financial markets and the banking sector, an important gap still exists in the empirical literature about liquidity and its measuring. Most studies focus on bid-ask spreads in the market and thus on the measurement and determinants of market liquidity. However, only few studies aim to identify determinants of funding liquidity measured by any liquidity ratio which reflects in part the lack of interest, until recently at least, of some regulators [2]. [8] is the only study which focus on liquidity of foreign owned banks in the Visegrad countries and also in some other countries from Central and Eastern European economies (Bulgaria, Estonia, Latvia, Lithuania, Romania, and Slovenia) in the period 1994-2004. They found out that bank liquidity is influenced (influence of factors are in parenthesis) by size of banks (-), capital adequacy (+), interbank rate (+), deposit rate (-), growth rate of GDP (-) and GDP per capita (-).

[1] analyzed 57 UK-resident banks in period 1985Q1-2003Q4. They found out that bank liquidity is a function of the probability of the support from a lender of last (-); short term interest rate (-); growth rate of GDP (-); and interest margin (-). Bank profitability, loan growth and size of the bank are not statistically significant. [6] focused on 1107 banks from 36 emerging economies in 1995-2000. Bank liquidity is mainly influenced by the exchange rate regime (higher liquidity in pure floating, currency board and dollarized economies), capital adequacy (+), lending interest rate (+), the share of public expenditures on GDP (+) and financial crisis (-). On the contrary, size of the bank, the growth rate of GDP and the rate of inflation do not matters. The focus of [12] is on 5066 European banks in 1998-2004. Bank liquidity is determined by interbank interest rate (+), risk-free interest rate (+), size of the bank (-) and monetary policy interest rate (-).

[13] found out that on bank liquidity in Latin America and Caribbean countries in period of January 1970-December 2004 was influenced by business cycle (-),volatility of the cash-to-deposit ratio (-), money market interest rate (+ in some countries, - in some other countries), financial crisis (+/-). Liquidity of 457 German state-owned savings banks in 1997-2006 has been investigated by [18]. Monetary policy interest rate (-) and level of unemployment (-) are the most important determinants. On the contrary, bank profitability, size of the bank and savings quota was not statistically significant. Determinants of liquidity of Egyptian banks in 1983-1996 are a bit specific: economic cycle (+), discount rate (+), rate of depreciation of the black market exchange rate (+), impact of economic reforms (-) and number of violent political incidents (+) [9]. [7] studies data of all US commercial banks in 2006Q1-2009Q2. Bank liquidity is positively influenced by financial crisis and negatively by size of the bank. [5] focus on same banks almost in the same period (2005-2009) and confirmed negative impact of size of the bank and positive impact of financial crisis. Moreover, bank liquidity is influenced also by anticipation of future expected loan losses (+), capital adequacy (-) and core deposits (-).

Liquidity of 112 Australian depository institutions in the period June 2002-March 2012 is analyzed by [10]. They found out that liquidity is determined by economic cycle (-), size of the institution (-), lending growth (+), profitability (+), interbank interest rate (+) and interest margin (-). Finally, determinants of liquidity of 27 banks active in Romania in 2002-2010 were investigated by [14]: according to their results, bank liquidity is affected by loan loss provisions (+), funding costs (+), unemployment rate (+) and interbank interest rate (-).

The selection of variables was based on the above cited relevant studies. We considered whether the use of the particular variable makes economic sense in the Visegrad countries. For this reason, we excluded from the analysis variables such as political incidents. We also considered which other factors could influence the behavior of banks in the interbank market. The limiting factor then was the availability of some data. The list of used variables can be found in Tab. 1.

variable	definition	source
CAP	the share of equity on total assets of the bank	annual reports
NPL	the share of non-performing loans on total volume of loans	annual reports
ROE	return on equity: the share of net profit on banks´ equity	annual reports
TOA	logarithm of total assets of the bank	annual reports
FIC	dummy variable for financial crisis (1 in 2009, 0 in rest of the period for CR and SK, 1 in 2008 and 2009, 0 in rest of the period for PL and HU)	own
GDP	growth rate of gross domestic product growth (GDP volume % change)	IMF
INF	inflation rate (CPI % change)	IMF
IRB	interest rate on interbank transactions	IMF
IRL	interest rate on loans	IMF
IRM	difference between interest rate on loans and interest rate on deposits	IMF
MIR	monetary policy interest rate	IMF
UNE	unemployment rate	IMF
EUR	exchange rate CZK(HUF, PLN, SKK)/EUR (yearly average)	Oanda Corp.

Table 1 Variables definition

We considered four bank specific factors and nine macroeconomic factors. We do not have an exact expectation of the impact of these factors on the loan to asset ratio because their impact was different in the above cited studies. The macroeconomic data were provided by the International Financial Statistics of the International Monetary Fund (IMF). The data on average exchange rates were provided by Oanda Corporation. The bank specific data were obtained from the unconsolidated balance sheet and profit and loss data of banks´ annual reports. We used unconsolidated balance sheet and profit and loss data over the period from 2000 to 2011.

	00	**01**	**02**	**03**	**04**	**05**	**06**	**07**	**08**	**09**	**10**	**11**
Czech Republic												
total number of banks	40	38	37	35	35	36	37	37	37	39	41	44
number of observed banks	14	15	16	16	16	15	13	13	12	12	13	12
share of obs. banks on total assets (%)	75	77	76	76	74	71	74	74	68	70	70	77
Hungary												
total number of banks	40	41	39	38	35	34	37	38	36	35	35	35
number of observed banks	13	18	23	24	26	29	28	27	26	23	21	13
share of obs. banks on total assets (%)	72	74	84	86	87	88	88	87	88	88	87	83
Poland												
total number of banks	73	71	62	60	54	54	51	50	52	49	49	44
number of observed banks	15	23	26	31	34	34	31	30	31	30	27	19
share of obs. banks on total assets (%)	60	70	73	88	84	84	82	80	79	79	78	74
Slovakia												
total number of banks	23	21	20	21	21	23	24	26	26	26	29	31
number of observed banks	11	13	13	13	13	14	12	12	12	11	10	10
share of obs. banks on total assets (%)	46	58	54	59	59	65	62	65	69	67	65	66

Table 2 Data availability

Table 2 shows more details about the sample. We do not include data from building societies, mortgage banks and from specialized banks like Českomoravská záruční a rozvojová banka, Slovenská záručná a rozvojová banka, Česká exportní banka, Exim banka, Magyar Fejlesztési Bank or Bank Gospodarstwa Krajowego which focus on very special financial products and services. The panel is unbalanced as some of

banks do not report over the whole period of time. As it includes significant parts of all banking sectors (not only by a number of banks but mainly by their share on total banking assets), we used fixed effects regression.

4 Results

We used an econometric package EViews 7. First of all, we tested stationarity. We used Levin, Lin and Chu test which proved that all time series are stationary on the level values. After tests of normality and multicollinearity, we proceed with the panel data regression analysis with fixed effects. First we included all explanatory variables which might have an effect on the dependent variable. The estimated equation is following (2):

$$LAR_{it} = \alpha + \beta_1 CAP_{it} + \beta_2 NPL_{it} + \beta_3 ROE_{it} + \beta_4 TOA_{it} + \beta_5 FIC_{it} + \beta_6 GDP_{it} + \beta_7 INF_{it} + \beta_8 IRB_{it} + \\ + \beta_9 IRL_{it} + \beta_{10} IRM_{it} + \beta_{11} MIR_{it} + \beta_{12} UNE_{it} + \beta_{13} EUR_{it} + \delta_i + \varepsilon_{it} \quad (2)$$

We estimated Equation 2 separately for each country. To reduce the number of explanatory variables, we used information criteria (Akaike, Schwarz and Hannan-Quinn). The aim was to find a regression model with a high value of the adjusted coefficient of determination in which all the variables involved are statistically significant.

The loan to asset ratio measures the share of illiquid loans on total assets. As it was mentioned above, it is an indirect way how to measure liquidity risk. As higher value of this ratio means lower liquidity, the results of the regression analysis have to be interpreted in reverse: positive sign of the coefficient means negative impact on liquidity and conversely.

Czech Republic			**Hungary**		
variable	coefficient	std. deviation	variable	coefficient	std. deviation
constant	240.627*	58.4428	constant	99.9082*	25.5418
CAP	-0.57282*	0.15979	CAP	-0.55497*	0.13125
NPL	-0.74412*	0.21766	EUR	-0.1912***	0.11419
ROE(-2)	0.23798*	0.08931	GDP(-2)	0.86891**	0.40591
TOA	-7.80602**	3.78591	UNE	1.78638**	0.90380
EUR	-3.21142*	0.61843			
adjusted R^2	0.766591		adjusted R^2	0.847060	
Durbin-Watson stat.	1.860734		Durbin-Watson stat.	1.843006	
total observ.	129		total observ.	199	

Table 3 Determinants of loan to asset ratio in the Czech Republic and in Hungary. The starred coefficient estimates are significant at the 1% (*), 5% (**) or 10% (***) level.

As it can be seen from Table 3, five factors influence liquidity of Czech banks. The explanatory power of the model is quite high. The positive influence of the share of capital on total assets is consistent with the risk absorption hypothesis which is related to banks´ role as risk transformers and emphasizes that higher capital improves banks´ ability to absorb risk and hence their ability to create liquidity, as it was proved by [4], [6], [8] and [14]. The depreciation of the Czech koruna leads to higher bank liquidity. This result is quite logical: probability of interbank transactions with foreign banks increases with the domestic currency depreciation. Increase of such transactions increases liquid assets and thus bank liquidity. The positive influence of non-performing loans on bank liquidity could be a sign of prudent policy of banks: they offset a higher credit risk with cautious liquidity risk management. The negative link between bank liquidity and its profitability measured by return on equity is consistent with the standard finance theory and also with results of [10]. However, the sign of the last regression coefficient does correspond neither to our expectations nor to findings of any other study: it seems that bank liquidity increases with size of the bank.

The estimated coefficients that fit best the regression model for Hungarian banks are presented also in Table 3. The explanatory power of this model is very high. As in case of Czech banks, liquidity of Hungarian banks increases with higher capital adequacy and with the depreciation of the Hungarian forint. A positive sign of the coefficient signals that banks provide more loans in periods of the stronger economic growth which reduces their liquidity, as in [1] and [13]. However, this impact is statistically significant with two years lag which enable us to make a different conclusion. As two years after the recession banks provide more loans (and two years after the expansion provide less loans and prefer to be more liquid a vice versa), it seems that liquidity tends to be

rather positively related to the business cycle, as in [9]. The last determinant is in accordance with [13]: the higher the rate of unemployment the higher the demand for loans is and thus the less liquid banks are.

The loan to asset ratio is determined by capital adequacy, interest margin, share of nonperforming loans and size of the bank in Poland. The explanatory power of the model is high (Table 4). The influence of capital adequacy on bank liquidity is opposite than it was for the Czech Republic and Slovakia. However, the negative influence of capital adequacy is consistent with the financial fragility-crowding out hypothesis which suggests that bank capital may impede liquidity creation by making the bank´s capital structure less fragile. A fragile capital structure encourages the bank to commit to monitoring its borrowers, and hence allows it to extend loans. Additional equity capital makes it harder for the less fragile bank to commit to monitoring, which in turn hampers the bank´s ability to create liquidity. Capital may also reduce liquidity creation because it crowds out deposits. It is also consistent with findings of [4]. Higher interest margin means higher interest income from lending activity and thus higher opportunity costs of holding liquid assets. Therefore, as in [1] and [10], bank liquidity is inversely related to interest margin. The reaction of Polish banks on worsening quality of credit portfolio (measured by the share of nonperforming loans) is two years lagged and consists in lower lending activity and more cautious liquidity risk management. Finally, small Polish banks are more liquid than large banks. This finding fully corresponds to the well-known "too big to fail" hypothesis: if big banks see themselves as "too big to fail", their motivation to hold liquid assets is limited because they rely on liquidity assistance of a Lender of Last Resort in case of a liquidity shortage. This is also in accordance with results of [8], [10] and [12].

Poland			**Slovakia**		
variable	coefficient	std. deviation	variable	coefficient	std. deviation
constant	-25.5821	17.6765	constant	90.3900*	6.52797
CAP	0.53717**	0.23403	CAP	1.18825*	0.31887
IRM	1.04975***	0.62818	GDP(-1)	-1.01415*	0.27598
NPL(-2)	-0.18231**	0.09233	IRM	-5.95185*	0.88808
TOA	7.84335*	1.88526	NPL(-2)	-0.2419***	0.13774
			UNE	-2.01276*	0.39837
adjusted R^2	0.804507		adjusted R^2	0.813069	
Durbin-Watson stat.	2.046988		Durbin-Watson stat.	1.847315	
total observ.	247		total observ.	98	

Table 4 Determinants of loan to asset ratio in Poland and in Slovakia. The starred coefficient estimates are significant at the 1% (*), 5% (**) or 10 % (***) level.

Determinants of the loan to asset ratio in Slovakia can be found in Table 4. Again, first we included all explanatory variables which might have an effect on the dependent variable and then we reduced them with the use of information criteria in order to find a regression model with a high value of the adjusted coefficient of determination in which all the variables involved are statistically significant. Two of them – capital adequacy and nonperforming loans – have the same impact as in case of Poland. The influence of interest margin is completely opposite. This may be consistent with the problem of credit rationing and credit crunch: increase in interest margin may be a sign of higher credit risk of borrowers and as a result, banks focus more on interbank transactions or trading with government securities which increases their liquidity. Although most studies assumed the negative lint between the business cycle and bank liquidity, the results show that during expansionary phases companies and households rely more on internal sources of finance and reduce the relative proportion of external financing and lower their debt levels which in turn increases bank liquidity. In recessions, households and corporations increase their demand for bank credit in order to smooth out the impact of a lower income and profits. The growth rate of GDP is statistically significant with one year lag which is consistent with the fact that companies and households need some time for accumulating profits and savings and therefore it takes some time for changes to be reflected in banks´ liquidity. Lagged values also more likely reflect earlier decisions [4]. The rate of unemployment can act as a proxy for general health of the economy, therefore the higher the rate the more liquidity is held by banks, in order to hedge against potential higher liquidity risk.

5 Conclusion

The aim of this paper was to identify determinants of the loan to asset ratio of commercial banks in the Visegrad countries. This ratio is determined by capital adequacy, exchange rate development, share of nonperforming loans, size of the bank and bank profitability, two years lagged in the Czech Republic. In Hungary, loan to asset

ratio is influenced by bank capital adequacy, exchange rate development, rate of unemployment and by growth rate of gross domestic product, two years lagged. Loan to asset ratio of Polish banks is affected by the share of nonperforming loans, two years lagged, size of the bank, interest margin and bank capital adequacy. Finally, following factors are important for the development of loan to asset ratio of Slovak banks: growth rate of gross domestic product, one year lagged, interest margin, rate of unemployment, share of nonperforming loans, two years lagged and bank capital adequacy.

Although the Visegrad countries have a lot in common, it is evident that different factors determined the loan to asset ratio in individual countries. The only factor that is important in all analyzed countries is capital adequacy. Nevertheless, the link between capital adequacy and bank liquidity differs among countries: liquid banks are solvent, too, in Czech and Hungarian banking sector. This is consistent with the risk absorption hypothesis. On the contrary, insolvent banks are more liquid in Poland and in Slovakia, which confirms the financial fragility-crowding out hypothesis. The direction of influence of exchange rate depreciation and share of nonperforming loans is the same for all countries where these factors were statistically significant. However, direction of influence of the growth rate of GDP, the rate of unemployment, the interest margin and size of the bank differs among countries. Financial crisis, inflation, interbank interest rate, interest rate on loans and monetary policy interest rate have no statistically significant effect on the loan to asset ratio in any of the Visegrad countries.

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On quantile optimization problem with censored data

Petr Volf[1]

Abstract. The stochastic optimization problem is, as a rule, formulated in terms of expected cost function. However, criterion based on averaging does not take in account possible variability of involved random variables. That is why the criterion considered in the present contribution uses selected quantiles. Moreover, it is assumed that the stochastic characteristics of optimized system are estimated from the data, in non-parametric setting, and that the data may be randomly right-censored. Therefore, certain theoretical results concerning estimators of distribution functions and quantiles under censoring are recalled and then utilized to prove consistency of solution based on estimates. Behavior of solutions for finite data sizes is studied with the aid of randomly generated example.

Keywords: optimization, censored data, product-limit estimator, empirical quantile.

JEL classification: C41, J64
AMS classification: 62N02, 62P25

1 Introduction

Let us consider an optimization problem with utility function $\varphi(y, v)$, where v are input variables from certain feasibility set $\boldsymbol{V}$ and values y are random, results of a random variable (or vector) Y with distribution function F. Standardly, corresponding stochastic optimization problem can be formulated as $\sup_v E_F\varphi(Y, v)$, where E_F stands for the expectation w.r. to F. If F is known, we actually deal with a "deterministic" optimization case. However, criterion based on averaging does not take in account possible variability of r.v. Y and is actually reasonable for optimizing over long time period. Even then the variability of solution can be large. That is why the present paper is devoted to optimization of quantiles of random criterion $Z(v) = \varphi(Y, v)$. Alternatively, we can be interested in a kind of multi-objective optimization, simultaneously reducing also variability of solution (measured by variance, or certain inter-quantile range).

Further, our information on probability distribution could be non-complete. Either, known distribution type depends on unknown parameters. Or, we have to employ nonparametric estimates of F. Then, as a rule, the estimates are plugged into objective function. Hence, we have to analyze both possible bias and increased variability of obtained solution (compared to an ideal solution when F is known). An investigation of usage of empirical (estimated) characteristics in stochastic optimization problems started already in 70-ties. A number of papers has dealt with these problems, let us mention here just Kaňková (2010) with an overview and a number of other references.

In the present paper we consider even more complicated case when distribution function F should be estimated from the data censored randomly from the right side. Such situation is quite frequent in the analysis of demographic, survival or insurance data. The lack of information leads to higher variability of estimates and, consequently, to higher uncertainty of optimal solutions. The approaches to statistical data analysis in cases when the data are censored or even truncated are provided by a number of authors. The most of results were derived in the framework of statistical survival analysis and collected in several monographs (cf. Kalbfleisch and Prentice, 2002, or Andersen et al, 1993).

The main objective of the present paper is to study the increase of uncertainty of results of quantile optimization problem when the censoring is causing growing variability of non-parametric estimate of F. Therefore, in the next section, certain theoretical properties of estimates under random right censoring will be recalled. We shall consider the product-limit estimator as a generalization of the empirical distribution

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function, and corresponding estimator of quantiles. Their properties in cases with and without censoring will be compared. In Section 3 the consistency of solution employing estimated quantiles is proven. Finally, in Section 4 a simple example deals with optimal maintenance schedule, properties of obtained 'sub-optimal' solution are illustrated with the aid of simulations.

2 Estimators of distribution and quantile functions

Let us consider a continuous-type random variable Y characterizing for instance a random time to certain event. Let another continuous random variable U be a censoring variable, both be positive, continuous and mutually independent. Further, let $f(y)$, $g(u)$, $F(y)$, $G(u)$, $\overline{F}(y) = 1 - F(y)$, $\overline{G}(u) = 1 - G(u)$ denote the density, distribution and survival functions of both variables. It is assumed that we observe just $X = \min(Y, U)$ and $\delta = 1[Y \leq U]$, i. e. δ indicates whether Y is observed or censored from right side. The data are then given as random sample $(X_i, \delta_i, i = 1, \ldots, N)$. Notice that the case without censoring is obtained when $G(t) \equiv 0$ on region where $F(t) < 1$. Let us remark here that in some cases we can deal, for instance, with the logarithm of time. Then the domain of data can be the whole R_1.

A generalization of empirical distribution function is the well known Kaplan–Meier "Product Limit Estimate" (PLE) of survival function. Let us first sort (re-index) the data in increasing order, $X_1 \leq X_2 \leq \cdots \leq X_N$, then the PLE of $\overline{F}(t)$ has the form

$$\overline{F}_N(t) = \prod_{i=1}^{N} \left(\frac{N-i}{N-i+1} \right)^{\delta_i \cdot 1[X_i \leq t]}. \tag{1}$$

Again, notice that when all $\delta_i = 1$, we obtain the empirical survival function. The following proposition is due to Breslow and Crowley (1974):

Proposition 1. *Let $T > 0$ be such that still $\overline{F}(T) \cdot \overline{G}(T) > 0$. Then the random process*

$$V_N(t) = \sqrt{N} \left(\frac{\overline{F}_N(t)}{\overline{F}(t)} - 1 \right) = \sqrt{N}\, \frac{F(t) - F_N(t)}{\overline{F}(t)} \tag{2}$$

converges, on $[0, T]$, when $N \to \infty$, to Gaussian martingale with zero mean and variance function

$$C(t) = \int_0^t \frac{\mathrm{d}F(s)}{\overline{F}(s)^2\, \overline{G}(s)}. \tag{3}$$

Here, $F_N(t) = 1 - \overline{F}_N(t)$. In other words, $V_N(t)$ converges in distribution on $[0, T]$ to the process $W(C(t))$, where $W(\cdot)$ denotes the Wiener process. The asymptotic variance function can be estimated by its empirical version:

$$C_N(t) = \sum_{i=1}^{N} \frac{N\delta_i}{(N-i+1)^2} \cdot 1[X_i \leq t],$$

which is consistent in probability, uniformly w.r. to $t \in [0, T]$ (see again Breslow and Crowley, 1974).

Let us now recall also properties of empirical quantiles. 'True' p-quantile, for any $p \in (0, 1)$, is defined as $Q(p) = \min\{x : F(x) \geq p\}$, and is obtained as a unique solution of equation $F(x) = p$ provided F is strictly increasing. Empirical quantile is then defined as $Q_N(p) = \min\{x : F_N(x) \geq p\}$. Let now $p \in (0, F^{-1}(T))$, where T is from Proposition 1. Notice that $Q_N(p)$ is well defined only if $F_N(x) \geq p$ for some x, therefore with probability tending to 1 when $N \to \infty$. The following statement can be found for instance in Andersen et al (1993), Ch.IV.3 .

Proposition 2. *Let $f(x) > 0$ in the neighborhood of $Q(p)$. Then the empirical quantile $Q_N(p)$ is P-consistent and asymptotically normal, namely, for each $c < 1/2$*

$$N^c \cdot (Q_N(p) - Q(p)) \to^P 0, \quad \sqrt{N}(Q_N(p) - Q(p)) \to^d N(0, S(p))$$

and asymptotic variance equals

$$S(p) = \frac{(1-p)^2 \cdot C(Q(p))}{f(Q(p))^2}.$$

It follows that the variance of $(Q_N(p) - Q(p))$ can be estimated by

$$\frac{S_N(p)}{N} = \frac{(1-p)^2 \cdot C_N(Q_N(p))}{N \cdot f_N(Q_N(p))^2}, \tag{4}$$

which is complicated by inevitable estimation of density function, as a rule with the aid of kernel method.

If we denote $D_N(t) = V_N(t)/(1 + C(t))$, then for the case without censoring we obtain that $C(t) = F(t)/\overline{F}(t)$ and $D_N(t) = \sqrt{N}(F(t) - F_N(t))$ leading to standard Kolmogorov–Smirnov statistics. Notice also that then we obtain a well known result $asvar[\sqrt{N}(Q_N(p) - Q(p))] = \frac{p(1-p)}{f(Q(p))^2}$.

Further, from (3) it is also seen that the variance in the case with censoring (when $\overline{G}(t) \leq 1$) is larger than without it (i.e. when $\overline{G}(t) = 1$ on whole $[0, T]$).

3 Criterion based on quantiles, consistency

Let the optimization problem be now formulated as maximization of a p-quantile of distribution of random variable $Z(v) = \varphi(Y, v)$, for some selected $p \in (0, 1)$. If function $\varphi(y, v)$ is monotone increasing in y for each v, i.e. when there exists its inverse function $\varphi^{-1}(z, v)$, also increasing in z, then the distribution function of $Z(v)$, for fixed v, is $F_Z(z, v) = F(\varphi^{-1}(z, v))$. Therefore also quantiles of $Z(v)$ can be expressed as function of quantiles of Y, namely $Q_Z(p, v) = \varphi(Q(p), v)$ and optimal v depends directly on $Q(p)$. In general, however, connection between distribution and quantiles of variables Y and $Z(v)$ is not so straightforward and has to be analyzed, for instance with the aid of simulation.

Let the following assumptions hold:

A1. Let $p \in (0, F^{-1}(T))$ and let $f(x) > 0$ in a neighborhood of $Q(p)$.

A2. Function $\varphi(y, v)$ is bounded, increasing and continuous in a neighborhood of $y = Q(p)$, uniformly w.r. to $v \in \boldsymbol{V}$.

A3. $\boldsymbol{V}$ is compact and $\varphi(Q(p), v)$ is continuous in $v \in \boldsymbol{V}$. Hence, it is continuous uniformly in $\boldsymbol{V}$.

Further, denote $v^* = \arg\max_v \varphi(Q(p), v)$, $\varphi^* = \varphi(Q(p), v^*)$, $v_N^* = \arg\max_v \varphi(Q_N(p), v)$, $\varphi_N^* = \varphi(Q_N(p), v_N^*)$. When $\boldsymbol{V}$ is compact, at least one v^* exists, while v_N^* is defined with probability tending to 1.

Proposition 3. *When assumptions A1, A2, A3 hold, then*

1. $\varphi_N^* \to \varphi^*$ *in probability,*
2. *there exists a.s. a (random) subsequence* $N(k) \subset \{N\}$ *and* $\bar{v} \in \{v^*\}$ *such that* $v_{N(k)}^* \to \bar{v}$ *a.s.*

Proof.

i) From notation above, it follows that a.s. $\varphi_N^* \geq \varphi(Q_N(p), v^*)$ and $\varphi^* \geq \varphi(Q(p), v_N^*)$.

ii) Further, P-consistency of $Q_N(p)$ and A2 imply that, in probability, $\varphi(Q_N(p), v^*) \to \varphi^*$ and also $\varphi_N^* - \varphi(Q(p), v_N^*) \to 0$.

From i) and ii) assertion 1 follows, namely $\varphi_N^* \to \varphi^*$ in P. Moreover, it is also seen that

iii) $\varphi(Q(p), v_N^*) \to \varphi^*$ in P.

iv) The existence of converging subsequence $v_{N(k)}^*$ follows from compactness of $\boldsymbol{V}$. Then the uniform continuity of $\varphi(Q(p), v)$ ensures that $\varphi(Q(p), v_{N(k)}^*) \to \varphi(Q(p), \bar{v})$ in P.

This, together with iii), yields that $\varphi(Q(p), \bar{v}) = \varphi^*$ a.s.

Thus, except convergence of optimal values we showed also existence of a random sequence of solutions converging towards the set of optimal solutions $\{v^*\}$. If v^* is unique, then $\bar{v} = v^*$ a.s.

4 Example

Let us consider the following rather simple example of optimization problem (see also Volf, 2012): A component of a machine has its time to failure Y given (modeled) by a continuous-type probability distribution with distribution function, density, survival function F, f, $\bar{F} = 1 - F$, respectively. The cost of repair after failure is C_1, the cost of preventive repair is $C_2 < C_1$. For the simplicity we assume that only complete repairs, 'renewals', are provided, i.e. after each repair the component is new (exchanged) or as new. Let τ be the time from renewal to preventive repair, we wish to select an optimal value of τ.

Let us, as a criterion function, consider the proportion of component availability time to the unit of cost, namely

$$\varphi(y,\tau) = \frac{y}{C_1} \quad \text{if} \quad y \le \tau, \quad \varphi(y,\tau) = \frac{\tau}{C_2} \quad \text{if} \quad y > \tau.$$

4.1 Optimization of the mean

Let us first search for an optimal τ, from a reasonable closed interval $\boldsymbol{T}$, maximizing the mean

$$\phi_F(\tau) = E_F\varphi(Y,\tau) = \int_0^\tau \frac{y}{C_1}\mathrm{d}F(y) + \frac{\tau}{C_2}\overline{F}(\tau).$$

Optimal solution can be found directly, by solving equation $\mathrm{d}\phi_F(\tau)/\mathrm{d}\tau = 0$. In our case

$$\frac{\mathrm{d}\phi_F(\tau)}{\mathrm{d}\tau} = \frac{\tau}{C_1}f(\tau) + \frac{1}{C_2}\Big(\overline{F}(\tau) - \tau f(\tau)\Big).$$

Assume that the distribution of Y is Weibull, with parameters $a = 100, b = 2$, i.e. its survival function is $\overline{F}(t) = \exp\left(-\left(\frac{t}{a}\right)^b\right)$, corresponding numerical characteristics are $\mathrm{E}Y \sim 89$, $\mathrm{sd}(Y) \sim 46$, $\mathrm{median}(Y) \sim 83$. Further, let the costs be $C_1 = 10$, $C_2 = 1$. When the distribution function F is known, there exists an unique optimal solution with

$$\tau^* = a\Big(\frac{C_1}{(C_1 - C_2)b}\Big)^{1/b} = 74.5356$$

and maximal mean of working time per cost unit $\phi_F(\tau^*) = 44.7644$.

We can also compute directly the distribution of random variable $Z(\tau) = \varphi(Y,\tau)$, with property $Z(\tau) = \frac{Y}{C_1}$ if $Y \le \tau$ and $Z(\tau) = \frac{\tau}{C_2}$ if $Y > \tau$. If $Y \sim \mathrm{Weibull}(a,b)$ and certain τ is selected, the conditional distribution of random variable $Z(\tau)|Z(\tau) \le \frac{\tau}{C_1}$ has distribution function

$$F_Z(z) = \frac{1 - \exp(-(\frac{z \cdot C_1}{a})^b)}{(1 - \exp(-(\frac{\tau}{a})^b)}, \tag{5}$$

i.e. $Z(\tau)$ has on interval $(0, \tau/C_1)$ Weibull distribution with parameters $(a/C_1, b)$ and $Z_\tau = \tau/C_2$ with probability $P(Y > \tau)$. Therefore, we can compute variance and standard deviation. Namely, for optimal τ^* $\mathrm{sd}(Z(\tau^*)) = 34.5$. If we wish to reduce it, we have to accept certain trade-off. For instance, $\tau = 52$ yields approximately $\mathrm{E}Z(\tau) = 40.4$, so that just by 10% smaller value than the maximum, while standard deviation is reduced to $\mathrm{sd}(Z(\tau)) = 20.7$.

4.2 Optimization of quantiles

Let us return to the criterion based on a quantile. Above, in (5), the distribution of Z_τ has been derived, the quantiles $Q_Z(p,\tau)$ follow immediately from it. Figure 1 shows their form, from two different points of view. It follows that if we wish to maximize certain α-quantile of Z_τ, optimal $\tau^*(\alpha)$ should be such that $P(Y > \tau^*(\alpha)) = 1 - \alpha$, i.e. $\tau^*(\alpha)$ is the α quantile of the distribution of Y. Guaranteed value reached by Z with probability $1 - \alpha$ is then $\tau^*(\alpha)/C_2$. It is a consequence of the form of function $\varphi(y,\tau)$.

For instance for $\alpha = 0.1$ and $Y \sim$ Weibull(100,2) we obtain that $\tau^*(\alpha) = 32.4593$ and 90% guaranteed value of Z is $\tau^*(\alpha)/C_2 = 32.4593$, too, because $C_2 = 1$. If we wish to achieve a higher value of Z with sufficiently large probability, we can select for instance $\tau = 45$. As it corresponds roughly to 18%-quantile of Y, such a choice guarantees that $P(Z(\tau) = 45) \sim 0.82$. On the other hand, if we take τ^* maximizing the mean $E_F\varphi(Y,\tau)$, it guarantees that $P(Z(\tau) = 74.5) \sim 0.57$ and $P(Z(\tau) < 7.45) \sim 0.43$. It is seen that even here a kind of trade-off, with the use of multi-criteria approach, is reasonable.

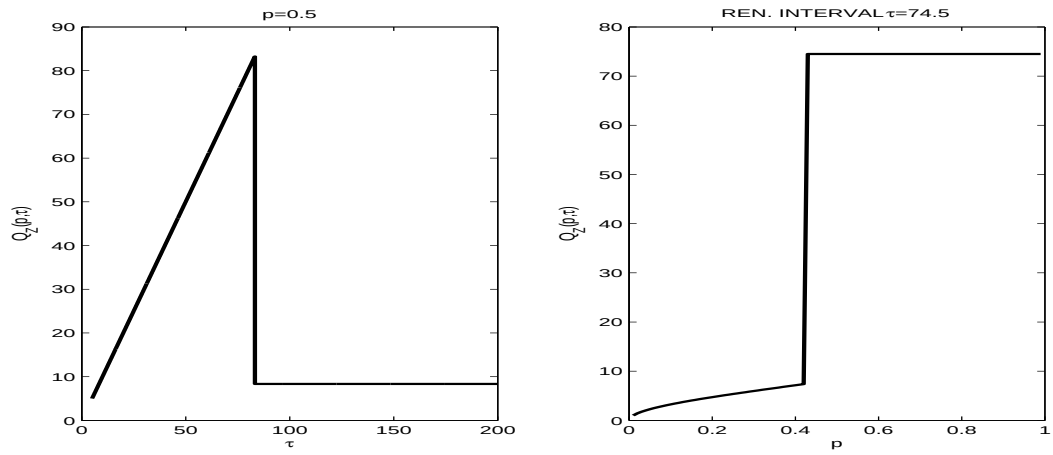

Figure 1 Quantiles $Q_Z(p,\tau)$ of variable $Z(\tau)$, left as a function of τ for given $p = 0.5$, right as a function of p for given $\tau = \tau^*$

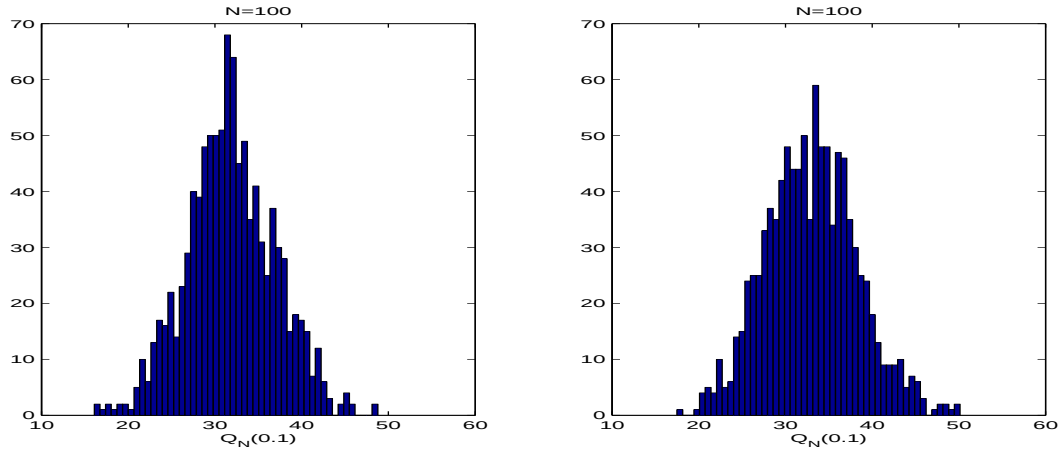

Figure 2 Histogram of generated sample quantiles $Q_N(p)$, for $N = 100$ and $p = 0.1$, computed from non-censored (left) and censored (right) data cases

4.3 A numerical study

In this part we provide a numerical study where it is assumed that the distribution of variable Y is estimated from data. In both considered cases (without or with censoring) $M = 1000$ samples of $N = 100$ and $N = 300$ observations Y_i are generated from the Weibull distribution specified above. In cases with censoring, censoring variables U_i have uniform distribution on $[0, 250]$, hence with survival function $\overline{G}(u) = (250 - u)/250$ (value 250 corresponds roughly to 0.998 quantile of distribution of Y). The rate of censoring is then about $36\,\% \sim EY/250$.

It is assumed that the type of distribution of Y is not known and therefore F is estimated non-parametrically with the aid of the product-limit estimator (i.e. as the empirical distribution function in the case without censoring). Thus, M estimates $F_N^{(m)}, m = 1, ..., M$ are obtained, from each the empirical quantile is computed, for given p. Figure 2 displays histograms of these M estimated quantiles, for $N = 100$, $p = 0.1$, the cases without censoring are plotted in the left subplot, the right subplot shows estimates obtained from censored data. Incorrect specification of the quantile shifts slightly both the guaranteed value and its probability, either decreases the value and increases probability, if $Q_N(p) < Q(p)$, or, in the opposite case, increases the value and decreases its probability.

It is well seen how the variability in the right subplot has increased due to censoring. Table 1 compares sample means and standard deviations computed from M=1000 values $Q_N^{(m)}(p)$ with true $Q(p)$ and standard deviations (denoted 'as-std') obtained as square roots of approximate variances $S(p)/N$, with $S(p)$ from Proposition 2. It is well seen how sample characteristics approach theoretical values.

p=0.1 N	Q(p)	non-cens.: as-std	sample mean	sample std		censored: as-std	sample mean	sample std
100	32.46	5.135	32.11	5.077		5.378	33.16	5.283
300	32.46	2.962	32.96	2.926		3.105	32.61	3.090
p=0.5								
100	83.26	6.006	82.82	5.826		6.907	83.53	6.990
300	83.26	3.467	83.34	3.466		3.988	83.34	3.961

Table 1 Comparison of theoretical and sample-based characteristics of empirical quantiles for $p = 0.1$ and $p = 0.5$, $N = 100$ and $N = 300$

5 Conclusion

We have studied the impact of variability of statistical estimates to uncertainty of solution in a stochastic optimization problem formulated via certain quantiles of utility function. We compared two cases, namely that the stochastic characteristics of the problem were estimated, in a non-parametric way, from fully observed or from randomly right-censored data. Therefore, theoretical properties of estimators of distribution function and quantiles from censored data were recalled, in order to compare them with the behavior of estimates in real situations. Such a comparison was performed with the aid of a simple optimization problem example and randomly generated data. Simultaneously, the convergence of solutions based on estimated quantiles to optimal solution was proven.

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Modeling and Optimization of Business Services Center

Jan Voráček[1], Tomáš Cerman[2]

Abstract. Computational modeling and simulation of enterprise processes proposes a promising alternative to mathematical and statistical methods. Simulations also handle complex flows of operational data satisfactorily, but instead of explicit formulas, they use specific structural representation of analyzed processes or workflows. Related schemes are interactively composed from fixed set of objects and resultant behavior can be either directly visualized or statistically evaluated. This paper introduces theoretical foundations of traditional queuing systems and compares their application possibilities with simulated solutions. General considerations are extended on process improvement and optimization and illustrated with example of real logistical processes. Obtained results confirmed high industrial potential of this innovative framework.

Keywords: discrete events simulation, operations management

JEL Classification: C63
AMS Classification: 90B22, 90B50

1 Introduction and theoretical foundations

Modern management consistently searches for innovative techniques and tools, capable to evaluate complex situations in a short time and transparent way. Purely formal methods, however, cannot simultaneously process all vital attributes of socio-technical systems, including their nonlinearity, non-stationarity, multidimensionality or subjectiveness. Dynamic conceptual modeling, followed be computational simulation of underlying phenomena, is convenient for well-defined tasks with tangible variables, appearing especially on the operational level of management hierarchy. These tasks, generally known as queuing systems or waiting lines, are capable to model complicated configurations of job shops or supply chains. Presented research combines theoretical resources and tools from neighboring areas of management science, computer science and strategic management in order to sufficiently represent and improve non-trivial industrial processes. Its main goal was to propose an instantly applicable framework to managers, searching for fast and transparent - even if only suboptimal - answers to their day-to-day operations performance planning and control questions.

1.1 Queuing nodes and networks

Queuing node is a basic functional element of systems, where any kind of waiting is unavoidable. First theoretical frameworks for systematic handling of independent sequential events were introduced and applied, e.g., by Poisson [1], Markov [2] or Erlang [3]. Independent and memoryless occurrences of events, compliant with discrete Poisson distribution, are fully characterized by their mean arrival rate over a given period of time λ and, wherever applicable, also by exponentially distributed mean service time μ. Analytical solution of queuing nodes requires enumeration of corresponding stationary distributions, leading, e.g. to the mean values of the following performance variables:

- Waiting, service and total times in system, i.e. W_q, W_s and $W=W_q+W_s$,
- Amounts of waiting, served and all objects in system, i.e. L_q, L_s and $L=L_q+L_s$,
- Utilization of server/system $\rho = \lambda/n\mu$, where n is the total amount of servers in system and $\rho < 1$.

In simple or simplified real-world applications, standalone nodes are combined to more complex structures with channels (parallelization of servers) and phases (serialization of servers). According to such taxonomy, isolated queuing node is a single-channel, single-phase entity. Analogically, several parallel servers, followed by linear sequences of other servers establish multi-channel, multi-phase structure. Heavily interconnected serio-parallel topologies with nontrivial routing logic, which cannot be expressed just in terms of channels and phases, form queuing networks. First explicitly introduced and still widely applied networking paradigm was published by Jackson [4]. Majority of compound queuing architectures has only stationary solutions. Their transient properties, determining quality of provided service, are usually derived from simplified formal representations or, more frequently, simulated.

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1.2 Simulation of discrete events

Simulation is imitation of system behavior by means of convenient computational approximation. In contrast to numerical methods, simulation does not require existence of mathematic formulas and can enumerate just annotated structural diagrams of given problem. Such paradigmatic shift is beneficial for socio-technical problems, formally rigorous representation of which is either impossible or even undesirable. It is important to mention that there are fundamental qualitative differences between analytic and simulation ways of process analysis. The latter approach is more engineering oriented, i.e. instead of performing calculations, user combines reusable structural elements through technical features of selected software platform. Such solution naturally lacks generality and is biased by method and tool. On the other hand, computational modeling of complex problems is faster and feasible for majority of involved stakeholders. Consequently, realization of simulated solutions must follow specific procedure, supporting smooth transformation of qualitative modeling artifacts into their computational version. Applicable modeling methodology, described, e.g. in Gharajedaghi [5] includes the following main stages:

- Conceptualization, i.e. collection and formalization of related knowledge, information and data,
- Creation of executable model from conceptual resources, its calibration, validation and verification,
- Design and evaluation of experiments, including analyses of sensitivity, scenarios and policies,
- Iterative improvement and structural extension.

Presented research belongs to the family of discrete-event simulations (DES), collecting dynamic problems, discrete in time and values, stochastic and usually event-driven. DES models are composed from predefined library elements like servers, dispatchers, queues, delays or synchronization points. Beyond the numeric processing, this framework allows also realistic visualization of processes in native layouts. Thus the managers can interactively inspect animated flows of goods, customers or transport and balance available resources with respect to local indicators of operational performance (waiting or throughput time, utilization of servers, number of objects in system etc.). To achieve desired behavior, simulated problems can combine multiple types of inputs, including real data, empirical or parametric statistical distributions. Good experimental practices suggest execution of several independent and sufficiently long-lasting replications with initial throwaway warm-up period. Obtained numerical results are statistically processed and visualized. Details can be found in excellent DES monographs written by Banks et al. [6] or Kelton et al. [7]. Proximity between analytically solved queuing problem and its simulated counterpart can be demonstrated on example, searching for the best configuration of service station among the following three applicable layouts:

- Single (markovian) queue, single (markovian) server (M/M/1 in Kendall notation), λ = 10/min, μ = 12/min, $L_q = \lambda^2/\mu(\mu - \lambda)$, $W = L_q/\lambda + 1/\mu$, $\rho = \lambda/\mu$,
- Two queues, two servers (2 x M/M/1), λ = 5/min, μ = 12/min, $L_q = \lambda^2/\mu(\mu - \lambda)$, $W = L_q/\lambda + 1/\mu$, $\rho = \lambda/\mu$,
- Single queue, two servers (M/M/2), λ = 10/min, μ = 12/min, $L_q = \lambda^3/\mu(4\mu^2 - \lambda^2)$, $W = L_q/\lambda + 1/\mu$, $\rho = \lambda/2\mu$.

Selected performance parameters can be either calculated or simulated in SIMIO [8]. Results, summarized in Table 1, show that there are no significant differences between the both approaches. As supposed, M/M/2 ordering represents the most efficient option.

Parameters and models		**M/M/1**		**2xM/M/1**		**M/M/2**	
		Analytic	*Simulated*	Analytic	*Simulated*	Analytic	*Simulated*
L_q	[cust.]	4.2	*4.6*	0.6	*0.6*	0.2	*0.3*
W	[s]	30.0	*32.0*	8.6	*8.6*	6.1	*7.2*
ρ	[%]	83.3	*82.6*	41.7	*40.7*	41.7	*39.5*

Table 1 Comparison of rounded analytic and simulated results of sample problem

1.3 Process improvement

Processes are input-output oriented complex sequences of atomic functions, establishing standardized customer-to-customer flows, such as order to cash or procure/purchase to pay. Processes are naturally cross-functional, utilize company-wide resources and institutionalize its core, qualitative and managerial practices. There are many process-related methodologies [9], discussion of which is beyond the scope of this paper. Every process must be thoroughly quantified by means of diversified set of Key Performance Indicators (KPIs), comprehensively characterizing related quantitative and qualitative metrics. Besides direct process planning and monitoring, selected KPIs are also interlinked with company's strategic performance architecture. Because of number, complexity and mutual diffusion of company processes, it is impossible to assign set of unique and unrelated KPIs to every process. Instead, designers select particular indicators concerning (i) *core process functions* (percentage of processes completed in time, average time to complete task or sum of costs of terminated realizations), (ii) *service level agreement* (percentage of service requests completed in time, total cost

of service delivery or availability of service) and (iii) *service quality and efficiency* (cycle time and length, numbers of customer complaints, late tasks or process alerts, rework ratio or customer satisfaction). Industrial processes are structurally similar to queuing networks, because single entities (customers, raw materials, documents or money) are also moving between different service nodes, waiting in queues for free resources or returning for rework. Tangible and intangible resources are shared according to internal process logic and flows of objects are multiply dispatched and merged before the endpoint is reached. This makes processes convenient for conceptual modeling, followed by DES solution, where earlier versions are gradually improved and optimized according to standardized methodologies, like Lean, Six Sigma, Toyota Production System, Just-In-Time, Theory of Constraints, Total Quality Management, ISO standardization or benchmarking [10,11]. In the rest of this text we focus only on simplified characteristics and application of Lean and Six Sigma principles. *Lean* continuously escalates value for customer in continuously repeating phases, involving value identification, value stream integration, flow creation, pull establishment and perfection achievement. Non-value adding procedures (waste) like transportation, storage or unnecessary bureaucracy are gradually removed or minimized, which leads to costs reduction and both efficiency and quality improvement. Employment of Kanban cards guarantees smooth bidirectional flows of goods and information. *Six Sigma* strives to keep variation of selected KPIs within the range of six standard deviations between their means and nearest specification limits, which guarantee zero-defects operations. With growing process variability, sample distributions move from expected locations towards upper or lower limits and probability of defects increases. Continuous development of process sigma contributes to process quality and capability. In context of process improvement, Six Sigma is implemented through DMAIC methodology, including problem Definition, collection of necessary Measurements, Analysis of this data, Improvement of existing techniques based on achieved results and Control of updated processes to be sure that changes were applied properly. If Lean eliminates waste and Six Sigma reduces variability, their right combination boosts throughput (less steps) and quality (less defects) simultaneously. As optimal process costs lie around the intersection of waiting costs and costs of service capacity curves, minimizes the joint Lean Six Sigma application also this indicator.

2 Case study: processes in business service center

Global companies strive to concentrate the majority of transactional and supportive activities in dedicated unit, called Business Services Center (BSC). Modern BSCs are highly scalable and reconfigurable offshore structures, profiting from uniqueness, specialization, economies of scale and lower labor price. Presented case demonstrates daily operations of export department, collecting orders from two channels, negotiating production and arranging either container or truck transport. This agenda is split among three teams, order taking (OTt), transport planning (TPt) and logistic administration (LAt). Orders from different channels are undertaken by OTt and entered to corporate information system. Depending on whether the order will be dispatched from stock or made to order, TPt checks and provides the corresponding Lead Time (LT). If customer requires also Point of Sales (POS) items, i.e. miscellaneous sales-supporting materials, OTt reserves them, finalizes order and notifies customer and manufacturer or distributing warehouse. When the product becomes available, TPt takes care for all practical aspects of transport. In parallel, LAt processes necessary transport documents and after loading maintains the rest of business case. Responsibilities are concretized in process diagrams in figure 1. Duration of every random activity x is represented with triangular distribution, probability density function of which is $f(x|a,b,c)$, where a and b specify the range of approximated variable and c is its most likely value. In contrast to the normal distribution, triangularly determined temporal samples are always nonnegative. Incoming orders can enter BCS either via Electronic Data Exchange (EDI) channel, providing always batch of 13 orders with about 25:75 truck to container ratio and distributed with parameters (0.16, 0.2, 0,19) or individually (marked as non-EDI channel, incorporating phone, mail or personal delivery) with distribution (0.1, 0.2, 0.16) and approx. 50:50 truck to container ratio. Data also shows that EDI to non-EDI ratio is 40:60. Converting rates to daily amount of orders for 8-hours shifts we obtain the following distributions: (i) *EDI orders* (100, 125, 119), (ii) *non-EDI orders* (48, 96, 77), (iii) *container orders* (124, 173, 157) and (iv) *truck orders* (24, 48, 38). Detailed distributions of internal activities, derived from industrial data, are listed in table 2 altogether with the following experts' judgments/probabilities:

- Occurrence probability (OP) quantifies existence of event with respect to the particular input channel. This metrics, e.g., says that there are two non-EDI inputs - one for truck and one for container (activities 1 and 5) because sum of their OPs is 1- and only one EDI container order (event 18),
- System probability (SP) quantifies existence of events with respect to the whole system. Extending the previous example, total contribution of the both non-EDI inputs is sum of their SPs, i.e. 40%. Announced 25:75 truck to container ratio is reflected with SPs of purely truck-related activities like 7 or 9. Containers proportion is visible either as a sum of events 5 and 11 or just from activity 19.

Beyond the described internal activities, BSC processes must accommodate also external activities, including manufacturing, transportation and administrative transactions, marked as delays in figure 1 and listed in table 3.

No.	Event	Type	Input	Team	OP	SP	Distribution
1	Lead time checked, order inserted manually	T	n-EDI	TP	0,5	0,20	(1, 5, 2)
2	Order in system, customer informed	T,C	n-EDI	OT	0,5	0,20	(1, 3, 2)
3	POS arranged	T,C	n-EDI	OT	0,025	0,01	(2, 10, 5)
4	Order inserted manually	C	n-EDI	OT	0,5	0,20	(1, 3, 2)
5	Lead time checked, delivery date estimated	C	n-EDI	TP	0,5	0,20	(2, 4, 3)
6	(Truck) transport arranged, system updated	T	n-EDI	TP	0,1	0,05	(8, 30, 15)
7	Slot booked	T	n-EDI	TP	0,5	0,25	(1, 5, 2)
8	Special documents arranged	T	n-EDI	LA	0,025	0,01	(1, 2, 1)
9	All (truck) documents arranged and sent	T	n-EDI	LA	0,5	0,25	(1, 4, 1)
10	(Container) transport arranged, system updated	C	n-EDI	TP	0,5	0,25	(5, 10, 8)
11	All (container) documents arranged and sent	C	n-EDI	LA	0,5	0,25	(8, 30, 12)
12	EMCS registered	T,C	n-EDI	LA	0,1	0,05	(2, 3, 2)
13	Proof of delivery received	T,C	n-EDI	LA	1	0,50	(1, 4, 2)
14	Transportation invoice paid	T,C	n-EDI	LA	0,1	0,05	(2, 4, 2)
15	Business case archived.	T,C	n-EDI	LA	1	0,50	(1, 3, 2)
16	Lead time checked. Delivery date estimated	C	EDI	TP	1	0,60	(2, 4, 3)
17	POS arranged	C	EDI	OT	0,05	0,03	(2, 10, 5)
18	Order in system, customer informed	C	EDI	OT	1	0,60	(1, 3, 2)
19	(Container) transport arranged, system updated	C	EDI	TP	1	0,75	(5, 10, 8)
20	All (container) documents arranged and sent	C	EDI	LA	1	0,50	(8, 30, 12)
21	Proof of delivery received	T,C	EDI	LA	1	0,50	(1, 4, 2)
22	Transportation invoice paid	T,C	EDI	LA	0,1	0,05	(2, 4, 2)
23	Business case archived	T,C	EDI	LA	1	0,50	(1, 3, 2)

Legend. *Type*: Truck or Container, *OP*: Occurrence Probability, *SP*: System Probability. All distributions are triangular.

Table 2 List of analyzed activities, related resources and probabilities

2.1 Process performance and its improvement

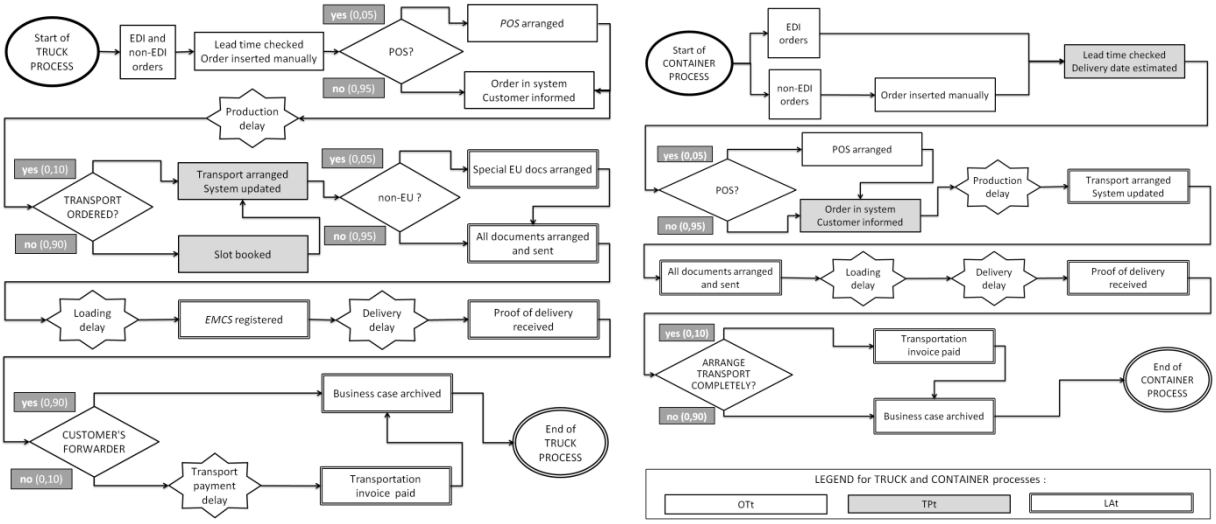
In accordance with methodology, proposed in chapter 1.2, we converted available structural and temporal information to valid computational SIMIO model and made a set of simulations, mimicking actually used allocation of resources, which expects 3 officers in OTt, 6 in TPt and 9 in LAt. Performance indicators were adopted from practice and besides the traditional quantities, including means of waiting time, time spent in system, utilization and costs, there are three quality related metrics:

- Standard performance indicator (SPI) is a proportion orders, waiting in any queue less than 30 minutes,
- Defects pre Million of Opportunities (DPMO) summarizes all defects, i.e. orders, spending in system more than 70 minutes,
- Process sigma is a Six Sigma-standardized measure of process capability. This ratio is derived from DPMO by dividing minimal distance between sample mean and either lower or upper specifications limit by sample standard deviation. Recommended target value for non-manufacturing processes varies between 4 and 5.

Costs are proportional to time and equal for all administrative procedures. Basic time unit is one minute, which is convenient for office operations but rather misleading in case of external activities. Total simulation time is 120 000 minutes, i.e. one single-shift working year. Simulated results are summarized in the first two data columns of table 4. First column shows a hypothetical case of "unlimited" resources and serves as benchmarking standard for real experiments. There are no doubts that immoderate resources (8 OTt, 8 TPt, 12 LAt) imply erroneous operations – our costly configuration really guarantees 100% SPIs, zero DPMO and maximal level of process sigma. Also other indicators are outstanding with natural exception of total costs.

External activities and probabilities	SP	Distribution of service time	
		Currently applied model	Improved model
Production	1	(0, 12000, 9600)	(9600, 12000, 9600)
Container loading	0.75	(480, 2880, 1920)	(480, 960, 480)
Truck loading	0.25	(0, 2880, 1440)	(480, 960, 480)
Waiting POD	1	(1920, 28800, 5280)	(1920, 28800, 5280)
Transport payment	0.15	(11000, 33000, 22000)	(11000, 33000, 22000)

Table 3 List of external processes (delays) and corresponding system probabilities



a) Truck process b) Container process

Figure 1 Process diagrams for the both presented processes

Currently practiced configuration (3 OTt, 6 TPt, 9 LAt) achieves performance, listed in the second data-column of table 4. We can notice cca. 20% increase in the both temporal indicators, compensated with more than 30% costs savings. Utilization is below the recommended 80% limit, process sigma indicates reasonable maturity and standard performance is also excellent.

Our next considerations followed two basic strategies – managers could either concentrate on *quality* or emphasize financial aspects and reduce *costs*. The first orientation expects adoption of convenient process improvement methods; the second implies modification of the most significant costs drivers. In accordance with Six Sigma methodology, we concentrated on statistical parameters of particular activities and evaluated their system probabilities, durations and shapes of distributions. Table 2 shows, that there are only two major candidates for future improvement: activities *(Truck) transport arranged, system updated* (no. 6) and *All (container) documents arranged and sent* (no. 11 and 20). Because system probability of truck process is negligible in comparison with the container one, we proposed organizational redesign of the later, resulting in balanced distribution (10, 23, 17). Principles of Lean manufacturing identify and minimize non-vale adding activities across the whole process/business case. From this point of view, it is evident that especially long-lasting processes outside BSC are convenient for "leanizatation". Our first proposal was to replace the whole production to stock with production to order. Pull strategy saves inventory and manipulation costs, allows full profit from economies of scale, makes lead times more deterministic and establishes smoother production flow.

Operational performance for different parameterizations of simulation model		**Configuration (OTt, TPt, LAt)**			
		Hypothetical redundant (8, 8, 12)	**Currently applied (3, 6, 9)**	**Lean SixSigma improved (3, 6, 9)**	**Improved and optimized (2, 5, 8)**
Mean time in system [s]		32.2	40.0	38.1	46.6
Mean waiting time [s]		4.9	6.3	6.0	7.2
Costs in millions [CZK]		8.2	5.4	5.4	4.6
Mean utilization [%]	Total	37.8	57.5	53.5	60.5
	OTt	16.7	42.6	40.0	53.1
	TPt	43.1	58.7	54.0	59.8
	LAt	53.6	71.1	66.4	68.7
Standard performance indicator [%]	Total	100	97.7	99.8	98.2
	OTt	100.0	100.0	100.0	97.7
	TPt	100.0	99.3	99.8	97.8
	LAt	100.0	99.4	99.5	99.2
DPMO [orders]		0	23 390	8 831	95 977
Process sigma [-]		6.0	3.5	3.9	2.8

Table 4 Comparison of selected performance indicators for different configurations of simulated BSC model. Duration: 1 working year, computational step: 1 minute.

As a consequence, transportation can be planned more precisely, i.e. with lower dispersion, shortened loading delays and proof of delivery acquisition times. What we cannot affect anyhow are payment terms, fixed legally and administratively outside the processes. Adjusted distributions are listed in table 4, column "Lean Six Sigma improved model". Costs-centered considerations can be applied primarily to the salaries, i.e. optimization in this case means the minimization of number of team members, preserving acceptable performance. We did simple one-shot three-dimensional step-based search for minima of improved model, insisting in gradual change of every variable until the desired behavior is reached. Because sample mean daily arrival rate $\lambda_S = 196$, we initially searched for the lowest number of OTt members, capable to process more than λ_S orders per day. When found, we fixed this team size and repeat the same procedure for remaining teams. Finally we got a costs-suboptimal configuration (2 OTt, 5 TPt, 8 LAt) performing according to the last column of table 4. The goal was met and costs were reduced on about 15%. Although mean time spent by order in system is higher, SPI is still very high and utilization acceptable. On the contrary, process quality and capability evidently suffer from diminished resources and managers must decide, whether approximately ten times more defects is really yearly saved salaries worthwhile. Our last note concerns the existing SPI level. Although we keep its original level, i.e. waiting time in any queue up to 30 minutes during all experiments, it is evident that information value of such frozen metrics is low. In practice all thresholds of performance indicators must be reconsidered altogether with related improving activities. As a part of validation testing, we realized also parametric sensitivity analysis of all internal activities and identified operation *POD received* as the major BSC bottleneck, resulting to the longest queue of unfulfilled requirements. Beyond the pure monitoring, management can handle discovered risks either with [Drum, Buffer, Rope] principle, introduced in the Theory of Constraints [12] or train members of other teams in LAt agenda, to be able to undertake their operations when necessary. Experiments with different scenarios, incorporating realistic variations of incoming orders, delayed payments or pending proofs of delivery confirmed parametric stability and robustness of presented model.

3 Conclusion

We claimed the role of discrete events simulations in modern management of operations. In contrast to analytical approaches, our solution can quickly and satisfactorily process complex operational layouts and enhances managerial decision making with interactivity and representational clarity. Consequently, beyond the provision of purely computational solution, simulations can serve also as a supportive platform for creative thinking and iterative institutionalization of organizational knowledge.

Our paper also suggests, how seemingly complex methodologies like Lean Six Sigma or theoretically demanding adjustments like (sub-)optimization can be straightforwardly implemented and evaluated. Because of transparency and practical nature of presented case we believe that prospective industrial adopters could find analogies with own businesses and consider pilot evaluation of discrete events simulation framework in companies.

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Business Cycle Analysis: Comparison of Approaches to Cycle Measurement

Lenka Vraná[1]

Abstract. In the context of recent economic recession public attention was focused on the business cycle analysis and possibility of forecasting the cycle movements.
One of the methods used for the business cycle analysis is based on the study of composite indicators which combine several individual economic indexes. The indexes can be divided into groups of leading, coincident and lagging ones with regard to the reference time series (usually GDP or industrial production index). The dependency between cycles of the individual indicators and the reference series can be determined with cross correlations and it is essential to adjust the time series first, so we get rid of the trend and seasonal component.
The traditional approach (used by OECD and other organizations) is using algorithms like Hodrick-Prescott filter for the decomposition of the time series. We compare this methodology with smoothing the series of growth rates and with direct usage of the year-to-year growth rates. We discuss how these methods affect the cross correlations and the selection of the individual indexes for the composite indicators.

Keywords: business cycle, growth cycle, composite indicators, cycle measurement.

JEL Classification: E32
AMS Classification: 91B84

The traditional business cycle, the recurring fluctuations of economic levels, consists of sequences of expansions and contractions in economic activity. The recessions (or the larger downturns – depressions) mean the decline in absolute level of economic output. Most European economies experience rather slowdowns (declines in growth rates) than the real declines in the level. These slowdowns and speedups are referred to as growth cycles. Nowadays usually the growth cycles are analyzed and the slowdowns are often treated like recessions. However, as Zarnowitz and Ozyildirim [10] point out, all recessions involve slowdowns but not all slowdowns involve recessions and therefore the growth cycles are more numerous than the business cycles. In this paper we analyze the growth cycles and we show how the dating of the turning points can be affected by this choice.

This paper describes the different approaches to the cycle measurement used for construction of composite indicators. The goal of this paper isn't to create composite indicators for Czech economy. Section 1 provides brief explanation of OECD methodology including the description of Hodrick-Prescott filter. Section 2 offers alternatives to the OECD method of determining the cycle component and section 3 compares all the described algorithms. All sections are supplemented with the analysis of 3 economic indicators to illustrate the algorithms: index of industrial production, import and composite confidence indicator.

1 Composite indicators and OECD methodology

One of the methods how to analyze business cycle (or growth cycle) is to create system of composite indicators. The composite indicators consist of several individual economic indicators and enable to monitor the state of business cycle better than just by analyzing the individual time series. The construction of composite indicators in the Czech Republic usually follows OECD methodology with only slight alterations; see Czesaný and Jeřábková [4] or Tkáčová [9].

1.1 Pre-selection

When constructing the composite indicators the eligible individual series has to be selected first. According to OECD methodology [5] only long time series of indicators that have justified economic relationship with the reference series, broad coverage of economic activity, high frequency of observations (preferable monthly), that were not subject to any significant revisions and that are published soon should pass the p*re-selection phase*.

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The selection of individual indicators is highly dependent on the reference time series. Usually GDP or index of industrial production is used as reference series. The GDP should respond to the cyclical movements better but it is quarterly statistics and it is necessary to convert it to the monthly estimates. OECD had used the industrial production index until March 2012 and then switched to the adjusted monthly GDP. We will use the index of industrial production because it shows strong co-movements with GDP series and it is available monthly so we won't need to recompute the reference series.

1.2 Filtering

The second phase of the composite indicator construction is called the *filtering*. The main task of this stage is to decompose the individual time series. The series have to be seasonally adjusted and the trend component has to be removed. OECD has used Hodrick-Prescott filter to de-trend the series since December 2008.

Hodrick-Prescott filter divides the series into two parts (τ_t - trend component and c_t - the cyclical component) and optimizes expression

$$\min_{\tau_t}\left[\sum_t \left(y_t - \tau_t\right)^2 + \lambda \sum_t \left(\tau_{t+1} - 2\tau_t + \tau_{t-1}\right)^2\right]. \tag{1}$$

It minimizes the difference between the trend and the original series and smoothes the trend as much as possible at the same time. The λ parameter prioritize the latter from the two contradictory goals – the higher the λ, the smoother the trend.

Hodrick-Prescott filter deals with the series as with the system of sinusoid and it keeps in the trend only those with low frequency (high wave length). According to OECD the business cycles last 10 years at maximum therefore the fluctuations with lower wave length should be kept in the cycle component. Ravn and Uhlig [8] recommend to set the λ parameter equal to 129 600 to do so for monthly series. Nilsson and Gyomai [6] confirm that Hodrick-Prescott filter gives clear and steady turning point signals[2].

After the trend component is estimated it is subtracted from the original data set (this is called the *deviation cycle* then).

After the cycle components of all the individual indicators are found, turning points are detected. Not every peak or trough of the cycle is considered as the turning point though. For more information how to reveal the turning points see Bry and Boschan [2].

1.3 Evaluation

The cycle components of all the individual indicators are compared to the reference series. OECD uses several methods how to evaluate their relationship: the average lead (lag) times between the turning points, cross correlations and number of extra and missing cycles. Then the selected individual indicators are divided into groups of leading, coincident and lagging ones and the composite indicators are created.

The leading composite indicator should be able to predict future states of economic activity. The coincident indicator serves mainly to confirm the hypothesis about the state the economy is currently in. It also may replace GDP or industrial production index as the reference series for the evaluation of the individual indicators (this approach is used in the USA by the Conference Board). The lagging indicator should certify the cycle behavior and the dating of the turning points.

After *evaluation* phase OECD continues with *aggregation* of the individual series and *presentation* of the results. These stages are not described in this paper because they don't relate closely with the topic. For more information about the rest of the process see OECD [5].

1.4 Example: Hodrick-Prescott filter

We selected 3 individual economic indicators to illustrate the usage of the algorithms described in this paper: the index of industrial production, the import and the composite confidence indicator. Each of these time series is available from January 2002 to August 2012 on monthly basis and was seasonally adjusted by the Czech Statistical Office so we focus mainly on the detrending. The indicators were selected arbitrary not as the result of the pre-selection phase.

The industrial production index was chosen because it shows strong co-movements with the GDP and (unlike GDP) is available monthly and published sooner. The import was selected because of its unclear link to the

[2] Nilsson and Gyomai use the Hodrick-Prescott filter twice: first with high λ to find the trend and then with low λ to smooth the cycle component.

Czech economic cycle. While Czesaný and Jeřábková [4] classify it as one of the lagging indicators, it is included in the set of leading indicators by Tkáčová [9]. As the Czech economy is small and open, there could be significant relationship between the business cycle and the cycle component of the import. The composite confidence indicator (also known as economic sentiment indicator) is according to the Czech Statistical Office [3] calculated as a weighted average of confidence indicators in industry, construction, trade, in selected services and of the consumer confidence indicator. Confidence indicators are very often included in the composite leading indicators.

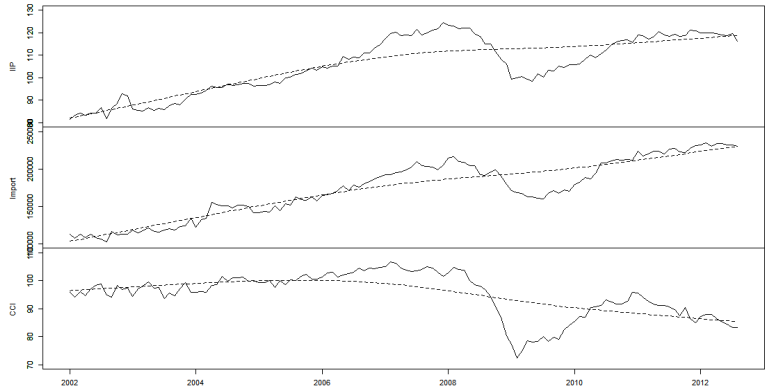

Figure 1 Index of industrial production (IIP), import and composite confidence indicator (CCI) with trend estimations

The figure 1 shows the development of the index of industrial production, the import and the composite confidence indicator. The rapid drop that occurs in 2008 is visible in all three time series. The chart contains the estimated trend component as well. The trend was computed with Hodrick-Prescott filter according to (1) with the λ parameter equal to 129 600 (so it contains only the fluctuations with wave length higher than 10 years and all the other fluctuations are kept in the cycle component).

The obvious similarity of the deviation cycles of the three selected indicators (shown on the figure 2) suggests that the import as well as the composite confidence indicator could be consistent with the cyclical behavior of the whole economy (measured here by the industrial production index).

Figure 2 Deviation cycle of index of industrial production (IIP), import and composite confidence indicator (CCI)

2 Different approaches to the de-trending of the time series and their consequences

There are other options how to remove the trend component from time series except the Hodrick-Prescott filter (or any other band-pass filters). The simplest one is to use growth rates.

The month-to-month growth rates tend to be unstable and therefore they need to be smoothed. Zarnowitz and Ozyildirim [10] discuss how smoothing with fairly long moving averages affects the results of the business cycle analysis in a negative way, because the procedure runs a certain risk of distorting the patterns and the timing of turning points. Instead of moving average we use the Hodrick-Prescott filter as described above (however only to smooth the series). In this case it is sufficient to use very low values of the λ parameter so we would get rid of the high frequency fluctuations and the cycle movements would remain with the series. After some experimentation we choose λ equal to 30.

To avoid the erratic movements of growth rates (and the need of smoothing them) it is possible to compute growth rates over longer periods, in this paper we use year-to-year changes. These series show the similarity to the shape of cycle component calculated with the previous methods (from the original series with Hodrick-Prescott filter and by smoothing the month-to-month growth rates).

2.1 Time shift of the turning points

As was mentioned above dates of the business cycle turning points may not coincide with dates of the growth cycle turning points. The dates of the growth cycle turning points are identified from the deviation cycle which was described above. The turning points of the cycle computed from the month-to-month series differs even more as the cycle has slightly distinct interpretation.

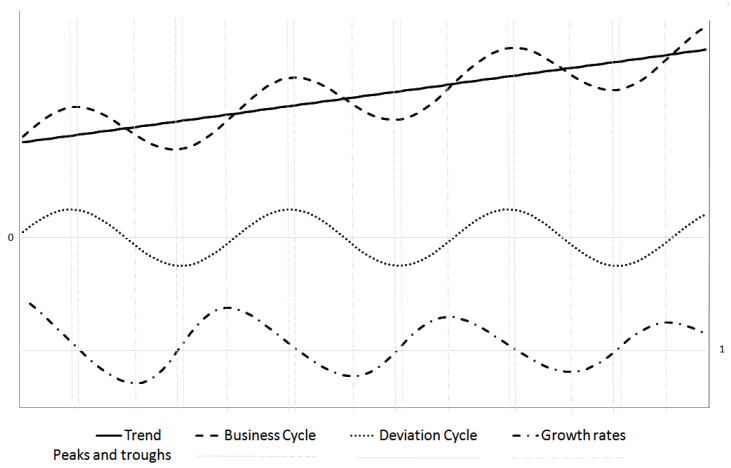

Figure 3 Schema of turning points dates of business cycle,
deviation cycle (on the main axis) and growth rates (on the secondary axis)

Figure 3 shows the timing of the turning points of the business cycle, the deviation cycle and the growth rates. The vertical lines highlight the dates of the turning points. As the deviation cycle is created by subtracting the trend component from the business cycle it leads the business cycle in peaks and lags in troughs. The growth rates start to decrease when the business cycle is still rising however its growth tends to slow down. When the growth rates fall bellow one the business cycle is in its maximum and starts to decline and vice versa. In other words the dates of the peaks and troughs of the growth rates correspond to the inflection points of the business cycle. This has to be considered when interpreting the results of cycle analysis based on the growth rates.

2.2 Example: Smoothed month-to-month growth rates

Figure 4 shows the month-to-month growth rates of the index of industrial production, the import and the composite confidence indicator. The time series are smoothed by the Hodrick-Prescott filter according to (1) with λ parameter set to 30. The low λ guarantees that only the fluctuations with very high frequency are excluded and the time series is smoothed. The smoothed series itself represent the cycle component. Also the drop in 2008 is visible as well in the smoothed series.

Figure 5 compares the timing of the deviation cycle with the cycle computed by smoothing the growth rates of the index of industrial production (the charts of the import and the composite confidence indicator would be very similar and therefore they won't be presented here). The latter seems to be in lead; however this is caused by the different interpretation of these two cycles as was mentioned above. The most obvious drops in the deviation cycle match with the periods when the smoothed growth rates were below one.

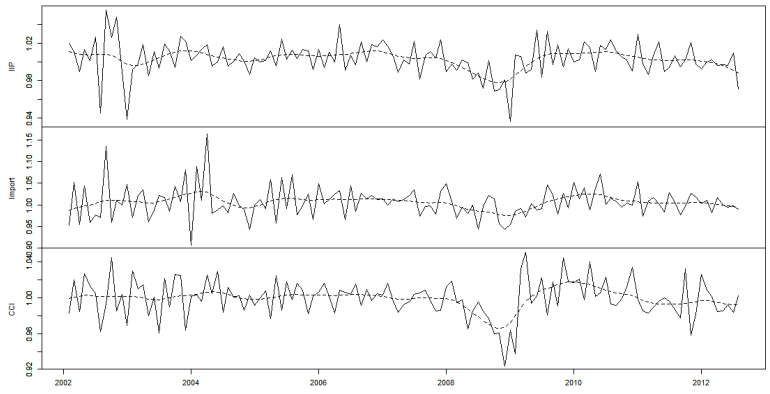

Figure 4 Month-to-month growth rates and smoothed month-to-month growth rates of index of industrial production (IIP), import and composite confidence indicator (CCI)

2.3 Example: Year-to-year growth rates

The advantage of the year-to-year growth rates is that they are much less erratic than the month-to-month growth rates and so they don't need to be smoothed or otherwise adjusted. The figure 6 compares the deviation cycle and the year-to-year growth rates of the index of industrial production and proves that the timing of these two series is much more similar then in the previous case.

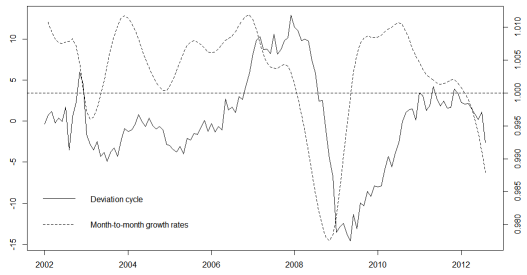

Figure 5 Comparison of deviation cycle and smoothed month-to-month growth rates of industrial production index

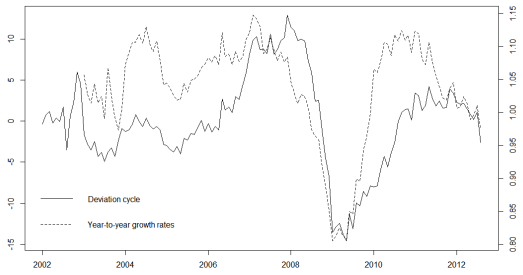

Figure 6 Comparison of deviation cycle and year-to-year growth rates of industrial production index

3 Evaluation of the relationship between indicators

We presented three algorithms how to measure the cycle component of the time series: deviation cycles, smoothed month-to-month growth rates and year-to-year growth rates. We showed that the timing of the turning points may differ when using these methods. It is possible that also the dependencies between the cycle components of the individual indicators are changed.

In the first section we mentioned several evaluation methods how to determine the relationship between the reference series and the economic indicator. We use cross correlations to assess the influence of the described algorithms. The cross correlations measure the linear dependency between the reference series and individual indicator with applied time-lag. Then the maximum of absolute value of cross correlation is found and the individual indicator is included into one of the composite indicators (leading, coincident or lagging) according to the time-lag where the maximal value appeared.

Table 1 proves that maximums of cross correlations occur in the same time-lag for the deviation cycles and the year-to-year growth rates. According to these two methods the import would be included into the coincident composite indicator and the composite confidence indicator into the leading composite indicator. On the other hand when using the month-to-month growth rates the import would be included into the lagging and the composite confidence indicator into the coincident composite indicator. This shows that with month-to-month growth rates not only the turning points of the cycles are affected but also the method interprets the relationships between the economic indicators differently.

Table 1 Cross correlations between index of industrial production and import or composite confidence indicator with applied time-lag

	t-5	t-4	t-3	t-2	t-1	t	t+1	t+2	t+3	t+4	t+5
Deviation cycle											
Import	0.661127	0.730080	0.787996	0.846698	0.886445	**0.916873**	0.901211	0.890431	0.851692	0.795020	0.725795
CCI	0.760244	0.810098	0.841704	0.868856	**0.879717**	0.859643	0.820470	0.756998	0.685356	0.600052	0.498510
Month-to-month growth rates											
Import	0.412395	0.511734	0.610386	0.700181	0.771535	0.816594	0.859626	**0.863546**	0.829779	0.761484	0.664725
CCI	0.562657	0.663062	0.751862	0.820612	0.862096	**0.871547**	0.856434	0.804603	0.720344	0.609471	0.479253
Year-to-year growth rates											
Import	0.583460	0.671306	0.738754	0.807571	0.860423	**0.892476**	0.874678	0.874933	0.828707	0.757909	0.656686
CCI	0.672138	0.742332	0.794910	0.837820	**0.856281**	0.847744	0.821346	0.764822	0.692962	0.597306	0.488362

However it is necessary to keep in mind that the cross correlations are not the only measure for the component selection and should be used only in the combination with other criteria (e. g., average lead/lag times between turning points).

4 Conclusion

In this paper basics of the OECD methodology of composite indicators construction were presented. We focused on the stage of the time series decomposition where Hodrick-Prescott filter is used by OECD. We compared this method with usage of smoothed month-to-month growth rates and year-to-year growths rates. We discussed influence of these algorithms on the timing of the cycle turning points and on the dependencies between the selected indicators (the import and the composite confidence indicator) and the reference time series (the industrial production index) measured here by the cross correlations. We proved that OECD's deviation cycles and the year-to-year growth rates give similar results which are distinct from the results of the smoothed month-to-month growth rates.

There are of course several extensions that should be performed on this research, especially verification of the results with other evaluation methods than the cross correlations.

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Time-varying network structure of cross-correlations among stock markets of CEE-3 and Germany

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Abstract. The paper presents an analysis of stock market networks based on return correlations in the Czech Republic, Hungary, Poland (the CEE-3 markets) and Germany. Germany is selected as a major developed market for its proximity to the CEE-3 countries. The graph structures are formed by constructing the vertex sets from stocks traded on each market with edge weights based on correlations from standardized residuals on each return series. The evolution of these structures in time is studied within the time-frame of the financial crisis, allowing for the analysis of the development of inter-market linkages. The increase of such linkages in time of crisis might be viewed as a support for the contagion hypothesis.

Keywords: stock market networks, minimum spanning trees, contagion

JEL Classification: L14, G1
AMS Classification: 62P20, 94C15, 91G70

1 Introduction

In this paper we investigate the time-varying structure of correlations among stocks traded in CEE-3 markets – the Czech Republic, Hungary and Poland, as well as a developed market represented by Germany. To capture the structure of co-movements in stock returns we utilize the methods used in analyzing graph networks.

The study of networks has a long history in many fields, including operations management and operations research. In the last twenty years, the approach has been used in various areas dealing with complex systems [1], social networks, food webs [7], the World Wide Web [2], the internet [16] and others.

Our main interest however deals with the network analysis applied to stock markets. The generally used methodology follows the work of Mantegna [12], who proposed a distance based on the correlations between individual stock returns, satisfying the axioms of a metric distance (such as symmetry and triangle inequality). This distance can in turn be used to construct a weighted undirected graph with the set of vertices corresponding to the stocks analyzed and edge weights given by the calculated distance.

The next step in the analysis usually consists of choosing a suitable subgraph of the created network. As the correlation (and thus also the distance) is defined between all of the stocks, the created structure presents a complete graph on given vertices. To reduce the complexity of the network, the most common choice is to consider a minimum spanning tree, or MST[4]. The MST results in a subgraph retaining all vertices, but reducing the $N(N-1)/2$ edges from the complete graph to $N-1$. Several alternative methodologies to the use of MSTs on correlation matrices have been proposed, some focusing on the selection of subgraphs, such as in the case of Planar Maximally Filtered Graphs (PMFG) [18], [19], others dealing with the basis for calculating of edge weights, such as in partial correlation planar maximally filtered graphs (PCPG) [10].

The research conducted using the methodology based on MSTs on financial markets is numerous. For example, Coelho et al. [4] explored the stock market integration between 53 countries by creating MSTs from stock index returns over the period of 1997 – 2006. They conclude that by analyzing the evolution of MST properties over time, it can be seen that the trees become more compact, suggesting increasing interrelation between the markets. The study was based on the calculation of mean occupational layer and survival rates, proposed by Onnela et al. [14, 15]. A similar methodology based on MSTs was used by Gilmore et al. [8] to analyze twenty government bond market indices for developed North American, European, and Asian countries, also finding an increase in co-movements. Also studying government bond yield rates, Dias [5] performed an analysis of MSTs

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[4] For an algorithm used to create an MST, see [17].

of 19 European countries between April 2007 and October 2010. Similar methodology has also been used to analyze networks among currencies and commodities [13].

2 Data and methodology

Our dataset spans the stock markets of three Central and Eastern European markets (CEE-3) and a single developed market. The CEE-3 markets include the Czech Republic, Hungary and Poland. These markets have had important economic as well as political links in the past, and are members of the so-called Visegrad group (V4). The only country missing from V4 is Slovakia, which was excluded because of the very low liquidity and trade volume, lagging behind the CEE-3 countries.

Germany was chosen as a representative for the developed countries. First, it is the geographically closest major market to the CEE-3 countries. There is also real economic rationale (like the structure of mutual trade on the macroeconomic level and ties between companies on microeconomic level) that might give rise to relationships between stocks traded on the respective exchanges. Yet another reason for selecting Germany is the smaller time-lag of German stock market with the CEE-3 exchanges, as the trading occurs in the same time-zone. To further mitigate the non-synchronous trading effects [3], we use daily prices to calculate weekly returns and perform the analysis on weekly data.

Although several previous studies used market indices, we use data on individual stock trading from each market. Using stock data from several markets allows us to visualize the dynamics of inter-market linkages, particularly the clustering of stocks by the country of origin. As for the selections of stocks, our sample includes the constituents of the leading local stock indices, traded between January 2003 and October 2012. Daily closing prices were obtained from the Thomson Reuters Datastream.

We have checked the stationarity of each return series with a version of the KPSS as proposed by Hobijn et al. [9][5]. At significance level of 0.1 we were unable to reject the null of mean stationarity for all of the analyzed series[6]. As most financial time series exhibit both autocorrelation and heteroscedasticity, we went on to fit a suitable ARMA(p, q) – GARCH(1, 1) model, to account for both effects. The ARMA parameters were evaluated up to 5 lags. As for the specification of the GARCH part, the following models were considered: GARCH, AVGARCH, NGARCH, GJR-GARCH, APARCH, NAGARCH, TGARCH, FGARCH and CSGARCH. All selected models were required to show no significant autocorrelation and conditional heteroskedasticity in standardized residuals in up to 25 lags (about 5% of the sample), tested by the Ljung-Box test. Of all models matching the above mentioned conditions, the optimal model was chosen according to the Bayesian information criterion.

The standardized residuals obtained from the modeling stage have then been used for the calculation of rolling Pearson product-moment correlation coefficients, with windows size of 52 weeks and drift of 1 week. The use of overlapping rolling correlations allows for the analysis of the network dynamics, although it induces greater persistence with increasing windows size [11].

The rolling correlation coefficients have subsequently been converted to distances using the formula by Mantegna [12]:

$$d_t(i, j) = \sqrt{2(1-\rho_{ijt})} \tag{1}$$

where $d_t(i,j)$ is the calculated distance between stocks i and j ($i, j = 1, 2, \ldots, N;\ N = 50$) at time t, $t = 52, 53, \ldots, 509$ (time index starts at 52 because of the 52-week rolling correlation window) and ρ_{ijt} is the rolling correlation coefficient between standardized residuals of stock i and j at time t. The calculated distances allow for the creation of minimum spanning trees, thus we have obtained 458 MSTs. Henceforth we denote the MST at time t as MST_t, it's vertex set as V_t and set of edges E_t. Note that the vertex set remains the same and $|V_t| = N$ for all t.

To analyze the way the graphs evolve in time, we have used two measures, as defined in [14, 15], namely the *normalized tree length* $L(t)$ and *mean occupational layer* $l(t)$ defined as follows

$$L(t) = \frac{1}{N-1} \sum_{(i,j)\in E_t} d_t(i, j) \tag{2}$$

[5] For the estimation of the long-run variance, we used kernel weights calculated from quadratic spectral with Newey-West automatic bandwidth selection procedure.
[6] The test results are excluded for brevity.

$$l(t) = \frac{1}{N}\sum_{i=1}^{N} \mathrm{lev}(v_{i,t}) \tag{3}$$

In (3), $\mathrm{lev}(v_{i,t})$ is the level of the vertex $v_{i,t}$. It is defined as the number of vertices along the path connecting $v_{i,t}$ and $v_t^C \in V_t$, the central vertex for MST_t, excluding $v_{i,t}$. The central vertex can be selected in a number of ways [14, 15]: a) by the highest vertex degree (number of incident edges), b) by the vertex for which the sum of correlations corresponding to the incident edges is the highest, or c) by the centre of mass, corresponding to a vertex, which (when used as a central vertex) produces the lowest mean occupational layer $l(t)$.

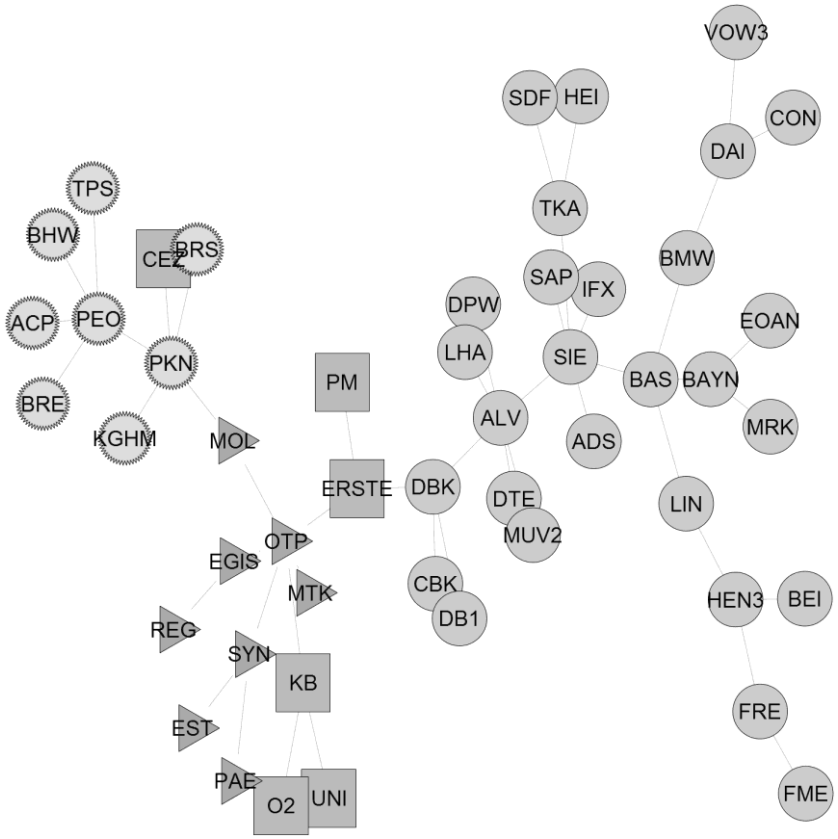

Figure 1 Minimum spanning tree for the whole sample

3 Results

3.1 Whole sample properties

Before we analyze the time-varying properties of the MTSs, we first start by describe some aspects over the whole sample. The list of analyzed companies, as well as fitted ARMA-GARCH models is shown in Table 1. A MST constructed from correlations estimated over the whole sample period can be seen in Figure 1. First, we may note that the vertices are fairly well clustered by country of origin – there is only one edge between a German stock and other countries (the articulation vertex being DBK). Similarly, the Polish and Hungarian vertices are linked among themselves, the only exception being the Czech Republic.

As this is a MST constructed on the whole sample, it tells us nothing about the dynamics of the relationships. It gives us an overall picture of almost ten years, which is why we also pursue the analysis on rolling correlations. Nevertheless, the overall MST does show some properties that give nice economic interpretation. For example, consider the links within the group consisting of OTP (Hungary), ERSTE and KB (Czech Republic), DBK and CBK (both Germany). All of these vertices correspond to banks, and the MST edges between them exist despite being from different countries. In a sense, the MST shows that the vertices that are used to link country clusters are made primarily of banks.

Ticker	Name	Market	Model specification	Ticker	Name	Market	Model specification
ERSTE	Erste group bank	CZE	ARMA(1,1)-IGARCH(1,1)	BMW	Bayerische Motoren Werke	DEU	ARMA(1,1)-IGARCH(1,1)
PM	Philip morris CR	CZE	ARMA(1,1)-GARCH(1,1)	BAYN	Bayer	DEU	ARMA(1,1)-IGARCH(1,1)
CEZ	ČEZ	CZE	ARMA(1,1)-CSGARCH(1,1)	BEI	Beiersdorf	DEU	ARMA(1,1)-IGARCH(1,1)
KB	Komercni banka	CZE	ARMA(3,1)-GJRGARCH(1,1)	CBK	Commerzbank	DEU	ARMA(1,1)-IGARCH(1,1)
UNI	Unipetrol	CZE	ARMA(1,1)-GJRGARCH(1,1)	CON	Continental	DEU	ARMA(3,2)-IGARCH(1,1)
O2	Telefónica CR	CZE	ARMA(1,1)-GARCH(1,1)	DAI	Daimler	DEU	ARMA(1,1)-CSGARCH(1,1)
EGIS	Egis pharmaceuticals	HUN	ARMA(1,1)-GARCH(1,1)	DBK	Deutsche Bank	DEU	ARMA(3,1)-GJRGARCH(1,1)
EST	Est media	HUN	ARMA(1,1)-APARCH(1,1)	DB1	Deutsche Boerse	DEU	ARMA(1,1)-GJRGARCH(1,1)
MOL	MOL	HUN	ARMA(1,1)-APARCH(1,1)	DPW	Deutsche Post	DEU	ARMA(1,1)-IGARCH(1,1)
MTK	Magyar telekom	HUN	ARMA(1,2)-APARCH(1,1)	DTE	Deutsche Telekom	DEU	ARMA(1,1)-GARCH(1,1)
OTP	OTP bank	HUN	ARMA(3,3)-IGARCH(1,1)	EOAN	E.ON	DEU	ARMA(1,1)-IGARCH(1,1)
PAE	PannErgy	HUN	ARMA(1,1)-CSGARCH(1,1)	FME	Fresenius Medical Care	DEU	ARMA(1,1)-CSGARCH(1,1)
REG	Richter Gedeon	HUN	ARMA(2,1)-IGARCH(1,1)	FRE	Fresenius SE & Co KGaA	DEU	ARMA(1,1)-IGARCH(1,1)
SYN	Synergon	HUN	ARMA(1,1)-APARCH(1,1)	HEI	HEICO Corporation	DEU	ARMA(3,2)-GARCH(1,1)
KGHM	KGHM	POL	ARMA(1,1)-GJRGARCH(1,1)	HEN3	Henkel AG & Co.	DEU	ARMA(1,1)-IGARCH(1,1)
PEO	Bank Polska Kasa Opieki	POL	ARMA(2,1)-IGARCH(1,1)	IFX	Infineon Technologies	DEU	ARMA(1,1)-IGARCH(1,1)
PKN	Polski Koncern Naftowy Orlen	POL	ARMA(2,3)-APARCH(1,1)	SDF	K+S Aktiengesellschaft	DEU	ARMA(1,1)-EGARCH(1,1)
TPS	Telekomunikacja Polska	POL	ARMA(1,1)-APARCH(1,1)	LIN	Linde Aktiengesellschaft	DEU	ARMA(1,1)-IGARCH(1,1)
ACP	Asseco Poland	POL	ARMA(1,1)-IGARCH(1,1)	LHA	Deutsche Lufthansa	DEU	ARMA(2,3)-APARCH(1,1)
BHW	Bank Handlowy w Warsza-wie	POL	ARMA(1,1)-IGARCH(1,1)	MRK	Merck KGaA	DEU	ARMA(3,3)-APARCH(1,1)
BRE	BRE Bank	POL	ARMA(2,3)-IGARCH(1,1)	MUV2	Munich RE	DEU	ARMA(1,1)-GJRGARCH(1,1)
BRS	Boryszew	POL	ARMA(1,2)-EGARCH(1,1)	SAP	SAP	DEU	ARMA(1,1)-GARCH(1,1)
ADS	Adidas	DEU	ARMA(1,1)-IGARCH(1,1)	SIE	Siemens Aktiengesells-chaft	DEU	ARMA(1,1)-IGARCH(1,1)
ALV	Allianz	DEU	ARMA(1,1)-IGARCH(1,1)	TKA	ThyssenKrupp AG	DEU	ARMA(1,1)-IGARCH(1,1)
BAS	BASF	DEU	ARMA(1,1)-CSGARCH(1,1)	VOW3	Volkswagen	DEU	ARMA(2,2)-APARCH(1,1)

Table 1 List of analyzed companies and fitted ARMA-GARCH models

One more aspect that can be observed on the level of the entire sample is the relationship of normalized tree length $L(t)$ and the average correlation at time t. One could immediately see that there is reason for these measures to be related – normalized tree length in essence computes the average distance in an MST, which is calculated from correlation. However, when calculating the average correlation, there are $N(N-1)/2$ averaged correlation coefficients. With normalized tree length, there are only $N-1$ distances, which is an order of magnitude smaller. Hence, their relationships might not be so clear. To measure this relation, we calculated the correlation between $L(t)$ and average correlation coefficient at time t over the whole sample. The correlation turns out to be about -0.9761, which is very strong. The negative sign is to be expected, given the construction of distances in Equation (1). The result is similar to the original finding of Onnela [14, 15], despite our use of weekly, instead of daily data.

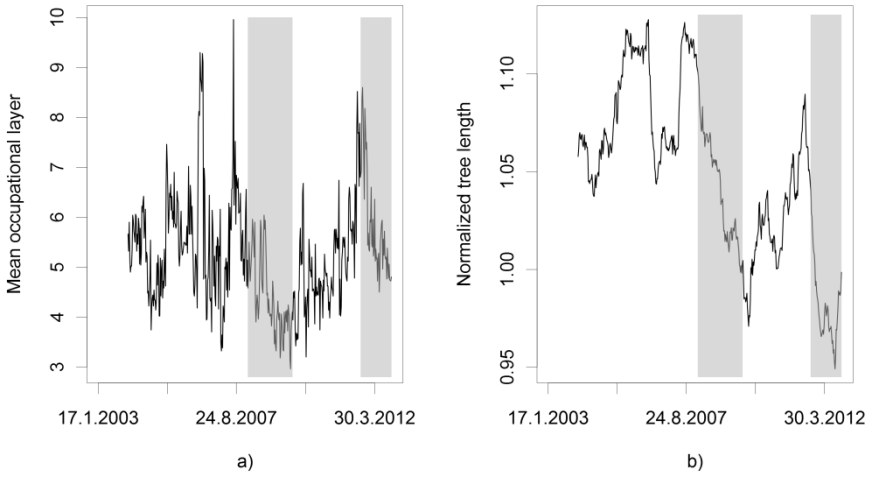

Figure 2 The evolution of a) mean occupational layer, b) normalized tree length

3.2 Time-varying properties

After describing the graph properties of the sample as a whole, we employ the mean occupational layer and normalized tree length to describe the time-varying character of the network structure. The evolution of both measures is shown in Figure 2.

In case of the mean occupational layer, we have calculated all three variants mentioned previously. The values in Figure 2 have been obtained by defining the central vertex as one with the maximal vertex degree. The shaded areas represent recessions in the Euro area, dated by the Centre for Economic Policy Research. There have been two recessions present in our sample – one starting with first quarter of 2008 and ending with the second quarter of 2009, and the other starting in the third quarter of 2011.

The figure showing the mean occupational layer can be seen as illustrating a decrease of $l(t)$ during the recession periods. This behavior makes economic sense – lower value of $l(t)$ may be interpreted as a decrease in the average layer at which the vertices reside, as seen from the central node. That would mean that the tree is becoming more compact in such periods. When using the alternative options for the selection of central vertices for the purposes of $l(t)$ calculation, the results are less pronounced for the 2008 – 2009 recession, but are still quite visible for 2011.

The outcome is also supported by the evolution of normalized tree length. During both recession periods, we see a marked drop in $L(t)$. As the correlation of $L(t)$ and the average correlation coefficient is very strong, the result seems to point out the general tendency of stock returns becoming more correlated in times of crisis, which is known as contagion [6]. The obtained results seem to support this hypothesis.

4 Conclusion

By using the methodology generally employed in the analysis of stock market networks, we have analyzed the rolling correlations of standardized residuals of stocks traded in CEE-3 countries, as well as Germany.

Our methodology differed slightly from the one cited in most empirical works. First, instead of continuous daily returns, we have analyzed the standardized residuals from a fitted ARMA-GARCH model, taking into account autocorrelation and conditional heteroscedasticity that is frequently present in financial time series. Second, in order to mitigate the effect of non-synchronous trading, we have used weekly instead of daily returns.

The results on the sample as the whole were interesting in two respects. Even though the construction of the analyzed series differed from other empirical works, the constructed MSTs still retained much of the information on stock return correlations, thus providing rationale for the use of MSTs. An MST in itself presents a rather drastic reduction of graph edges from the initial complete graph, and the question of MST validity is an important one. It could also be seen that over the whole sample, the vertices exhibited strong country clustering. The vertices incident with edges connecting country clusters corresponded mostly to banks, which provides an interesting insight from the economic point of view, particularly with respect to the ongoing European sovereign-debt crisis.

The simple analysis conducted on the mean occupational layer and normalized tree length suggest, that the MSTs change in recession times – particularly, the trees become more dense and compact, with the average number of vertices needed to reach a central vertex becoming smaller, similarly to the average distance, which also decreases. These results seem to support the so-called contagion hypothesis, which predicts an increase in stock return correlation in times of crisis.

Acknowledgements

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Modeling claim severity via h-likelihood

Alicja Wolny-Dominiak[1]

Abstract. The aim of this paper is to apply hierarchical generalized linear model (HGLM) in property/casualty insurance classification ratemaking. Classification (a'priori) ratemaking focuses on the risk segmentation via rating variables and outlining criteria to consider when using a certain risk characteristic as a rating variable. The extension is a'posteriori ratemaking, in which claim experience of the policyholder is taken into account. At present, the typical statistical technique used in classification ratemaking is the generalized linear model (GLM). In GLM we can only deal with fixed effect in linear predictor. But for some insurance data, there is a need to treat one or more parameters as random effects, e.g. in situation when the independent assumption is not satisfied. In this paper we consider HGLM model for claim severity with gamma distribution. As the method of estimation we use an extended likelihood for canonical scale h-likelihood [8]. In all calculations the real-world insurance data are used [4].

Keywords: classification ratemaking, claim severity, GLM, HGLM.

JEL classification: C21, C20
AMS classification: 62J12

1 Introduction

The classification ratemaking is an important part of data analysis in insurance business in which the construction of a fair tariff structure is done. The goal of this classification is partition of all policies in particular portfolio into homogeneous classes according to rating variables. Within every class, all policyholders pay the same premium. In practice, insurance company calculates the tariff which has to fulfill the following general principles stressed in [11] :

- the tariff has to be as correct as possible in relation to different risk groups;
- the structure of the tariff has to be such that the calculation of the insurance premium is quite straightforward;
- the factors influencing the tariff have to be few enough and the structure of the tariff has to be the simple (e.g. linear or multiplicative) function.

Keeping these principles, to design classification rating plans, actuaries use the generalized linear models (GLM) technique. In GLM model, the dependent variable Y is usually the claim severity defined as the total claim amount divided by the number of claims. Additional information is the claim frequency-modeling in similar way in [13]. As Y variable is positive and skewed to the right, we focus on gamma claim severity distribution. Tariff variables $X_1, \ldots, X_p$ are usually categorical with few categories like e.g. gender or a large number of categories like e.g. spatial variables. We consider the extension of GLM model, in which the random effect is included outside fixed effects. This paper is organized into three sections, except introduction. In Section 2. and Section 3. we describe models for gamma claim severity, with fixed and mixed effects respectively. Section 4. includes a case study for insurance data with simple procedure of creating the tariff structure. We use cross-sectional data set.

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2 GLM gamma claim severity - fixed effects

Consider the independent observations $y_1, \ldots, y_n$ of claim severity with gamma pdf of the form

$$f\left(y \mid \mu, \nu\right) = \left(\frac{\nu}{\mu}\right)^{\nu} \frac{y^{\nu-1} e^{-\frac{\nu y}{\mu}}}{\Gamma(\nu)}, \quad y_i > 0, \tag{1}$$

with first two moments $E(y) = \mu$ and $V(y) = \frac{\mu^2}{\nu}$. Assume the fixed-effect model as follows

$$y_i = \mu + \sum_j \beta_j x_{ij} + \epsilon_i, \tag{2}$$

where $Var(\epsilon_i) = \sigma^2$, $i = 1, ..., n$. It can be expressed in matrix notations

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}, \tag{3}$$

where $\boldsymbol{\beta} = (\mu, \beta_1, ..., \beta_p)$ is a vector of fixed-effect parameters and μ can be interpreted as base claim severity. The variance matrix $\boldsymbol{V}$ can be written as $\boldsymbol{V} = Var(\boldsymbol{y}) = \sigma^2 \boldsymbol{I}$, under the assumption that observations are uncorrelated. In survey sampling non-diagonal $\boldsymbol{V}$ matrices are also considered, as in [14]. We assume multiplicative effects thus log-link function is used. The structure of GLM gamma claim severity is then in form

- $\boldsymbol{y} = \boldsymbol{\mu} + \boldsymbol{\epsilon}$, where $\boldsymbol{\mu}$ and $\boldsymbol{\epsilon}$ are vectors of expected means of the observations and residuals respectively;
- $\boldsymbol{\eta} = \boldsymbol{X}\boldsymbol{\beta}$ - the linear predictor of rating variables connected with $\boldsymbol{\mu}$ via log-link function;
- link function - $\boldsymbol{\eta} = \log(\boldsymbol{\mu})$;
- y is gamma distributed with $E(y) = \mu$, $Var(y) = \phi\mu^2$ and dispersion parameter $\phi = \frac{1}{\nu}$.

In this model all structural parameters $\beta_1, \ldots \beta_p$ are fixed effects estimated directly from data through IWSL method, see in [8]. But in classification ratemaking there are few reasons to treat one or more parameters as random variable and then these parameters are random effects modeled themselves by certain model with hyperparameters. The first reason is independence of a sample $y_1, \ldots, y_n$ as the assumption in GLM model. Taking the spatial rating variable, this assumption is usually doubtful [2]. Another reason is that particular rating variable can take, for example, a large number of categories and setting the parameter separately for each category would be justified, it is preferable to treat it as a random effect. Finally, the data can have a panel structure and then portfolio of policies is repeated measurement over time. In the next part of the paper the model with both: fixed and random effects will be consider. In the literature one can find different names for this type of model: mixed effects model, random effect model, hierarchical model.

3 HGLM gamma-gamma claim severity - fixed and random effects

Generally speaking, hierarchical models are used in the situation when the data are grouped (because of the random effect in the model) into homogenous groups, also called subjects, with a similar expected value of Y. Few examples shown in [7] for insurance business are as follows :

- Expected workers compensation claims for exposures with various NCCI class codes;
- Expected loss ratio relativities for a personal auto carriers various state territories.

The excellent motivation of using hierarchical models generally in insurance are presented in [2]. We examine models suitable in classification ratemaking called HGLMs (hierarchical GLM). HGLM is an extension of GLM by including random effects in the model. It means that one or more rating variables have the structural parameters treated as a random variable. This variable follow the exponential family distributions (not only normal distribution as in LMM or GLMM). Using the knowledge of the characteristics of insurance data, in our study we consider HGLM claim severity with one random effect, in which fixed and random effects are gamma distributed (shortly gamma-gamma model). The general structure of the model is

- $\boldsymbol{y} = \boldsymbol{\mu} + \boldsymbol{\epsilon}$;
- $\boldsymbol{\eta} = \boldsymbol{X\beta} + \boldsymbol{Zv}$ (no more the linear predictor of rating variables as in GLM);
- log-link $\boldsymbol{\eta} = \log(\boldsymbol{\mu})$;
- log-transformation $v(u) = log(u)$;
- the random effect $\boldsymbol{u}$ is gamma distributed with $E(u) = \psi$, $Var(u) = \rho\psi^2$ and dispersion parameter λ.

In this matrix notation $\boldsymbol{y}_{(p\times 1)}$ is a vector of responses, $\boldsymbol{X}_{(n\times p)}$ is a design matrix for fixed effects, $\boldsymbol{Z}_{(n\times q)}$ is a design matrix for random effects, $\boldsymbol{\beta}_{(p\times 1)}$ is a vector of fixed effects and $\boldsymbol{u}_{(q\times 1)}$ is a vector of the possible realisations of random effect (simplified we use v instead of $v(u)$). To estimate fixed effects and random effect jointly the classical likelihood inferences can be applied. However, it requires extension of likelihood function due to the presence of random effect v as in [3]. This extended likelihood must take into account three components: unknown gamma distribution parameters, unobservable random quantities v and observed data y and we defined log-extended likelihood for claim severity in HGLM gamma-gamma as

$$\begin{aligned} l_E &= \log L(\beta_1, \ldots, \beta_p, v; y, v) = \log L(\beta_1, \ldots, \beta_p, v; y|v) + \log L(v) = \\ &= [C_1 - (\frac{\epsilon_i}{\mu_i} + \log \mu_i + (\nu - 1)\log(\epsilon_i)] + [C_2 - (\frac{v_i}{\psi_i} + \log(v_i))\nu_v + (\nu_v - 1)\log(v_i)], \end{aligned} \tag{4}$$

where $\boldsymbol{\epsilon} = \boldsymbol{y} - \boldsymbol{X\beta} - \boldsymbol{Zv}$ and C_1, C_2 are constants equal to $C_1 = \nu log(\nu) - log(\Gamma(\nu))$ and $C_2 = \nu_u log(\nu_u) - log(\Gamma(\nu_u))$. The extended log-likelihood l_E (4) is equivalent to h-likelihood, as we assume canonical scale $v(u) = log(u)$ in HGLM gamma (see for details [8], p.p. 112), and in further considerations we denote l_E as h. In order to find the solution of

$$\begin{aligned} \frac{\partial h}{\partial \beta_i} &= 0, \quad i = 1, \ldots, p \\ \frac{\partial h}{\partial v} &= 0, \end{aligned} \tag{5}$$

the iterative algorithm has to be applied. The first possibility is to use Newton-Raphson algorithm directly to find maximum value of the function (4) and the other is to apply modified IWSL dealing with two GLM models (for fixed and random effects) as in [8]. These two algorithms are implemented in R program, in `HGLMMM` implemented by Molas [9] and in `hglm` implemented by [10]. We estimate the HGLM gamma-gamma claim severity and construct the vector of tariff (in case of two rating variables it could be matrix) defined as $t = \exp(\eta)$. The first algorithm is proper for any scale and in consequences the convergence is slower. As we assume canonical scale, the second algorithm, which is quicker, is applied.

4 Estimation HGLM gamma claim severity via h-likelihood - the case study

In our case study, the automobile insurance data set taken from [4] is analyzed. This data set is based on one-year vehicle insurance policies taken out in 2004 or 2005. There are 67856 policies, of which 6.8% had at least one claim. There are followings rating variables in the model:

- numclaims - number of claims;
- claimcst - claim amount;
- veh.body - vehicle body;
- veh.age - age of vehicle: A (youngest), B, C, D;
- gender - gender of driver: M, F;
- area - driver's area of residence: A, B, C, D, E, F;
- agecat - driver's age category: A (youngest), B, C, D, E, F.

In our investigation we model claim severity Y in different groups including interactions between groups according to categories of rating variables. As the shape of vehicle has a large number of categories and it is hard to estimate effects for every body separately, we treat the structural parameter for variable *veh.body* as random variable. The average claim severity for few most popular shapes are presented in 1

The vehicle body	% in portfolio	The average claim amount
SEDAN	31,92%	1361.07
HBACK	27,34%	2296.27
STNWG	25,37%	2474.06

Table 1 The average claim severity in *veh.body* groups

Distributions of claim severities in each group of shapes Figure 2 shows.

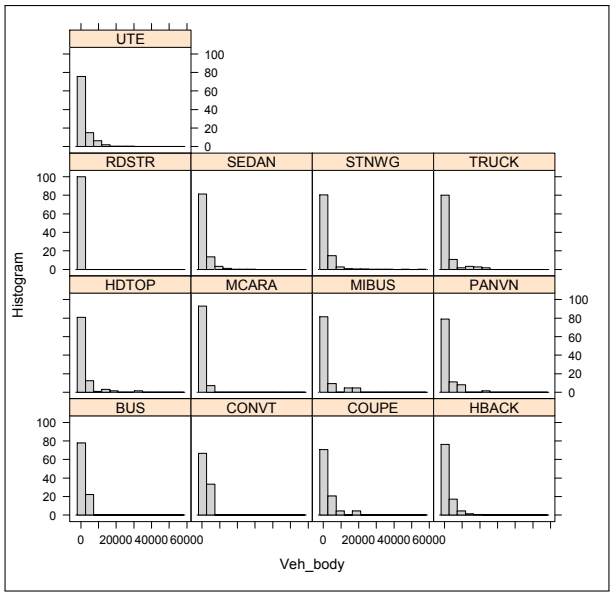

Figure 1 Residual plot - Gamma quantiles vs. residual quantiles

We observe that the claim severity is positive and skewed to the right for every vehicle body, as can be seen in Figure 2. Therefore in further analysis the gamma claim severity is assumed. Generally, in mixed effect model we are not interested in estimated value of particular realization of random effect, but rather in the inferences about the distribution and the dispersion of those effects. Obviously, in our case there is no information about all possible shapes of vehicle, but only about few realizations. However, in the interpretation of results the tariff are expected. Therefore we focus on the levels of vehicle body effects rather than on the analysis of variance components. We propose to construct tariff structure based on two values: values of estimated mixed effects and fitted value in HGLM gamma-gamma claim severity model. The procedure may be formulate in four steps

- select the tariff variables;
- estimate the structural parameters of the model;
- estimate fitted values for claim severity;
- construct the tariff structure.

We start with all steps simultaneously directly by estimation HGLM gamma-gamma severity model, in which the base claim severity is 1958.63. Remaining results are shown in Fig. 2.

The vehicle body	Tariff ($\boldsymbol{t}$ vector)	s.e.
SEDAN	0.9818	0.0887
HBACK	1.0056	0.089
STNWG	1.0596	0.0765
UTE	1.0463	0.0493
HDTOP	0.9900	0.0717
TRUCK	0.9522	0.0888
COUPE	1.0584	0.0799
PANVN	1.0104	0.0786
MIBUS	0.9920	0.0898
MCARA	0.9305	0.0487
BUS	0.9332	0.0497
CONVT	1.0456	0.716
RDSTR	0.9943	0.0638

Table 2 Estimated tariff for *veh.body*

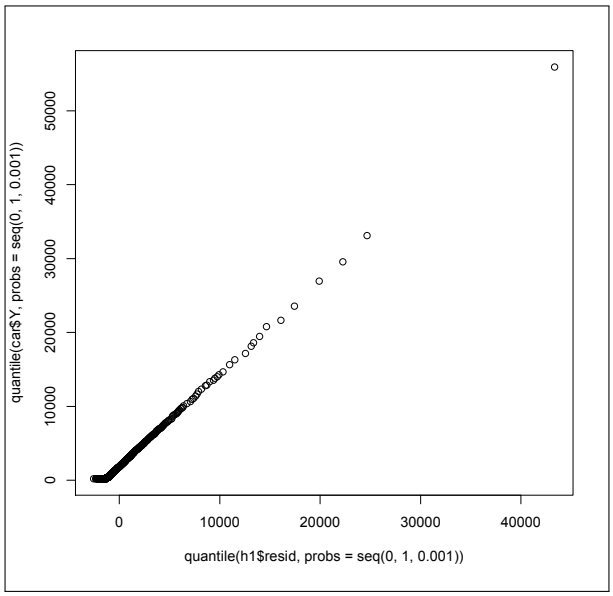

Figure 2 Histogram of claim severity grouped by *veh.body*

In the last step we create the vector of fitted claim severity in every combination of rating variable. As there are 3744 combinations in portfolio, we introduce only tariff cells for random effect *veh.body*

$$\mathbf{t} = (\mathbf{0.98}, \mathbf{1.01}, \mathbf{1.06}, \mathbf{1.05}, \mathbf{0.98}, \mathbf{0.95}, \mathbf{1.06}, \mathbf{1.01}, \mathbf{0.99}, \mathbf{0.93}, \mathbf{0.93}, \mathbf{1.05}, \mathbf{0.99})$$

For the estimated tariff in general we observe slight deviation of the base tariff equal to one and thus a slight change of the base claim severity. Thus ultimately we can expect that for other unobserved shapes the effect will be similar.

5 Conclusions

The presented idea of tariff cells estimation using mixed model is specially helpful for the case of the new policyholder, when there is insufficient information about his/her rating variables (as in our e.g. unlisted vehicle shape). Obviously the unresolved problem is the assortment of random effect distribution, if not normal. The solution is to diagnose few different models with some additional information about random effect. If any HGLM model is not satisfied (e.g. when it is rejected during verification) then non-parametric model may be applied, e.g. isotonic regression algorithms in the spirit of [5]. This could lead to improved MSE of constrained estimates versus unconstrained ones in a way similar to the effect described in [6]. Such an approach might constitute an interesting alternative to the generalized linear modelling. In ongoing researches PAVA algorithm, see in [1], is under consideration.

Acknowledgements

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The monitoring of consumers behavior on the field of the bioproducts by using the correspondence and dependence analysis

Martina Zámková[1], Martin Prokop[2]

Abstract. This article primarily aims to analyze the factors that influence young respondents when buying organic food by using appropriate statistical methods. In the questionnaire survey analysis we usually obtain categorical data. Contingency tables are easy way to illustrate the data relations. With respect to the character of the data we use suitable tests of the independence. Using analysis of contingency tables we will evaluate interesting dependence between frequency of organic food purchases and other indicators. Furthermore we will identify the target group for marketing strategies.
Correspondence analysis is a multivariate statistical technique. It is conceptually similar to principal component analysis, but applies to categorical rather than continuous data. In a similar way as a principal component analysis, it provides means of displaying or summarizing a set of data in two-dimensional graphical form. The graphical representation of results from correspondence analysis is commonly done with so-called symmetric maps. Using correspondence analysis of observed data we will identify what the respondents most often link various organic labeling with and we will propose a better flow of information.

Keywords: bioproducts, correspondence analysis, dependence analysis

JEL Classification: C30
AMS Classification: 62H99

1 Introduction

The main objective of the article is, based on the results obtained from the research, to identify factors that could influence the younger generation to more frequent purchases of organic food. Furthermore, next goal is to formulate recommendations to producers and traders of organic products, on the basis of statistical data processing, how to make young Czechs become regular customers of bioproducts for many years.

The article (Zagata [11]) deals with consumers' beliefs and behavioral intentions towards organic food, evidence from the Czech Republic. Research has revealed that organic consumers share beliefs about positive health effects, environmentally friendly production and better taste of organic food. The article (Malá, Malý [7]) is focused on an analysis of the phenomena appearing in the implementation of the transition from classic conventional technology in the production of agricultural food products to an ecological manner of farming.

What motivates Czech consumers to buy organic food? This task is described in (Urban, Zvěřinová, Sčasný [10]). First, the authors analyze the factors that affect the intention of Czech consumers to purchase organic food using the theory of planned behavior (TPB). Second, they employ an extended TPB model that introduces descriptive norms as an additional factor of behavioral intention.

2 Materials and Methods

1 289 respondents participated in the Czech Republic in marketing research, which was conducted through a questionnaire survey. The questionnaire was distributed electronically. Data collection was proceeding from 21st November 2012 to 21st December 2012. More women (70.8%) than men (29.2%) participated in the study. Typical respondent's age ranged from 16 to 25 years. Regarding education, respondents were mostly college students and therefore most frequently reported highest level of education was - high school graduation (88.7%).

It can be stated that this was not a representative sample with respect to the population, but for the fulfilment of the objectives of this article a sample of young respondents is entirely appropriate. It must address especially

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young people to buy organic food more and more frequently, because young people are potential customers for many years. (Zámková, Prokop [12])

In the questionnaire survey analysis we usually obtain categorical data and easy way to illustrate the data relations are *contingency tables*. With respect to the character of the data we use suitable tests of the independence. According to (Řezanková [9]) in the case of contingency table of the type $r \times c$ (*r* is the number of rows, *c* is the number of columns) we usually use statistics: $\chi^2 = \sum_i \sum_j \frac{(n_{ij} - e_{ij})^2}{e_{ij}}$. Alternatively $G^2 = \sum_i \sum_j n_{ij} \ln \frac{n_{ij}}{e_{ij}}$, e_{ij} is an expected and n_{ij} real frequency. We use the statistic χ^2 in Pearson´s chi-square test, G^2 in likelihood-ratio test. These two statistics have asymptotically $\chi^2_{(r-1)(c-1)}$ distribution with the presumption of the independence. For further details see (Hindls [6]).

Previous tests can be used in the case of high expected frequencies in the contingency table (more than 5 for each field), see (Hendl [5]). In some studies this rule is not so strict, it is enough to have at most 20% of frequencies less than 5 but all of them more than 1, see (Agresti [1]). According to (Anděl [2]) if frequencies are too small, we can use Fisher´s exact test or we can calculate simulated p-value of χ^2 statistic.

Correspondence analysis (CA) is a multivariate statistical technique. It is conceptually similar to principal component analysis, but applies to categorical rather than continuous data. In a similar manner to principal component analysis, it provides a means of displaying or summarising a set of data in two-dimensional graphical form.

All data should be nonnegative and on the same scale for CA to be applicable, and the method treats rows and columns equivalently. It is traditionally applied to contingency tables - CA decomposes the chi-squared statistic associated with this table into orthogonal factors. Because CA is a descriptive technique, it can be applied to tables whether or not the chi-square statistic is appropriate.

According to (Nenadič, Greenacre [8]), as in principal component analysis, the idea in CA is to reduce the dimensionality of a data matrix and visualize it in a subspace of low-dimensionality, commonly two- or threedimensional. The data of interest in simple CA are usually a two-way contingency table or any other table of nonnegative ratio-scale data for which relative values are of primary interest. The CA solution was shown by (Greenacre [4]) to be neatly encapsulated in the singular-value decomposition (SVD) of a suitably transformed matrix. To summarize the theory, first divide the $I \times J$ data matrix, denoted by *N*, by its grand total *n* to obtain the so-called correspondence matrix $P = N/n$. Let the row and column marginal totals of *P* be the vectors *r* and *c* respectively, that is the vectors of row and column masses, and D_r and D_c be the diagonal matrices of these matrices. The computational algorithm to obtain coordinates of the row and column profiles with respect to principal axes, using the SVD, is as follows:

1. Calculate the matrix of standarized residuals: $S = D_r^{-\frac{1}{2}}(P - rc^T)D_c^{-\frac{1}{2}}$
2. Calculate the SVD: $S = UD_\alpha V^T$ where $U^T U = V^T V = I$
3. Principal coordinates of rows: $F = D_r^{-\frac{1}{2}} U D_\alpha$
4. Principal coordinates of columns: $G = D_c^{-\frac{1}{2}} V D_\alpha$
5. Standard coordinates of rows: $X = D_r^{-\frac{1}{2}} U$
6. Standard coordinates of columns: $Y = D_c^{-\frac{1}{2}} V$.

The total variance of the data matrix is measured by the inertia, see, e.g., (Greenacre [4]), which ressembles a chi-square statistic but is calculated on relative observed and expected frequencies:

7. Inertia: $\phi^2 = \sum_{i=1}^{I} \sum_{j=1}^{J} \frac{(p_{ij} - r_j c_j)^2}{r_i c_j}$.

The rows of the coordinate matrices in (3)-(6) above refer to the rows or columns, as the case may be, of the original table, while the columns of these matrices refer to the principal axes, or dimensions, of the solution. Notice that the row and column principal coordinates are scaled in such a way that $FD_r F^T = GD_c G^T = D_\alpha^2$, i.e.

the weighted sum-of-squares of the coordinates on the *k*-th dimension (i.e., their inertia in the direction of this dimension) is equal to the principal inertia (or eigenvalue) α_k^2, the square of the *k*-th singular value, whereas the standard coordinates have weighted sum-of-squares equal to 1: $XD_r X^T = YD_c Y^T = I.$ The implementation of the algorithm follows (Blasius, Greenacre [3]).

The graphical representation of results from CA is commonly done with so-called symmetric maps. In that case, the row and column coordinates on each axis are scaled to have inertias equal to the principal inertia along that axis: these are the principal row and column coordinates. Depending on the situation, other types of display are appropriate. This can be set with the scaling option map in the plotting functions for CA. Software UNISTAT and STATISTICA was used for processing of primary data.

3 Research results

Monthly household income of respondents varies mostly around 20 000 to 40 000 CZK, while most respondents live in a four-person household. Young people who were addressed do then their purchases mostly in municipalities with a population of 2 000 (23.9%), but the representation of other groups is relatively balanced. Recommendations can be directed to all organic food sellers all over the Czech Republic.

When asked: *What is the organic food in your opinion, what do you recall under this title, what do you associate with this term?* the respondents answered as follows. Respondents most often appropriately marked (almost 90%), that they are ecologically grown products, without chemistry. However, an interesting fact is that more than 50% of the respondents associate organic food most often with the fact that the food is expensive. It resulted from the answers to the next question that 74.5% of respondents believe that organic foods are healthier than conventional food and 74.7% think they are of better quality, but only 39% of respondents consider organic food more flavoursome and only 31.5% think they are more attractive than conventional food.

To the question: *Is organic food part of assortment in your favourite store?* 64.2% of respondents answered that yes. The finding that 23.6% of respondents haven't looked for organic food in the store so far is surprising. Only 47.7% of respondents are persuaded about sufficiently recognizable organic food in stores. Whereas according to respondents the best recognizable organic products are in chains – Tesco, Albert, DM drugstore and Billa. Only 43.3% of young people think that organic foods are sufficiently publicized, and surprisingly 17% of respondents have not even seen advertising for organic food. If addressed people saw advertising, it was mostly in the magazine and newspaper.

The question: *Do you buy organic food in your household?* respondents answered as follows: it is clear that there is a huge amount of young people who don't buy organic food (43.4%). It is therefore necessary to do basic steps to convince young people that buying organic food makes sense. If respondents don't buy organic food, they stated that the main obstacle for them is the financial aspect - the higher price of organic food (35.9%). Furthermore, they denoted, that organic food is not attractive for them (20.5%) and that they don't believe that the food is organic, non-chemical, better (19.8%).

If respondents buy organic food, they make their purchases of organic food mostly in hypermarkets and supermarkets (23%). The second most frequent response was health food and organic food stores (16.8%), approximately 7.1% of respondents attend farms and farmers markets. A relatively large part of respondents then stated that they grow the organic food by themselves (13.5%). The research also shows that most respondents spend money on organic fruit and vegetables, as well as organic milk and dairy products, as well as organic flour, cereals, and also for organic meat and sausages. Domestic food production predominates in the available assortment of organic products.

To the question: *How often do you buy organic food in your household?* respondents most frequently responded that about once per month. And to the question: *How much do you approximately spend on buying organic food per year in your household?* respondents answered: it is thus evident that the Czechs don't spend too much on organic food a year (if buying it), the amount invested in organic food ranges mostly up to 500 CZK, often only up to 100 CZK.

4 Correspondence and dependence analysis

We tried to find out whether household income affects the amount of purchased organic products in the analysis of contingency tables. So we watched the dependence between the questions: *Monthly income of your household*. and *Do you buy organic food in your household?* Using Pearson's chi-square test of independence we tested the relationship between the frequency of organic food purchases and size of household income. During testing of independence we found the value $\chi^2 = 59.54$ with significance $p < 0.001$. I.e. for a 5% risk we reject independ-

ence between the frequency of organic foods purchasing and household income. We can therefore talk about the dependence between the amount of income and frequency of organic food purchases.

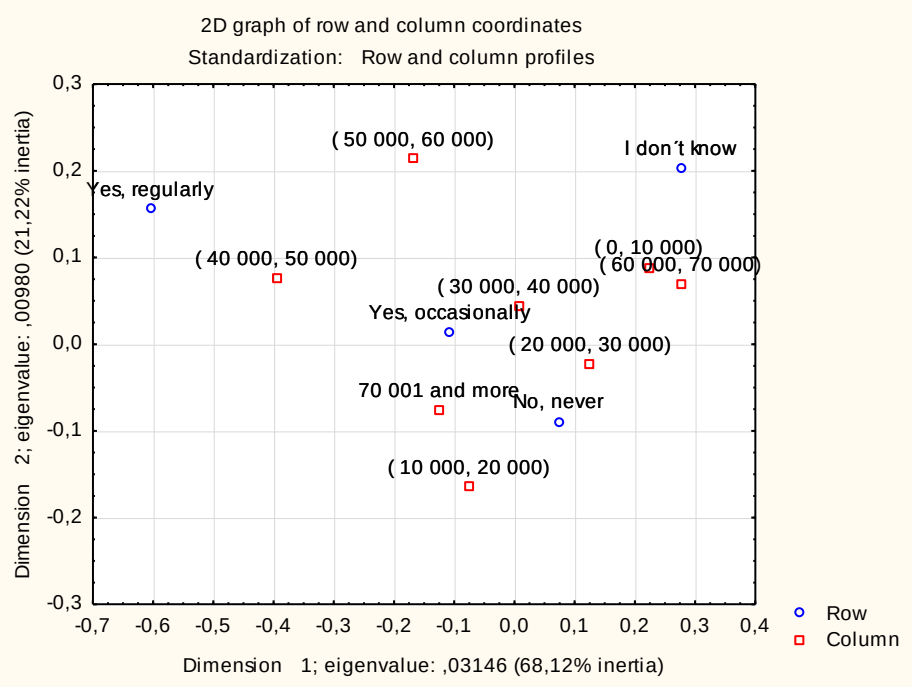

Figure 1 Correspondence analysis - *Monthly income of your household.* and *Do you buy organic food in your household?*

The results of the correspondence analysis (see Fig.1) came out likewise the analysis of contingency, when the respondents with incomes from 40 000 to 60 000 CZK most often stated, that they buy organic food regularly. Respondents with incomes up to 10 000 CZK usually don't follow whether they buy organic food. That is logical, considering the lower education of the respondents.

Furthermore, we were interested in whether sex influences the amount of purchased organic food. So we watched the dependence between the questions: *Your sex*. and *Do you buy organic food in your household?* It is clear from the contingency that women buying organic food regularly and occasionally are in our sample 428 (i.e. 46.9%) and only 139 men (i.e. 36.9%). Using Pearson's chi-square test of independence we tested the relationship between the frequency of purchases of organic food and sex of respondents. During testing we found the value of independence $\chi^2 = 21.03$ with significance level $p < 0.001$. I.e. for a 5% risk we reject independence between the frequency of purchasing organic food and sex. We can therefore say that the willingness to buy organic food is dependent on sex. Women in our sample buy organic food more often than men, but this can be caused by the fact that women do shopping in general more often in the Czech Republic.

From modified contingency table (Tab. 1) we can see, that shopping place to buy organic food also depends on sex ($\chi^2 = 36.49$ with significance level $p < 0.001$). From the table of row relative frequencies it follows that men buy organic food more often in hypermarkets and supermarkets, at the farms and farmers markets, whereas women buy bio products more in health food and organic food stores, in drugstores.

Row %	Hypermarket, supermarket	Health food and organic food stores	Farms, farmers markets	I grow them myself	Smaller stores	Drugstore	Pharmacy	No answer
Man	30.0%	9.8%	8.8%	14.1%	2.7%	0.8%	0.5%	33.4%
Woman	20.1%	19.9%	6.8%	13.4%	2.4%	3.6%	0.7%	33.2%

Table 1 Contingency table – *Where do you buy organic food?* and *Your sex.*

Our next idea was whether an advertisement for organic food affects the willingness to buy these products. We monitored answers to the following questions: *Do you buy organic food in your household?* and *Have you ever seen advertising for organic food?* Again using Pearson's chi-square test of independence we tested the relationship between the frequency of purchases of organic food and the influence of advertising. During testing we

found the value of independence $\chi^2 = 26.5$ with $p = 0.0889$. I.e. we don't reject independence between the frequency of purchasing organic food and watching advertising for organic food. Surprisingly advertising for organic food doesn't affect respondents to purchase more frequently, which is also evident from the frequency column, where respondents watching advertising buy organic food more likely and respondents not watching advertising vice-versa, but the difference is very small (about 4 percentage points).

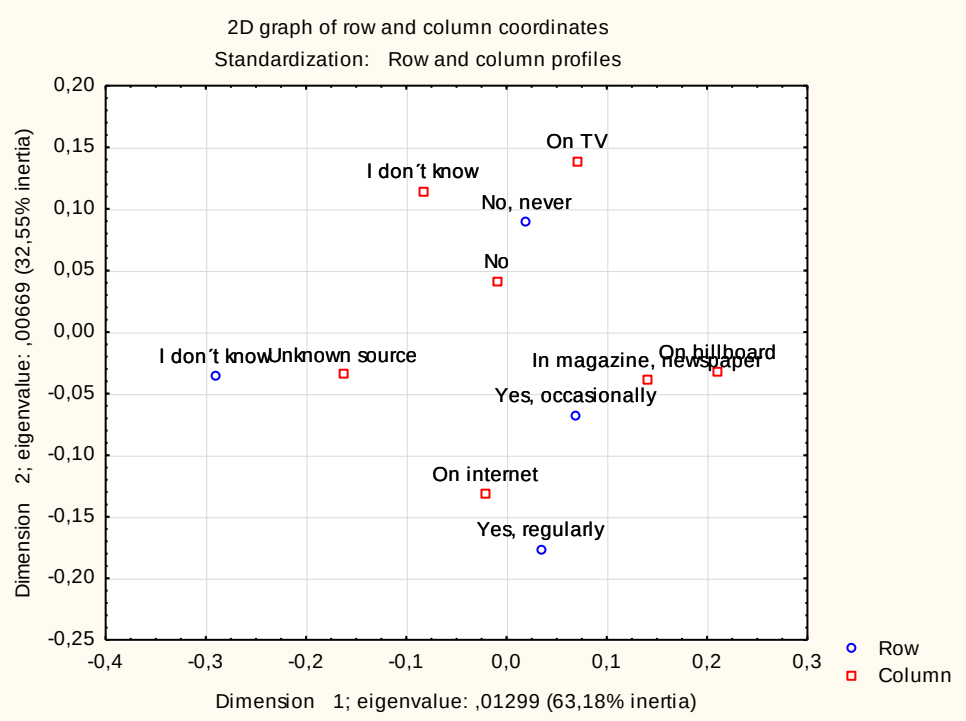

Figure 2 Correspondence analysis - *Do you buy organic food in your household?* and *Have you ever seen advertising for organic food?*

It is clear from the correspondence analysis (see Fig. 2) that if respondents are influenced by advertisements on the Internet or in magazines, newspapers or eventually on billboards, they buy organic food occasionally or regularly. It is furthermore evident that respondents, who haven't seen advertising or they don't know about it, often don't buy organic food at all or don't follow that it is organic.

It is thus evident that advertising for organic food either watched in magazines, newspapers, on television or on billboards or on the internet is not indeed a factor that would influence the respondents too much when buying organic food.

Furthermore we attempted to find out whether gender affects the amount that respondents spend on organic food. So we watched the dependence between the questions: *Your sex*. and *How much do you approximately spend for buying organic food in your household per year?* According to Pearson's chi-square test of independence we tested the relationship between the size of the amounts spent on organic food and sex of respondents. During testing we found the value of independence $\chi^2 = 20.16$ with $p = 0.0052$. I.e. for a 5% risk we reject independence between the size of the amount spent on the purchase of organic food and sex. It was therefore proved dependence on sex.

It is furthermore apparent from the analysis of contingency table that with the increasing population the frequency of purchases of organic food slightly increases. For example, in the cities of 50 001 to 200 000 inhabitants the most of respondents (22%) buy organic food several times a month. On the contrary, in the towns with the population up to 2 000 inhabitants 34.4% of the respondents do not buy organic food at all.

In the age category up to 25 years for 63.5% of respondents, who don't buy organic food, the barrier is the financial aspect and therefore higher price of organic food. For the age category over 25 years the financial aspect is less important, it is important for only 44.2% of respondents.

From further analysis it is clear, that if respondents are convinced about healthier organic food, 59.3% of them really buy it. Respondents, who don't consider organic food healthier, don't almost buy it (only 22.3%).

5 Conclusions

In this paper we tried to assess the shopping behavior of young respondents on the field of the organic products. It is clear from the research that there is a considerable amount of young people, who don't buy organic food. In the age group up to 25, most respondents stated that the biggest obstacle is the higher price of organic food. Those, who buy them, regard organic food as healthier than conventional food. At the same time there is a higher frequency of buying organic food in larger cities than in the villages.

It is noticeable from the correspondence analysis that respondents with incomes from 40 000 to 60 000 CZK often stated that they buy organic food regularly. And respondents with incomes up to 10 000 CZK usually don't follow the organic food's origin. It is furthermore seen that women buy organic food more often than men.

Dependence of the frequency of buying organic food on watching advertising didn't come out statistically significant. However, it results from the correspondence analysis that if respondents are influenced by media advertising, they more often state that they buy organic food sometimes or regularly.

It is necessary to take steps enabling us to address the young generation, because these are the customers who will make their purchases for many years. It is then apparent from the analysis of contingency tables that the target group is mainly young women who make purchases more frequently than men and are also usually willing to spend a higher amount of money on organic food.

It can be identified from the results of the research that young people are relatively well informed what organic food is and how it differentiates from classic food. It is interesting that more than 50% of the surveyed young people associate the term organic food with the food which is expensive. For this reason, we recommend to traders organization of various discount actions also on organic food and to attract young customers on the basis of special discounts on organic food. Similarly, we can recommend organizing more farmers markets and similar events where young people can gain personal experience with organic products and become a fan of them.

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The Role of Confidence Indicator in Determination of Consumption Expenditures in Selected EU Member Countries

Jana Závacká[1]

Abstract. The article is oriented on the role of consumer confidence indicator in determination of consumption expenditures in European territory. The confidence indicator was already confirmed as a coincident indicator of consumption. It was also approved as a coincident indicator within the context of The Rational Expectation Permanent Income Hypothesis (REPIH) because of its ability to predict future income. Although the conception of REPIH was already rejected on UK data, the significance of confidence indicator for explanation of GDP or consumer expenditures variability was accepted.
For the analyses of significance of confidence indicator for prediction consumption spending in European territory EU harmonized consumer confidence indicator is used. The estimation are made with panel data from selected EU member countries: Czech Republic, Slovakia, Italy, Spain, Portugal, Germany, France, Netherlands, Austria, Denmark and Belgium. According to the common harmonized system of consumer tendency survey managed by the European Commission the time series of indicators are fully comparable across the countries. The wide range of countries was selected to show the sensitivity of the effect of consumer confidence indicator on consumption expenditures in different economic environment. The paper confirms the coincidence of confidence indicator with consumption in the European territory, with variable effect within the selected countries.

Keywords: rational expectations, EU harmonized consumer confidence indicator, panel data.

JEL Classification: E21, C33
AMS Classification: 62P20

1 Introduction

The formation of consumer decision about his expenditures is a long time discussed topic, mainly because expenditures on consumption by households create a significant share on the gross domestic product in the country. The recent approaches to consumer expenditures forming are often oriented on testing the Rational Expectation Permanent Income Hypothesis (REPIH), defined by Hall [5]. The approach is based on Friedman [5] conception of permanent income, saying that it is the permanent income (e.g. expected future stream of incomes) rather than current income, which determines the consumption spending. REPIH extends the conception for rational expectations. Rational expectations are understood as expectations based on all actual available information to consumer. To express the rational expectation formation by a measurable variable the confidence indicators are broadly in use in recent years. Although REPIH was rejected by Acemoglu and Scott [1] and Flavin [4], Acemoglu and Scott [1] confirmed that the confidence indicator is a coincident indicator for consumption predicting the consumption growth even when conditioning upon other macroeconomic variables. The predictive ability of confidence indicator for changes in consumer spending was approved also by Bram and Ludvigson [2], Carrol, Fuhrer and Wilcox [3], Souleles [7]. The analyses were made on US and UK data.

The main task of this paper is to estimate the influence of confidence indicator on consumer expenditures in European territory. We took advantage of the European Union common statistic indicator, the consumer confidence indicator, which is harmonized for all member countries and reported by each of them. Thanks to the harmonization, we could estimate the effect of confidence indicator by using panel data of EU member countries. More, the estimations for individual countries are made too. The harmonization enables us to compare effects within the group. By virtue of segmentation of European territory, the panel data enable us to make estimations in various economic environments, various national cultures.

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The paper is organized as follows. A short introduction of EU harmonized consumer confidence indicator and the formulation of testing equations are described in the part 2. In the part 3 there are a description of panel dataset and estimation results. Part 4 concludes.

2 Methodology

Within the European territory the EU harmonized consumer confidence indicator is measured. This indicator consists of four questions, which are: (1) Expected change in financial situation of household over the next 12 months; (2) Expected change in general economic situation over next 12 months; (3) Expected change in unemployment over the next 12 months; (4) Expected change in savings of household over next 12 months. The possible answers to all questions are: a lot better, a little better, the same, a little worse, a lot worse. The confidence indicator is then equal to balance of positive and negative answers with double weights on the extreme. As the indicator is partly oriented on opinion of household about its financial situation (question (1), (4)) we assume that it could coincide with the REPIH as an expression of rational expectation of household about the future change in its stream of incomes.

Analogically to Acemoglu and Scott [1] we estimate the consumption spending through the equation

$$\Delta \ln C_t = const + \beta_1 \Delta \ln GDP_t + \beta_2 ICC_t + \varepsilon_t, \tag{1}$$

where C_t is real consumption per capita, GDP_t real gross domestic product per capita, ICC_t the EU harmonized consumer confidence indicator and ε_t error term. Moreover, we test the significance of confidence indicator using panel data through the similar equation allowing for the fixed effects β_{0i} (fixed effects are applied here to distinguish among possible different "equilibrium rates" of growth in different countries):

$$\Delta \ln C_{it} = const + \beta_{0i} + \beta_1 \Delta \ln GDP_{it} + \beta_2 ICC_{it} + \varepsilon_{it}. \tag{2}$$

Formulation of consumer decision of expenditures expressed in such form (difference of logarithms) generally means that we suppose that the rate of growth of consumer spending is equal to some constant (in equilibrium case the equilibrium growth) adjusted for the effect of the rate of growth of personal disposable income and future expectations (expressed as a consumer indicator with positive value for improvement and negative otherwise). Because of the difference the error term could be in the form of MA(1), as already alert Acemoglu and Scott [1].

3 The empirical estimation

3.1 The data

The analysis was made with the quarterly data of EU member countries: Czech Republic (1996Q1-2012Q4), Slovakia(1999Q2-2012Q4), Italy(1996Q1-2012Q4), Spain(1996Q1-2012Q4), Portugal(1996Q1-2012Q4), Germany(2000Q1-2012Q4), France(2003Q1-2012Q4), Netherlands(1996Q1-2012Q4), Austria(1996Q1-2012Q4), Denmark(2000Q1-2012Q4) and Belgium(2000Q1-2012Q4). The choice was made according to the length of available time series data of EU harmonized consumer confidence indicator. The variables entering the panel data set are as follows.

- C_t is real consumption expenditure per capita, counted as final consumption expenditure of households (expressed in national currency, chain-linked volumes, reference year 2005,seasonally adjusted data, source: Eurostat) and divided by population that year (time series of population available only in yearly data from Eurostat) ;
- GDP_t is real gross domestic product per capita, counted as GDP at market prices (expressed in national currency, chain-linked volumes, reference year 2005,seasonally adjusted data, source: Eurostat) and divided by population that year (time series of population from Eurostat);
- ICC_t is EU harmonized consumer confidence indicator (transformed from monthly data by average in the quarter, source: OECD).

Following the statistical tests of significance the dummy variable EA for Slovakia was constructed, representing the change for euro currency from 1991Q1. As a part of analysis additional time dummy variables were applied in case of necessity. All time series, expressed in the form entering equation (1) or (2), could be consid-

ered as stationary according to Phillips Perron test or Dickey Fueller test. The analysis was made in program Stata 10.

The estimated mean values of all variables entering regression (the log-difference of consumption spending per capita, the log-difference of GDP per capita, the consumer confidence indicator) for each country with the variation of observations of log-differences of consumption expenditures could be seen in the Figure 1. The estimated mean values from panel data are 0.0035 for log-difference of GDP, 0.0028 for log-difference of consumption and -11.0403 for consumer confidence indicator.

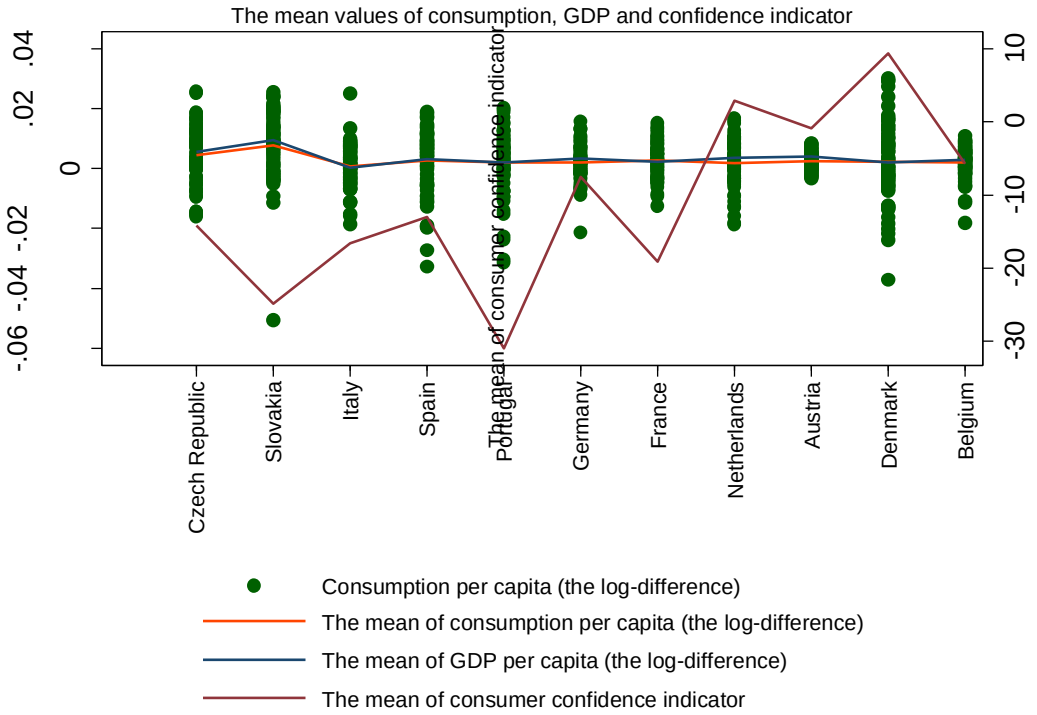

Figure 1 The mean values of consumption, GDP and confidence indicator

The mean rate of growth of consumption expenditures as well as GDP, expressed by the difference of logarithms, is similar for all countries in the analysis. On the other hand, the mean of consumer confidence indicator varies among observed countries a lot. The positive mean of consumer confidence indicator is only in Denmark and Netherlands. What could be surprising is the fact, that southern countries (Italy, Spain and Portugal), fighting with high rate of unemployment in last years, accompanied with high indebtedness, are not the countries with obviously lower consumer confidence indicator (for example we can compare it with Slovakia, with higher rate of GDP and consumption expenditures, entering Eurozone and having low mean of consumer confidence indicator anyway). The explanation is of course straightforward. Firstly, it is the mean for last 15-20 years and the countries are fighting with economic problems just in few past periods. Secondly, the initial level of economy in these countries were probably lower in 1996 than in countries like Germany, France or Denmark, so despite the last economic challenges these countries could pass through bigger development accompanied with higher expectations expressed by higher level of consumer confidence. But thirdly, the formation of rational expectation about future development could be also country specific stemming from the character of the nation. The soft nature of the confidence indicator has to be taken into the consideration.

3.2 Estimations

The estimation results of equation (1) for each country and panel regression (2) are in the Table 1 and Table 2 (standard errors in parenthesis, * significant on 5%, ** significant on 1% level of significancy). For the technical reasons the use of time dummy variables was in some cases necessary[2]. In most cases heteroscedasticity and autocorrelation between residuals were identified (this is in accordance with the warning of possible MA(1) process). For that reason the Newey-West standard errors technique for estimations were applied. In case of

[2] The time dummy variables: 1993Q3, 2005Q1, 2007Q1,2011Q1, 2011Q2 in case of panel data; 1997Q2 in case of Italia; 2007Q1 in case of Germany; 1996Q4 in case of France; 2006Q1, 2008Q1, 2008Q2, 2011Q1 in case of Netherlands; 2011Q1 in case of Belgium. All dummy variables were significant on 1% level of significancy.

Spain the high level of multicolinearity between consumer confidence indicator and log-difference of GDP was estimated. More, by use only the log-difference of GDP as a regressor the estimated constant by this term was higher than 1. In combination with detected autocorrelation and heteroscedasticity the model (1) for Spain was considered without any further transformations as inappropriate. The correlation between consumer confidence indicator and log-difference of comsumption expenditures was estimated as insignificant at the 5% level of significance in case of Austria. Therefore consumer confidence indicator was not considered as a regressor in this model. In case of panel data the cross-sectional dependence among residuals could not be rejected. According to tests of insignificance of time dummy variable some time variables have to be used. Hausman specification test confirms fixed effects as suitable in the model. The estimations were made by use of Driscoll-Kraay standard errors, with respect to detected cross-serial dependence and heteroscedasticity in error terms.

Explanatory variable	**Panel data (All)**	**Czech Republic (1)**	**Slovakia (2)**	**Italia (3)**	**Portugal (5)**
$\Delta\ln GDP_t$	0.3408**	0.2456**		0.4178**	0.5959**
	(0.071)	(0.0919)		(0.0932)	(0.1103)
ICC	0.0002**	0.0003**	0.0002*	0.0003**	0.0003**
	(0)	(0.0001)	(0.0001)	(0.0001)	(0.0001)
Const.	0.0048**	0.0072**	0.0168**	0.0043*	0.0103**
	(0.0007)	(0.0016)	(0.0013)	(0.0017)	(0.0026)
EA	-0.0078**		-0.0137**		
	(0.0026)		(0.0017)		
N	653	67	55	67	67
R^2	0.3874	0.2286	0.7343	0.6128	0.6509

Table 1 Estimation results from equations (1) and (2)

Explanatory variable	**Germany (6)**	**France (7)**	**Netherlands (8)**	**Austria (9)**	**Denmark (10)**	**Belgium (11)**
$\Delta\ln GDP_t$		0.713**	0.5203**	0.2105**	0.5135**	0.5197**
		(0.1459)	(0.0556)	(0.0435)	(0.1471)	(0.0512)
ICC	0.0001*		0.0002*		0.0002	
	(0.0001)		(0.0001)		(0.0001)	
Const.	0.0031**			0.018**		
	(0.0008)			(0.0003)		
N	52	67	67	67	52	67
R^2	0.3457	0.5638	0.7918	0.2652	0.3202	0.6506

Table 2 Estimation results from equation (1)

As we can see in the Table 1 and Table 2, the estimation results are strongly country specific. From the panel data the effect of consumer confidence indicator on decision of consumption spending was approved as significant at 5% level of significance, with the estimated coefficient of 0.0002. If we consider the mean value of consumer confidence indicator estimated from panel data -11.0403, such level of confidence indicator would decrease the quarterly growth rate of consumption spending (approximated by log-differences) by 0,22%. The common constant growth of quarterly consumer expenditures in panel data was estimated as 0.48%, corrected with 34.08% of quarterly rate of growth of GDP. The coefficient of determination in panel data is about 34.57%. We expect that such a low number is a result of wild economic development in last years connected with country specific conditions as well as country specific effects of consumer confidence indicator.

Regression analyses made for each country vary in values of estimated coefficients as well as significancy of applied regressors. The log-difference of GDP, representing here the quarterly growth of actual personal income,

plays the crucial role of determination of consumer expenditures growth in most cases. In case of Germany and Slovakia, the constant term may supply the effect of the growth of GDP as an estimated constant level of growth. The proper interpretation of all results would demand deeper analysis of economic development in all considered countries, which is not in the scope of interest of this article. What we are interested in is the significance and the size of the effect of rational expectation in forming decision about consumer expenditures. Except France, Austria and Belgium the consumer confidence indicator is highly significant in the 1% level of significance in all regressions, with the quite stable estimated effect around 0.0002, corresponding with the estimation gained from the panel dataset. We can conclude that whatever the size of the level or the mean level of the consumer confidence indicator in a country is, we can according to the obtained results assume that the effect of the indicator is stable among the countries. This conclusion mainly supports the Keynes hypothesis about importance of the level of confidence in the society for the economic development. More, it is in accordance with the results of Acemoglu and Scott [1], Bram and Ludvigson [2], Carrol, Fuhrer and Wilcox [3], Souleles[7], confirming the importance of confidence indicators as a partial explanation of consumption expenditures movement. Unfortunately, we could not compare the estimation coefficients with the other studies, because they were made with different confidence indicators, constructed in a different way with the different questionnaire.

4 Conclusion

The intention of this article was to estimate the importance and the size of the effect of confidence indicator in forming consumer decision about expenditures. By use of EU harmonized consumer confidence indicator the significance was approved on the panel data and in the seven of ten examined countries. The size of the effect was estimated as relatively stable value of 0.0002. This conclusion supports the importance of the level of confidence in society for economic development.

Through the analyses the high variability of mean values of consumer confidence level among countries was observed. It demands for appropriate and careful interpretation. By use of consumer confidence indicator it should be always taken into consideration the psychological character of the indicator, expressing the common subjective perceptions about future development of economy compared to actual situation. The relative expression emphases the importance of comparison to actual state, leading to possible positive values in case of crises or negative values in case of boom. The long-term continuous improvement of economy could be reflected in positive rational expectations even if the actual state of economy in the country is worse than in their neighbor countries. Another aspect of confidence indicator is that the indicator could be country specific. It would be an objective of sociology to determine if forming positive view of future development differs among nations.

For forgoing research it has to be also mentioned that the use of log-difference of GDP as a substitute of quarterly growth of personal income could be definitely improved. Unfortunately, there were no available time series of disposable income for all examined countries. The real GDP allowed work with more data. The time series of GDP could be ideally replaced by time series data of real disposable income or transformed by use of income taxes in all countries. The changes in income tax policies or government transfers to household could improve the estimated results. Nevertheless the significance of consumer confidence indicator was already achieved with the GDP approximation.

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Possibilities of Time Input-Output Tables[1]

Jaroslav Zbranek[2], Jaroslav Sixta[3]

Abstract. The importance of the input-output analysis has been increasing in the Czech Republic in recent years. The use of input-output tables is spreading among experts because of the possibilities for creation of different models. These models are aimed at various fields of economic and social life. Our paper is focused on the time input-output analysis of the Czech economy. The time input-output tables were not usually compiled in the Czech Republic. Special attention is devoted to the hours worked. The analysis is focused on the domestic production and its use because the value of imported goods and services is not directly connected with hours worked in the Czech Republic. The emphasis is put on the labor intensity development and the comparison with the development of capital endowment of labour. Besides, the development of real gross value added is taken into account. The labor intensity is the time spent by the production of the product according to product classification CZ-CPA. Such use of the physical units for the input-output analysis has evident ad-vantages in comparison with widely used monetary approach for the elimination of different price levels. The analysis focuses on the Czech economy between 1990 and 2010.

Keywords: time input-output tables, national accounts, gross value added

JEL classification: C82, J22, J30, O11
AMS classification: 65C20

1 Introduction

Input-output tables are one of the possible tools for the description of the national economy. While monetary input-output tables are expressed in monetary units, time input-output tables show allocation of the number of hours worked in the production of products. The time input-output tables could be seen as an extension of the monetary input-output tables. Such tables could be used as one of tools for verification of deflation. Of course it is assuming the time input-output tables are well compiled. More about deflation is written in [1]. Moreover, as it is written in [1] or in [6] it is an old dream of economic science to describe economical activities using non-monetary units. Time is one of such metrics. Time represents a significant link between final output and living standard of the population. The time spent with labour is an input into the final production. The final production is intended to meet the demand. Time spent for leisure activities is one of the factors for measuring living standard. This paper is focused on the time spent on final output and it brings significant analytical potential. Such analyses have a long tradition in Germany, while in the Czech Republic this issue had been never dealt with in a detail.

Compilation of the symmetric time input-output tables consists of several steps. Compilation of the time supply and use tables is the first step. We can use time supply and use tables for some analysis connected with allocation working time fund among production of products in each industry. In comparison to monetary supply and use tables, the time supply and use tables are compiled in the simplified form and describe allocation of the number of hours worked among commodities in each industry. Simplified supply and use tables (expressed in hours worked) are used for the compilation of symmetric time input-output tables. These tables serve as a base for derivation of input-output models connected with working time during the process of production.

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The data from monetary SIOTs can be used for evaluation of the development of structure of national economy in monetary units; especially in a view of material and other inputs into the production. Also the comparison of hours worked in time input-output tables allows the evaluation of the development. It is focused on the amount of labour time needed for the production of products. Linking both of these approaches gives the possibility to evaluate the development of labour intensity connected with each group of products.

The aim of the paper is the evaluation of the development of structure of national economy with focus on the development of labour intensity in production of products. Besides it offers a description of the development of capital endowment of labour in the Czech Republic between 1990-2010.

2 Possibilities of analysis based on the time input-output tables

The evaluation of development of labour intensity according to products is partly based on the results of compiled time input-output tables (TIOT). The Methodology for compilation of the TIOT and its final form for the Czech Republic is described in [9]. The compilation of the mentioned TIOTs is based on the methodology written in [1], [5] and [6]. The foundations for use of the input-output tables focusing on human labour put probably Stäglin for example in [4]; instead of hours worked number of employees were used. It should be noted that his aims were different because he focused on the analysis of the situation of individual professions on the labour market. As it is pointed out in the mentioned resources, it should be taken into account that also the compiled TIOT for the Czech economy could be and probably are partly distorted due to human capital. For example the same output can be done for different time by different people according to different level of education or length of practice. Number of hours worked does not reflect the quality of work.

The number of hours worked is one of possible approaches to description of labour inputs to the production process. The labour intensity is a bit different approach to the assessment of labour productivity. More about the labour productivity in the Czech Republic is written in [2] and [7].

Subsequent goal is the evaluation of the development of capital endowment of labour between 1990–2010. This evaluation is based on the economic theory. The indicator called capital endowment of labour provides the information on amount of capital which can be used by employees in their work. For these purposes, net stocks of fixed assets published are used. Next data source is the number of hours worked broken down by commodity classification (CZ-CPA). It is obvious that the partial aim in this context is the transformation of the indicator net stocks of fixed assets from the industries to commodities.

3 Methodology

The methodology for evaluation of the indicator labour intensity development is overtaken from [8]. At the level of the national economy, it does not have a sense to measure output expressed in physical units. It is because of the level of aggregation of the commodity classification used. Therefore it is common that the real development according to commodity and industrial classification is estimated and published (volume indices).

The evaluation of development of the labour intensity is based on the equation 1 where the numerator is the index of number of hours worked spent on the production of product i during the year t in comparison to a base year. The denominator indicates the volume index for particular product i in the year t in respect to a base year.

$$i_{LI,i,t} = \frac{i_{T,i,t}}{i_{q,i,t}} \tag{1}$$

The evaluation capital endowment of labour mentioned in the section 2, as one of the aims, needs also some methodological background. As it was written, the evaluated indicator is based on the economic theory. Therefore the methodology of calculation of the capital endowment of labour is taken over from [3]. In line with [3] for the calculation of the average capital endowment of labour the equation 2 is used.

$$k = K/L \tag{2}$$

Also the equation 3 which is used to evaluation of the development of the capital endowment of labour is taken over from [3]. It is clear that the evaluation is based on the relative change of the average capital endowment of labour between two selected years.

$$g_k = \frac{K_1/L_1 - K_0/L_0}{K_0/L_0} \tag{3}$$

Possible results of the application of the equation 3 and the key to interpretation of these results is shown in the table 1.

Situation	Designation	Description
$\mathbf{g_k < 0}$	**Decrease of capital endowment of labour**	When the consumption of fixed capital is higher than gross fixed capital formation and the amount of hired labour is increasing or is not changing.
$\mathbf{g_k = 0}$	**Capital expansion**	When the capital endowment of labour is not changing and capital stock is changing according to amount of hired labour.
$\mathbf{g_k > 0}$	**Capital deepening**	When the capital endowment of labour is increasing. Gross fixed capital formation is increasing faster than the amount of hired labour.

Source: Soukup [3]

Table 1 Possible changes of the average capital endowment of labour

As it was written in the section 2 before the evaluation of the capital endowment of labour the data for net stocks of fixed assets has to be transformed from the industries to commodities. For this purpose the methodology described in [10] was used. This methodology of transformation is based on the structure of the square matrix of output from the Supply and use tables and it is connected with the symmetrisation of the Output matrix. It should be noted that this method is more or less simplified as it is used in [10] for the indicator of number of hours worked.

$$\mathbf{K_C} = \mathbf{K_I}(\mathbf{V^T})^{-1} diag(\mathbf{q}) \tag{4}$$

The transformation is done by the equation 4. This equation is result of three matrices. In the equation K_C means the vector of net stocks of fixed assets broken down by commodities according to CZ-CPA classification. The K_I means the vector of net stocks of fixed assets broken down by industries according to CZ-NACE classification. The $(V^T)^{-1}$ is the inverse output matrix where the rows are broken down by commodity classification and the columns are broken down by industry classification. The third matrix $diag(q)$ is diagonal output matrix. This matrix ensures the assumption that the national economy is based on the homogenous units.

4 Results

The table 2 describes the development of labour intensity. Labour intensity decreased while the real output increased during the whole period. At this leve,l the development of capital endowment of labour is also decreasing but this indicator is in positive numbers during the whole period. Of course, it seems that between 2005 and 2010 there is a capital expansion.

When we look on the development connected with products of agriculture, forestry and fishing (**A**), there is quite stable output that is about 65% in comparison with the output of these products in 1990. Labour intensity development shows the decreasing tendency from 2000 onwards. It is probably caused by the movement of the employees from the industry of agriculture to other industries. Also the development of capital endowment of labour shows a capital deepening during the whole time.

Special attention should be paid to manufactured products (**C**) due to the importance for the Czech economy. The figures on the development of capital endowment of labour are only in 2010 negative. It is

	Labour intensity development				Volume indices of output				Development of capital endowment of labour			
CZ-CPA	**1995**	**2000**	**2005**	**2010**	**1995**	**2000**	**2005**	**2010**	**1995**	**2000**	**2005**	**2010**
Total	**93.6**	**75.9**	**56.9**	**49.5**	**103.2**	**123,7**	**160.2**	**188.1**	**0.11**	**0.06**	**0.02**	**0.00**
A	85.3	85.2	63.0	54.9	65.3	62.5	65.4	66.5	0.11	0.02	0.03	0.01
B	63.4	61.6	48.2	43.0	78.8	59.2	57.1	54.4	0.26	0.16	0.04	0.02
C	93.3	63.8	43.1	33.6	88.1	125.8	181.2	216.6	0.14	0.09	0.04	−0.07
D	49.4	30.1	24.1	20.3	166.8	187.3	213.5	219.7	0.34	0.16	−0.05	0.13
E	65.9	59.7	59.6	52.6	124.1	156.0	180.4	203.0	0.32	0.19	0.02	−0.09
F	95.8	86.3	71.8	66.3	130.3	120.3	145.4	167.7	0.59	0.17	0.01	0.09
G	131.6	117.9	88.0	71.1	87.4	104.5	139.4	175.2	−0.40	0.27	0.24	−0.11
H	90.9	70.3	58.1	47.7	114.3	129.3	160.0	187.2	0.10	0.21	0.06	0.02
I	52.9	59.7	56.3	72.6	248.9	244.0	230.8	216.8	−0.18	−0.13	0.05	0.21
J	31.2	20.5	15.4	15.8	279.6	489.9	680.1	851.8	−0.56	0.14	−0.09	0.05
K	113.5	111.1	73.0	65.6	260.9	326.4	404.7	517.6	0.37	0.06	0.01	0.02
L	194.9	144.6	81.4	80.5	108.7	135.2	170.1	234.8	−0.23	−0.05	0.00	0.00
M	52.6	48.9	42.4	41.5	191.5	213.0	280.7	324.4	0.48	0.03	0.01	0.00
N	63.6	86.6	63.0	57.7	168.3	142.8	180.5	204.4	0.32	0.59	0.10	0.17
O	64.6	58.5	56.0	50.1	121.1	131.6	151.9	158.0	0.29	0.04	0.04	−0.03
P	82.5	77.5	75.6	61.3	112.8	113.9	127.5	140.8	0.20	0.09	0.00	0.02
Q	117.4	133.2	110.1	114.0	90.1	73.0	86.7	90.6	0.07	0.08	−0.05	−0.02
R	91.5	71.7	62.9	64.9	102.2	130.6	148.5	159.9	0.37	−0.07	0.09	0.08
S-U	129.8	131.6	128.5	157.1	93.3	91.9	88.2	90.7	−0.15	0.05	0.14	−0.16

Source: own calculations

Table 2 Development of labour intensity, output and capital endowment of labour in the period 1990–2010

CZ-CPA	Name of product
A	Products of agriculture, forestry and fishing
B	Mining and quarrying
C	Manufactured products
D	Electricity, gas, steam and air conditioning
E	Water supply; sewerage, waste management and remediation services
F	Constructions and construction works
G	Wholesale and retail trade services; repair services of motor vehiclesand motorcycles
H	Transportation and storage services
I	Accommodation and food services
J	Information and communication services
K	Financial and insurance services
L	Real estate services
M	Professional, scientific and technical services
N	Administrative and support services
O	Public administration and defence services; compulsory social security services
P	Education services
Q	Human health and social work services
R	Arts, entertainment and recreation services
S-U	Other services

Source: Czech Statistical Office

Table 3 The list of commodities connected with Table 2

obviously due to economic recession which began in 2009. We can see capital deepening of these products in the period before 2009. Also the output corresponds to this development of capital endowment of labour. Moreover, it could be said that human labour is partly replaced by capital. It show also figures about the development of labour intensity which were decreasing in the whole period.

In the last years it is quite often spoken about the situation in construction (**F**). In the table 2 we can see the decrease of labour intensity during the whole period. Despite the recession in 2010, the capital endowment of labour is not negative though investments are in decline.

It seems the situation connected with products of wholesale and retail trade services (**G**) shows similar properties as by the manufactured products. There is the decrease of labour intensity, the increase of output and only the specific development of capital endowment of labour. There is almost no investment in the period 1990–1995 according to this indicator. For the next ten years there is a state of capital deepening which follows again the decrease of the capital endowment of labour. It is also probably due to economic recession.

The information about development in connection with financial and insurance services is also realistic. We can see the increase of labour intensity for the first ten years. It means the output is made by higher hiring of human labour. The next ten years is in the sign of the decrease of labour intensity. It is probably due to the development in the information and communication technologies. There is an obvious increase of importance of financial sector from the point of view of output. The figures on capital endowment of labour show higher investment in the period 1990–1995. Since the 1995 we can see only moderate capital deepening.

In the same way it could be evaluated also the development of the rest of commodities. There is not enough space to do this in this paper.

5 Conclusion

The paper is focused on the methodology of usage of data from time input-output tables. It focuses on the commodity structure of the working time and its development. From this point of view, the results of the analysis described in this paper show quite realistic view of the economic development. Comprehensive view on the three estimated indicators represents a basis for evaluation of the development connected with a particular group of products. Based on this information, we can evaluate whether investments into the capital assets can be substituted by human work or human work can be replaced by machines.

Nevertheless it should be noted that the results of application of the method provide only a kind of additional information to ordinary published information in supply and use tables. Alternatively it can be used the development of gross value added in comparison with the labour intensity development. Nevertheless it seems that some weaknesses of the methods could be identified while analysing specific products in detail. Nevertheless it seems that some weaknesses of the methods could be identified when analysing specific products in detail.

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Categorical data imputation under MAR missing scheme

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Abstract. Traditional missing data techniques of imputation of the MAR (missing at random) schemes focus on prediction of the missing value based on other observed values. In the case of continuous missing data the imputation of missing values often focuses on regression models. In the case of categorical data, usual techniques are then focused on classification techniques which sets the missing value to the 'most likely' category. This however leads to overrepresentation of the categories which are in general observed more often and hence can lead to biased results in many tasks especially in the case of presence of dominant categories.
We present original methodology of imputation of missing values which results in the most likely structure (distribution) of the missing data conditional on the observed values. The methodology is based on the assumption that the categorical variable containing the missing values has multinomial distribution. Values of the parameters of this distribution are than estimated using the multinomial logistic regression.

Keywords: Missing data, Categorical data, Multinomial regression.

JEL Classification: C35
AMS Classification: 62J99

1 Introduction

Popular methods for a completion of (individual) observation as for example mean imputation, regression imputation or maximal likelihood imputation are usually focused on imputation of a continuous variable. Those methods mostly classify the missing values as "most likely" or "expected" values. Overview of those methods can be found for example in [5]. List of methods for imputation of categorical variable is less extensive. In the case of categorical data, usual techniques are then focused on classification techniques which sets the missing value to the 'most likely' category (see [6]). This however leads to overrepresentation of the categories which are in general observed more often and hence can lead to biased results in many tasks especially in the case of presence of dominant categories.

The aim of the paper is to introduce multinomial logistic regression as very effective tool for missing data imputation. Motives for using this technique could be described by the following three requirements:

1. to impute data set in form which can be re-used for variety of different analysis and applications; this means single imputation is required,
2. to impute data in the most detailed level; optimally on individual observation level
3. to impute data in a way that will respect "expected" ratios of categories in general.

In the following text the methodology and its specific features will be described.

2 Missing data typology

In this article the widely renowned typology of missing data structures developed in [4] will be adopted. Rubin considered the missingness as a probabilistic phenomenon, i.e. a set of random indicator variables R indicating non-missingness of a particular observation was considered. Also the partition of the complete dataset Y_{com} into set of observed values Y_{obs} and set of missing values Y_{mis}, i.e.

$$Y_{com}=(Y_{obs}, Y_{mis})$$

was considerd. Missing data are called missing at random (MAR) in the case where the distribution of the missingness does not depend on Y_{mis}, i.e. when

$$P(R| Y_{mis}) = P(R|Y_{obs}).$$

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This is the case where 'MAR allows the probabilities of missingness to depend on observed data but not on missing data'. A special case of the MAR is then MCAR (missing completely at random), where the probabilities of missingness do not depend on the observed data either:

$$P(R|Y_{com}) = P(R).$$

If MAR is violated, data are missing not at random (MNAR).

The task solved within the paper

In this article a methodology for a specific task is developed which can be however reused in many similar tasks. Conditions of the solved task could be described as:

1. We have a univariate pattern of categorical data, i.e. data where several variables are completely observed (X_{obs}) and one variable contains missing values. This can be schematically expressed as in [5]:

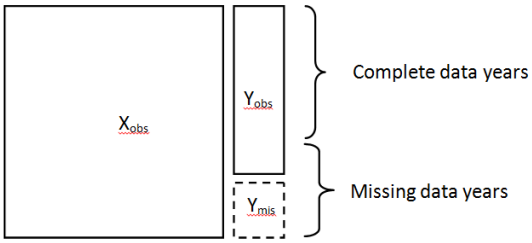

Figure 1 Data set structure

2. Data are observed over certain period of time and for some (major) part of the period data are either complete or with negligible amount of missing data. These years will be referred as 'complete data years'.
3. For the outstanding years, the amount of missing data is rather large and MAR is not guaranteed. These years will be referred as 'missing data years'.
4. The trends observed during the complete data years are relevant for the predictions for the missing data years.
5. Observations are assumed independent.

The time structure of data set according to missing data

From the time point of view three types of missing data position could be distinguished. The first is situation where we have complete information from some moment (year) but before this time missing data occur. This situation illustrates Figure 3a. In such a situation the aim is to reconstruct data before some point in time.

The second example is situation where data are complete, however, from some moment in time some (or all) data are missing. This situation is in Figure 3b and the aim in such a situation is to estimate the missing information for that period after any concrete moment. The third type is a situation where we have missing data "in the middle" of the time period, i.e. for some (limited) period of time the information is partially or completely missing. Figure 3c describes that situation. The aim is to bridge this part, estimate the missing data respecting trends before and after this missing period.

Figure 3a Structure of the data set – missing data at the beginning of the analyzed time period

Figure 3b Structure of the data set – missing data at the end of the analyzed time period

Figure 3c Structure of the data set – missing data for a limited time during the analyzed time period

3 The imputation algorithm

In the following text, we will mean by **determinants** the original or rediscretized variables that have a significant impact on the distribution of the variable containing missing data. (The significance is measured over the years with complete data.) By **profile** we then mean a group of data with the same combination of values of the determinants. We will assume time points t for which complete data years are the time points $t \leq c$ and missing data years are $t > c$.

The basic steps of our imputation algorithm:

1. Find determinants of the missing data structure within the observed variables.
2. Define profiles of observations with missing data based on the values of the observed determinants. These profiles have different distribution of the incomplete variable.
3. Estimate probabilities of each category of the missing variable for each profile.
4. Based on the probabilities, find "appropriate" count of missing observations of each category in each profile and distribute these counts to each individual in the profile.

3.1 Multinomial logistic regression application

If we assumed that data are independent (independence of the observations) the categorical variable containing the missing values (Y) follows for a given profile the multinomial distribution. This fact immediately suggests using the multinomial logistic regression on the complete data years ($t \leq c$) as the methodology for finding the determinants (X or subset of X) of the structure of Y (as the response variable) and predicting the expected probabilities of each category of the response variable for each profile of data at each time point (for both $t \leq c$ and $t > c$). This requires assessing the time variable as covariate and assuming some (possibly polynomial) trend. That is the probability distribution of the categories of Y for each profile in each year $P(Y|X,t)$ is fitted as the outcome of the regression analysis (steps 1-3 of the above outlined imputation algorithm).

3.2 Partially missing data

Based on the above described analysis we obtain the predicted distribution of the variable containing the missing data (Y) also for the years containing missing data ($t > c$) for each profile and each time point (conditioning on X and t will be leaved out in this section for simplicity). However, for these years we may have some amount of observed data (supposing partially missing data in the data set). Therefore we can estimate two distributions of missing values, first based on complete data years and second based on missing data years:

1. First distribution as the prediction based on the complete data years $P(Y = i)$ for each category $i = 1,\ldots,k$ and a given profile and each time point.
2. Second distribution fitted based on the observed data $P(Y = i| R=0)$ in the missing data years, i.e. distribution conditional on the fact that an observation is not missing.

Besides these distributions, we can also estimate the probability of missing values $(P(R=0))$. The (marginal) distribution $P(Y=i)$ equals

$$P(Y=i) = P(Y=i,R=0) + P(Y=i,R=1),$$

where $P(Y=i,R=0)$ (or $P(Y=i,R=1)$) is the (joint) probability that the observation is certain category and is missing (or is not missing respectively) which equals

$$P(Y=i,R=0) = P(Y=i|R=0)\, P(R=0) \text{ and}$$

$$P(Y=i,R=1) = P(Y=i|R=1)\, P(R=1).$$

We can write for the distribution of the observations that are missing (i.e. for which we already know that R=0) as:

$$P(Y=i|R=0) = [\, P(Y=i) - P(Y=i|R=1)\, P(R=1)\,] \,/\, P(R=0).$$

3.3 Finding the appropriate count of missing observations of each category

Let us assume one particular profile of the data in a given year. Based on the above described regression analysis we can get the estimated distribution of the categories of Y for the missing observations, denoted as $P(Y=i|R=0) = p_i,\ i=1,\ldots,k$ for this given profile and year. Furthermore we know that in this profile and year, there is certain amount of missing data n. Given the probability distribution of the categories of the response variable and the number of missing observations we still need to determine how many of the missing observations correspond with each category (step 4 of the above outlined imputation algorithm). Note that it is required that the missing

values are imputed on the individual level and hence we need to determine counts (integers) of missing observations for each educational category.

Normally the expected value would be the first choice for the predictions as it yields predictions with the lowest least square error. The expected value of the multinomial distribution in a particular category i is simply the count of missing observations (in the particular profile in the particular year) times its probability, i.e.

$$E(y_i) = n\, p_i\,, \quad i=1,\ldots,k.$$

However, the expected values are generally real numbers (not necessarily integers). Therefore we suggest using the maximum likelihood criterion where the maximization is performed only on the discrete (integer) numbers. This means finding such y_i , $i=1,\ldots,k$ that the joint distribution $P(y_1,y_2,\ldots,y_k \,|p_1,p_2,\ldots,p_k,\, n)$ is maximized. This in fact means we are looking for the mode of the multinomial distribution.

Mode of the multinomial distribution

There is no closed form formula for the mode of the multinomial distribution. There are however several iterative algorithms developed for this task. See for example [2], [1] or [3]. In our computations we selected the **Finucan's algorithm** published in [1].

Distribution of estimated data on the individual level

Having found the mode of the multinomial distribution for a particular profile we have a vector of counts (integers) of missing values of each category of the variable of the concern which has the highest probability. Within the profile, these counts may be 'assigned' randomly to the individuals as all individuals of the given profile have the same probability vector p_i, $i=1,\ldots,k$ of being in i-th category.

4 Discussion

The proposed method of estimation of missing data could be used in many spheres of application. In this paper we demonstrated the algorithm on (completely or partially unknown) education structure of a population. Education attainment could be taken as a typical example of categorical data. Moreover, when studying the population, this type of data is relatively often incomplete. Other example could be e.g. the marital status, age profile, etc.

The described algorithm is based on the assumption of continuous trend in the data within the missing data years. It corresponds with situation where data are missing because of some administrative changes etc. which does not affect the trend in the data. Application of the described method in situations where this condition is not fulfilled (e.g. where the missingness of the data is at least partially related to some changes affecting also the long-term trend – wars, etc.) would mean some sort of simulation of "unaffected" development – how the structure (partially or completely missing) would have developed if there had not been any interruption of the trend.

Furthermore the estimates of the differences between the distributions $P(Y=i)$ and $P(Y=i|R=1)$ may suggest the (non)randomness in missingness 'mechanism.' The results in integer form have the advantage that it allows imputing data on individual level. (Such imputation is however not unique).

5 Illustrative example

As an illustrative example we assumed educational attainment of a studied population as the variable containing missing data. For simplicity we assume that data are complete up to 2009 and data are completely missing in 2010 and 2011. The variable has 3 categories (low, middle and high education). There is only one other variable (X) which is the gender. Therefore we only have two profiles in each year. The probabilities estimated with the multinomial logistic regression are displayed in the Figure 4.

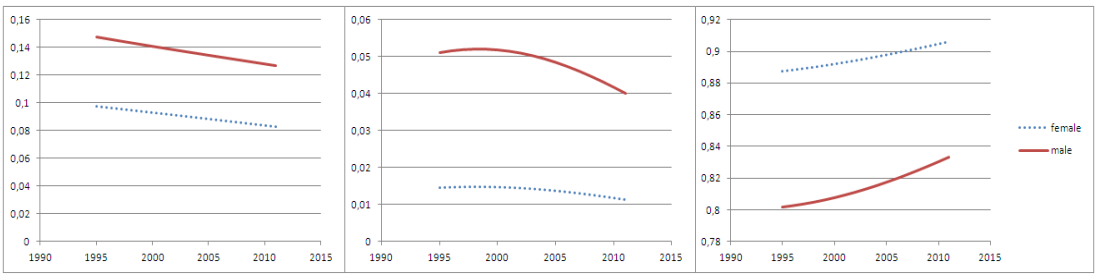

Figure 4 Estimated probabilities of the educational categories.

For illustration we will assume only the year 2011 where we know that we assume that we are missing 100 females and 50 males. The probabilities, expected counts and the mode of the multinomial distribution are contained in the Table 1.

2011	Count missing	Probabilities			Expected counts			**Mode**		
		Low	Middle	High	Low	Middle	High	**Low**	**Middle**	**High**
Female	100	0,906	0,083	0,011	90,62	8,26	1,12	**91**	**8**	**1**
Male	50	0,833	0,127	0,040	41,65	6,35	2,00	**42**	**6**	**2**

Table 1 Estimated probabilities, expected counts and the mode of the multinomial distribution.

Based on these results, we will impute 91 females with low education, 8 females with the middle education and 1 female with high education and analogously for males. As there is no more information available, these counts are distributed within each profile randomly.

6 Conclusion

Aim of this paper was to introduce multinomial logistic regression as very effective tool to missing data imputation. To the authors' knowledge the combination of the multinomial regression and mode searching algorithm was used for the first time for the missing data imputation task. The outcome of the proposed algorithm follows expected structure of the variable containing the missing values.

As a by-product the outcomes of the intermediate steps of the algorithm may be used for further analyses such as analyses of the dependencies (determinants) of the variable of our concern, or analysis of the missingnes mechanism.

Future steps in the research will be to proof this method in some other practical situation. Demographic data (with incomplete information about the education attainment occurring in the latest years of the involved time period –as in the Figure 3b) were used for the very first verification of the model and first results seem to be acceptable. Next part of the research will be to find more datasets with missing data, both MAR and MNAR and with different structure of missing data from the time point of view (length of missing, time of missing) and to prepare more detailed analysis of complemented data files.

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Game theory and real options decision-making hybrid models under random demand

Zdeněk Zmeškal[1]

Abstract. The real options methodology is approach reflecting a risk and flexibility in decision -making. There are two possibilities in decision-making conditions concerning a game competitor: a non-intelligent (against nature) or an intelligent. There is in the paper described and investigated a hybrid game real options decision model. We suppose the underlying random process is quantitative demand defined by geometric Brown´s process. Furthermore, we consider the non-cooperative games based on Nash - Cournot equilibrium optimum. Illustrative example of flexible investment under game competition is presented.

JEL Classification: C6, C7, C 44, C53, C72, C73, F2, F21, G1, G11, G15, G2, G21

AMS Classification: 91B25, 91G20, 91G50, 91G60, 91G80

1 Introduction

Investment decision-making and valuation of company assets are basic and crucial problems of financial management. Typical features of an investment modelling are risk (uncertainty, randomness) and flexibility (contingent option). These aspects are included fully in the real option methodology as a generalised approach. We can present publications concerning real options, see e. g. [1-8], [11], [16], [20-33]. Decisions are sometimes influenced by other market participants. The aspect is modeled by game theory methods. There exist various types of market structure. Usually, perfect market is supposed and prices are not influenced by a demand quantity and price clearing exist. However, other structures are determined by quantity and prices. We distinguish e. g. monopoly, duopoly and oligopoly market. Market equilibrium is based on Nash equilibrium principle. There are examples of game theory application, see [9], [15], [17-19]. Relatively new approaches which are investigated are combination of the real options and game theory, see [9, 10], [12, 13].

The objective of the paper is to describe and verify an investment problem under hybrid conditions, real option (risk and flexibility) and game (non-cooperative competition). We suppose a quantity demand as the underlying random variable and monopoly and duopoly structure. Simplified illustrative example is presented.

2 Valuation procedure of an American real option

The generalised principle of valuation is called the martingale principle see e.g. [14]. The principle is being defined so that a value has to be equal to the expected future value, implying the random process is without trend. In the case of the risk neutral approach this category is ratio of random value and risk-free asset, so after rearranging

$$V_t = e^{-r\cdot dt}\cdot \widehat{E}\left(V_{t+dt}\right), \tag{1}$$

here V_t is value, r is risk-free rate, dt is time interval, $\widehat{E}\left(V_{t+dt}\right)$ is risk neutral expected value. We can gain the same result for complete market under replication valuation strategy or arbitrage principle.

One of the basic approaches of derivatives valuation under complete market is replication strategy, see e.g. [20]. Having derived the replication strategy, we suppose a compact (effective) market, asset-bearing the incomes (dividends, coupons, etc.) proportional to an asset price. The replication strategy is based on creation a portfolio from underlying asset *S* and risk-free asset *B* so, for every situation the derivative value is to be replicated; it means a derivative value equals a portfolio value.

Portfolio value in appraising a moment t is $\Pi_t \equiv a\cdot S_t + B_t = f_t$; portfolio value in a moment $t+dt$ for growing price is $\Pi_{t+dt} \equiv a\cdot S^u_{t+dt} + B_t\cdot e^{r\cdot dt} = f^u_{t+dt}$; the portfolio value in a moment for declining price, $\Pi_{t+dt} \equiv a\cdot S^d_{t+dt} + B_t\cdot e^{r\cdot dt} = f^d_{t+dt}$;

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where Π_t is portfolio value, S is underlying asset value, a is a amount of underlying asset, B is risk-free asset value, f is derivative value, r is continuous risk-free rate, u (d) are indexes of growth (fall) of underlying asset, S^u_{t+dt}, S^d_{t+dt} are their prices in up-movements (down-movements).

By solution of three equations for variables a, B, f_t, we can get a general formula for derivative price,

$$f_t = e^{-r\cdot dt}\cdot\left\{ f^u_{t+dt}\cdot\left[\frac{e^{r\cdot dt}\cdot S_t - S^d_{t+dt}}{S^u_{t+dt}-S^d_{t+dt}}\right] + f^d_{t+dt}\cdot\left[\frac{S^u_{t+dt} - e^{r\cdot dt}\cdot S_t}{S^u_{t+dt}-S^d_{t+dt}}\right]\right\}. \tag{2}$$

This is the general formula for derivative price valuation by the replication strategy, which should be written as follows,

$$f_t = e^{-r\cdot dt}\cdot\left[f^u_{t+dt}\cdot(\hat{p}) + f^d_{t+dt}\cdot(1-\hat{p})\right], \text{ or } f_t = e^{-r\cdot dt}\cdot\hat{E}\left(f_{t+dt}\right). \tag{3}$$

$$\text{Here } \hat{p} = \frac{e^{r\cdot dt}\cdot S_t - S^d_{t+dt}}{S^u_{t+dt}-S^d_{t+dt}}, \tag{4}$$

implies the risk-neutral probability of up-movement and $\hat{E}\left(f_{t+dt}\right)$ is the risk-neutral expected value.

The derivative price is determined as a present value of expected value in a following period. This probability can be considered neither a market growth nor a subjective probability. Due to (3) the derivative price is equal to the present value of risk-neutral expected value of subsequent period, which coincides with generalised martingale principle, see (1).

We can express the underlying asset price, under the proportional continuous income c, due to Geometric Brown's process as follows,

$$S^u_{t+dt} = S_t\cdot U;\quad S^d_{t+dt} = S_t\cdot D, \text{ where } U = e^{\sigma\cdot\sqrt{dt}},\ D = e^{-\sigma\cdot\sqrt{dt}},\ (1+C)^{-1} = e^{-c\cdot\sqrt{dt}}, \tag{5}$$

then after substitution to (4) and after re-arranging we get particular risk-neutral probability formula

$$\hat{p} = \frac{\frac{1+R}{1+C} - D}{U-D}. \tag{6}$$

This formula can be generalised substituting $\frac{1+R}{1+C} = G$ as follows $\hat{p} = \frac{G-D}{U-D}$.

3 Non-cooperative game methodology

Game theory concerns of the problems considering decision of other players. We concentrate in the paper on the non-cooperative games. The crucial term is Nash equilibrium principle, see [19]. This principle is applied in the oligopoly (equilibrium) optimal market production computation, under continuous decision variants. Analogically, this principle is possible to apply in investment variant decision under two person bi-matrix games.

3.1 Nash-Cournot oligopoly and monopoly equilibrium game

We suppose a duopoly market structure. Quantity x_i resp. x_j of production of company *i-th*, resp *j-th*, determine profit of the company $\pi_i(x_i)$, being difference of revenue $\mathrm{Re}v_i(x_i,x_j)$ and cost $C_i(x_i)$,

$$\pi_i(x_i) = \mathrm{Re}v_i(x_i,x_j) - C_i(x_i). \tag{7}$$

Revenue is given as follows, $\mathrm{Re}v_i(x_i,x_j) = p_i(x_i,x_j)\cdot x_i$, and depends on the price $p_i(x_i,x_j)$,

$p_i(x_i,x_j) = a - b_i\cdot(x_i + x_j)$,

cost is given, $C_i(x_i) = c_i\cdot x_i + d_i\cdot x_i^2 + FC_i$, here c_i, d_i are variable cost coefficients, FC_i is fixed cost.

Optimal and equilibrium quantities depend on decision of opposite side and are given by reaction functions. Under convexity of revenue function and cost function, profit is of concave function. Due to Nash equlibrium principle, these assumptions lead to one optimal solution.

$$\frac{\partial\pi_i(x_i)}{\partial x_i} = \frac{\partial p_i(x_i,x_j)}{\partial x_i}\cdot x_i + p_i(x_i,x_j) - \frac{\partial C_i(x_i)}{\partial x_i} = 0, \tag{8}$$

$$\frac{\partial\pi_j(x_j)}{\partial x_j} = \frac{\partial p_j(x_j,x_i)}{\partial x_j}\cdot x_j + p_j(x_j,x_i) - \frac{\partial C_j(x_j)}{\partial x_j} = 0. \tag{9}$$

Solving equations (8), (9) system then generalised formula of companies quantity is following,

$$x_i = \frac{(a-c_i)\cdot(2b_j+2d_j)-b_i(a-c_j)}{(2b_i+2d_i)\cdot(2b_j+2d_j)-b_i\cdot b_j}, \tag{10}$$

$$x_j = \frac{(a-c_j)\cdot(2b_i+2d_i)-b_j(a-c_i)}{(2b_j+2d_j)\cdot(2b_i+2d_i)-b_j\cdot b_i}. \tag{11}$$

Special results for simplified conditions are as follows.

If $(b_i \neq b_j)\wedge(d_i = d_j = 0)$, then $x_i = \frac{(a-c_i)\cdot 2b_j - b_i(a-c_j)}{3\cdot b_i\cdot b_j}$.

Assuming, $(b_i = b_j = b)\wedge(d_i \neq d_j)$, then $x_i = \frac{b(a-2c_i+c_j)-(a-c_i)2d_j}{(2b+2d_i)(2b+2d_j)-b^2}$.

The most simplified version, assuming $(b_i = b_j = b)\wedge(d_i = d_j = 0)$ is $x_i = \frac{a-2c_i+c_j}{3b}$.

Monopoly equilibrium is following:

$$x_i = \frac{a-c_i}{(2b_i+2d_i)}, \tag{12}$$

$$x_j = \frac{a-c_j}{(2b_j+2d_j)}. \tag{13}$$

3.2 Two person bi-matrix Nash equilibrium game

The equilibrium point of the bi-matrix game, assuming discrete variants, is given in a pure strategy and generally in a mixed strategy. We suppose, that strategy of *i-th* player $s_{m,i} \in S_i$, and *j-th* player $s_{n,j} \in S_j$. Then $s^*_{m,i}$ is pure Nash equilibrium strategy and best response of *i-th* player, if $\pi_i(s^*_{m,i}, s^*_{n,j}) \geq \pi_i(s_{m,i}, s^*_{n,j})$. Analogically, for the *j-th* player $\pi_j(s^*_{n,j}, s^*_{m,i}) \geq \pi_j(s_{n,j}, s^*_{m,i})$. We can find Nash pure equilibrium maxmin strategy,

$$\{s^*_{m,i}, s^*_{n,j}\} = \left\{\arg\max_{s_{m,i}\in S_i}\min_{s_{n,j}\in S_j}\pi_i(s_{m,i}, s_{n,j}); \arg\max_{s_{n,j}\in S_j}\min_{s_{m,i}\in S_i}\pi_j(s_{n,j}, s_{m,i})\right\}. \tag{14}$$

3.3 Hybrid real option and game theory

The hybrid problem consists of combination of the real option flexibility method and equilibrium payoff function of players, given by a best response function. We suppose application of the binomial option pricing model for development of underlying variable (asset) and valuation by backward under uncertainty and flexibility, according to (3) as present value of the expected value of subsequent period, using risk neutral probability. Payoff of players is stated in coincidence with duopoly Nash Cournot equilibrium due to (10), (11), (12), (13). The best discrete strategy is given in a way of bi-matrix game, due to (14).

Procedure steps of hybrid game real option model

a) Determination of the random factor (asset) due to Geometric Brown´s process.
b) Computation player's Cournot-Nash equilibrium payoff due to market structure type, oligopoly (10), (11), monopoly (12), (13) depending on demand level.
c) Computation bi-matrix Nash equilibrium of player's strategy (invest, abandon), due to (14).
d) By backward recurrent procedure from the end of binomial tree to the beginning for state *s* and time *t* the player's strategy value due to real option method (3) is calculated.

4 Investment game real option model application

Two investors (players) are considering investment in the oligopoly or monopoly market. Investment is prepared for two years, and in the second year is necessary to decide if invest or abandon the investment possibility. The necessary pre-emption cost is 50. Random price function and demand influenced by $\tilde{a}$, due to geometric Brown´s process, is given by the formula:

$$p_i(x_i, x_j) = \tilde{a} - b_i\cdot(x_i + x_j). \tag{15}$$

The profit (payoff) of non-cooperative game is of three variants:

(i) both investor invest, oligopoly market (10),(11);
(ii) one investor invest, the second abandon, monopoly market (12),(13);
(iii) both investors abandon , not-operating market.

Value of the game is evaluated in investment decision by NPV criterion, for the *i-th* and *j-th* player in 2 years:

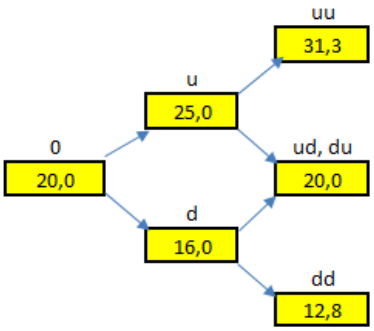
$$NPV_j = \frac{\pi_j\left(s^*_{n,j}, s^*_{m,i}\right)}{k} - INV_j \qquad NPV_i = \frac{\pi_i\left(s^*_{m,i}, s^*_{n,j}\right)}{k} - INV_i\,, \tag{16}$$

here k is a cost of capital, INV are investments.

Bi-matrix Nash game equilibrium strategy is calculated for investors according to (14). Value of the player's game flexible strategy is calculated due to valuation formula (3).

4.1 Input data

We suppose a random parameter a, at beginning $a_0=20$, index $U=1{,}25$, and $D=0{,}8$, risk-free rate $R=0{,}1$, probability $p=0{,}667$. Other input data for firms profit calculation are in Fig. 1.



parameter	investor i	investor j
b	0,5	0,5
c	2	4
d	0,2	0,3
FC	20	30
INV	110	120

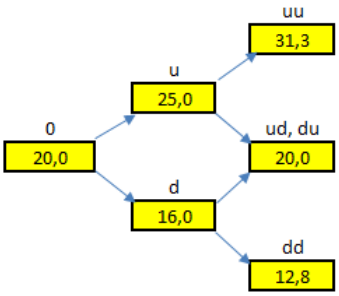
Fig 1 Binomial tree of *a* parameter and input parameters

4.2 Results

There are in Tab. 2 calculated results of the market equilibriums, profit and NPV. Furthermore, Fig. 2 presents payoff of bi-matrices game due to equilibriums states (uu, ud=du, dd) (14). Fig. 3 shows calculations of the NPV in the beginning, $NPV_i=843{,}9$, $NPV_j=256{,}4$. Project values after inclusion of pre-emptive cost are $V_i=793{,}9$, $V_j=206{,}4$. It is apparent, that for both firms pre-emption is effective. Therefore recommendation for firms is to start preparation the projects.

state			node uu		node ud,du		node dd	
firm			i	j	i	j	i	j
demand	coef.	a	31,3	31,3	20,0	20,0	12,8	12,8
oligopoly	production	x	16,7	11,8	10,5	6,7	6,5	3,5
market	price	p	17,0	17,0	11,4	11,4	7,8	7,8
	profit	π	174,5	81,8	56,5	6,3	9,3	-20,3
		NPV	1635,4	698,0	454,7	-57,3	-16,8	-323,3
monopoly	production	x	29,3	27,3	18,0	16,0	10,8	8,8
market	price	p	16,6	17,6	11,0	12,0	7,4	8,4
	profit	π	236,7	118,5	77,2	21,2	15,0	-14,5
		NPV	2256,7	1065,1	662,0	92,0	39,9	-265,1

Tab. 2 Market structure games results



uu		firm j			
		invest		abandon	
firm i	Invest	1635,4	698,0	2256,7	0,0
	abndon	0,0	1065,1	0,0	0,0

ud, du		firm j			
		invest		abandon	
firm i	Invest	454,7	-57,3	662,0	0,0
	abndon	0,0	92,0	0,0	0,0

dd		firm j			
		invest		abandon	
firm i	Invest	-16,8	-323,3	-265,1	0,0
	abndon	0,0	39,9	0,0	0,0

Fig 2 Binomial tree of *a* parameter and markets bi-matrix game equilibriums

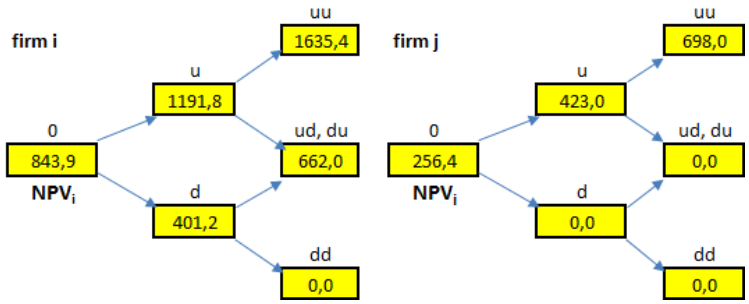


Fig. 3 Binomial tree valuation for investors (firm i, firm j)

5 Conclusions

The hybrid real option game model under random demand was proposed and investigated in the paper. Simplified model was presented and calculated. We can conclude that hybrid model can reflects more realistically decision-making conditions in comparison without considering behaviour of the market participants. Introduced model could be considered to be a generalised approach for decision-making reflecting several aspects together: risk (randomness, uncertainty), flexibility (contingent option decision) and competiveness (behaviour of other players).

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A note on proportional rationing in a PQ duopoly

Jan Zouhar[1]

Abstract. In a PQ oligopoly, firms pick prices and quantities simultaneously, and unlike with the traditional Cournot and Bertrand models, market clearing is not imposed. It is thus necessary to specify the rules of how demand is assigned to individual firms when the market does not clear, known as rationing rules. One of the popular approaches is proportional rationing, which is justified using a notion of randomly ordered consumers coming to the market. Such a process would render the resulting firm-specific demand a random variable, which is ignored in the existing deterministic models. In the paper, we (a) formalize the notion of randomly-ordered consumers into a new stochastic version of the proportional rationing scheme, (b) derive the probabilistic properties of firm-specific demand under this new scheme, and (c) study whether the results of the stochastic and deterministic versions are consistent.

Keywords: oligopoly, homogenous production, proportional rationing, PQ duopoly.

JEL classification: D43
AMS classification: 91B24

1 Introduction

Since the early attempts at modelling the oligopoly market by Cournot and Bertrand, economists have been aware of the fact that solutions to oligopoly models crucially depend on the strategic variables that oligopolists are assumed to decide about. Even though in many real-life settings either price or quantity can be identified as the sole strategic variable, in many other instances one has to admit that firms do in fact decide about both prices and quantities simultaneously. This is the situation that is studied in the framework of a PQ oligopoly, which is perhaps the most natural extension of the models studied by Cournot and Bertrand. Just like the Cournot and Bertrand oligopolies, a PQ oligopoly is a static one-shot non-cooperative game with perfect information, where firms sell homogenous production to price-taking consumers and try to maximize individual profits. In a PQ oligopoly, however, firms pick their prices and quantities simultaneously, which can lead to over-supplying or under-supplying the market. This is a significant distinction from Cournot and Bertrand, where markets always clear.

Despite the PQ oligopoly being a straightforward extension to the traditional oligopoly models, other concepts of incorporating both decision variables, seemingly more complicated ones, had preceded the PQ oligopoly in the academic literature – e.g. the extensive-form vesion of the price-quantity game of Kreps and Scheinkman [7] or the model of Davidson and Deneckere [3] that involves capacity constraints. The reason for this, presumably, is that it is fairly difficult to obtain equilibrium results in the PQ oligopoly: as Friedman [5] showed, the PQ oligopoly game does not have any pure-strategy equilibria under fairly general conditions. The first characterization of the mixed-strategy Nash equilibrium (NE) in a PQ oligopoly was discovered by Gertner [6], albeit for the duopoly case only. His work remained unpublished and this line of research was largely abandoned by active scholars in the field in late 1980s; however, in recent years, several researchers have taken up this research path again – see e.g. [8],[2], where a version of Gertner's model with discretized variables is analyzed.[2]

In a PQ oligopoly, as well as in any other oligopoly model that does not impose market clearance, one has to carefully state the assumptions about *rationing*, which refers to the process that determines how firm-specific demand functions are derived from the industry demand. Note that these assumptions are a part of Cournot and Bertrand models as well. For instance, in the classical Bertrand model, the unique lowest-priced firm covers the whole industry demand at its price, leaving nothing for the remaining

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[2]In fact, as far as I know, the term "PQ oligopoly" was coined by the author of the former of the two papers.

oligopolists, and in case of equal lowest prices, the corresponding industry demand is split evenly among the lowest-priced firms.[3] Nonetheless, rationing assumptions are especially important in case the market does not clear: if the lowest-priced firm does not produce the entire industry demand at that price, some of the consumers that were not served are likely to purchase the good from one of the higher-priced firms, thus creating the *spillover* (or *residual*) *demand*.

Most researchers use one of the two popular rationing schemes known as *parallel rationing* and *proportional rationing*. As the title suggests, this paper focuses on the latter. For two firms with different prices, proportional rationing works as follows: if the low-priced firm supplies only a fraction ϕ of the industry demand at the low price, the high-priced firm faces a demand of $(1-\phi)$ times the industry demand at its price. Gertner [6, p. 62] and Binmore [1, p. 308] mention that this is consistent with a demand function that represents a large number of consumers coming to the market in a random order, each only interested in buying a single unit of the good and willing to pay anything less than or equal to their *acceptance threshold*. It is worth noting that this process automatically renders firm-specific demand a random variable – a fact that has been ignored in the existing research into NE outcomes of PQ duopolies so far.

The aim of this paper is to (a) formalize the notion of randomly-ordered consumers into a new version of the proportional rationing scheme, (b) derive the probabilistic properties of firm-specific demand under this new scheme, and (c) analyze whether the explicit incorporation of randomness in the model changes the current results obtained from the deterministic version of proportionate rationing.

2 The model

We consider a static one-shot non-cooperative game in a duopoly with homogeneous production. The duopolists, firms 1 and 2, simultaneously pick prices and quantities, and both are trying to maximize their individual profit. In the sequel, we use i, j to denote either $1, 2$ or $2, 1$. The profit of firm i can be expressed as

$$\pi_i = p_i \min\{q_i, d_i\} - \gamma_i(q_i), \tag{1}$$

where p_i, q_i are the price and production quantity picked by firm i, d_i is the quantity demanded of firm i's goods, and γ_i is firm i's cost function. Note that total cost depends solely on q_i, implying that free disposal of unsold goods is assumed. A strategy consists in specifying both price and production quantity. We allow for mixed strategies – thus, formally, a strategy of firm i is a random vector[4] (P_i, Q_i). Firm-specific demand, eventually realized as d_i, depends on the following: (a) both firms' prices and quantities, (b) the *industry demand* function, and (c) the rationing scheme.

Industry demand. Industry demand is defined as follows. We assume there are n consumers, each willing to buy 1 unit of production in case the price does not exceed his/her *acceptance threshold*, denoted t. Therefore,

$$\delta(p) \;=\; \text{the number of consumers with } t \text{ at or above } p \;=\; \sum_{c=1}^{n} \mathbf{1}_{[p,\infty)}(t_c), \tag{2}$$

where $\mathbf{1}$ is the indicator function. Depending on the distribution of acceptance thresholds in the population of consumers, the industry function defined in this way can take on all sorts of shapes. For instance, if the distribution is uniform over a certain interval, the resulting industry demand is approximately linear; if it is positively skewed, industry demand resembles a convex function.[5]

Rationing scheme. The rationing scheme defines the process of how possible demand spillovers are created. The n consumers come to the market in a random order. Each consumer c decides whether and from whom to demand the good according to the following rules:

(i) if no price is less than or equal to c's acceptance threshold, c leaves without purchase;

[3] The inelegant property of this rationing scheme is the inherent discontinuity of demand in points where prices are equal. This motivates the use of models of differentiated oligopoly, originally conceived by Dixit [4] and later used in several papers related to the PQ oligopoly, such as [3] and [5].

[4] Throughout the text, we use capitals to denote random variables, and their lower-case counterparts to indicate specific values.

[5] Obviously, the industry demand from (2) is a step function; as such, it can be neither linear nor convex. However, its smoothed version may have these properties.

(*ii*) c demands the good from the firm with the lowest price; if there are multiple lowest prices, he/she picks among them at random with equal probabilities,

(*iii*) if the selected firm has nothing left to sell, c leaves this firm out of further consideration and proceeds again from step (*i*).

After consumer c leaves, either with or without a purchased good, the next consumer turns up and proceeds in an analogous fashion. Once all n consumers have left, the quantity demanded of firm i, d_i, is determined as the total number of consumers that have come and demanded the firm's goods (no matter whether they were served or not). It is worth noting that the rationing process defined above is not restricted to a duopoly model; it is applicable to oligopolies with an arbitrary number of firms.

3 Probabilistic properties of firm-specific demand

Due to the random order of consumers in the rationing scheme, the eventual firm-specific demand d_i is an outcome of a random variable, D_i. If prices are different and the low-priced firm i covers the entire industry demand at its price, then $D_i = \delta(p_i)$ and $D_j = 0$ with probability 1. In case the low-priced firm produces less then the entire industry demand, it sells out all its production, and with a non-zero probability some of the unserved customers will recalculate their plans and demand the good from the high-priced firm. In other words, the spillover effect is likely to set in, and the high-priced firm faces residual demand that is positive with a non-zero probability. The actual probability distribution of this residual demand is given in Proposition 1.

Proposition 1. *If $p_i > p_j$ and $q_j < \delta(p_j)$, residual demand for firm i's goods follows the hypergeometric distribution:*

$$D_i \sim \text{Hypergeometric}(\delta(p_j), \delta(p_i), \delta(p_j) - q_j), \tag{3}$$

meaning that

$$\mathbb{P}\{D_i = d_i\} = \frac{\binom{\delta(p_i)}{d_i}\binom{\delta(p_j)-\delta(p_i)}{\delta(p_j)-q_j-d_i}}{\binom{\delta(p_j)}{\delta(p_j)-q_j}}, \qquad \textit{and} \qquad \mathbb{E}D_i = \frac{\delta(p_j)-q_j}{\delta(p_j)}\delta(p_i). \tag{4}$$

Proof. At price p_j, there are $\delta(p_j)$ consumers willing to buy from firm j (denote this set of consumers as A), and $\delta(p_i)$ of them are willing to accept price p_i as well (set B). Consumers are coming at random order, first of q_j them buying from firm j. The residual demand is formed by those out of the remaining $\delta(p_j)-q_j$ consumers who belong to B. As all orderings of consumers are equally likely, the situation is akin to making $\delta(p_j) - q_j$ random draws without replacement from population A [of size $\delta(p_j)$] containing a subset of successes B [of size $\delta(p_i)$]. The number of successes has thus the distribution posited above. □

In case prices are equal, the probability distribution of D_i is more complicated to derive. The reason is that both kinds of spillover effects can be present, from firm i to firm j and the other way around, depending on the results of the random selection process of individual buyers [see rule (*ii*) above]. In Proposition 2, we derive the characterization of the support of the joint distribution of D_i and D_j, given the prices, production quantities and industry demand. (It is worth noting that the results in Proposition 2 do not hinge on that the buyers pick evenly-priced firms with equal probabilities; the result holds as long as the probabilities are non-zero for both firms, see the proof of the proposition below.)

Proposition 2. *If $p_i = p_j = p$, the support of the distribution of (D_i, D_j) is made up by all integral points lying in set S defined as the set of all 2-tuples $(d_i, d_j) \in [0, \delta(p)]^2$ that satisfy the inequalities*

$$d_i \geq \delta(p) - q_j, \tag{5}$$

$$d_j \geq \delta(p) - q_i, \tag{6}$$

$$d_i + d_j \geq \delta(p), \tag{7}$$

with the further requirement that at least one of these inequalities is tight in each point of S.

Proof. Firstly, note that if there are $\delta(p)$ buyers willing to buy at price p, firm-specific demand has to be an integer between 0 and $\delta(p)$. Next, if $q_j < \delta(p)$ and more than q_j customers will first come to buy from firm j, then all coming after the q_jth one will be turned down, and will subsequently demand the

good from firm i as well. Thus, (5) has to hold. Swapping i and j in the previous argument gives (6). The requirement in (7) is straightforward – there are $\delta(p)$ buyers, and each will demand the good from i, j or both.

Finally, we have to prove that at least one of the inequalities has to be tight in any (d_i, d_j) having a positive probability. If $d_i + d_j > \delta(p)$, some buyers have demanded the good from both firms, i.e. one or both firms must have run out of supplies. Moreover, one of them must have been the first to run out supplies; if this was firm i, then necessarily $d_j = \delta(p) - q_i$, because all consumers but the q_i that purchased the good from firm i demanded the good from firm j (either directly or through the spillover effect). Analogously, if firm j is the first to run out of supplies, $d_i = \delta(p) - q_j$. In other words, if (7) is strict, one of (5),(6) has to be tight, which completes the proof. □

In Proposition 3, we derive the actual probability mass function of the joint distribution of D_i, D_j.

Proposition 3. *If $p_i = p_j = p$, the joint distribution of D_i, D_j is given by a probability mass function that can be expressed as*

$$\mathbb{P}\{(D_i, D_j) = (d_i, d_j)\} = \\ = \begin{cases} 2^{-\delta(p)} \binom{\delta(p)}{d_i} & \text{if } d_i + d_j = \delta(p), \\ 2^{-\min\{\delta(p), q_i+q_j\}} \sum_{k=0}^{\delta(p)-d_i} \binom{k+q_i-1}{k} \binom{\min\{q_j, \delta(p)-q_i)\}-k}{\delta(p)-d_i-k} & \text{if } d_j = \delta(p) - q_i, \\ 2^{-\min\{\delta(p), q_i+q_j\}} \sum_{k=0}^{\delta(p)-d_j} \binom{k+q_j-1}{k} \binom{\min\{q_i, \delta(p)-q_j)\}-k}{\delta(p)-d_j-k} & \text{if } d_i = \delta(p) - q_j, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

for any $(d_i, d_j) \in \{0, 1, \ldots, \delta(p)\}^2$.

Proof. First note that conditions in the first three cases of (8) correspond to (5)–(7); it is thus easily verified that (8) gives a non-zero probability for all points in the support (derived in Proposition 2), and zero otherwise.

At price p, there are $\delta(p)$ consumers coming and demanding the goods from firms i and j. It is useful to define

$$\sigma_c = \begin{cases} 1 & \text{if } c\text{th consumer decides to go to firm } i \text{ first,} \\ 0 & \text{if } c\text{th consumer decides to go to firm } j \text{ first,} \end{cases}$$

and $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_{\delta(p)})$. As the inital decisions made by individual consumers are independent, an arbitrary sequence $\boldsymbol{\sigma}$ happens with a probability of $2^{-\delta(p)}$. Next, we verify the correctness of the formulas in (8) for the different cases:

Case $d_i + d_j = \delta(p)$. This condition requires that no consumers are turned down by either firm, i.e. there are no spillover effects. Therefore, any sequence $\boldsymbol{\sigma}$ with d_i 1s and $[\delta(p) - d_i]$ 0s produces the same outcome, and there are $\binom{\delta(p)}{d_i}$ such sequences.

Case $d_j = \delta(p) - q_i$. This condition implies that i was the first firm to have run out of supplies (or, more precisely, no consumers purchased from i through the spillover effect). The corresponding $\boldsymbol{\sigma}$ sequences have no more than q_j 0s preceding the q_ith 1. The number of different sequences that have k 0s before the q_ith 1 is given by

$$\binom{k+q_i-1}{k} \underbrace{2^{\delta(p)-q_i-k}}_{\text{the last } \delta(p)-q_i-k \text{ elements are irrelevant}}.$$

However, we also require that quantity demanded from i equals d_i; this includes both the direct and the spillover demand. If $d_i = \delta(p) - x$, it means that there are x consumers who both (a) decided to go to firm j first and (b) were not turned down (i.e. they indeed purchased the good). In case $q_i + q_j \geq \delta(p)$, this corresponds with sequences having x 0s in total; if $q_i + q_j < \delta(p)$, this corresponds

with x 0s in the first $q_i + q_j$ elements of the sequence, since after $q_i + q_j$ consumers have come in, firm j will have sold out all its production. Therefore, the number of suitable sequences that have k 0s before the q_ith 1 *and* produce the firm-specific demands (d_i, d_j) is

$$\underbrace{\binom{k+q_i-1}{k}}_{k \text{ 0s before the } q_i\text{th 1}} \overbrace{\binom{\min\{\delta(p), q_i+q_j\}-q_i-k}{x-k}}^{x-k \text{ 0s in the first } q_i+q_j \text{ elements and after } q_i\text{th 1}} \underbrace{2^{\max\{0,\delta(p)-q_i-q_j\}}}_{\text{the last } \delta(p)-q_i-q_j \text{ elements are irrelevant}}.$$

Summation across all $k \leq x$ and a slight rearrangement gives the formula in (8).

Case $d_j = \delta(p) - q_i$. The result is obtained easily from symmetry; it suffices to swap the i and j subscripts in the previous argument. □

4 Does randomness matter?

In this section, we demonstrate that the traditional deterministic model of proportional rationing is not fully consistent, in terms of the expected equilibrium outcomes, with the random-ordering model – despite the notion of randomly ordered customers being used as the justification for the former. As a benchmark, we use Gertner's [6] model. Gertner defines firm-specific demand as follows:

$$d_i = \begin{cases} \delta(p_i) & \text{if } p_i < p_j, \\ \dfrac{\delta(p_j) - \min\{q_j, d_j\}}{\delta(p_j)}\,\delta(p_i) & \text{if } p_i > p_j, \\ \max\left\{\delta(p_i)/2, \delta(p_i) - q_j\right\} & \text{if } p_i = p_j. \end{cases} \tag{9}$$

Gertner's findings for his model are as follows. There is no equilibrium in pure strategies. A mixed-strategy NE exists, and its characterization differs with varying returns to scale. With constant or decreasing marginal cost, firms produce the entire industry demand at their price, the price has a continuous distribution on a certain interval, and expected profits are zero. With increasing marginal cost, the equilibrium distribution of prices has a mass point at the upper support, where the firm produces half the industry demand, and is atomless below this point, where the firm produces more than a half, but less than the entire industry demand. (For details, see [6].)

Ideally, if the Gertner's model and our model from §2 are consistent, the expected profit π_i is the same when calculated both by plugging (9) into the profit formula (1), and by taking the expectation of π_i with respect to the firm-specific demand, for any combination of prices and quantities. However, as we argue below, this may not be the case in general.

Firstly, it is obvious that the definition of d_i in (9) is supposed to give the expected value of firm-specific demand; this is the case for $p_i > p_j$, as can easily be verified by comparison with (4), but not for $p_i = p_j$, as can be easily seen from Proposition 2. This is not as serious a problem in Gertner's treatment, as the equilibrium price distributions are mostly atomless, and therefore $\mathbb{P}\{P_i = P_j\}$ is zero. (With increasing marginal cost, the price distribution has a mass point, but the corresponding quantities are such that there is no spillover demand.) It might, however, change the results significantly in case we switch from continuous prices and quantities to discrete ones, as in [8] and [2].

Secondly, even if the formula for the case $p_i = p_j$ is replaced with $\mathbb{E}D_i$, the resulting profit will in general be different from the expectation of the profit in the model where D_i is treated as random. To see why, note that in the profit equation sales are calculated as the minimum of production and demand. Broadly speaking, it matters whether the expectation is taken before or after the minimum operator. More precisely, it is not difficult to show that

$$\mathbb{E}\big(\min\{q_i, D_i\}\big) < \min\{q_i, \mathbb{E}D_i\} \quad \Leftrightarrow \quad \mathbb{P}\{D_i < q_i\} > 0 \ \text{ and } \ \mathbb{P}\{D_i > q_i\} > 0, \tag{10}$$

which implies that using (9) in the profit formula will underestimate the expected profit whenever the probabilities on the right in (10) are non-zero. It is thus natural to ask whether these probabilities are in fact non-zero in the region of interest, i.e. near the equilibrium combinatoin of strategies. Gertner's equilibrium results for increasing marginal cost suggest the answer is positive: the NE prices and quantities

[in any realization of the NE mixed strategy (P_i, Q_i)] always satisfy $\delta(p_i)/2 \leq q_i < \delta(p_i)$ in his model. This automatically guarantees that if firm i is the high-priced one, $\mathbb{P}\{D_i < q_i\} > 0$ (since D_i can be as low as $\delta(p_i) - q_j < \delta(p_i)/2 \leq q_i$). Moreover, $p_i = \delta(p_i)/2$ at the upper support of the price distribution, which makes it likely that $\mathbb{P}\{D_i > q_i\}$ may indeed be non-zero.[6]

5 Conclusions

In this paper, we proposed a stochastic rationing scheme in PQ oligopolies based on a notion of randomly ordered consumers with varying acceptance thresholds, which is often used as a justification for the well-known deterministic proportional rationing scheme. First, we derive the probabilistic properties of our rationing scheme for the duopoly case. Second, we show that expected sales and profits determined from our stochastic model are not fully consistent with those obtained from the deterministic proportionate rationing scheme, suggesting that the justification for deterministic proportional rationing is not quite correct. Whether or not this substantially changes equilibrium results remains an open question that should be addressed in future research.

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[6] Note however that analytic solution to Gertner's problem is not known, and numeric experiments that would verify this are not easily carried out; e.g., it is neccessary to turn the problem into a discrete-price one to run the simulation, and even for a relatively coarse price scale, the computational burden is huge, see [8].

Nascent entrepreneurs in the Czech Republic: demographic characteristics and failure hazards.

Jan Zouhar[1], Martin Lukeš[2]

Abstract. Using longitudinal data on new entrepreneurial activity collected in 2011 and 2012 in the Czech Republic, we study the impact of several demographic characterstics, namely age, gender, region and education, on the risk of venture disbandment (failure), and the way this risk evolves with time. Unlike most extant studies, we focus on the total effect of demographic characteristics, meaning that we deliberately do not control for other factors that mediate their effect, such as human, social, and financial capital. Drawing on the longitudinal design of our study, we employ the methods of duration analysis to describe the failure hazard. Our results suggest that (i) those who live in a large city are less prone to fail, (ii) the effect of age changes its sign, with failure hazard being minimized around the 43$^{\text{rd}}$ year, (iii) education does not affect failure hazard significantly, and (iv) failure hazard falls monotonically with time.

Keywords: nascent entrepreneurs, duration analysis, Czech Republic.

JEL classification: L26,O31
AMS classification: 62P20

1 Introduction

Recent years have seen a growing interest in research on entrepreneurial dynamics. Arguably, the most extensive research project into this area is the *Global Entrepreneurship Monitor* (GEM), an on-going international project initiated in 1999 that has grown to comprise 'national teams' in more than 100 countries, including the Czech Republic (see [5]). While GEM focuses on all stages of entrepreneurial activity, the current paper deals solely with *nascent entrepreneurs* (NE), i.e. those actively involved in an on-going but not yet operational start-up effort in which they are going to be (co-)owners. A highly influental research program in this area, the *Panel Study of Entrepreneurial Dynamics* (PSED), was started in late 1990s and gathered data on NE in the U.S.; its research protocol was later adopted for parallel studies undertaken in Argentina, Canada, Greece, the Netherlands, Norway, and Sweden. Indeed, a dominant portion of recent empiric research on NE relies on datasets of PSED and its descendants (see [2]). In our study, we use longitudinal data on NE in the Czech Republic, who were first contacted using a randomized selection procedure in 2011 and then interviewed again in 2012; the design of our survey, as described in §3, was heavily inspired by both PSED and GEM.

While the particular topics of interest vary considerably among researchers in the NE area, Davidsson and Gordon [2] note that they can be roughly divided into three categories: 'characteristics of nascent entrepreneurs', 'antecedents and characteristics of the new venture creation process', and 'explaining new venture creation process outcomes'. Our study classifies into the *outcome-explaining* category: we analyze the probability that an entrepreneur gives up on his/her start-up efforts, i.e. the risk that a venture disbands (referred to as a *failure* henceforth). In particular, we focus on two aspects of failure hazard: (i) the way it evolves with time, and (ii) to what extent it is affected by demographic characteristics of the NE, namely age, gender, region and education.

As for (i), we are primarily interested in whether the failure hazard increases or decreases with time elapsed since the conception of the NE's start-up efforts; the former would suggest that in the early stage of the start-up, NE's are not able to fully recognize the potential pitfalls of their ventures, and dwell on their business ideas despite them being founded on overconfidence. The analysis of failure hazard as a

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function of time is enabled by the longitudinal design of our survey, and we employed the methods of duration (or survival) analysis. We consider this methodology very well suited for the outcome-explaining category of NE research, yet we found it to be rarely used, the rare exceptions being [8] and [3].

Our motivation for (ii) requires a more detailed explanation. Most of the existing research that falls into the 'explaining new venture creation process outcomes' category focuses on much more subtle factors of NEs' success/failure than we do; these include for instance

- *human capital*, the most important aspects of which are typically found to be previous start-up and industry experience and psychological factors, such as entrepreneurial self-efficacy, the NE's attitude towards risk and fear of failure,
- *social capital*, which can comprise acquaintances ranking from other entrepreneurs to potential suppliers/customers and other related professionals (lawyers, banks etc.),
- *financial capital*, including household income and money invested in the venture so far,

and a number of *other* (observable) *factors*, such as growth expectations, size of the entrepreneurial team, or the status of the new venture's ownership. A thorough discussion of both the underlying theory and empiric results is well beyond the scope of this text, we refer the reader to [7], [2]. A vast majority of these studies concluded that, after controlling for the factors listed above, demographic characteristics do not play a significant role in explaining the NEs' outcomes. This finding is hardly surprising. Even if we admit that demographic factors such as gender or region have a significant impact on NEs' outcomes, this impact will automatically be thought of as *mediated*, rather than *direct*. A large portion of influence of the ('primary') demographic characteristics will necessarily be transmitted through the ('secondary') factors listed above – as depicted in Figure 1. E.g., it can be argued that experience gathers with age, women may (on average) possess different attitudes towards risk, and NE's living in large cities may exhibit a different type of social capital. It therefore comes as no surprise that controlling for the 'secondary' factors renders the influence of the demographic characteristics insignificant. We therefore think it is of interest to investigate the *total*, i.e. both direct and mediated, effect of the demographic factors. (Note, however, that the effect of each demographic factor is estimated *holding the remaining demographic factors fixed*.)

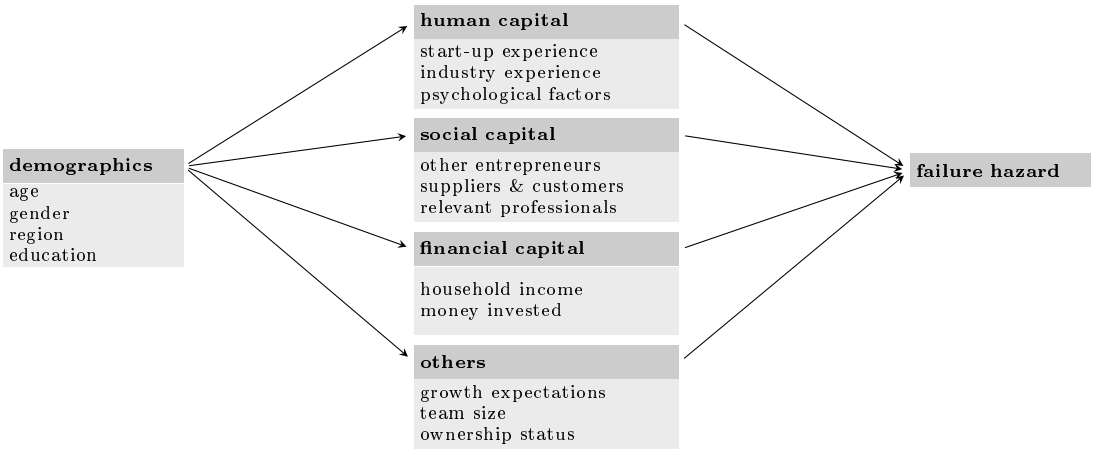

Figure 1: Transmission of influence of demographic factors.

The rest of the paper is organized as follows: §2 specifies the econometric models used in our study (which rank among standard duration analysis models, see e.g. [4]) and introduces the notation that is used later in the text, §3 briefly comments on the origin of our data, §4 discusses the estimation procedure and results, and §5 reports on the sensitivity of our results to alternative model specifications.

2 Model specification

The *failure time* T is modeled as a random variable with cdf $F(t) \equiv \Pr\{T \leq t\}$ (called the *failure function*); quite often, it is convenient to work with the *survivor function*, $S(t) \equiv 1 - F(t)$, instead of the cdf. In the exposition below, we treat the cdf as differentiable (which simplifies statements of certain probabilities), and denote the pdf of T as $f(t)$. In our regression analysis, we are interested in

the properties of T *conditional on* (time-constant) demographic covariates, collectively denoted as $\boldsymbol{x}$. The *conditional* failure function, survivor function and pdf will be denoted as $F(t|\boldsymbol{x}), S(t|\boldsymbol{x})$ and $f(t|\boldsymbol{x})$, respectively. Regression models do not explicitly estimate the shape of F, S or f; instead, they focus on the *hazard function* defined as

$$\lambda(t,\boldsymbol{x}) \equiv \lim_{\Delta t \downarrow 0} \frac{\Pr\{t \le T < t+\Delta t | T \ge t, \boldsymbol{x}\}}{\Delta t} = \frac{f(t|\boldsymbol{x})}{1-F(t|\boldsymbol{x})} = \frac{f(t|\boldsymbol{x})}{S(t|\boldsymbol{x})}.$$

The survivor function can be retrieved from the hazard function, as $S(t,\boldsymbol{x}) = \exp\left[-\int_0^t \lambda(u,\boldsymbol{x})\,\mathrm{d}u\right]$.

For the sake of estimation, we need to make parametric assumptions about the hazard function. In our analysis, we mostly restricted ourselves to the (most common) class of *proportional hazard models* (PH). A proportional hazard function has the form

$$\lambda(t,\boldsymbol{x}) = \lambda_0(t)\exp(\boldsymbol{x}\boldsymbol{\beta}) = \lambda_0(t)\nu, \tag{1}$$

where λ_0 is the *baseline hazard* function, common to the whole population, which is scaled by the person-specific, time-invariant factor $\nu \equiv \exp(\boldsymbol{x}\boldsymbol{\beta})$. Note that (1) can also be written as

$$\log\lambda(t,\boldsymbol{x}) = \boldsymbol{x}\boldsymbol{\beta} + \log\lambda_0(t), \tag{2}$$

and β_j measures the semielasticity of the hazard with respect to x_j. Below we briefly describe the particular PH models used in the present study:

- **Cox model**: Cox [1] formulated a partial likelihood method for estimating the $\boldsymbol{\beta}$ parameters in the PH model without the need to specify λ_0; this is very useful in case one is only interested in the β's and does not want to know the shape of the baseline hazard.
- **Weibull model**: In the Weibull model, conditional on $\boldsymbol{x}$, T has a Weibull distribution, and the baseline hazard is specified as $\lambda_0(t) = \alpha t^{\alpha-1}$. If $\alpha < 1$, the hazard is monotonically decreasing with time (referred to as *negative duration dependence*), if $\alpha > 1$, the hazard increases with time, i.e. exhibits *positive duration dependence*.
- **Gompertz model**: The Gompertz model has a baseline hazard function given by $\lambda_0(t) = \exp(\gamma t)$. Plugging this into (2) yields $\log\lambda(t,\boldsymbol{x}) = \boldsymbol{x}\boldsymbol{\beta} + \gamma t$, so logged hazard is a linear function of time. The direction of duration dependence corresponds with the sign of γ.

As is often the case, the time variable is interval-censored in our study: the respondents reported only the month of the conception of their entrepreneurial activities and the month of disbandment (where applicable). In general, there are two approaches to using interval-censored data: one can either (i) use the continuous-time methods despite censoring (= ignore the problem), which typically does not distort results too much in case the censoring scale is fine enough, or (ii) switch to methods that treat interval-censoring explicitly, which should be prefered if the censoring scale is rather coarse. In our case, the average failure time of those who failed was 12.9 months; even though we think that continuous-time methods can be used in such a setting without much worry, we decided to use a method with explicit interval-censoring in parallel, to double-check the results. We briefly outline the main concepts of this method in the next paragraph.

Assume from now on that the censoring intervals are $[0,1), [1,2), \ldots$, which is the case in our study in case we pick months for time units. The conditional probability of failing within the current interval at time $t = 0, 1, \ldots$ is called the *interval hazard function*, defined as

$$h(t,\boldsymbol{x}) \equiv \Pr\{t \le T < t+1 | T \ge t, \boldsymbol{x}\} = 1 - \frac{S(t+1|\boldsymbol{x})}{S(t|\boldsymbol{x})} = 1 - \exp\left[-\int_t^{t+1} \lambda(u,\boldsymbol{x})\,\mathrm{d}u\right].$$

Plugging in the PH assumption (1) gives $h(t,\boldsymbol{x}) = 1 - \exp\left[-\exp(\boldsymbol{x}\boldsymbol{\beta})\int_t^{t+1}\lambda_0(u)\,\mathrm{d}u\right]$, and if we define $H_t \equiv \log\int_t^{t+1}\lambda_0(u)\,\mathrm{d}u$ for $t = 0, 1, \ldots$, we can write

$$h(t,\boldsymbol{x}) = 1 - \exp\left[-\exp(\boldsymbol{x}\boldsymbol{\beta} + H_t)\right], \qquad \text{or}$$
$$\log\left[-\log h(t,\boldsymbol{x})\right] = \boldsymbol{x}\boldsymbol{\beta} + H_t.$$

The $\log[-\log(\cdot)]$ transformation is often referred to as the *complementary log-log transform*; hence the discrete-time PH model is often referred to as the *cloglog model*. In the estimation, one can either treat

H_t's as parameters to be estimated, or let H_t follow a particular parametric function, e.g. $H_t = \delta t$, which reduces the number of parameters significantly. We opted for the latter approach, and report the results for two alternative models:

- **cloglog 1 model**: $H_t = \delta t$,
- **cloglog 2 model**: $H_t = \theta \log(t)$.

3 Data

The initial sample was gathered in the Czech Republic in summer 2011 using a randomized procedure (see [5] for details). A representative sample of 387 nascent entrepreneurs answered questions related to their entrepreneurial activity. Some questions were identical with those used in GEM and/or PSED, while some others were newly created (see [5] for details). Out of this sample, 186 individuals were identified as nascent entrepreneurs, and agreed to be contacted a year later. We contacted these individuals again in summer 2012, twelve months after the first interview, and asked them to provide detailed information on how their entrepreneurial effort develops. Out of the sample of 186 nascent entrepreneurs, 82 participated in the follow-up interview and answered a set of questions focused on the development of their entrepreneurial activity (see [7] for details). In the language of duration anaylysis, our data were collected using *stock sampling with a follow-up*.

Dependent variables. Respondents reported time elapsed (in months) since the conception of the start-up efforts (in the first interview), their failed/persistent status during the second interview, and month of venture disbandment (if applicable). Because of the under-representation of short-lived ventures induced by stock sampling, we need to work with both *total time elapsed* and *time elapsed at the first interview* in the estimation; the failed/persistent status serves as the *censoring indicator*.

Independent variables. Gender is included in the model through the conventional *female* dummy. Education is classified into the *elementary*, *secondary* and *tertiary* categories and coded into respective dummies; *elementary* is taken as the base category. As regards respondents' regions, we mainly wanted to distinguish those living in larger cities from the rest; as a guidance, we followed the results in [6] that suggest that it is the two largest cities that stand out with respect to entrepreneurial activity in the Czech Republic. Therefore, we include a dummy indicating that a person lives in either *Prague or Brno*. *Age* is expressed in years, measured at the beginning of the NE period; we include a quadratic term to account for a nonlinear relationship (often attributed to the effect of age on various economic outcomes).

4 Results and discussion

Table 1 shows the regression results. All models were estimated using the maximum likelihood (ML) method (or partial ML in case of the Cox model). For likelihood specifications that properly handle the continuous-time PH hazard functions with stock sampling, see [4]. The discrete-time cloglog model is correctly estimated as a generalized linear model with a cloglog link function, after one transforms the data set in the following way: for each respondent, one observation is created for each month between the two interviews that he/she spent in the nascent phase, together with a dummy variable indicating the failed/persistent state. Hence the higher number of observations for discrete-time models (there were, nevertheless, observations on the same 81 NEs in the dataset). Our main findings are summarized below:

- *The results are quite consistent across the different model specifications.* As we expected, because of the relatively fine scale of interval grouping, the difference between the continuous-time models and the discrete-time models is very small – with respect to the magnitude of both the point estimates of the β's and their standard errors. What we find especially assuring is the correspondence of the estimated effects of the explanatory variables in the non-parametric Cox model and its parametric counterparts; this suggests that either the results are very robust to the choice of the baseline hazard, or our choice of baseline hazard is correct (§5 contains further discussion on this topic).
- *Age significantly affects failure hazard: hazard falls with age up until the 43^{rd} year and rises onwards.* The statistical significance of the squared term suggests that the effect of age is indeed non-linear. Point estimates imply that hazard as the function of age is u-shaped, with the turning point being very similar in all models, ranging from 42.5 to 42.8.

	continuous-time models			discrete-time models	
	Cox	Gompertz	Weibull	cloglog 1	cloglog 2
age	-0.359^{**}	-0.298^{**}	-0.237^{*}	-0.314^{**}	-0.264^{**}
	(0.146)	(0.131)	(0.125)	(0.131)	(0.127)
$age^2/100$	0.420^{**}	0.349^{**}	0.278^{*}	0.368^{**}	0.308^{*}
	(0.178)	(0.163)	(0.160)	(0.162)	(0.160)
female	−1.083	−0.978	−0.924	−0.989	−0.935
	(0.671)	(0.645)	(0.645)	(0.645)	(0.644)
Prague or Brno	-1.968^{*}	-2.000^{*}	-1.910^{*}	-2.032^{*}	-1.945^{*}
	(1.089)	(1.078)	(1.071)	(1.080)	(1.072)
secondary educ.	0.380	0.0215	−0.232	0.0840	−0.101
	(0.954)	(0.885)	(0.849)	(0.892)	(0.870)
tertiary educ.	−0.0836	−0.420	−0.711	−0.361	−0.596
	(1.372)	(1.332)	(1.297)	(1.338)	(1.313)
hazard shape†		−0.0478	0.601	-0.0558^{*}	-0.627^{*}
		(0.0299)	(0.301)	(0.0313)	(0.364)
n	81	81	81	930	930
$\log L$	−51.84	−36.12	−36.94	−73.42	−74.04
BIC	130.0	107.4	109.0	201.5	202.8

Standard errors in parentheses; * $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$.
† Estimates of baseline hazard parameters, Gompertz: $\hat{\gamma}$, Weibull: $\hat{\alpha}$, cloglog 1: $\hat{\delta}$, cloglog 2: $\hat{\theta}$ (see §2).

Table 1: Proportional hazard models – regression results.

- Technically, the effect of gender is insignificant in all regressions at the conventional 5 % level. However, note that (i) the two-tailed p-values were ranging from 0.11 to 0.15, which is very close to significance for the one-tailed test, and (ii) the point estimates are consistent across all models. Therefore, we are in fact tempted to think that the statistical insignificance may be solely due to our relatively small sample size. All in all, we conclude that *our results regarding the effect of gender are not conclusive.*
- *NEs living in a large city are less likely to terminate their ventures.* For one-tailed tests, the *Prague or Brno* dummy is significant at the 5 % level in all models (two-tailed p-values range from 0.060 to 0.071), and the point estimates are again consistent.
- *Education does not affect failure hazard in any significant way.* Not only the education dummies are far from significant, also the signs of their coefficients vary across the alternative models. This extends the typical finding that education does not matter in case that traditional success/failure factors have been controlled for (for a possible explanation of this phenomenon, see [6]).
- *Hazard function exhibits negative duration dependence, i.e. hazard falls monotonically with time.* Even though the statistical significance is rather marginal, all parametric models showed this type of relationship (in the Weibull model, $\hat{\alpha} < 1$, in the remaining parametric models, the direction of duration dependence corresponds with the sign of the shape parameter, see §2).

5 Alternative model specifications

Hazard functions with an 'inverted-u' shape. Our parametric model specifications (all but the Cox model) assumed that hazard evolves monotonically with time. In other words, they did not allow for a 'u' or 'inverted-u' shape of the hazard (as a function of time). The latter shape was suggested by Delmar and Shane [3], who ran a semi-parametric piecewise-exponential regression on a Swedish dataset. To check whether our monotonic functional form could be overly restrictive, we tried out several alternative model specifications that allow for the inverted-u shape. Firstly, we used a piecewise-exponential model (as that in [3]) with hazard rate being assumed constant within each year of the nascent entrepreneurship

existence; the point estimates indicated that hazard was indeed monotonically decreasing with years in our data. Next, we used two parametric hazard functional forms that allow for the inverted-u shape, namely the log-logistic and lognormal hazard models. [These models are not of the PH type; rather, they belong to the class of *accelerated failure time* (AFT) models (which we did not describe here).] Both of these models were outperformed by our PH models in terms of the Schwarz information criterion (BIC), and the 95 % confidence interval for the shape parameter in the log-logistic model was $(-4.4, 1.86)$, providing no support for the inverted-u shape hypothesis (the log-logistic hazard function has an inverted-u shape only if the shape parameter is less than 1; in the lognormal model, the inverted-u shape is built in the model specification, and thus it cannot be tested statistically).

Unobserved heterogeneity. With both parametric PH models, we also ran the alternative specifications that account for *unobserved heterogeneity* (or *frailty*) with either gamma or inverse-Gaussian distribution (see e.g. [4], Ch. 4). Both the point estimates of the β's and their standard errors hardly changed (the changes typically occured around the third significant digit) and the shape parameter of the frailty distribution was insignificant.

6 Conclusions

We believe that the contribution of this paper to NE research is twofold. Firstly, the number of longitudinal datasets of our type that have been collected and analyzed so far is rather small; therefore, we contributed by bringing the first empiric results from the Czech Republic, a country with many specifics of its own. Secondly, we emphasized and demonstrated the suitability of duration analysis methodology in the outcome-explaining category of NE research.

We are aware that the present study has several limitations that mostly stem from the small size of our sample, which limited (i) the number of explanatory variables we could effectively use and (ii) the reliability of both our estimates and the subsequent hypothesis tests. We aim to improve on this issue in our future research, as the data-collection process is bound to continue in the following months.

Acknowledgements

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Markov chain model of web browsing

Martina Zouharová[1]

Abstract. The paper deals with a pricing model of banner advertising. We focus on customers' usage pattern while navigating through the websites. That allows us to quantify the exposure effect of banner ads placed on the particular sites, and consider some psychological aspects influencing customers' recall of a banner ad, and so its effectiveness. By employing a Markov chain approach, we analyze real, customer-based clickstream data collected by one of the Czech discount voucher websites. We briefly discuss some advantages of the Markov chain approach compared to pricing models based on click-through rate or cost-per-thousand value. We conclude with comments on the appropriateness of our methodology as a tool for measuring banner advertising effectiveness.

Keywords: Marketing, banner ads, Markov chains.

JEL Classification: C44
AMS Classification: 90C15

1 Introduction

Internet advertising has become a fundamental part of creating marketing strategies. According to a recent report from GroupM, global internet ad expenditures reached $99bn in 2012 and are expected to hit $113.5bn this year. These enormous values provide a great opportunity for many websites owners to keep their sites profitable. When advertisers consider particular options of banner ads placement, they call for measurable indicators that would help them decide whether their investments are lucrative.

When websites sell their advertising space, they generally use one of the three basic models to measure the value that a banner ad placement could yield to the advertiser:

1. *interaction-based* model, which applies the *click-through rate* defined as the percentage of people who transfer from an advertisement to the advertiser's website;
2. *outcome-based model*, which considers the clicks on ads leading to a particular *outcome activity* such as a user's registration, a realized phone call, purchase etc.;
3. *exposure model*, where ads are usually priced according to the *cost-per-thousand* value, which is the number of visits to a particular site, where the ad is placed.

The click-through rate constitutes an important pricing metric in online business, mainly due to its simple calculation. However, many empiric studies have suggested that the suitability of this of this model as a tool for evaluating effectiveness of advertising is debatable (see e.g. [2], [5], [18]). The main reason is that there is little correlation between click-through rates and sales conversion rates.

It could seem that the outcome-based model could eliminate these shortcomings. However, since typical click-through rates are very small in magnitude (see [17], [6]), a user's click and a subsequent outcome action occur simultaneously on a very rare basis. In reality, there typically is a delay between ad exposure and product search, choice or purchase [3]. Therefore, the outcome-based model doesn't represent a very good indicator of banner ads effectiveness either.

With respect to what has been said, we employ the exposure model in our paper, with the belief that mere exposure to banner ads has a positive effect on building brand awareness, brand loyalty, and the demand the firm's products in general. This is in line with many other studies (e.g. [5], [10], [17]) that illustrate this impact by empiric research. Within the exposure model, we discuss two alternative exposure criteria: (i) how often a given banner ad is displayed to a user, and (ii) the total time of the user's exposure to the banner ad. Both of these aspects have a significant impact on advertising recall, brand recognition and brand awareness, as shown in [3] and [12]. In effort to capture the aspects mentioned, we employ Markov chain approach. We follow Pitkow and Pirolli's work [8] in this respect. We just don't identify each page with one state as they do, because today's web complexity doesn't allow this procedure any more. Thus, to get an acceptable amount of states, we classify pages into several categories. We do so in a similar way as Montgomery [13].

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The rest of the paper is organized as follows. In section 2, we describe our dataset. In section 3, we briefly introduce the methodology, which our analysis is based on. Next, in section 4, we deal with our case study with practical comments on the estimates of our mode. Finally, we conclude by discussing the suitability of our methodology as a tool for measuring banner advertising effectiveness.

2 Data

We obtained the data in a web log format from one of the Czech discount voucher websites.[2] The log files contained all requests made to the web server from 27 September 2012, 17:00, to 29 September 2012, 23:00. In total, log files included 900,020 page requests. As the log file contains an identifier of the user that requested the particular page, we were able to associate each user's click with a particular session, where the session represents a sequence of page requests such that each request is issued by the same user. In agreement with some previous studies [4], [19], we used a timeout-based method to group requests into sessions, which is a simple rule saying that if the time between consecutive requests exceeds 30 minutes, the more recent request is assumed to be a part of a new session. Organizing clicks into sessions allows us to collect basic descriptive statistics about the sessions, such as the average session length (app. six clicks in our case). The average length of session could be a good indicator of several pertinent metrics, such as how satisfied a user is with the retailer's site content and performance. The distribution of user session length is shown by the Figure 1. Note that the logged frequency is approximately linear in the number of clicks, indicating that the empiric distribution is close to the exponential one.

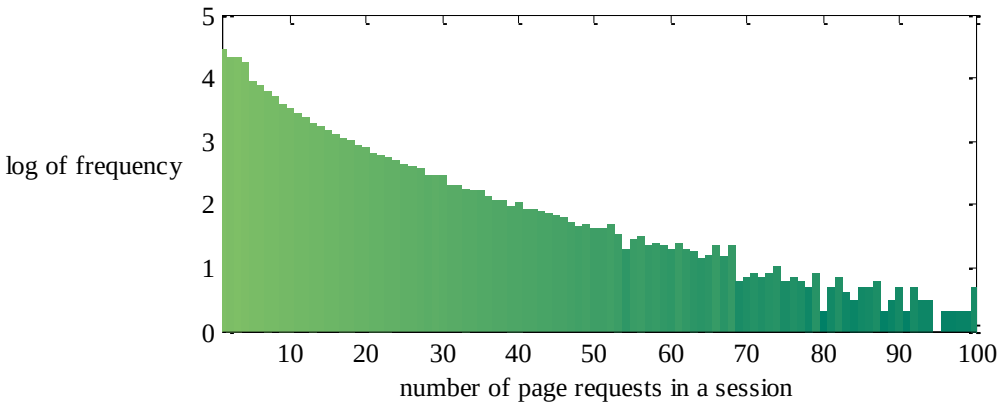

Figure 1 Histogram of sessions by page request counts. Vertical axis shows the base 10 logarithm of session frequencies by the page request counts (horizontal axis).

Next, we are able to calculate probabilities that user starts a session in a particular state. It could be used as an indicator of where the user came to the pages from or as a weighted vector for next calculations, as we do so later.

3 Methodology

3.1 Exposure criteria

In general, we can distinguish two different metrics of the exposure of users to the banners placed on a website:

- total *number of pages* containing the banner visited by a user;
- total *duration* spent by a user on pages containing the banner.

Depending on the selected metric, different modeling frameworks need to be used. In the former case, we employ a discrete-time Markov chain (DTMC) that accounts for any page requests, both inter- and intra-category. In the latter case, we use a continuous-time Markov chain (CTMC) model that does not treat intra-category page requests as state transitions, since the focus is on the durations spent in individual categories, not the number of different pages visited within a category. (This reduced the number of observed transitions to 482,606.)

[2] The website owner requested that we do not disclose the name of the website.

3.2 Discrete-time model

In the discrete-time model, we employ a DTMC, the steps of which represent individual page requests. Therefore, duration in terms of time spent on individual pages is not treated in this model. The states of the DTMC indicate the current category of the website that is being viewed by the user. Recall that a DTMC is fully described by its transition matrix, conventionally denoted $\mathbf{P} = [p_{ij}]$. Here, the transition probabilities are

$$p_{ij} = \Pr\{\text{next page visited in category } j \mid \text{current page in category } i\}.$$

Transition probabilities are calculated as relative frequencies of the transitions from the respective state observed in our data. As shown in [8], this is a consistent estimator of transition probabilities.

Each session terminates in an absorbing state (state 9, *end of session*, see below). Therefore, our DTMC model describes an *absorbing Markov chain*. Thus we're also interested in its *fundamental matrix* denoted as $\mathbf{N} = [n_{ij}]$, where n_{ij} represents the average number of a user's visits to a particular category j, given that he/she starts the session in category i. The fundamental matrix is calculated as

$$\mathbf{N} = (\mathbf{I} - \mathbf{Q})^{-1}, \tag{1}$$

where $\mathbf{Q}$ denotes the matrix formed by those rows and columns from $\mathbf{P}$ that correspond with transient (i.e. non-absorbing) states, and $\mathbf{I}$ is the identity matrix of appropriate rank. For a more detailed description of a fundamental matrix and calculations outlined above, see [8], [16].

3.3 Continuous-time model

CTMC's are most conveniently described by the so-called *generator matrix* (also *intensity matrix*, *q-matrix*), here denoted $\mathbf{Q} = [q_{ij}]$, defined as (see e.q. [8], [16])

$$q_{ij} = p'_{ij}(0_+) = \begin{cases} \lim_{t\to 0+} \dfrac{p_{ij}(t)}{t} & \text{if } i \neq j, \\ \lim_{t\to 0+} \dfrac{p_{ii}(t)-1}{t} & \text{if } i = j. \end{cases} \tag{2}$$

The transition matrix for time interval of length t, $\mathbf{P}(t)$, can be recovered from $\mathbf{Q}$ using the fact $\mathbf{P}(t) = \exp(t\mathbf{Q})$, where exp is the matrix exponential function. We estimated the $\mathbf{Q}$ matrix as follows. First, we estimated diagonal elements using the fact that time spent in the current state of a CTMC before the first transition has the exponential distribution with parameter $-q_{ii}$, see e.g. [16]. Therefore, the expected time to first transition in each state equals $-1/q_{ii}$, and thus q_{ii} is consistently estimated using the mean of the observed duration times for state i, $\bar{d}_i$:

$$\hat{q}_{ii} = -1/\bar{d}_i. \tag{3}$$

Next, we employed the following property of the $\mathbf{Q}$ matrix (see [8]): the probability that the first transition from the current state i leads to the state j is $-q_{ij}/q_{ii}$. These probabilities are again consistently estimated using the corresponding relative frequencies in the sample, and multiplying by $-\hat{q}_{ii}$ gives the estimate $\hat{q}_{ij}$.

In order to calculate the exposure duration, one needs to know both the mean duration spent in the individual states and the average number of times these states are entered. The latter can be calculated using the so-called *jump chain* associated with the studied CTMC. The jump chain is itself a discrete-time Markov chain each step of which is represents one transition of the original CTMC (no matter how long it actually took for this transition to realize). The transition matrix of the jump chain (referred to as the *jump matrix*). The main difference between the jump chain of the CTMC model and the DTMC from section 3.2 is that in the jump chain, the intra-category transitions are not accounted for. In other words, the diagonal entries in the jump matrix are zero in all rows representing non-absorbing states. The off-diagonal elements in these rows represent the first-transition probabilities calculated above as $-q_{ij}/q_{ii}$.

4 Results

4.1 Discrete-time model

Since the work of Pitkow and Pirrolli [15] was published in 1999, discrete-time Markov chains models have often been used to model clickstream data, but to the best of our knowledge, not as a metric for a pricing model for advertising. Contrary to Pitkow and Pirolli's model, where each page was supposed as a unique state, we

follow the approach of Montgomery et al. [13], where the pages are classified into several categories to define the state space of Markov chain. Our categories were chosen based on the design and hierarchy of the website. In accordance with the purpose of our analysis, we were interested just in such pages where banner ad space is offered. The resulting categories are evident from the transition matrix in Table 1.

		1	2	3	4	5	6	7	8	9
1	User login	.0556	.5621	.0287	.1827	.0027	.0012	.0101	.0030	.1539
2	Region catalogue	.0578	.2452	.1367	.2476	.0035	.0020	.0061	.0061	.2951
3	Product catalogue	.0023	.0341	.6993	.1826	.0073	.0032	.0002	.0003	.0707
4	Product view	.0054	.0222	.0363	.3616	.4387	.0085	.0104	.0189	.0981
5	Product details	.0035	.0275	.0479	.2145	.0428	.4359	.0312	.0423	.1544
6	Product more details	.0074	.0425	.0511	.2918	.0316	.0770	.0527	.0673	.3786
7	Product discussion	.0107	.0562	.0076	.5261	.0219	.0168	.1293	.0177	.2137
8	Product ext. link	.0037	.0378	.0084	.5476	.0393	.0341	.0137	.0737	.2417
9	End of session	0	0	0	0	0	0	0	0	1

Table 1 Transition matrix for the DTMC model

Firstly, let us use the transition matrix to answer the question about average number of pages containing the banner that are visited by a user in one session. For this purpose, we calculated the fundamental matrix of the DTMC by using (1). To obtain average amount of page requests in one category made by a user in one session, the fundamental matrix was left-multiplied by a vector of probabilities that user starts a session in a particular state (these probabilities were estimated using relative frequencies of initial states, see Table 2).

	1	2	3	4	5	6	7	8
Relative frequency (%)	3.74	39.72	2.13	52.35	0.86	0.16	0.50	0.54
Avg. clicks	0.115	0.814	1.009	2.311	1.106	0.559	0.117	0.152

Table 2 Relative frequencies of initial categories and average page requests made by a user in a session

The average page request calculation can be decomposed in levels; each provides useful information that could be of certain interest to advertisers. Let us consider that in more details. What does the transition matrix say about customers' browsing habits and what does it mean for banner ads effectiveness?

The region catalogue (RC) as well as the product view (PV) could be considered as a kind of a homepage. Discount vouchers are placed there regardless of the topics they refer to (such as travelling, courses etc.). The only difference between the two categories is that RC displays only discounts from a particular region, indicated by the user's hometown. The homepage status of the categories corresponds with the fact that 52 % of users start their sessions with the product view and 39 % with the region catalogue. If the user remains in the same category (RC or PV) after one click, it means that he requested a next list of discounts at the same page. As we can see from the transition matrix, a lot of customers do so (24 % in the case of RC, 36 % PV, see the elements p_{22}, p_{44}). While in the case of the PV category 44 % of customers click on one displayed discounts to see more details, in the case of the RC category almost 30 % of customers just see the first page of discounts briefly and finish the session (see the element p_{45} and p_{29}).

The product catalogue (category no. 3) provides discounts ordered according to the particular topics such as travelling, courses, holidays, culture etc. The low values in the third column can indicate that customers' browsing is not directed in most cases. This aspect could have an important impact on the recognition and recall scores for ads, since some papers suggest that recall and recognition are higher when subjects are browsing aimlessly [14]. In other words, there is an assumption that the aimlessness of customers' browsing influences the efficiency of banner ads in a positive way. The average clicks calculated based on the transition matrix and shown in Table 2 tell us that a typical user visiting the website sees banner ads placed in the PV category roughly twice a session, and banner ads in the product catalogue category as well as the category with details about a particular product are viewed once per an average session. The important aspect of this basic statistic lies in the connection with browsing habits described by a typical customer's path.

4.2 Continuous-time model

In this section, we focus on time spent by a user in a particular category during a one session. This information could provide another perspective to the exposure effect.

		1	2	3	4	5	6	7	8	9
1	User login	-1.3526	.8051	.0411	.2617	.0039	.0017	.0145	.0043	.2204
2	Product catalogue	.0408	-.5329	.0965	.1748	.0025	.0014	.0043	.0043	.2083
3	Region catalogue	.0056	.0824	-.7266	.4412	.0176	.0077	.0005	.0007	.1708
4	Product view	.0093	.0383	.0627	-1.1027	.7577	.0147	.0180	.0326	.1694
5	Product details	.0032	.0251	.0438	.1961	-.8753	.3986	.0285	.0387	.1412
6	Product more details	.0024	.0138	.0166	.0949	.0103	-.3002	.0171	.0219	.1231
7	Product discussion	.0056	.0292	.0039	.2732	.0114	.0087	-.4522	.0092	.1110
8	Product ext. link	.0018	.0185	.0041	.2678	.0192	.0167	.0067	-.4531	.1182
9	End of session	0	0	0	0	0	0	0	0	0

Table 3 The generator matrix

To calculate a generator matrix, we employed the method described in section 3.3. The resulting q-matrix **Q** is shown in Table 3. The off-diagonal elements in a particular row are proportional to probabilities that a particular category will be visited in next step, given that the current state corresponds with the row's heading. The diagonal elements represent the rate parameter for an exponential distribution of the waiting time in the given row; in other words, they refer to average time spent by a user in a particular category in a single visit to that category.

In order to be able to evaluate a category's exposure duration in total, we need to know how many times the category is visited. For this purpose, we calculate (based on the simple calculation stated in section 3.3) the fundamental matrix of the jump chain. This fundamental matrix can be interpreted in a similar way as the fundamental matrix in the case of the DTCM model; its elements refer to the average number of visits to a category made by a user in one session. However, there is one important difference: intra-category transitions are not included in this case. To find out the average total exposure duration during a session conditional on the initial state, it suffices to multiply all columns of the fundamental matrix by the average times spent in the categories.

	1	2	3	4	5	6	7	8
Avg. time (minutes)	0,911	4,031	1,275	1,690	0,884	1,285	1,263	1,224

Table 4 Average time spent by a user in a particular category during a session

As shown in Table 4, users spent (on average) the largest portion of time is spent viewing the region catalogue – 4 minutes is much more than the time spent in any other category.

There is an important question concerning the functional form of the impact that exposure duration has on banner ad effectiveness. There are many works trying to specify this function form ([11], [9], [7]). It is widely agreed that this impact not constant over time. For instance, some works propose an inverted u-shape function [1], implying that at a certain exposure level, the "annoying effect" comes to pass; in this case, long exposure times may led to a decrease in the banner ad's efficiency. In general, the function that models the effect of exposure duration should probably be plugged into the calculation of the exposure criterion for pricing. We would like to address this issue in more detail in our future research.

5 Conclusions

In our paper, we described methods that can be used to analyze real customer-based clickstream data. By using a Markov chain approach, we provided an insight into customers' browsing habits, and calculated some statistics, which could be used as a tool for the pricing of advertisements. Contrary to the conventional pricing techniques based on click-through rate or the value of cost-per-thousand, our statistics refer to typical behavior of a typical customer during one session. Moreover, we showed that the Markov chain model allows us to con-

sider some psychological aspects influencing customers' recall of a banner ad. To conclude, we find the Markov chain approach as an appropriate tool for evaluating the "websites' attractiveness" from an advertisers' point of view, and thus an appropriate way for pricing banner advertisements.

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On parametric bootstrap and alternatives of MSE

Tomasz Żądło[1]

Abstract. Mean squared error (MSE) is the key statistics used in survey sampling to measure accuracy. What is more, it is widely accepted to call an estimator or predictor with smaller MSE as better. We propose some alternatives of MSE which may be use to compare accuracy instead of MSE or to give additional information on accuracy. Parametric bootstrap estimators of these statistics are also proposed. In the simulation study based on the real data from Polish Agricultural Census the problem of accuracy of some predictors of subpopulation total is studied. The biases and MSEs of the parametric bootstrap estimators of the proposed statistics are compared with parametric bootstrap estimator of root MSE (RMSE).

Keywords: parametric bootstrap, small area estimation, mean square error

JEL Classification: C83
AMS Classification: 62D05

1 Introduction

In many applications of statistics and econometrics the properties of different methods are compared based on statistics which measure some type of variability such as accuracy, risk or goodness-of-fit. For example, to compare goodness of fit of two models, we can use among others coefficient of determination, residual variance, Bayesian Information Criterion or Akaike's Information Criterion (see e.g. [7]). To compare risks of two portfolios, we can use among others variance, semivariance, value at risk or conditional value at risk. To compare prediction accuracy, mean squared error of prediction is mainly used as in [10] or [13] but its modifications such as heteroskedasicity adjusted mean squared error are also considered for example in [5]. But in the case of both design-based and model-based approaches in survey sampling the accuracy of estimation or prediction is compared mainly based on mean squared error as considered in [11]. For example,

Definition 1. (Cassel at al. [2], p. 39) *For a given design p, an estimator* $t_1(\mathbf{y})$ *is said to be at least as good as another estimator* $t_2(\mathbf{y})$ *if* $MSE(t_1(\mathbf{y})) \leq MSE(t_2(\mathbf{y}))$ *for all* $\mathbf{y} \in R_N$. *If, in addition, strict inequality holds for at least one* $\mathbf{y}$, *then* $t_1(\mathbf{y})$ *is said to be better than* $t_2(\mathbf{y})$.

If both sampling design and superpopulation model are taken into account, MSE is also used to compare accuracy (Cassel at al. [2], p. 93). Alternative definitions of accuracy in survey sampling are rare in the literature but are considered in e.g. [3].

What is important the MSE is the mean of squared errors which usually have strong positively-skewed distribution (see the distribution of modulus of prediction errors for different predictors on the Figure 1 and note that the skewness of the distribution of squared errors is stronger). This is the reason that we should be interested not only in the mean. For example in the case of estimating indexes the whole distribution of the statistics may be of interest as studied in [1].

2 Accuracy of prediction

In the paper the problem of measuring accuracy will be considered for model approach in survey sampling based on the quantiles of the distribution of absolute prediction errors. From the population Ω of size N a sample s of size n is drawn (not necessarily at random). We assume that population data obey assumptions of the general linear mixed model:

$$\begin{cases} \mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{v} + \mathbf{e} \\ E(\mathbf{e}) = \mathbf{0} \\ E(\mathbf{v}) = \mathbf{0} \\ D^2\begin{bmatrix} \mathbf{v} \\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \end{cases} \quad (1)$$

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where $\mathbf{Y}=\left[Y_1,Y_2,...,Y_N\right]^T$, $\mathbf{X}$ and $\mathbf{Z}$ are $N\times p$ and $N\times h$ respectively matrices of values auxiliary variables, $\boldsymbol{\beta}$ is a $p\times 1$ vector of unknown parameters, $D^2(\mathbf{Y})=\mathbf{V}=\mathbf{R}+\mathbf{ZGZ}^T$ is a variance-covariance matrix which depends on some unknown in practice variance parameters $\boldsymbol{\delta}=\left[\delta_1,...,\delta_q\right]^T$. Let $\mathbf{Y}=\left[\mathbf{Y}_s^T \quad \mathbf{Y}_r^T\right]^T$ where $\mathbf{Y}_s$ is $n\times 1$ vector for sampled elements and $\mathbf{Y}_r$ is $(N-n)\times 1$ vector for population elements which are not in sample.

Let $\hat{\theta}=\hat{\theta}(\mathbf{Y}_s)$ be a predictor of a function of random variables $\theta=\theta(\mathbf{Y})$, and let $U=\hat{\theta}-\theta$ be a prediction error. For this setup, Mean Squared Error is defined as $MSE(\hat{\theta})=E(\hat{\theta}-\theta)^2=E(U^2)$. We propose an alternative of the MSE given by *p*-th quantile of absolute prediction error:

$$Q_p(|U|)=Q_p(|\theta-\hat{\theta}|) \tag{2}$$

To estimate (2) we propose to use parametric bootstrap (see [4] and [6]). The procedure will be as follows:

(i) based on the sample we estimate parameters of the model (1) using REML method under normality and we obtain estimates $\hat{\boldsymbol{\beta}}$, $\hat{\boldsymbol{\delta}}$,

(ii) then, we construct bootstrap superpopulation model: $\mathbf{Y}^*=\mathbf{X}\hat{\boldsymbol{\beta}}+\mathbf{Z}\mathbf{v}^*+\mathbf{e}^*$, where $\mathbf{v}^*\sim N(\mathbf{0},\mathbf{G}(\hat{\boldsymbol{\delta}}))$, $\mathbf{e}^*\sim N(\mathbf{0},\mathbf{R}(\hat{\boldsymbol{\delta}}))$,

(iii) based on *B* realizations of $\mathbf{Y}^*$ denoted by $\mathbf{Y}^{*(b)}$, where *b*=1,...,*B*, we compute *B* values of $\hat{\theta}^*=\hat{\theta}(\mathbf{Y}_s^*)$ which will be denoted by $\hat{\theta}^{*(b)}=\hat{\theta}(\mathbf{Y}_s^{*(b)})$ (based on the sample with the same indices as in the original population) and *B* values of $\theta^*=\theta(\mathbf{Y}^*)$ denoted by $\theta^{*(b)}=\theta(\mathbf{Y}^{*(b)})$,

(iv) finally, we compute parametric bootstrap estimator of (2) as *p*-th quantile of absolute prediction error in the bootstrap distribution:

$$\hat{Q}_p(|U|)=Q_p(|U^*|)=Q_p(|\hat{\theta}^*-\theta^*|). \tag{3}$$

In the next paragraph the problem will be specified for small area estimation problem even if the normality assumed in (ii) is not met.

3 Special case

In the paragraph, we will introduce from theoretical point of view the problem which will be considered in the simulation study. Let the population be divided into *D* subpopulation (domains) each of size N_d (where $d=1,2,...,D$). Let the sample in the *d*-th domain be denoted by $s_d=\Omega_d\cap s$ and its size by n_d. The domain of interest is denoted by Ω_{d*}. We consider the problem of prediction of subpopulation totals $\theta_d=\sum_{i\in\Omega_d}Y_i$ (*d*=1,2,...,*D*). We assume the following random coefficient model as in [8], which is special case of (1):

$$Y_{id}=\beta_d x_{id}+e_{id}=\beta x_{id}+v_d x_{id}+e_{id} \tag{4}$$

where $i=1,2,...,N$, $d=1,2,...,D$, $\beta_d=\beta+v_d$, $v_d\overset{iid}{\sim}(0,\sigma_v^2)$, $e_{id}\overset{iid}{\sim}(0,\sigma_e^2)$. Let us introduce three predictors of subpopulation total $\theta_d=\sum_{i\in\Omega_d}Y_i$ (their MSEs were studied earlier in [12]) to consider the problem of measuring prediction accuracy based on alternatives of MSE. Empirical Best Linear Unbiased Predictor (EBLUP) under (4) is given by (see [12]):

$$\hat{\theta}_d^{EBLUP}=\sum_{i\in s_{d*}}Y_i+\hat{\beta}\left(\sum_{i\in\Omega_{rd*}}x_i\right)+\hat{b}_{d*}^{-1}\hat{\sigma}_v^2\left(\sum_{i\in\Omega_{rd*}}x_i\right)\left(\sum_{i\in s_{d*}}x_i^2\right)\left(\hat{\beta}_{d*}-\hat{\beta}\right) \tag{5}$$

where $\hat{b}_d=\hat{\sigma}_e^2+\hat{\sigma}_v^2\sum_{i\in s_d}x_i^2$, $\hat{\sigma}_e^2$ is restricted maximum likelihood estimator σ_e^2, $\hat{\sigma}_v^2$ is restricted maximum likelihood estimator σ_v^2, $\hat{\beta}=\left(\sum_{d=1}^{D}\hat{b}_d^{-1}\sum_{i\in s_d}x_iY_i\right)\left(\sum_{d=1}^{D}\hat{b}_d^{-1}\sum_{i\in s_d}x_i^2\right)^{-1}$ and

$$\hat{\beta}_{d*}=\left(\sum_{i\in s_{d*}}x_iY_i\right)\left(\sum_{i\in s_{d*}}x_i^2\right)^{-1} \tag{6}$$

We also study properties of the indirect predictor (see [12]):

$$\hat{\theta}_d^{IP}=\sum_{i\in s_{d*}}Y_i+\tilde{\beta}\sum_{i\in\Omega_{rd*}}x_i \tag{7}$$

where $\tilde{\beta} = \left(\sum_{i \in s} x_i^2 \right)^{-1} \sum_{i \in s} x_i Y_i$, and direct predictor given by (see [12]):

$$\hat{\theta}_d^{DP} = \sum_{i \in s_{d*}} Y_i + \hat{\beta}_{d*} \sum_{i \in \Omega_{rd*}} x_i \tag{8}$$

where $\hat{\beta}_{d*}$ is given by (6).

In the next paragraph we will study the accuracy of predictors (5), (7) and (8) in the sense of MSE as well as quantiles of distribution of absolute prediction errors.

4 Simulation analysis

The simulation prepared in R (see [9]) is based on the real data on *N*=1921 farms from 3 municipalities (Bolesław, Gręboszów and Mędrzechów) from Polish Agricultural Census in 1996. The population is divided into *D*=31 cities and villages. From the population stratified random sample is selected of size *n*=192 with approximately proportional allocation among *D*=31 subpopulations. The variable of interest is the crops area (in ha) while the auxiliary variable is the total area of farms (in ha). We assume the model (4) and two cases:

- N case - normality of random components v_d and e_{id} ,
- E case - shifted exponential distribution of random components with the same variances as in N case

Number of iteration equals 1000, in each step of simulation B=500 bootstrap realizations of bootstrap distributions are generated. Three predictors presented in the previous paragraph are studied: EBLUP given by (5), IP given by (7) and DP given by (8). Simulated distributions of absolute prediction errors of the predictors together with values of root mean square errors (RMSE) are presented on Fig. 1. Let us compare RMSEs of these predictors for N and E case (presented on the Fig. 1 by symbol "■") for one out of *D*=31 subpopulations. RMSEs of DP, EBLUP and IP for N case are 1887.7 ha, 1761.7 ha, 4007.5 ha respectively while for E case: 1928.9 ha, 1788.4 ha, 4040.7 ha respectively. Differences between RMSEs for these cases are small but the differences in distributions of absolute prediction errors presented on the Fig. 1 are larger.

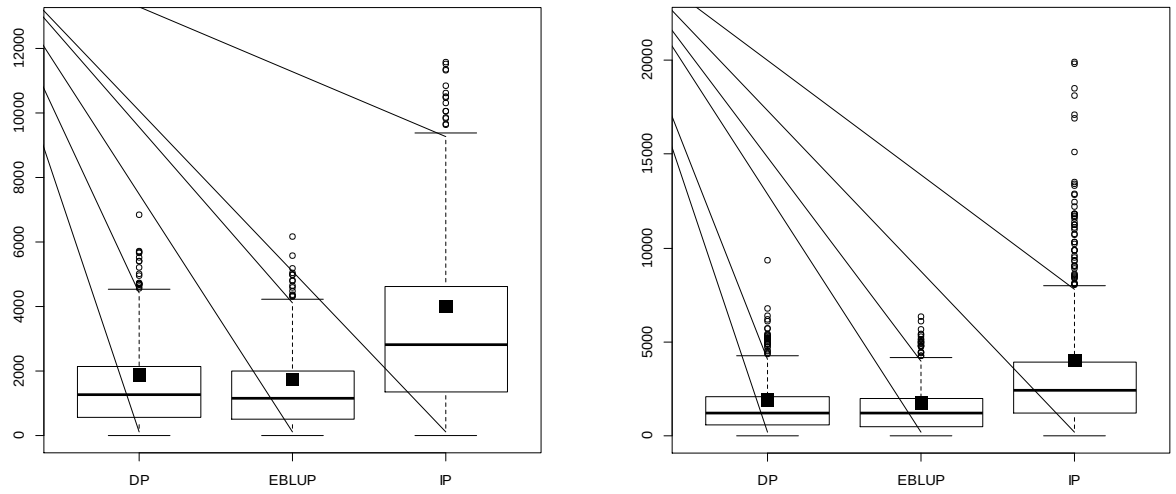

Figure 1 Distribution of absolute prediction errors (in ha) of three predictors of subpopulation totals in Bolesław city for N (on the left) and E (on the right) cases (■ – values of RMSEs)

In the simulation RMSEs and 0,5-th, 0,75-th, 0,9-th and 0,95-th quantiles of absolute prediction errors denoted by MeAE, Q75AE, Q90AE and Q95AE of three predictors are computed. Based on the statistics we have compared accuracy of three predictors for *D*=31 domains. EBLUP has smaller RMSEs than both ID and DP for all of domains, and DP has smaller RMSEs than IP for both N and E case. Then we have defined better predictor as the one with smaller appropriate quantile of absolute prediction error. Comparing accuracy of these 3 predictors for 2 cases (N and E) for *D*=31 domains based on 4 quantiles of absolute prediction errors ($3 \times 2 \times 31 \times 4 = 744$ comparisons) in almost 99% cases (in 735 out of 744 cases) we obtain the same conclusions as in the case of accuracy comparisons based od RMSEs.

Although for the studied data MeAE, Q75AE, Q90AE and Q95AE give similar results in accuracy comparisons, they allow to obtain more information about accuracy (see the Fig. 2 for EBLUP) because they inform not only on average value of prediction errors but also on quantiles of absolute prediction errors. On the Figure 2 we present distributions over domains of MeAE, Q75AE, Q90AE and Q95AE where lines match values for the same subpopulations. For example, the biggest relative RMSE, MeAE, Q75AE, Q90AE, Q95AE for N case are obtained for the same subpopulation (the values are matched by one line on the left part on the figure 2) and equal 8.6%, 5.8%, 9.5%, 13.7%, and 16.8%. It means that we know more about the accuracy than based only on the value of RMSE.

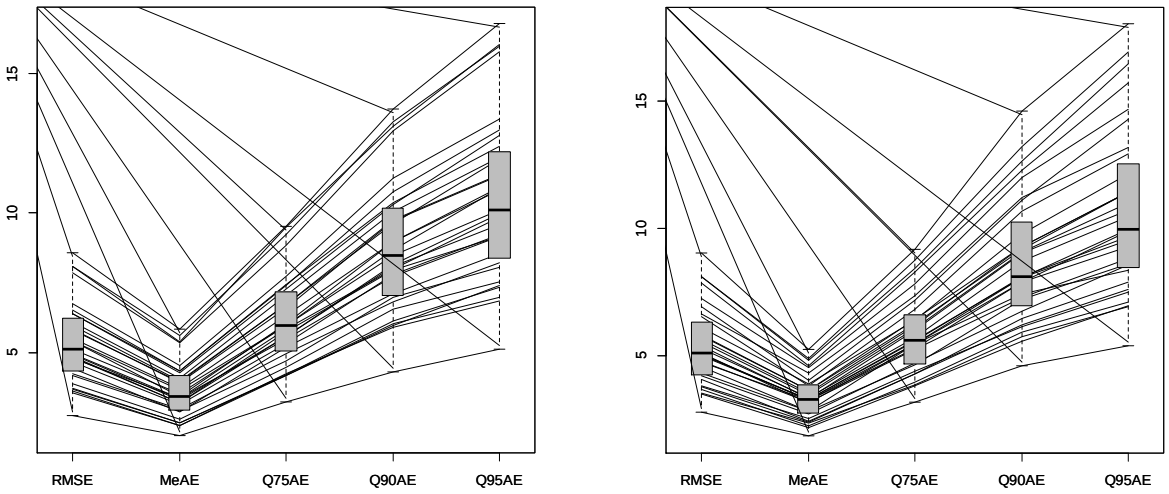

Figure 2 Relative accuracy (in %) of EBLUP for N (on the left) and E (on the right) cases for *D*=31 domains

What is more, in the simulation we study properties of MeAE, Q75AE, Q90AE and Q95AE parametric bootstrap estimators based on the normality assumption (see left parts of Fig. 3 and Fig. 4). To check the performance of the these parametric bootstrap estimators for the misspecified model, we use them assuming that normality of random components holds while the true distribution is shifted exponential (see right parts of Fig. 3 and Fig. 4).

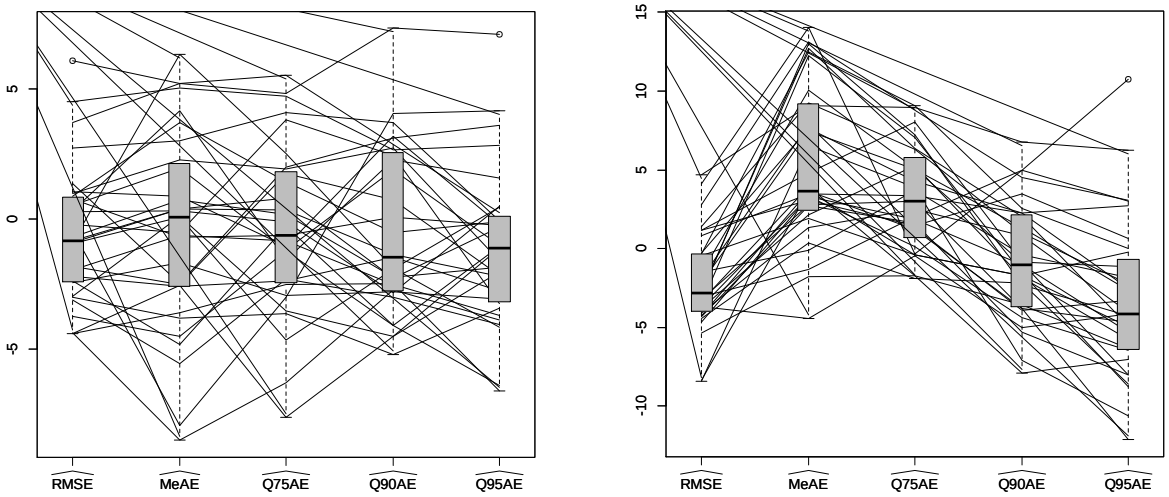

Figure 3 Relative biases (in %) of accuracy estimators of EBLUP for N (on the left) and E (on the right) cases for *D*=31 domains

Distributions over subpopulations of relative biases of accuracy estimators for N cases are around zero. In the case of model misspecification (for E case) biases are not large. Estimators of RMSE and Q95AE on average underestimates while estimators of MeAe, Q75AE and Q90AE overestimates on average true values. On the Fig. 4 accuracy of parametric bootstrap estimators of RMSE, MeAe, Q75AE, Q90AE and Q95AE. Values are quite similar but on average estimators of RMSE are more accurate than estimators of MeAe, Q75AE, Q90AE and Q95AE.

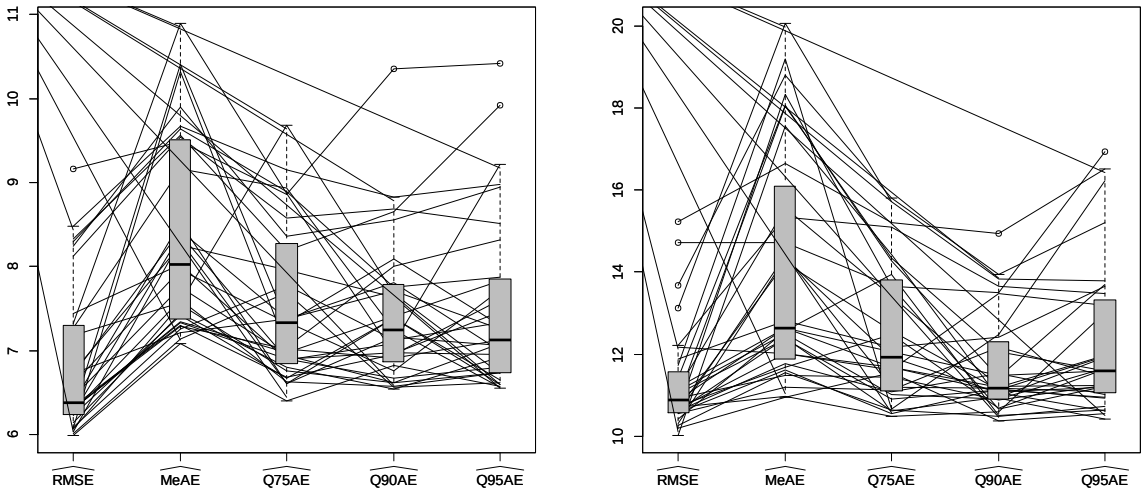

Figure 4 Relative RMSEs (in %) of accuracy estimators of EBLUP for N (on the left) and E (on the right) case *D*=31 domains

5 Conclusions

In the paper model approach in survey sampling is considered. Indicators of accuracy alternative to mean squared error are proposed together with their parametric bootstrap estimators. The accuracy of the estimators of the proposed accuracy indicators is analyzed in the Monte Carlo simulation study based on the real data from Polish Agricultural Census.

It is shown that the proposed alternatives of the MSE defined as quantiles of absolute prediction errors give additional information on the accuracy of the considered small area predictors. It allows to describe the accuracy defined by distribution of prediction errors more precisely comparing with root of mean of squared errors. Moreover, it is shown in the simulation study that parametric bootstrap estimators of the proposed statistics have small relative biases and similar accuracies comparing with the parametric bootstrap MSE estimator. We hope that the proposed statistics will be used by statisticians, especially survey statisticians, together with MSE to give detailed information on accuracy of predictors in business practice.

Acknowledgements

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The impact of exchange rate changes on inflation in the V4 countries in the process of economic transition

Libor Žídek[1], Daniel Němec[2]

Abstract. Our contribution focuses on the role of the exchange rate changes in the V4 countries during the transition process towards a market economy. Regarding the variety of exchange rate regimes implemented in the V4 countries at the start of the economic transition, we are especially interested in the degree of exchange rate pass-through to the domestic inflation in these countries. The respective countries followed different exchange rate strategies. The fixed exchange rate regime was applied in Czechoslovakia. Crawling peg was used in Hungary and Poland. And floating and fixed exchange rate with large band were applied in the new century. We have compared the impacts of the different exchange rate regimes on the price stability during the transformation process. The effects were examined using country specific SVAR models and corresponding historical shock decompositions. We found out that the exchange rate indeed played important role in the price stability but the specific impact is highly individual.

Keywords: V4 countries, exchange rate pass-through, structural vector autoregression model, economic transition, inflation.

JEL Classification: C22, F31

AMS Classification: 91B84

1 Introduction

The Central European countries decided to follow different exchange rate strategies during the transformation process. Our goal is to detect the impact of the different exchange rate regimes on the inflation development in the V4 countries (the Czech Republic, the Slovak Republic, Hungary and Poland) in the period between the beginning of 1993 and the end of 2007. The first date was selected due to availability of data and the second due to the start of the financial crisis that would affect the overall results. We suppose that the different exchange rate strategies had different impacts on inflation development. Foremost we expect that the fixed exchange rate should have only a limited impact on inflation – indeed, this type of exchange rate should stabilize it. The crawling peg system should be inflationary because small devaluations are supposed to cause growth in imported prices. And we expect that the impact of the floating regime will be ambiguous because there are usually periods of both appreciation and depreciation. The countries applied different regimes in different times. We split the overall analyzed period into subperiods according to the applied exchange rate regime. And we use the country specific Structural Vector Autoregression models (SVAR models) to estimate structural shocks. Using SVAR simulation procedures we are able to decompose the development of inflation into the specific structural shocks contribution. This method allows us to evaluate the impact of the historical exchange rate shocks on inflation. Our approach differs from those one presented by Mirdala [4], [5] due to fact that he has evaluated the exchange rate pass-through to domestic prices using the forecast error variance decomposition. He was thus not able to observe the real historical consequences of the exchange rate changes in the first half of the 1990s. Although the SVAR approach assumes possible interdependencies among all of the modeled endogenous variables, we will focus on one sided causality from nominal exchange rates changes to price level changes

2 Exchange rates in transformation process

In this section we first of all describe the different exchange rate regimes and consequently write about their application in the transformation process. The Central European countries used generally three different exchange rate regimes. The first was the fixed exchange rate regime. There is no general agreement on the definition of the fixed exchange rate because definitions diverge about the existence and the size of the potential fluc-

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tuation zone. [3] We will consider the exchange rate as fixed even if there was a fluctuation zone. Keeping a fixed exchange rate was the main goal of the respective central bank in the specific period and it was supposed that a stable exchange rate should stabilize inflation pressures at the same time. The flip side of this exchange rate system was a parallel real appreciation of the currency. The other con is that the fixed exchange rate regimes are in the long run unstable and can lead to currency crisis as in the Czech Republic in 1997. The second exchange rate regime – crawling peg – tried to avoid or soften this real appreciation by small periodic monthly devaluations of the currency. These devaluations were supposed to balance the impact of higher growth in prices in the domestic economy. The positive side is transparency for the market subjects. The flip side of this regime should be inflation that is caused by devaluations. It means that this exchange rate regime tries to soften real appreciation but it causes it as the side effect of devaluations. The third regime that was applied later on in the period is floating. There should not be any general impact of the regime on inflation. The depreciation of the exchange rate should have of course an inflationary impact and contrary wise.

What was the specific development of the exchange rate regimes? We will consider only the main changes in the development. The reformers decided for the fixed exchange rate regime at the beginning of the transformation process in the Czech Republic (Czechoslovakia). A small fluctuation zone (+/- 0.5%) was introduced in May 1993. And the zone was widened in February 1996 to +/- 7.5 %. The fixed exchange rate system collapsed in May 1997 in the currency crisis. The managed floating has been in use since that time which is close to de facto free floating. The trend in this period was towards nominal appreciation. Generally, the real exchange rate was appreciating for the whole period Frait, Komárek [1]. We can see the development of both nominal as well as real exchange rate in the following charts – Figure 1.[4]

Slovakia had the same basis of the exchange rate as the Czech Republic. But the Slovak crown was allowed to move in a small fluctuation zone +/- 1.5 since February 1993 and the currency was devaluated by 10 % in July of that year. In 1996 and 1997, the fluctuation zone was widened in three steps to +/- 7 %. And the currency regime changed to managed floating in October 1998. The country became a member of ERM II in October 2005. The country joined the Eurozone after two revaluations at the beginning of 2009. The overall trends in the nominal and real exchange rates were similar to the development of the Czech crown.

Figure 1 Development of the nominal and real exchange rates (index, 1993M1 = 100, indirect quote)

Hungary used the fixed exchange rate regime at the beginning of the transformation process. But the exchange rate was irregularly devaluated and the fluctuation zone gradually widened from +/- 0.3 % in July 1992 to +/-2.25 % in December 1994. The overall exchange rate regime was changed to crawling peg in March 1995 after a 9 % devaluation. Regular devaluations in the system gradually decreased till October 2001, when the central bank decided for the fixed exchange rate system with a fluctuation zone +/- 15 %. The system changed to managed floating only in May 2008. We can see in the following chart that the nominal exchange rate signifi-

[3] For instance the term fixed exchange rate is in the Czech literature used even in situation of broad fluctuation zone +/- 7.5 % (for example Singer [6]). On the contrary the definition of the IMF says that fixed exchange rate is only within margins +/- 1 % (see for example [2]).

[4] The data source is International Financial Statistics (IFS) provided by the International Monetary Fund. The observed variables for all V4 countries are real and nominal effective exchange rate (index).

cantly depreciated during the 1990s and consequently oscillated. The real exchange rate appreciated in the first years but then it was highly stable during the crawling peg period. Since 2003, the trend was again towards real appreciation.

Poland applied the fixed exchange rate regime at the beginning of the transformation process Žídek [4] . It was changed to the crawling peg regime at the end of 1991. Consequently there were additionally irregular devaluations of Zloty. The monthly devaluations connected with crawling peg were gradually declining. Meanwhile, the fluctuation zone was widened up to +/- 15 % in March 1999. Zloty has been in the regime of free floating since April 2000. The currency significantly nominally depreciated in the 1990s. Consequently, there were periods of appreciation and depreciation. In real terms Zloty appreciated nearly for the whole period regardless of the exchange rate regime.

3 Data and methodology

The main purpose of our paper is to estimate the impact of exchange rate changes on domestic inflation. To study the determinants of the domestic price level changes in the economy we use the methodology proposed by Ito and Sato [3] tuned in a way to incorporate the particularities of the V4 transition economies. In order to examine the sources of inflation in the transition period, we will use a framework that includes the possibility of mutual interactions of the key macroeconomic variables. Although the main question is to reveal the direct influence of the exchange rate on domestic inflation, it is also possible that domestic inflation may affect the exchange rate. This type of causality will be thus examined using the vector autoregression (VAR) analysis which is a useful approach allowing interaction between the exchange rate and domestic macroeconomic variables.

Our VAR analysis starts with an estimation of a reduced form model. Obtained residuals are then transformed to the structural shocks using Cholesky decomposition. Effects of structural shocks of various macroeconomic variables on inflation are thus investigated using the structural VAR model (SVAR model). Identified structural shocks are incorporated into simulation exercises that will decompose the actual trajectories of our observed variables into the particular shock components. The resulting historical shock decomposition points out the main factors standing behind the development of inflation in the transition period.

Our VAR model is set up using the vector of five endogenous variables[5]:

$$x_t = \left(\Delta OIL_t, gap_t, \Delta m_t, \Delta NEER_t, \Delta p_t\right) \tag{1}$$

where ΔOIL_t denotes changes in the natural log of oil prices, gap_t is the output gap, Δm_t denotes changes in the natural log of money supply, $\Delta NEER_t$ is the growth rate of nominal effective exchange rate and Δp_t is the difference of the natural log of the price index.

The data set used for estimation is from January 1993 to December 2007. The data source is International Financial Statistics (IFS) provided by the International Monetary Fund. The observed variables for all V4 countries are as follows:

- IPM: Index of industrial production, 2005=100;
- CPI: Consumer prices index, all items, 2005=100;
- PPI: Producer prices index, all commodities, 2005=100;
- M1: Monetary Aggregate M1, national currency;
- OIL: Crude oil prices (Brent – Europe), dollars per barrel;
- NEER: Nominal effective exchange rate (index), 2005=100, direct quote (increase means appreciation);

All variables (except NEER) were seasonally adjusted using the X12-ARIMA procedure. After seasonal adjustment, the variables were transformed using logarithmic transformation. Unit root tests proved the existence of unit roots in all variables. Except IPM, the variables were expressed in growth rates terms (i.e. logarithmic differences). Variables IPM were filtered using Hodrick-Prescott filter with the smoothing constant $\lambda = 14400$ $\lambda = 14400$. This procedure resulted in the corresponding gap_t variable. These transformations led to stationary variables. As a monetary policy variable (money supply), the base money (M1) is used. As price index variable, we used CPI in our final model. All the variables were selected in accordance with the arguments presented by Ito and Sato (2008) and are very similar to those used by Mirdala (2009). These variables express economic linkage between inflation and internal and external macroeconomic factors. Oil price inflation repre-

[5] We have constructed other models which were extended by foreign gap variable (output gap of Germany) and by inclusion of consumer price index and producer prices index. The main results remained relatively stable. We will thus present the results for the basic model with consumer prices index only. Due to lack of historical import prices indices, we were unable to model a link of inflation transmission among various sectors of economy (i.e. importers, producers and consumers).

sents possible external supply shocks, demand shock effects are included in the output gap, and money supply allows us to capture the effects of monetary policy on domestic inflation.

Identified structural shocks are based on the Cholesky decomposition of the variance-covariance matrix Ω of the reduced-form VAR residuals. The link between the reduced-form VAR residuals (u_t) and the structural shocks (ε_t) can be written as

$$u_t = S\varepsilon_t \tag{2}$$

where $u_t = \left(u_t^{oil}, u_t^{gap}, u_t^{m}, u_t^{neer}, u_t^{cpi}\right)$, $\varepsilon_t = \left(\varepsilon_t^{oil}, \varepsilon_t^{gap}, \varepsilon_t^{m}, \varepsilon_t^{neer}, \varepsilon_t^{cpi}\right)$ and S is the lower-triangular matrix derived given the covariance matrix Ω. The Cholesky decomposition of Ω implies $\Omega = PP'$ with a lower triangular matrix P. Since $\Omega = E(u_t u_t') = SE(\varepsilon_r \varepsilon_r')S' = SS'$, where structural disturbances ε_t are considered to be orthonormal, the matrix S is equal to P. The structural model defined by the equation (2) is identified due to fact that lower-triangular matrix S imposes enough zero restrictions $(k(k-1)/2)$, where k denotes the number of endogenous variables. The lower-triangular matrix S implies that some structural shocks have no long-lasting effect on the endogenous variables. The ordering is thus important and is based on the economic intuition. For example, we assume that changes in oil prices are influenced only by the oil price (supply) shock ε_t^{oil} itself.

4 Estimation results

The lag order of the VAR model was selected using the Akaike information criterion (AIC). The constant term was included in our model specification that might be justified by the fact, that there was a significant (i.e. non-zero) growth in the monthly CPI. The reduced form VAR models for all V4 countries have been estimated using the complete data set starting from 1993M2 to 2007M12[6]. Structural shocks were computed from the VAR residuals.

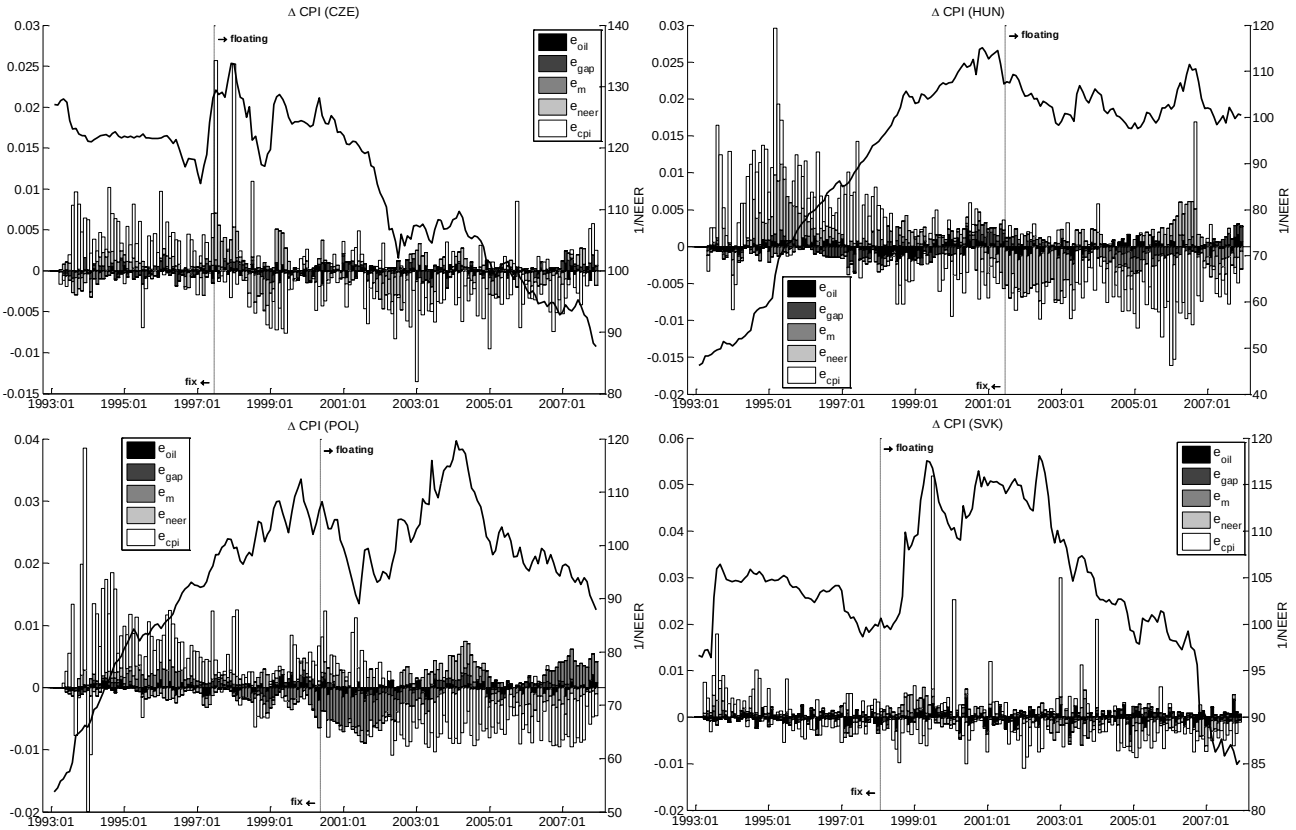

Figure 2 Historical shock decompositions – inflation

Figure 2 shows the historical shocks decomposition of monthly CPI inflation and the development of the NEER (indirect quotation where increase means depreciation). The corresponding structural shocks contribu-

[6] The first observation in 1993M1 was lost due to differencing procedure.

tions were computed using the identified and simulated SVAR model. Non-zero shocks were simulated with respect to the lag of reduced-form model (in 1993M4 for the Czech Republic and Slovakia or 1993M5 in case of Hungary and Poland). The subplots in Figure 2 contain a mark of the period of change (vertical line) of the exchange rate regimes. These time intervals are used in Table 2 to determine the relative importance of all shocks to the variance in inflation during the examined period. In particular, the ends of fixed exchange rate period are stated in each country as follow: 1997M5 for the Czech Republic, 2001M5 for Hungary, 2000M5 for Poland and 1998M10 for Slovakia. The remaining periods are considered as the regime of the floating exchange rate. We can notice in the following charts that in several periods exchange rate plays an important role in the explanation of inflation. An interesting period is for example spring of 1997 in the Czech Republic – after the currency crisis. The devaluation was causing inflation. Another interesting period is crawling peg in Hungary after 1995 that had an impact on the growth of the price level as well. On the contrary in Poland and Slovakia there were no periods with such obvious influence of the exchange rate.

Forecast error variance decomposition of inflation for all V4 countries is shown in Table 1. We should notice that the exchange rates were the most important inflation variables for all countries. But there are some obvious differences among the countries in terms of the extent of variability.

	ε^{oil}	ε^{gap}	ε^{m}	ε^{neer}	ε^{cpi}
CZE	0.0089	0.0224	0.0051	0.1467	0.8168
HUN	0.0151	0.0213	0.0190	0.1828	0.7618
POL	0.0085	0.0070	0.0507	0.0623	0.8715
SVK	0.0441	0.0061	0.0186	0.0439	0.8874

Table 1 Forecast error variance decomposition of inflation

Relative importance of structural shocks may be found in Table 2. The presented numbers are outputs of the regressions where dependent variable is the standardized inflation and the explaining variables are standardized structural shocks. The resulting R^2 is decomposed into individual contribution using the formula $R^2 = \sum_k b_k r_{yk}$ where b_k is the regression coefficient at the k^{th} structural shock (i.e. standardized coefficient in the regression with original variables) and r_{yk} is the correlation coefficient between the dependent variable (standardized inflation) and the k^{th} structural shock. All the computations were carried out for the periods of fixed exchange rate, floating exchange rate and the period of both regimes[7]. This approach allows us to distinguish the influence of exchange rate on inflation in periods mentioned previously.

	"1990s"				**"2000s"**				**"1990s" and "2000s"**			
	CZE	**HUN**	**POL**	**SVK**	**CZE**	**HUN**	**POL**	**SVK**	**CZE**	**HUN**	**POL**	**SVK**
	Fix	**Crawl peg**	**Crawl peg**	**Fix**	**Float**	**Fix with band**	**Float**	**Float**	**Both regimes**			
Till/from	1997	2001	2000	1998	1997	2001	2000	1998	1993 - 2007			
ε^{oil}	-0.21%	4.14%	0.58%	-1.08%	1.37%	-0.23%	5.26%	6.07%	1.04%	1.99%	1.63%	5.31%
ε^{gap}	7.68%	1.72%	3.55%	6.24%	3.33%	5.47%	8.33%	0.03%	2.82%	1.42%	1.92%	0.55%
ε^{m}	0.27%	-4.67%	16.81%	-1.36%	1.29%	1.76%	-5.52%	2.13%	0.67%	0.78%	11.76%	1.88%
ε^{neer}	11.46%	24.38%	-0.46%	1.31%	18.05%	15.51%	11.63%	3.10%	16.56%	23.83%	0.11%	3.09%
ε^{cpi}	80.80%	74.31%	67.38%	94.89%	75.95%	77.50%	80.31%	88.67%	78.92%	71.88%	78.25%	89.18%

Table 2 Explained historical variability of inflation

For the analysis of the results we divided our period as mentioned above. For easier recognition we roughly divided the period to the 1990s and the 2000s in Table 2. We can see the specific year of the switch of the system in the table. We consider periods of irregular devaluations in Poland and Hungary as part of the crawling peg system as well. Our conclusions are only to some extend in attune with our expectations. We generally supposed that the role of the exchange rate should be negative or limited in the periods of the fixed exchange rate

[7] We have omitted the months in 1993 due to the influence of initial zero-shocks conditions in historical simulation. Moreover, the negative R^2 are the results of insignificant influence of the shocks on inflation. It means that their explanatory power was simply negative due to the fact that the sum of individual R^2 has to be equal to the overall R^2.

regimes. But in the case of the Czech Republic, the model shows a reather strong impact of the exchange rate changes on the explanation of inflation. It can be caused by higher fluctuations when the Crown was moving in the bands. On the contrary, in Slovakia the impact was in accordance with our expectations. Secondly, we supposed that in the period of crawling pegs small devaluations should cause inflation and our conclusion for Hungary is fully in accordance with this expectation. On the contrary, in Poland general impact of the monetary aggregate was the main source of inflation and in our model, the exchange rate has even a negative impact.

Roughly with the turn of the century the exchange rate regimes changed. The impact of the exchange rate was in our view surprisingly important especially in the floating regimes in Poland and the Czech Republic; and in the Hungarian fixed exchange rate regime with a large band. On the contrary, in Slovakia the impact of the exchange rate was relatively small but it was significant. To be more specific, we have computed contribution of the nominal effective exchange rates to the year-on-year inflation in V4 countries. These computations are presented in Table 3. The periods are similar to those from Table 2. The column "Model" contains estimated intercept in the ΔCPI (i.e. inflation) equation of the SVAR model. This estimate corresponds to a part of the inflation which is not explained by the variability of the endogenous variables and may be this considered as the overall trend in price level development. We can see the inflationary impacts of the NEER changes in the 1990s and the mostly deflationary tendencies caused by the NEER changes in the 2000s.

	"1990s"		**"2000s"**		**"1990s and 2000s"**		**Model**
	ε^{neer}	ΔCPI	ε^{neer}	ΔCPI	ε^{neer}	ΔCPI	ΔCPI
CZE	0,47%	8,49%	-0,07%	3,65%	0,06%	4,81%	4,98%
HUN	1,29%	16,10%	-1,10%	5,46%	0,16%	10,97%	11,20%
POL	0,05%	16,09%	0,25%	2,81%	0,16%	8,62%	7,48%
SVK	0,15%	7,44%	-0,26%	6,42%	-0,12%	6,77%	7,18%

Table 3 Contribution of NEER changes to the average year to year CPI inflation

5 Conclusion

In our contribution, we have analyzed the relationship between exchange rates and inflation in the Central European countries. These countries applied different exchange rate strategies during the transformation process. We have found out that changes in exchange rates were a significant cause of inflation in some periods – for example the period of crawling peg in Hungary. On the hand, the impact was smaller than expected in other periods – for example Poland during the application of the same exchange rate system. Our conclusion is that exchange rate can play an important role in the development of inflation but the specific impact is highly individual.

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Construction of composite indicator based on factor analysis

Miroslav Žižka[1]

Abstract. The aim of this paper is to propose a methodology for determining the weights of the criteria and the subsequent construction of a composite indicator for the assessment of socio-economic status of municipalities. To determine the weights of sub-indicators entering the composite indicator the results of factor analysis were used. Weights of the indicators were set based on the proportion of variance explanation of a given indicator. Composite indicator was then created using the additive method of Weighted Sum Average. Sub-indicators include 27 variables whose criterion values were transformed to a scale from 0 to 1 according to the type of the variable, and then for each municipality value of the composite indicator was determined. Composite indicator serves as a comprehensive assessment of the state of the economic pillar of municipalities in processing spatial analytical data. The proposed methodology was then tested on a set of 6,249 municipalities in the Czech Republic. According to the values of the composite indicator, municipalities were divided into five groups, including mild to severe evaluation of the economic pillar. Due to the concentration of values of the composite indicator to a small interval, an irregular scale with uniform distribution segment of large frequency was applied.

Keywords: factor analysis, criteria weights, composite indicator, multicriteria decision making, weighted sum average.

JEL Classification: C38, R58
AMS Classification: 62H25

Introduction

Spatial analytical materials include three pillars of sustainable development – economic, social and environmental. Given the limited length of the paper, the text focuses on the economic pillar, which is obviously linked to the other two pillars. This is, among other things, suggested by indicators used to analyze the economic pillar. The aim of this paper is to design a composite indicator that can be used to assess the economic pillar of territorial analytical data of municipalities with extended powers. The evaluation of economic pillar is one of the typical problems of multi-criteria evaluation of alternatives, as the municipalities in the administrative district are seen from different angles. The original idea of the research was based on the assumption that it is not necessary to establish a single indicator synthesizing position of the municipality in various areas of concern, since each municipality does not have only weak factors, but also has strengths. However, the practice of spatial planning pointed to the need to create a composite indicator with which you can get broader perspective on the municipality with extended powers.

In practice, using composite indicators can be found, in particular, in evaluating regional disparities. For example, the team around A. Kutscherauer developed a set of integrated indicators for each sphere of regional disparities – social, economic and territorial – which they used to assess disparities between the regions of the Czech Republic (NUTS 3) [7]. The methodological approach to the evaluation of districts (LAU 1) from the perspective of rural areas using a composite indicator is specified in the article [3]. Villaverde et al. dealt with the problem of whether it is preferable to measure regional development and disparities by single or composite indicators. The results were tested on the EU regions at NUTS 2 level. They concluded that both approaches have their advantages and disadvantages and it cannot be generally determined which approach is better. A simple indicator of GDP per capita is suitable for measuring the degree of development. The development itself is a complex concept that requires quantification using a composite indicator [10].

To determine the composite indicator, certain problem areas should be solved. First of all, it is necessary to define the indicators used to assess the economic pillar (on this issue, see article [11]), further to set weights of individual indicators and choose the method of aggregation into a single composite indicator. Multicriterial assessment can be performed using various methods, see for example, article [2]. The present article deals mainly with the issue of determination of criteria weights. Scales indicators can be measured by different methods, such

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as the Czech Statistical Office extensive research [4] used the method of paired comparisons of indicators with the help of more than 80 experts.

Given that the contribution builds on previous research, whose main principle was to use factor analysis, not so frequently used method was utilized to determine the weights of the criteria. The method was based on the proportion of variance explained by the given factor. Setting the weights of sub-indicators uses the results of factor analysis already made. By means of this analysis seven factors that characterize the socio-economic level of municipalities have been identified. These were the factors: unemployment, domicile attractiveness, population density, age structure, civic amenities, branch structure and economic activity [11].

Factor analysis is used to construct composite indicators, particularly in the case of a large number of sub-indicators. In determining the weights, which each variable explains in various factors [1], the proportion of variance is taken into account. At the same time, each factor is weighted according to its contribution to the proportion of explained variance in the data file [9].

1 Methodology of composite indicator construction

Composite indicator is the indicator composed of several sub-indicators. The procedure of construction of a composite indicator involves a sequence of several steps according to [8]:

1. Theoretical framework.
2. Data selection.
3. Imputation of missing data.
4. Multivariate analysis.
5. Normalization.
6. Weighting and aggregation.
7. Uncertainty and sensitivity analysis.
8. Back to the data.
9. Links to other indicator.
10. Visualization of the results.

Due to a limited space and the fact that steps 1 to 5 have been carried out in the earlier stages of the project solution, the article focuses on step 6 that means setting the weights of the criteria and their aggregation.

The procedure can be divided into three basic stages. In the first stage, there were determined weights of sub-indicators, in the second stage, there were transformed criterial values of sub-indicators with respect to the direction of evaluation, and in the third stage, there was created a composite indicator.

The procedure for setting the weights of indicators included the following steps:

1. Obtaining a **factor loadings matrix** which was the result of factor analysis described in the article [11]. For the purposes of this article, we will mention the most important inputs and outputs of the factor analysis. All 6,249 municipalities in the Czech Republic were assessed using 27 indicators (see Table 1), which were reduced to 7 common factors (see above). Let us add that Varimax rotation was used. Factor loadings can be interpreted as correlation coefficients between the original variables and factors. They take values in the interval <-1, +1>. For this reason, their square is used to determine the weight.
2. Determination of **squared factor loadings** and their aggregation in each factor.
3. Calculation of **proportion of squares of factor loadings** on this sum. This proportion represents the share of the total variance, which is explained by the given factor [9].
4. **Weights of the individual indicators** were based on the proportion of variance explained by the given indicator across all extracted factors. Specifically, this means that the shares obtained according to step 3 are summarized across all factors.
5. **The sum of the weights** thus obtained is equal to the number of extracted factors.
6. If we require that the sum of the weights of indicators equals to one, a simple **normalization** is made: the weights specified in step 4 are divided by a number of factors.

In the second stage, a transformation of criterial values was carried out. This step seems to be necessary because of the type of indicators; maximization and minimization (see Table 1). Transformation procedure of the criteria values can be divided into the following steps:

1. **A linear utility function** of <0, 1> range was designed. This means that a municipality assessed according to certain indicators as the worst, received the benefit of 0 and vice versa, a municipality assessed according to certain indicators as the best obtained the value of 1. Other municipalities were then situated between the two extreme values.

2. **Transformed criterion values** for the parameters of a maximizing the type were obtained by equation (1), for indicators of a minimization by formula (2).

For the aggregation of sub-indicators and **creation of a composite indicator** simple additive method Weighted Sum Approach was used. This method works with ordinal variables [8]. The resulting indicator CI_i is based on the weighted sum of evaluation of individual municipalities. According to the resulting value CI_i, municipalities can be sorted in a descending order and thus determine their order in the economic pillar of territorial analytical data.

$$y_{ij}^{*} = \frac{y_{ij} - MIN(Y_j)}{MAX(Y_j) - MIN(Y_j)} \tag{1}$$

$$y_{ij}^{*} = \frac{MAX(Y_j) - y_{ij}}{MAX(Y_j) - MIN(Y_j)} \tag{2}$$

$$CI_i = \sum_{j=1}^{n} y_{ij}^{*} v_j \tag{3}$$

y_{ij}^{*} ... evaluation of a municipality *i* according to indicator *j*,

v_j ... weight of an indicator *j*.

2 Application of the proposed methodology

The above methodology was applied to a set of all municipalities in the Czech Republic. First, the weights of all 27 sub-indicators were determined (see the last column in Table 1). They are used to assess the economic pillar within the territorial analytical data. Data indicators for municipalities were obtained from the Czech Statistical Office and generally describe the reality of the year 2009, with certain exceptions, which are shown in Table 1 (abbreviation of each indicator has a year to which they relate in the last two positions). The values of sub-indicators were subsequently transformed to <0, 1> range, multiplied by weights and the final score of a municipality CI_i was calculated (see Table 2).

According to the values of composite indicator CI_i municipalities were sorted in descending order, i.e. from a top-rated municipality (in this case, the capital city of Prague). Due to a considerable extent of the data, only an excerpt is shown in Table 2 (selected indicators for the transformation and five best and worst rated municipalities). The final assessment of the municipalities was transferred to maps by means of GIS tools.

For a subsequent evaluation of the economic pillar in the practice, the most commonly used is two-stage (bad, good condition) or five-stage scale (poor, below average, average, above average, strong state), see [5]. The final assessment by the CI_i indicator should thus be classified using the above mentioned scales. Theoretically, the results can be divided evenly into intervals of the same width. The frequency chart (Figure 1) shows that 99% of the values are concentrated within the range of <0.3, 0.5>. The variability of the results is therefore very low. Concurrently, at the level of α = 5% we can reject the hypothesis according to Shapiro-Wilk test that the distribution follows a normal division. Therefore, it is preferable to use irregular scale with a uniform distribution of a section with large frequency [6]. This means that the segment of large frequency of <0.3, 0.5> is distributed evenly – due to the five-stage scale in three intervals – and the areas of the minimum occurrence of <0.2, 0.3) and <0.5, 0.7) are included in two separate intervals. The given method is commonly used in thematic cartography.

The first interval <0.20, 0.29), which can be interpreted as "poor rating" includes only 5 municipalities. In another interval <0.30, 0.37), there are located 727 municipalities whose economic pillar can be assessed as "unfavorable". Interval <0.37; 0.43) contains 4,544 municipalities with a neutral rating. Another interval <0.43; 0.50) covers 915 municipalities with a positive assessment of the economic pillar. Finally, the last interval <0.50, 0.70) includes 58 strong communities. Generally speaking, bigger municipalities have better evaluation, mainly due to the presence of amenities (schools, doctors).

Indicator	Type of indic.	Squared factor loadings							Weights before	Normalized weights
		F1	F2	F3	F4	F5	F6	F7		
PVEKC09	MIN	0.0096	0.3114	0.0044	0.2725	0.0077	0.0001	0.0142	0.3243	0.0463
ISTAR09	MIN	0.0000	0.0049	0.0008	0.2421	0.0058	0.0296	0.0036	0.1646	0.0235
IPMPR09	MIN	0.0010	0.7157	0.0001	0.0040	0.0159	0.0299	0.0005	0.3759	0.0537
IEZOB09	MIN	0.0001	0.0088	0.0014	0.5141	0.0029	0.0038	0.0001	0.3048	0.0435
OHUZA09	MAX	0.0000	0.0008	0.8724	0.0011	0.0001	0.0000	0.0006	0.6247	0.0892
SHUZA09	MAX	0.0007	0.0216	0.4665	0.0190	0.2601	0.0424	0.0059	0.4469	0.0638
HMROZ09	MIN	0.0008	0.0005	0.0012	0.0250	0.0172	0.1910	0.0137	0.1495	0.0214
MRNEZ09	MIN	0.8892	0.0002	0.0004	0.0029	0.0025	0.0062	0.0021	0.3410	0.0487
MDNEZ09	MIN	0.6906	0.0045	0.0004	0.0002	0.0000	0.0002	0.0048	0.2638	0.0377
MTLPP09	MIN	0.8299	0.0013	0.0002	0.0008	0.0015	0.0062	0.0013	0.3173	0.0453
PZAMO08	MAX	0.0004	0.0020	0.0000	0.0036	0.0001	0.0008	0.0086	0.0077	0.0011
ESPOP09	MAX	0.0216	0.0114	0.0000	0.0650	0.0002	0.0149	0.6512	0.3609	0.0516
SPODN09	MAX	0.0279	0.0100	0.0001	0.0538	0.0003	0.0010	0.6352	0.3399	0.0486
SUSLU09	MAX	0.0076	0.0188	0.0020	0.0014	0.0404	0.6037	0.0196	0.4228	0.0604
OPRVU09	MIN	0.0001	0.0045	0.0000	0.0018	0.0090	0.0462	0.0001	0.0353	0.0050
DBYTY09	MAX	0.0061	0.6496	0.0000	0.0017	0.0017	0.0123	0.0365	0.3460	0.0494
POBPL09	MAX	0.0346	0.1945	0.0006	0.0912	0.0930	0.0083	0.0000	0.1884	0.0269
KAHUZ09	MAX	0.0026	0.0010	0.0019	0.0384	0.0005	0.0021	0.5141	0.2633	0.0376
HOPEN09	MAX	0.0007	0.0040	0.0016	0.0353	0.2520	0.0007	0.1989	0.1756	0.0251
SUZEM09	MIN	0.0231	0.0296	0.0096	0.0449	0.0630	0.4160	0.0169	0.3473	0.0496
PRLEK09	MAX	0.0003	0.0000	0.0001	0.0002	0.8136	0.0055	0.0003	0.1968	0.0281
AMBUL09	MAX	0.0005	0.0014	0.0000	0.0003	0.7939	0.0050	0.0001	0.1925	0.0275
PFARM09	MAX	0.0000	0.0011	0.0010	0.0061	0.3376	0.0404	0.0058	0.1135	0.0162
VOLBY10	MAX	0.1129	0.0240	0.0015	0.1954	0.0671	0.0035	0.0228	0.1963	0.0280
MATSK09	MAX	0.0072	0.0104	0.0058	0.1122	0.3844	0.0045	0.0076	0.1741	0.0249
ZAKSK09	MAX	0.0000	0.0000	0.0025	0.0010	0.7362	0.0009	0.0004	0.1776	0.0254
STRSK05	MAX	0.0005	0.0139	0.0246	0.0018	0.3125	0.0729	0.0053	0.1492	0.0213
SUM		**2.6680**	**2.0460**	**1.3995**	**1.7358**	**4.2192**	**1.5483**	**2.1703**	**7.0000**	**1.0000**

Table 1 Squares of factor loadings and sub-indicators weights

Notes to Table 1: PVEKC09 - average age of living inhabitants in total, ISTAR09 - age index, IPMPR09 - intensity of natural and migration increment of inhabitants, IEZOB09 - economic loading on inhabitants index, OHUZA09 - general population density, SHUZA09 - specific population density, HMROZ09 - gross divorce rate, MRNEZ09 - registered unemployment rate, MDNEZ09 - long-term unemployment rate, MTLPP09 - job pressure rate, PZAMO08 - proportion of employees to the total number of inhabitants, ESPOP09 - proportion of active economic subjects to population aged between 15-64 years, SPODN09 - proportion of private entrepreneurs to population aged 15-64 years, SUSLU09 - proportion of active subjects in services, OPRVU09 - proportion of self-employed individuals to total number of economic subjects, DBYTY09 - number of completed apartments, POBPL09 - average floor space in one apartment, KAHUZ09 - capacity of mass accommodation facilities, HOPEN09 - proportion of beds in hotels and guest-houses to total mass accommodation facilities capacity, SUZEM09 - proportion of subjects active in agriculture and forestry, PRLEK09 - number of inhabitants per general practitioner's office for adults, children and youth, AMBUL09 - number of inhabitants per one outpatient medical facility, PFARM09 - number of chemists per 1,000 inhabitants, VOLBY10 - voter turnout, MATSK09 - number of inhabitants aged between 3-5 years per one nursery school, ZAKSK09 - number of inhabitants aged 6-14 years per one primary school, STRSK05 - number of inhabitants aged 15-19 years per one secondary school.

Code	Mun. Name	PVEKC09	ISTAR 09	IPMPR09	IEZOB09	OHUZA09	…	STRSK05	CI_i
554782	Praha	0.5806	0.9383	0.7521	0.8458	0.9742	…	0.3261	**0.6004**
555088	Havířov	0.5922	0.9407	0.7999	0.8334	1.0000	…	0.4220	**0.5873**
562971	Chomutov	0.6187	0.9539	0.7918	0.8473	0.6586	…	0.3259	**0.5468**
560383	Chodov	0.6513	0.9614	0.7981	0.8577	0.3838	…	0.3435	**0.5458**
567442	Teplice	0.5998	0.9470	0.7875	0.8298	0.8335	…	0.2299	**0.5425**
…									
551881	Sl. Pavlovice	0.6123	0.9524	0.7290	0.8027	0.0111	…	0.0000	**0.2995**
548871	Petrovičky	0.6477	0.9603	0.7469	0.5283	0.0087	…	0.0000	**0.2970**
592935	Březina	0.0545	1.0000	0.6169	0.0000	0.0000	…	0.0000	**0.2779**
595152	Zálesí	0.4784	0.8225	0.8211	0.8296	0.0093	…	0.0000	**0.2769**
553191	Vikantice	0.5596	0.9116	0.7996	0.9035	0.0034	…	0.0000	**0.2749**

Table 2 Example of final assessment of municipalities

Conclusion

The presented methodology of evaluating the economic pillar has a relatively wide use in the regional development. It can be used for the comparison of communities at different territorial levels: national (as presented in this article), regional (e.g. for the quantification of regional disparities) or in the administrative district of the municipality with extended powers (e.g. in processing territorial analytical data). It can also be used when comparing functionally defined sub-regions. For this purpose, there are two ways how to proceed. Either the evaluation of municipalities can be compared nationwide (the position of a municipality in the national scale, see Table 2) or methodology specified in Section 1 can be modified in the sense that it evaluates only a municipality in the given administrative district (the status of a municipality in the region). This means that the range of linear utility <0, 1> applies only to the municipalities in the given administrative district. The latter procedure should be followed when the characteristics of municipalities in the region are very similar. Finer differentiation results can be achieved by this method since a municipality assessed as average within the country may be assessed as a community with a stronger position in economically weak regions.

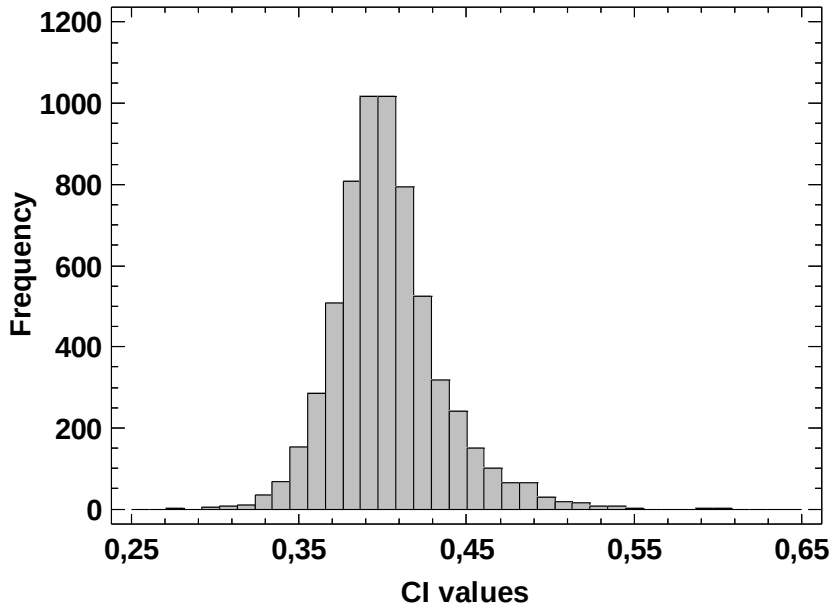

Figure 1 Histogram of CI values

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